Control of a Water Distribution Network

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***Abstract*—Around the world water distribution networks have a key role in cities' and industrial areas' infrastructure. Problems that arise in these networks include pipe leakages and excess energy consumption. A possible solution to lessen the effect of these issues is to control the pressure in the network. Pressure control in networks is also vital to ensure an end-users level of comfort. In general, water distribution networks have actuators, sensors and control systems that are geographically separated, thus, it is necessary to communicate measurements over a network. Networking issues, such as delays and packet losses may occur due to this. Since the network contains nonlinear components it needs to be linearized as the chosen control method is linear control. The linearized water distribution network is then modelled by applying electrical circuit theory to the network. Delays due to the communication system that are present are then included in the model. Integral action is used to extend the model in order to track a reference. A Linear Quadratic Regulator (LQR) controller is designed using state-space methods. The controller is then tested on the physical system to ensure it can track and maintain the necessary end-users pressure.**

Keywords— Water Distribution Networks; State-space; LQR Control; Linearization

I. INTRODUCTION

In developing cities and industrial areas around the world, water distribution networks play a key role in infrastructure. One of the important factors is to control the pressure in a water distribution network, such that all consumers have a satisfying water flow in their homes. Additional users in a network, leakages in pipes and many other reasons can cause pressure drops within the distribution network. On the other hand, having high pressure can also cause problems in pipes according to Barlow's formula [1]. Thus it is vital to have the pressure in the whole distribution network between a maximum and minimum allowed pressure. In any different network this maximum and minimum can be changed according to consumer's need and components available in that network.

The purpose of this paper is to design and implement a linear control strategy for controlling pressure in a water distribution network about a desired operating pressure. Improved pressure control in water distribution network can contribute to saving water by decreasing water leakage and the number of pipe bursts [2]. The chosen controller is a Linear Quadratic Regulator (LQR) controller.

The system used in this paper consists of two pressure management areas (PMAs) and the pressures in the system are regulated using three pumps. Each PMA has two outlets, each modelling a consumer district. The water distribution network has nonlinear characteristics [3].

In Section II, the water distribution network is explained, linearized and then the state-space model is derived. In Section III, the parameters of the system model are estimated using experimental data from the physical system. In Section IV, the delays present in the network are modelled. In Section V, the order of the model is reduced due to numerically unstable states. In Section VI, integral action is added by extending the state-space model. In Section VII, the LQR controller for the system is designed. The results obtained from network after implementing the controller is shown in Section VIII. These results are discussed in Section IX. The paper is concluded in Section X.

II. SYSTEM MODEL

A. Water Distribution Network

A water distribution network is a hydraulic network comprised of many two terminal components such as pipes, pumps and valves. The objective of the network is to distribute water from a source, to various outputs in PMAs while maintaining a desirable pressure at these outputs. The network may have multiple water sources, height differences within the network and multiple pumping stations. To analyze such a network, the various system components need to be modelled.

The network components are characterized by algebraic or dynamic relationships between two variables, the pressure drop, Δp, across a component, and the flow, q, flowing through that component [4].

1) Pipe Model: A large portion of the network comprises of pipes. Thus, it is important to derive a dynamic model that accurately describes the relation between the differential pressure across the pipe and the flow through it. The compact notation for the pipe model which can be applied to the k^{th} pipe component in the hydraulic network is expressed as

$$J_k \dot{q}_k = \Delta p_k - H_k |q_k| q_k - \zeta_k \tag{1}$$

In eq. (1) J_k represents the mass inertia of the water in the k^{th} pipe and is derived using Newton's Second Law, where water is assumed incompressible. It is calculated using

$$J_k = \frac{L_k \rho}{A_k} \tag{2}$$

where L_k is the length of the pipe, ρ is the density of water and A_k is the cross-sectional area of the pipe. Also in eq. (1), Δp_k is the pressure drop across the pipe and H_k represents the resistance in the pipe which includes both the surface resistance and the form resistance. These resistances can be calculated using their respective equations from [5]. The ζ_k term, represents the pressure difference due to the change in height.

Lastly, the units used to express the flow [m^3/s] and pressure [Pa] are converted to [m^3/h] and [Bar], respectively. These are the chosen units used throughout the paper.

2) Valve Model: The valves used in this paper have a variable opening degree (OD) and the pressure loss across a valve can be described as a function of its OD.

The valve model is derived similarly to the pipe model. The valve can be modelled as a pipe with its length, L_k, and height difference, ζ_k, assumed as zero, as shown in eq. (3).

$$0 = \Delta p_k - H_k |q_k| q_k \tag{3}$$

While it is possible to estimate the H_k parameter for the k^{th} valve, the valve's manufacturer provides a more accurate k_v parameter in the datasheets, which relates the flow through the valve to its pressure drop, as shown in eq. (4) [6].

$$q = k_v \sqrt{\Delta p} \tag{4}$$

The k_v parameter specifies the water flow, in m^3/h, through the valve at a pressure drop across the valve of 1 Bar [6]. This relation is given for a fully open valve.

3) Pump Model: In a hydraulic network, a pump creates flow by providing a positive pressure difference. A model that describes this positive pressure difference is given in eq. (5) and derived in [7].

$$\Delta p = a_{h0}\omega^2 + a_{h1} q \omega - a_{h2} q^2 \tag{5}$$

The differential pressure is given as a function of the flow, q, through the pump and the rotational speed, ω, of the pump. The rotational speed, ω, was normalized to a value between [0,1]. The pump parameters a_{h0}, a_{h1} and a_{h2} are found experimentally.

The diagram symbols for the kth pipe, valve and pump shows the pressure difference, $\Delta p = p_{in} - p_{out}$, across it and the flow, q, through it is given in Figure 1.

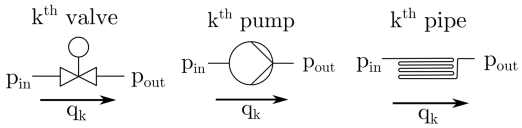

Figure 1: Pipe, valve and pump diagram symbol [8]

B. Test Network System

The diagram in Figure 2 shows the test setup system that was used. For this paper, datum 1, 2 and 5 were used, where datum 1 and 2 are the PMAs.

The outlets of the system are the valves C_{20}, C_{24}, C_{27} and C_{31}. The system is modelled with these valves fully open. The chosen input pumps of the system are C_2, C_{18} and C_{25}. The output pressures are measured using pressure sensors located before C_{24} and C_{31}.

Manual valves, C_{15}, C_{40}, and another, not labelled in the figure, that is connected to pump C_{32}, were closed. This is done such that no flow from C_{39} and pumps C_{16} and C_{32} are included in the model.

C. Linearization

The nonlinear components of the water distribution network need to be linearized in order to design a linear controller for the system. The pipe, valve and pipe models described in eq. (1), eq. (3) and eq. (5) are linearized using the Taylor Series approximation to the first order.

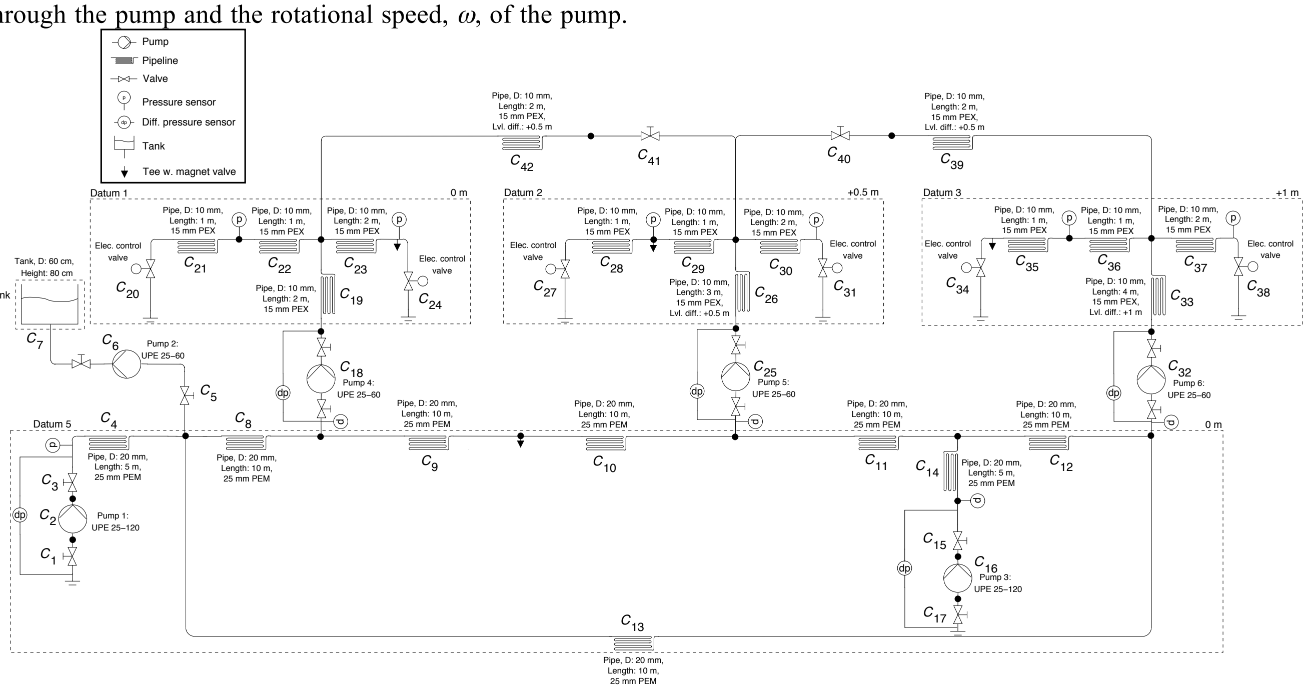

Figure 2: Diagram of the Water Distribution Network Test Setup

The method used is shown by considering a first order, nonlinear differential model, be given by

$$\dot{x} = f(x) + u \tag{6}$$

where $f(x)$ is a nonlinear function of x, and u is an additional term independent of x. To linearize eq. (6) at a chosen operating point $\{\bar{x}, \bar{u}\}$, the Taylor Series Expansion is used to describe $f(x)$ to the first derivative, such that values $f(x)$ can be approximated in the neighborhood of $\bar{x}$ as shown in eq. (7)

$$f(x) \approx f(\overline{x}) + f'(\overline{x})(x - \overline{x}) \tag{7}$$

At the steady-state operating point, the differential model is in the form

$$0 = f(\overline{x}) + \overline{u} \tag{8}$$

$$f(\overline{x}) = -\overline{u} \tag{9}$$

Using a linearized model about a given operating point, only the small deviations around the operating point are considered. Thus, x and u can be rewritten in a small signal form such that $x = \bar{x} + \tilde{x}$ and $u = \bar{u} + \tilde{u}$, where $\tilde{x}$ and $\tilde{u}$ are the small deviations around the operating point. Substituting these into the differential model, the following linearized model is achieved

$$\dot{\tilde{x}} \approx f(\overline{x}) + f'(\overline{x})\tilde{x} - f(\overline{x}) + \tilde{u} \tag{10}$$

$$\dot{\tilde{x}} \approx f'(\overline{x})\tilde{x} + \tilde{u} \tag{11}$$

The linearized model for the nonlinear resistance of the k_{th} pipe is given by

$$K_k = 2H_k|\bar{q}_k| \tag{12}$$

The chosen operating point for the system is such that the rotational speeds of the pumps are around 0.5. This allows for input speed to be varied in value and for there to be a flow through pipe C_{42} that is not only a pressure difference due to the height difference. With these characteristics, the operating point is chosen with the input speeds to the pump shown in Table I. The output pressures measured at the operating point are given in Table II.

Input Pump	Rotational Speed [ω]
C_2	0.54
C_{18}	0.52
C_{25}	0.5

Table I: Input Speed to pumps at operating point

Output Pressures	Pressure [Bar]
C_{24}	0.1254
C_{31}	0.921

Table II: Output pressures at operating point

D. Water Distribution Network Model

To model the network, the linearized equations for the pipe, valve and pump are used. These component equations describe a relationship between the flows through the components and the differential pressures across the components.

The method used to analyze the network is Kirchhoff's Circuit Laws. Kirchhoff's Circuit Laws can be used on hydraulic networks by replacing voltage and currents with pressures and flows, respectively. Kirchhoff's Current Law is used to describe continuity of nodal flow and Kirchhoff's Voltage Law is used to represent that the energy in a closed loop is conserved [9]. A pipe can be seen as an inductor, resistor and voltage source in series; as shown in eq. (1), the voltage source is equivalent to the constant pressure difference due to the height difference. Similarly, valves are analogous to resistors, as shown in eq. (3).

The analogy between electrical circuits and hydraulic circuits is illustrated in the example using Figure 3 and Figure 4. For the given hydraulic network in Figure 3, C_k represents the k^{th} component in the network. The hydraulic network can be translated to the electrical circuit shown in Figure 4.

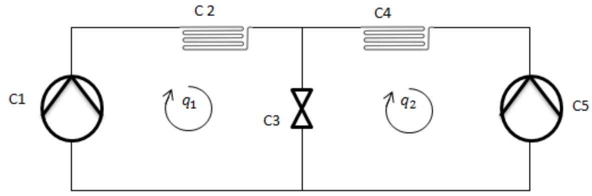

Figure 3: Two loop hydraulic network

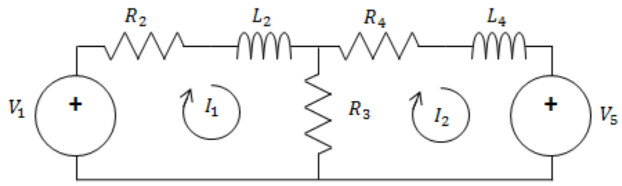

Figure 4: Two loop electrical circuit

In the two loop electrical circuit shown in Figure 4. KCL is applied and the following equations are derived

For the first loop,

$$V_1 - I_1R_2 - L_2\frac{dI_1}{dt} - (I_1 - I_2)R_3 = 0 \tag{13}$$

and for the second loop

$$-V_5 - I_2R_4 - L_4\frac{dI_2}{dt} - (I_2 - I_1)R_3 = 0 \tag{14}$$

In matrix form,

$$\begin{bmatrix} L_2 & 0 \\ 0 & L_4 \end{bmatrix}\begin{bmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{bmatrix} = -\begin{bmatrix} R_2 + R_3 & -R_3 \\ -R_3 & R_4 + R_3 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} V_1 \\ -V_5 \end{bmatrix}$$

This matrix can be translated back to the hydraulic network using the linearized component models. Thus, the equivalent model for Figure 3 is given as

$$\begin{bmatrix} J_2 & 0 \\ 0 & J_4 \end{bmatrix}\begin{bmatrix} \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} = -\begin{bmatrix} K_2 + K_3 & -R_3 \\ -K_3 & K_4 + K_3 \end{bmatrix}\begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} + \begin{bmatrix} \widetilde{\Delta p}_1 \\ -\widetilde{\Delta p}_5 \end{bmatrix}$$

E. State-Space Representation

The system equations derived from the loop equations can be expressed in the following form:

$$J\dot{\tilde{q}} = -K\tilde{q} + G\tilde{u} \tag{15}$$

Such that J matrix represents the inertia in the system and the K matrix represents the damping. Both J and K are positive definite and symmetric matrices. This is a property of using Kirchhoff's Circuit Laws [10].

To translate the system dynamics equation to state-space form, the J matrix must be invertible, therefore nonsingular. For a properly designed system this condition will be met as described in [11]. Thus, the state-space matrices are given by

$$A = -J^{-1}K; B = J^{-1}G; \quad (16)$$

$$C = -K_v; D = 0 \quad (17)$$

where $-K_v$ is the resistance terms across the output valves, C_{24} and C_{31}. Thus, the state-space representation of the system is

$$\dot{\tilde{q}} = A\tilde{q} + B\tilde{u} \quad (18)$$

$$\tilde{y} = C\tilde{q} \quad (19)$$

III. Parmeter Estimation

The state-space model derived for the system is not complete until all the parameters of the components are well-stated. Model parameters that have great uncertainty in its' values may need to be estimated using parameter estimation techniques.

To fit the state-space model to the physical systems response, the MATLAB System Identification Toolbox was used. Since the pipe lengths and the cross-sectional area of the pipes are well defined, the mass inertia of water matrix, J, is considered to be modelled sufficiently well. The K matrix is not sufficiently modelled as it is defined by the number, and characteristics of pipe bends as well as the roughness of the pipes. These values have greater uncertainties. As J is known, the remaining parameters can be identified from, the steady-state model given as

$$0 = -K\tilde{q} + G\tilde{u} \quad (20)$$

To estimate the correct K parameters, the system model was treated as a Grey-Box Model and the Gauss-Newton Least Squares method was used.

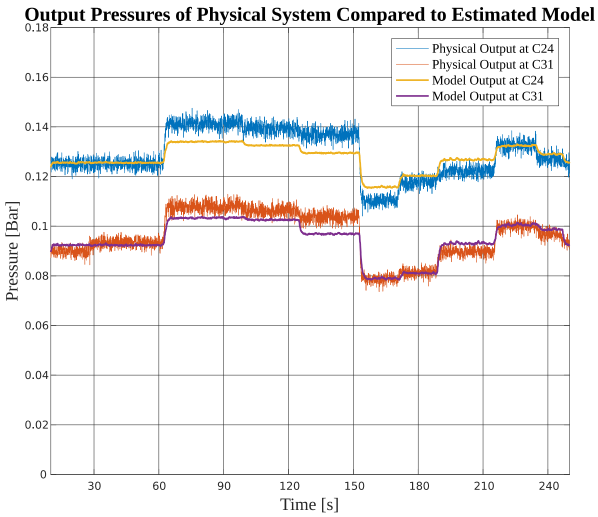

Figure 5: Output pressures of physical system compared to the state-space model

In Figure 5 the system's output is compared with the linearized model's output. Between 0 [s] and 60 [s], the operating point is given to both the physical system and state-space model. During this interval the model accurately tracks the system. From 60 [s] to 150 [s], a visible error can be seen between the model and physical system. This error is characteristic of the model as it represents the difference between the linear model, around the operating point, and the nonlinear system. After 150 [s], the model tracks the physical system adequately well with marginal errors shown between 190 [s] and 215 [s].

IV. Networking & Delay

The system used in this paper is a network control system where the controller obtains measurements from the sensors and sends data to the pumps. This introduces delays from the sensors to the controller and from the controller to the actuators. Delays due to internal signal processing in the actuators are significantly higher compared to others. The measured delay, T_d, in the communication between the control unit and the pump of the system is 0.7 [s].

Padè's Approximation is used to obtain a transfer function of the delay, such that the delayed input to the system is given by the equation

$$Z(s) = \frac{1 - \frac{T_d}{2}s}{1 + \frac{T_d}{2}s}U(s) \quad (21)$$

where Z(s) and U(s) are actuators signals with and without delay in the Laplace domain. Padè's approximation is used to linearize the delay function, hence, the first order approximation is used.

The delay is transformed into a state-space model in order to extend the original state-space system. By defining $x = z + u$,

$$\dot{x} = -\frac{2}{T_d}x + \frac{4}{T_d}u \quad (22)$$

$$y_c = x - u \quad (23)$$

gives the individual state-space equations for each pump connected to the system

The delayed model of the three pumps therefore has a state-space model with the matrices

$$A_c = -\frac{2}{T_d}I_3 \;\; B_c = \frac{4}{T_d}I_3 \,; C_c = I_3$$

$$D_c = -I_3$$

Thus, the extended state-space model has the matrices

$$A_d = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}; B_d = \begin{bmatrix} BD_c \\ B_c \end{bmatrix}; C_d = [C \; 0]$$

and the extended state-space model has the form

$$\dot{\tilde{x}}_d = A_d\tilde{x}_d + B_d\tilde{u}_d \quad (24)$$

$$\tilde{y}_d = C_d\tilde{x}_d \quad (25)$$

V. Model Order Reduction

To design a state feedback controller for the system, the system needs to be both controllable and observable. The system is considered controllable if the controllability matrix

$$\mathcal{C} = [B_d \; A_dB_d \; \ldots \; A_d^{n-1}B_d] \quad (26)$$

has full rank. The states of the system are observable if the observability matrix

$$O = [C_d \; C_d A_d \; ... \; C A_d^{n-1}]^T \tag{27}$$

has full rank.

To test the state-space model in eq. (24) and (25) for controllability and observability, the controllability and observability matrices were created using eq. (26) and (27).

The condition number of the controllability and observability matrices showed the matrices are ill-conditioned, thus, it is determined that the system is close to being uncontrollable. The method used to resolve this issue is to reduce the model by performing a balanced truncation on it [12]. The state-space model in eq. (24) and (25) containing nine states was reduced to a state-space

$$\dot{\tilde{x}}_r = A_r \tilde{x}_r + B_r \tilde{u}_r \tag{28}$$

$$\tilde{y}_r = C_r \tilde{x}_r \tag{29}$$

model with six states. The reduced system introduced a *D* state-space matrix with extremely low values, and is neglected.

The reduced order model is both controllable and observable. To validate the reduced order model, a unit step was given to the both systems and the responses are compared as shown in Figure 6.

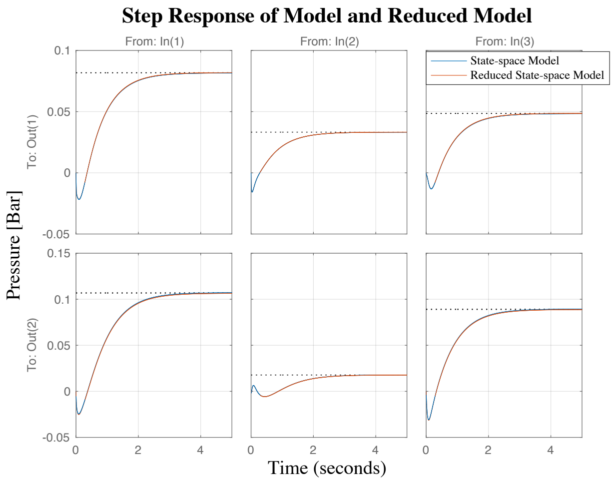

Figure 6: Step response comparing the state-space model with the reduced state-space model

From Figure 6 it can be seen that the step response of the reduced order model has an almost identical response to the step response of the system model. Therefore, the reduced order model is satisfactory to be used to design a controller.

VI. Intergral Action

In order to track a reference an additional integral term is added to the system, to provide a zero steady-state error. By adding integral feedback, the state-space model in eq. (28) and (29) is further extended; similarly to Section IV. The are two additional states integral states, one for each output. The integral states

$$\dot{\tilde{x}}_i = \tilde{y}_r - \tilde{r} = C_r \tilde{x}_r - \tilde{r} \tag{30}$$

represents the error between the reference signals and the outputs. The extended state equation is given in eq. (31)

$$\begin{bmatrix} \dot{\tilde{x}}_r \\ \dot{\tilde{x}}_i \end{bmatrix} = \begin{bmatrix} A_r & 0 \\ C_r & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_r \\ \tilde{x}_i \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} \tilde{u}_r + \begin{bmatrix} 0 \\ -I \end{bmatrix} \tilde{r} \tag{31}$$

Using the matrices as

$$\dot{\tilde{x}}_I = \begin{bmatrix} \dot{\tilde{x}}_r \\ \dot{\tilde{x}}_i \end{bmatrix}; A_I = \begin{bmatrix} A_r & 0 \\ C_r & 0 \end{bmatrix}; B_I = \begin{bmatrix} B_r \\ 0 \end{bmatrix}; C_I = [C_r \; 0]$$

the extended state-space model, with an integral term, has the form

$$\dot{\tilde{x}}_I = A_I \tilde{x}_I + B_I \tilde{u}_I + \begin{bmatrix} 0 \\ -I \end{bmatrix} \tilde{r} \tag{32}$$

$$\tilde{y}_I = C_I \tilde{x}_I \tag{33}$$

VII. Controller Design

The control method chosen for the state-space system is state feedback control. For the state-space system shown in eq. (32) and eq. (33) the control law is given by

$$\tilde{u}_I = F \tilde{x}_I \tag{34}$$

where $F = [F_s \; F_I]$ is the feedback gain matrix of the controller. F_s is the state feedback gain matrix and F_I is the integral feedback gain matrix.

A. LQR Control

The chosen state feedback control method is an LQR controller. The LQR method is designed such that the gains for a state feedback controller are chosen to optimize the cost function in eq. (35). This is done to balance system performance and the magnitude of inputs required to achieve a certain level of performance [13].

$$\mathcal{J} = \int_0^\infty \tilde{x}_I^T Q_x \tilde{x}_I + \tilde{u}_I^T Q_u \tilde{u}_I \, dt \tag{35}$$

where Q_x is a symmetric, positive semi-definite weight matrix and Q_u is a symmetric, positive definite weight matrix. The choice for the weight matrices determines the trade-off between the states and the control inputs [13].

Bryson's Rule is used to determine the cost function weights Q_x and Q_u [14]. The diagonal elements of the Q_x and Q_u matrices are chosen such that

$$Q_{x_{ii}} = \frac{1}{(maximum\ acceptable\ number\ of\ x_{ii})^2}$$

$$Q_{u_{ii}} = \frac{1}{(maximum\ acceptable\ number\ of\ u_{ii})^2}$$

The maximum acceptable number chosen was the values of the system's states and the inputs, at the operating point at which the system was linearized. The integral states, from Section VI, were given low weights as these states should not be penalized compared to the system's states.

For an LQR controller, the gain *F* is given by

$$F = Q_u^{-1} B_I^T P \tag{36}$$

where P is symmetric, positive definite matrix, with the same dimensions as A. The matrix, P, satisfies the *algebraic Riccati equation*

$$PA_I + A_I^T P - PB_I Q_u^{-1} B_I^T P + Q_x = 0 \qquad (37)$$

Once Q_x and Q_u are designed and the *algebraic Riccati equation* is solved and the outcome is P. Then the gain matrix, F, is calculated using eq. (36).

The system's closed loop response may be tuned by making adjustments to the Q_x and Q_u weight matrices.

B. Observer Design

The state feedback controller, F_s, designed in Section VII.A is only realizable if the states of the system can be measured either directly from the outputs or through an observer. As the system's states are not directly measurable from the outputs an observer is designed. The observer shall recover the system's states from the inputs and measured outputs of the system.

The observer matrix, L, is designed by placing the eigenvalues of the matrix $A_r + LC_r$. The L matrix is only stable if all the eigenvalues of $A_r + LC_r$ have negative real parts. The state equation of the observer takes the form

$$\dot{\hat{\tilde{x}}}_r = A_r\hat{\tilde{x}}_r + B_r\tilde{u}_r + L(C_r\hat{\tilde{x}}_r - \tilde{y}_r) \qquad (35)$$

where $\hat{\tilde{x}}_r$ are the estimated states of modeled system.

VIII. Results

The controller designed in Section VII is tested using the physical system. The physical system is a discrete-time system with a sampling time of 0.05 [s]. The controller has been discretized using the zero-order hold method. The physical system is controlled using MATLAB Real-time Simulink and the controller has been implemented in it as well.

The reference signal to the system has been stepped and the system's response is shown in Figure 7.

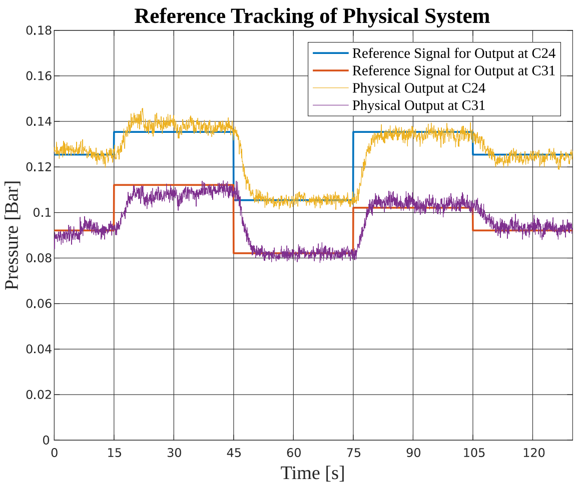

Figure 7: Reference Tracking of the Physical System

IX. Discussion

In Figure 7, the physical system's response to the reference signal is shown. It can be seen that system's outputs are able to track the reference signals accurately. Between 15 [s] and 45 [s] the outputs have minor errors. This inaccuracy is due to the reference signals being further from the operating point.

X. Conclusion

The components of the water distribution network have been modelled to show a relation between differential pressure and water flow. These models were linearized at the desired operating point. The network was modelled using electrical circuit theory and described in state-space form. The parameters with high uncertainties were estimated using a parameter estimation technique. The state-space model was extended due to delays in the communication system. The model was reduced using the balanced truncation method because the system is almost uncontrollable and unobservable. The state-space model was extended further to include integral action. A state feedback controller was designed using LQR and an observer was designed to estimate the system's states. The controller was tested on the physical system and the results show that the system can successfully track and maintain a reference signal in the neighborhood of it's operating point.

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