Pressure Regulation in Nonlinear Hydraulic Networks by Positive and Quantized Controls

Claudio De Persis and Carsten Skovmose Kallesøe

Abstract—**We investigate an industrial case study of a system distributed over a network, namely, a large-scale hydraulic network which underlies a district heating system. The network comprises an arbitrarily large number of components (valves, pipes, and pumps). After introducing the model for this class of networks, we show how to achieve semiglobal practical pressure regulation at designated points of the network by proportional control laws which use local information only. In the analysis, the presence of positivity constraints on the actuators (centrifugal pumps) is explicitly taken into account. Furthermore, motivated by the need of transmitting the values taken by the control laws to the pumps of the network in order to distribute the control effort, we study the pressure regulation problem using quantized controllers. The findings are supported by experimental results.**

Index Terms—**Circuit theory, hydraulic networks, nonlinear control, positive control, quantized control, robust control.**

I. Introduction

WE STUDY an industrial system distributed over a network, namely, a large-scale hydraulic network which underlies a district heating system with an arbitrary number of end-users. The problem consists of regulating the pressure at the end-users to a constant value despite the unknown demands of the users themselves. Since the focus is on a real industrial system, we are interested in controllers which can be easily implementable. The regulation problem is addressed for what is expected to be the next generation of district heating systems, where multiple pumps are distributed across the network at the end-users. In these new large-scale heating systems, the diameter of the pipes is decreased in order to reduce heat dispersion. The reduced diameter of the pipes increases the pressure losses which must be compensated by a larger pump effort. The latter can be achieved only with the multipump architecture [4]. Besides the reduced heat losses, having multiple pumps distributed across the network makes it robust to the failure of one or more pumps. However, this issue is not considered in the paper. Moreover, we do not take into account the problem of damping fast pressure transients due to water hammering, as this problem is not to be handled by our controller, but by well-placed passive dampers in the network. Preliminary results on the case study have appeared in [9], [11], and [10].

There is a large number of works devoted to large-scale hydraulic networks and more in particular to water supply systems. A recent paper with an extended bibliography on the modeling and control of hydraulic networks is [5], in which the emphasis is on "open" hydraulic networks, and modeling and control techniques essentially deal with linear systems. By open hydraulic networks, we mean networks whose topology is described by a tree and hence with no cycles (see Section II-C). These hydraulic networks are typically found in irrigation channels, sewer networks, and water distribution systems. Papers which deal with various control problems for open hydraulic networks include [25], [26], [34], and [33] and references therein.

In our application, however, the network has cycles. Similar networks and models arise for instance in mine ventilation networks and cardiovascular systems. These classes of systems are the motivation for the works [16], [22], and [23], where nonlinear adaptive controllers are proposed to deal with the presence of uncertain parameters. Large-scale ventilation systems are also considered in [35], where the use of a wireless sensor network is discussed. Other systems close to the one considered here are nonlinear *RLC* circuits (see, e.g., [19] and references therein).

In this paper, we derive the dynamic model for a general class of hydraulic networks with an arbitrary number of end-users. The precise expression of the constitutive laws of the components of the network is unknown and therefore the model is largely uncertain. Moreover, since the actuators are centrifugal pumps which can provide only a *positive* pressure, positivity constraints on the control laws must be taken into account. Relying on recent robust control design techniques for nonlinear systems [31], [17], [30], we design positive proportional controllers which guarantee semiglobal *practical* regulation.

Finally, we face an even more challenging control problem. For a correct implementation of the control laws, each controller, which is located at the end-user and which computes the control law based only on local information, is required to transmit the control values to "neighbor" pumps, i.e., auxiliary pumps which are found along the same circuit where the end-user lies. Due to physical constraints and the large-scale nature of the system, it is convenient to transmit information "sporadically." This motivated us to investigate the possibility to achieve the previous control objective (pressure regulation) by quantized controllers [24], [15], [6], [8]. These controllers take values

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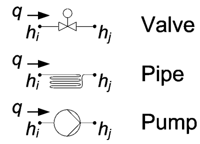

Fig. 1. Valve, pipe, pump, and their terminal points.

in a finite set (and therefore control values can be transmitted over a finite-bandwidth communication channel) and change their values only when certain boundaries in the state space are crossed. Controllers motivated by a similar need of being implemented in an industrial networked environment have been investigated in [33], as a result of an optimal control problem, and in [10], where binary controllers were employed. Our results are validated through experiments in a laboratory district heating system.

In Section II, the class of hydraulic networks of interest in this paper is introduced and the model is derived. In Sections III and IV, two different control strategies (positive proportional and quantized) are analyzed. Experimental results are discussed in Section V. Conclusions are drawn in Section VI.

II. MODEL

A. Hydraulic Networks

Hydraulic networks are connections of two-terminal components such as valves, pipes, and pumps (the symbols for valves, pipes, and pumps are depicted in Fig. 1). These components are characterized by algebraic or dynamic relationships between two variables, the pressure drop $\Delta h = h_i - h_j$ across the element, and the flow q flowing through the component. These relationships are introduced below.

1) Valves: The valves are normally viewed as pipe fittings. They can be modeled by a relationship between the pressure drop across the valve and the flow through it [27], that is,

$$h_i - h_j = \mu_k(K_{vk}, q_k) \tag{1}$$

where $h_i - h_j$ is the pressure across the terminals of the valve, q_k is the flow through the valve, and K_{vk} is a variable denoting the change of hydraulic resistance of the valve. Moreover, μ_k is supposed to be a continuously differentiable function which is strictly *monotonically increasing* and satisfies $\mu_k(K_{vk}, 0) = 0$ for all K_{vk}. In what follows, it will be useful to distinguish between valves in which the hydraulic resistance remains constant for all the times, and those in which K_{vk} ranges over a compact set of values. We shall refer to the latter valves as user-operated or *end-user valves*.

2) Pipe: The relationship describing the pipe is derived using the control volume approach [27]. If the fluid is assumed to be incompressible and the diameter of the pipe is constant along the pipe, the model for the kth pipe is

$$J_k \frac{dq_k}{dt} = (h_i - h_j) - \lambda_k(K_{pk}, q_k) \tag{2}$$

where $h_i - h_j$ is the pressure difference between the inlet and the outlet of the pipe, and q_k is the flow through the pipe. The function λ_k describes the pressure losses inside the pipe, which depend on the flow q_k and of the loss factor K_{pk}. The loss factor is a function of the friction factor and the dimensions of the pipe. The constant J_k depends on the mass density of the fluid and the pipe dimensions [27]. Finally, λ_k is a continuously differentiable function of its arguments and is strictly increasing in q_k with $\lambda_k(K_{pk}, 0) = 0$ for each $K_{pk} > 0$. Hence, it has the same properties as μ_k.

3) Pump Model: Models for pumps are derived in [20] and [21]. In this paper, we regard the pump as a device which is able to deliver a desired pressure difference $h_j - h_i$. Let the pressure delivered by the pump be denoted as

$$h_i - h_j = -\Delta h_{pk}.$$

The pressure difference Δh_{pk} delivered by the pump is viewed as a *control input* (see Section II-B).

It is important to stress that only the properties of being monotonically increasing and zero at the origin are known about the functions μ_k, λ_k. In particular, the precise expression taken by these functions is unknown. In addition, the values of the hydraulic resistance K_{vk} and of the loss factor K_{pk} are not available, although they are assumed to take values in a compact set.

B. Circuit Theory—Basic Notions

In what follows, we derive a model for a hydraulic network introduced above. Our derivation is based on graph-theoretic arguments employed in circuit theory (cf. [12]). We exploit the analogy between electrical and hydraulic circuits and replace voltages and currents with, respectively, pressures and flows. Then, valves and pipes can be seen as the hydraulic analog of (nonlinear) resistors and, respectively, inductors. Observe, however, that the pipe equation presents a drift term [see (2)] which is not generally present in inductors. Below, we recall basic facts about circuit theory. Although standard, they are useful to understand the derivation.

Networks are a collection of components which connect to each other through their two terminals. One can then associate with each terminal in the network a node and with each component an edge connecting the nodes, thus obtaining an undirected graph $\mathcal{F}$. Let a and, respectively, b be the number of nodes and edges of the graph. Since an edge represents a component, a flow and a pressure are associated with each edge of the graph. For each edge, a *reference direction* for the flow and a reference direction for the pressure is specified [12, pp. 3–5]. The reference direction for the pressure is denoted by the plus ("+") and minus ("−") signs near the nodes and the reference direction for the flow by an arrow. For edges which correspond to pipes and valves, we adopt *associated* reference directions, meaning that a positive flow enters the edge by the node marked with "+" and leaves it by the node marked with "−". On the other hand, for the edges which correspond to pumps, we adopt reference directions that are opposite from the associated reference directions [12, p. 24]. Our choice of reference directions is made explicit in Section II-C.

C. Standing Assumptions

As a first step towards the derivation of a model for the network, a set of *independent* state variables (that is, a set of flow variables which can be set *independently* without violating the Kirchhoff's node law) is identified. To this end, we assume the following.

Assumption 1: $\mathcal{F}$ is a connected graph.

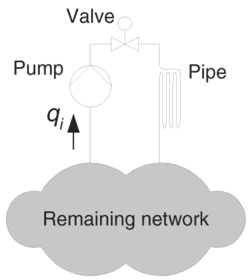

Fig. 2. Series connection associated with each end-user.

This means that, for each pair of nodes in the graph, there exists a path which connects them. Let $\mathcal{T}$ be the spanning *tree* of $\mathcal{F}$, i.e., a connected subgraph which does not contain any cycle and contains all the nodes of the graph [12, p. 477]. By Assumption 1 and [3, Corollary I.2.7, p. 10], the spanning tree always exists. The number of edges of $\mathcal{T}$ is $a-1$. By definition, adding to $\mathcal{T}$ any edge of the graph not contained in $\mathcal{T}$, i.e., a *chord* of $\mathcal{T}$, a cycle is obtained. We call the cycles obtained in this way *fundamental* cycles or loops, and we denote them by $\mathcal{L}_i$, with $i = 1, 2, \ldots, n$, and $n = b - a + 1$ the number of fundamental loops. Let $\mathcal{G}$ be the set of chords. It can be shown that the flows q_i through the chords in $\mathcal{G}$ form indeed a complete set of *independent variables*. In other words, each flow through the chord is independent of the flow through the other chords, while the flow through any other edges of the network which is not a chord depends linearly from the flows through the chords.

A second assumption is needed for the network under study. The discussion below helps us to better motivate the assumption.

As a first point, we observe that, since the control problem to be studied in the paper (Section III) concerns the regulation of pressure across valves at the end-users (i.e., end-user valves), and therefore of the flows through them, it is natural to choose as set of independent flows the flows at the user valves. Later on, we shall take these flows also as state variables of the system (see Section II-D).

As a second point, we remark that the district heating systems under consideration in this paper have a new structure that reduces the heat losses in the system by reducing the pipe diameters. The reduced pipe diameters create larger pressure losses throughout the system, meaning that much larger pump effort is needed. If this larger pump effort is to be implemented via a reduced number of pump stations not located at the end-users, the pressure at the points in the network where these pumps are located would be too large, and the pressures at the end-users would be very unevenly distributed, i.e., the end-users close to a pump stations will have very high pressures and the end-users far away from the pump station will have a very low pressure. To avoid this uneven distribution at the end-users, pumps are placed at the end-users (end-user pumps). Another motivation for this choice is that pumps should be installed where electricity is available.

The arguments above motivate us to introduce the following assumption.

Assumption 2: Each user valve is in series with a pipe and a pump (see Fig. 2). Moreover, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.

Remark: In the discussion preceding Assumption 2, we motivated the need to have end-user pumps. Conversely, one may wonder if other pumps different from the service pumps at the end-users are needed in the network. The answer is again positive since, to compensate for the large pressure losses, a large pump effort is needed, which cannot be provided by the end-users pumps alone. Rather, this is achieved by increasing the pressure at strategic points in the network via so-called booster pumps.

In what follows, we specify a reference direction for each edge of the graph in a manner which allows us to highlight a few important properties of the network (cf. Lemma 1). Moreover, we identify the direction of an edge with the reference direction of the flow through the component associated to that edge [12, p. 383], and, as a consequence, the graph associated with the hydraulic network becomes a *directed* graph. To state and prove Lemma 1, we need a few preliminary notions, which are introduced next.

The set of flows and pressures in the network must fulfill the well-known Kirchhoff's node and loop laws. Each fundamental loop has a *reference direction* given by the direction of the chord which defines the loop. Along any fundamental loop of the circuit Kirchhoff's voltage law holds, that is, $B\Delta h = 0$, where B is an $n \times b$ matrix called fundamental loop matrix such that [12, p. 481]

$$B_{ih} = \begin{cases} 1, & \text{edge } h \text{ is in } \mathcal{L}_i \text{ and directions agree} \\ -1, & \text{edge } h \text{ is in } \mathcal{L}_i \text{ and directions don't agree} \\ 0, & \text{edge } h \text{ is not in } \mathcal{L}_i \end{cases}$$

and Δh is the vector of pressure drops across the b components of the network.

Let each component of the network be denoted by the symbol c_i, with $i = 1, 2, \ldots, b$. Without loss of generality, we shall assume that the first $n = b - a + 1$ components correspond to chords of the graph, i.e., to user-pipes (see Assumption 2). The remaining $a - 1$ components (pipes, pumps, and valves) correspond to edges of the tree $\mathcal{T}$. Each fundamental loop $\mathcal{L}_i$ comprises a certain number of components and can therefore be described by the sequence $\mathcal{L}_i = \{c_{j_{i1}}, \ldots, c_{j_{ik_i}}\}$, where $1 \le j_{i1} \le n = b - a + 1$ and $b - a + 2 \le j_{i2}, \ldots, j_{ik_i} \le b$. Again, without loss of generality, we shall assume for each $i = 1, 2, \ldots, n$ that the chord of the fundamental loop $\mathcal{L}_i$ coincides with the component c_i of the network, that is, $j_{i1} = i$. With this choice, the fundamental loop matrix takes the form

$$B = (I_n \; F)$$

where I_n is the $(n \times n)$ identity matrix and F is a suitable $n \times (a-1)$ matrix of entries in the set $\{-1, 0, 1\}$.

Assumption 2 is very general and does not state anything about the structure of the distribution network, only about the structure at the end-users. In district heating systems, however, hydraulic networks have additional features to take into account:

Assumption 3: There exists one and only one component called the *heat source*. It corresponds to a valve[1] of the network, and it lies in all the fundamental loops.

The assumption appears to be very mild. In district heating systems (except extremely large-size district heating systems), it is typical to have only one common heat source which has to provide hot water to all the end-users. Hence, the heat source

[1]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

must lie in all the fundamental loops of the network. In what follows, we argue that as a consequence of the assumptions, the network must necessarily satisfy the following:

Lemma 1: Under Assumptions 1–3, it is possible to select the direction of the edges of the network in such a way that in the fundamental loop matrix $B = (I_n\ F), F = [F_{ij}]$ satisfies $F_{ij} \in \{0, 1\}$.

Proof: Consider the tree $\mathcal{T}$ obtained from $\mathcal{F}$ removing all of the chords in $\mathcal{G}$. If any additional edge is removed from $\mathcal{T}$ then, the resulting graph $\tilde{\mathcal{T}}$ is disconnected [3, Theorem 6], and it has two connected components. Each connected component does not contain any cycle (because otherwise $\mathcal{T}$ would not be a tree). Hence, each one of the two connected components is also a tree.

Remove from $\mathcal{T}$ the edge corresponding to the heat-source valve, denote it by e_{hsv} and let v_i, v_j be the nodes which correspond to the terminals of e_{hsv}. Since e_{hsv} lies in all of the fundamental loops, one of the two connected components of $\mathcal{T}$ will contain v_i and—for each end-user pipe—the node corresponding to one terminal of the end-user pipe, and the other one will contain v_j and—for each end-user pipe—the node corresponding to the remaining terminal of the end-user pipe.

Recall that, for each pair of distinct nodes in a tree, there exists a unique simple path (i.e., a path with no repeated nodes) connecting them (see, e.g., [32, Exercise 4.1.4]). Take the node $v_i(v_j)$, and identify the terminal of each end-user pipe which is connected to $v_i(v_j)$ via a simple path included in the connected component to which $v_i(v_j)$ belongs. Denote such terminals with the sign "$-$" ("$+$"). Observe that, by construction, $v_i(v_j)$ is connected to one and only one of the terminals of each end-user pipe.

Hence, in the first connected component, for each end-user pipe k, with $k = 1, 2 \ldots, n$, there exists a unique simple path connecting the negative terminal of the end-user pipe k and v_i. Such a path is included in the kth fundamental cycle. Then all the edges in the path have a natural direction, from the node corresponding to the negative terminal of the user-pipe k towards v_i. Similarly for the second tree, we consider the path connecting v_j and the positive terminal of the kth end-user pipe, and let the natural direction of the edges in the tree be the direction from v_j towards the nodes corresponding to the positive terminals. Finally, we let the direction of the kth chord be the direction which goes from the positive terminal to the negative one (in accordance with the associated reference direction rule recalled in Section II-B) and we let the orientation of e_{hsv} be from v_i to v_j. The reference direction of the kth fundamental cycle is given by the reference direction of the corresponding chord. In view of how we defined the directions of the edges in the two paths and in e_{hsv}, the directions of each edge along the kth fundamental circuit agree with the direction of the chord, that is, $B_{kh} = +1$ for each edge h in the fundamental cycle $\mathcal{L}_k$. Since this is true for each $k = 1, 2 \ldots, n$, i.e., for each fundamental cycle, the thesis follows. ■

Remark: The proof shows that, as a consequence of the assumptions, the network must necessarily take the structure illustrated in Fig. 3, where by $\mathcal{T}_f$ we denote the (forward) tree connecting the node v_j with all the positive terminals of the end-user pipes, and by $\mathcal{T}_r$ the (reverse) tree connecting the negative terminals of the end-user pipes with v_i. Observe that the two trees have a quite different physical role. $\mathcal{T}_f$ is the portion of the network which transports hot water from the heat source to the end-users, while $\mathcal{T}_r$ brings back the water which has been used by the end-users to the heat source.

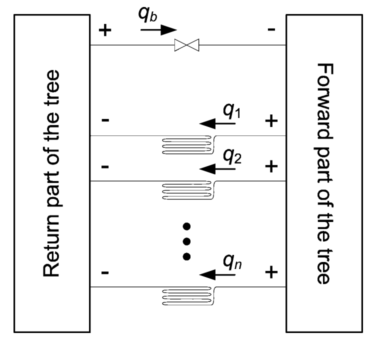

Fig. 3. Sketch of the network fulfilling Assumptions 1–3.

D. Model for Nonlinear Hydraulic Networks

We recall that, using Kirchhoff's current law, it is possible to establish the relation $q = B^T q_f$ [12, p. 482], where $q \in \mathbb{R}^b$ is the vector of the flows through each edge in the graph and $q_f \in \mathbb{R}^n$ is the vector of the flows through the chords in $\mathcal{G}$ [12]. The elements of q_f are called the *free flows* of the system and are *independent* variables.

The following result derives the dynamic model of the networks fulfilling Assumptions 1 and 2.

Proposition 1: Any hydraulic network satisfying Assumptions 1 and 2 is described by the model

$$J \dot{q}_f = f(K_p, K_v, B^T q_f) + u \tag{3}$$

where $q_f \in \mathbb{R}^n, u \in \mathbb{R}^n$ is a vector of n independent inputs, $J = J^T > 0$ is an $n \times n$ matrix, and $f(K_p, K_v, B^T q_f)$ is a continuously differentiable vector field.

Proof: Let $q = (q_f^T\ q_g^T)^T \in \mathbb{R}^b$ and $\Delta h = (\Delta h_f^T\ \Delta h_g^T)^T \in \mathbb{R}^b$ be the vectors of the flows and the pressure drops of each edge in the graph. In particular, q_f and Δh_f are the vectors of the flows through and of the pressure drops across each chord in $\mathcal{G}$, while the vectors q_g and Δh_g denote flows through and pressure drops across the edges of the graph which are not chords. Each component $i = 1, 2, \ldots, b$ of the network obeys the equation

$$\Delta h_i = J_i \dot{q}_i + \lambda_i(K_{pi}, q_i) + \mu_i(K_{vi}, q_i) - \Delta h_{pi}$$

where the terms in the equality are defined as follows. If: 1) the ith component is a pump, we have $J_i = 0, \lambda_i = 0$, and $\mu_i = 0$; 2) it is a pipe, $\mu_i = 0$ and $\Delta h_{pi} = 0$; 3) it is a valve, we have $J_i = 0, \lambda_i = 0$, and $\Delta h_{pi} = 0$. We can then collect together each component model to obtain

$$\Delta h = \mathcal{J} \dot{q} + \lambda(K_p, q) + \mu(K_v, q) - \Delta h_p \tag{4}$$

where $\Delta h_p = (\Delta h_{p1} \ldots \Delta h_{pb})^T$, $\mathcal{J} = \mathrm{diag}\{J_1, \cdots, J_b\}$ (with zero elements for each valve and pump component), $\lambda(K_p, q) = (\lambda_1(K_{p1}, q_1) \ldots \lambda_b(K_{pb}, q_b))^T$ and $\mu(K_v, q) = (\mu_1(K_{v1}, q_1) \ldots \mu_b(K_{vb}, q_b))^T$. Replacing the identities $B\Delta h = 0$ and $q = B^T q_f$ into (4), we obtain the following model:

$$0 = B\Delta h = B\mathcal{J}B^T \dot{q}_f + B\lambda(K_p, B^T q_f) + B\mu(K_v, B^T q_f) - B\Delta h_p$$

which we rewrite as

$$B\mathcal{J}B^T \dot{q}_f = - B\lambda(K_p, B^T q_f) - B\mu(K_v, B^T q_f) + B\Delta h_p \tag{5}$$

Setting $J = B\mathcal{J}B^T$, $f(K_p, K_v, B^T q_f) = -B\lambda(K_p, B^T q_f) - B\mu(K_v, B^T q_f)$, and $u = B\Delta h_p$, the model (3) is obtained. To complete the proof we need to show that $J = J^T > 0$, that f is a continuously differentiable vector field, and that $u = B\Delta h_p$ is a vector of independent inputs.

We start by showing that $J = J^T > 0$. Observe that J can be written as

$$J = (I_n \quad F) \begin{pmatrix} \mathcal{J}_f & 0 \\ 0 & \mathcal{J}_g \end{pmatrix} \begin{pmatrix} I_n \\ F^T \end{pmatrix} = \mathcal{J}_f + F\mathcal{J}_g F^T$$

where $\mathcal{J}_f = \mathrm{diag}\{J_1, \ldots, J_n\}$ and $\mathcal{J}_g = \mathrm{diag}\{J_{n+1}, \ldots, J_b\}$. As both $\mathcal{J}_f$ and $\mathcal{J}_g$ are diagonal, the matrix J is symmetric. Since all of the components corresponding to a chord in $\mathcal{G}$ are pipe elements by Assumption 2, all diagonal elements of $\mathcal{J}_f$ are strictly positive [see (2)], hence $\mathcal{J}_f > 0$. Next, we consider the term $F\mathcal{J}_g F^T$. The diagonal elements of $\mathcal{J}_f$ are all nonnegative, hence $x^T(F\mathcal{J}_g F^T)x \geq 0$ for all $x \in \mathbb{R}^n$, that is, $x^T J x = x^T \mathcal{J}_f x + x^T(F\mathcal{J}_g F^T)x \geq x^T \mathcal{J}_f x > 0$ for all $x \neq 0$. Hence, $J = J^T > 0$.

Next, we show that $f(K_p, K_v, q_f)$ is a continuously differentiable vector field. In fact, $f(K_p, K_v, B^T q_f) = -B\lambda(K_p, B^T q_f) - B\mu(K_v, B^T q_f)$ and $f(K_p, K_v, B^T q_f)$ is a continuously differentiable vector field because each entry is a linear combination of continuously differentiable functions.

Finally, we prove that each entry of $u = B\Delta h_p$ is an independent input. To this purpose, we show that each entry u_i can be controlled independently using one (and only one) of the pump pressures Δh_p at the end-users. Recall that we have chosen the pipe of the ith end-user to be the component c_i of the network, for $i = 1, 2, \ldots, n$. Without loss of generality, we also choose the component c_{n+i}, with $i = 1, 2, \ldots, n$, to be the pump at the ith end-user. Since the end-user pump is in series with the end-user pipe (see Fig. 2), the flow through the pump at the ith end-user is equal to the flow through the pipe at the ith end-user, i.e., $q_i = q_{n+i}$ for any $i = 1, 2, \ldots, n$. Recall now that $q = (q_f^T \; q_g^T)^T$, with $q_f \in \mathbb{R}^n$, and that $q = B^T q_f = (I_n \, F)^T q_f$. The latter and the property shown above that $q_i = q_{n+i}$ for any $i = 1, 2, \ldots, n$, prove that we can further partition the matrix F in $q = (I_n \, F)^T q_f$, to obtain the equality $q = (I_n \, I_n \, F')^T q_f$. Hence, $B^T = (I_n \, I_n \, F')^T$.

As we have chosen the first n components of the network to be the pipes at the n end-users, the first n entries of the vector Δh_p in (5) (that is the vector of the pressures delivered by the pumps present in the network) must be necessarily equal to 0. Furthermore, the successive n entries of Δh_p correspond to the pumps at the end-users. This implies that one can partition the vector Δh_p in the following way:

$$\Delta h_p = \left(\mathbf{0}_n^T \quad \Delta h_p^{eT} \quad \Delta h_p'^{\,T} \right)^T$$

where $\mathbf{0}_n$ is the n-dimensional vector of zero entries, $\Delta h_p^e \in \mathbb{R}^n$ is the subvector of the pressures delivered by the pumps at the end-users, and $\Delta h_p'$ is an $(b - 2n)$ subvector whose nonzero components coincide with the pressures delivered by all the remaining pumps in the network which are not pumps at the end-users (i.e., booster pumps). In view of this partition, $u = B\Delta h_p$ is given by

$$u = (I_n \quad I_n \quad F') \begin{pmatrix} \mathbf{0} \\ \Delta h_p^e \\ \Delta h_p' \end{pmatrix} = \Delta h_p^e + F'\Delta h_p'. \tag{6}$$

As the pressures delivered by the end-user pumps Δh_p^e are independent variables, so are the control inputs in u. This completes the proof. ■

The control system derived above is completed with a set of measured (and controlled) outputs. This set coincide with the set of the pressures across the user-valves, that is,

$$y_i = \hat{\mu}_i(\hat{K}_{vi}, q_{fi}), \qquad i = 1, \ldots, n \tag{7}$$

where $\hat{\mu}_i(\hat{K}_{vi}, q_{fi}) = \mu_{j_i}(K_{vj_i}, q_{fi})$ for $i = 1, \ldots, n$, and $j_1, \ldots j_n$ are the indexes of the components which correspond to the end-user valves.

A feature of the hydraulic networks which additionally satisfy Assumption 3 is the following, which is explored later in Section III:

Lemma 2: Under Assumptions 1–3, $q_f \in \mathbb{R}_+^n$[2] implies $-f(K_p, K_v, B^T q_f) \in \mathbb{R}_+^n$.

Proof: Recall that

$$q = B^T q_f = \begin{pmatrix} I_n \\ F^T \end{pmatrix} q_f$$

where from Lemma 1 all of the entries of F^T are nonnegative. Moreover, observe that we can assume without loss of generality that each column of F is nonzero. In fact, if this were not the case, it would exist an edge in the circuit through which the flow is always zero, no matter what the free flow vector q_f is. This means that the edge can be removed, and all of the conclusion would still hold. These facts imply that, if $q_f \in \mathbb{R}_+^n$, then necessarily $q \in \mathbb{R}_+^b$.

By definition, $f(K_p, K_v, B^T q_f) = -B\lambda(K_p, B^T q_f) - B\mu(K_v, B^T q_f)$, or $f_i(K_p, K_v, B^T q_f) = -\sum_{h=1}^b B_{ih}(\lambda_h(K_{ph}, q_h) + \mu_h(K_{vh}, q_h))$, with $B_{ih} \in \{0, 1\}$. Now, from the proof of Proposition 1, we know that there must exist a subset of indexes $\mathcal{H}_i \subseteq \{1, \ldots, b\}$ such that $f_i(K_p, K_v, B^T q_f) = -\sum_{h \in \mathcal{H}_i}(\lambda_h(K_{ph}, q_h) + \mu_h(K_{vh}, q_h))$, with $\lambda_h(K_{ph}, q_h) + \mu_h(K_{vh}, q_h) \neq 0$ for all $h \in \mathcal{H}_i$ and for all $q_h \neq 0$. More specifically, since the functions $\lambda_h(K_{ph}, \cdot), \mu_h(K_{vh}, \cdot)$ are strictly increasing for each value of the parameters K_{ph}, K_{vh} and zero at zero, and since $q \in R_+^b$, we have $f_i(K_p, K_v, B^T q_f) = -\sum_{h \in \mathcal{H}_i}(\lambda_h(K_{ph}, q_h) + \mu_h(K_{vh}, q_h)) < 0$. This completes the proof. ■

III. Proportional Controllers for Practical Regulation

We study here the problem of designing a set of controllers which regulates each output (the pressure drop at the end-user

[2] $\mathbb{R}_+^n$ denotes the positive orthant of $\mathbb{R}^n$, i.e., the set $\{q_f \in \mathbb{R}^n : q_{fi} > 0, i = 1, \ldots, n\}$.

valve) y_i to the positive set-point reference value r_i, with $r = (r_1, \ldots, r_n) \in \mathcal{R}$ ranging in a known compact set, namely $\mathcal{R} = \{r \in \mathbb{R}^n : 0 < r_m \leq r_i \leq r_M, i = 1, \ldots, n\}$ (although typically $r_1 = \cdots = r_n = 0.5$ [bar]). We want to control the system using a set of proportional control laws of the following form:

$$u_i = \begin{cases} -N_i(y_i - r_i), & y_i - r_i \leq 0 \\ 0, & y_i - r_i \geq 0 \end{cases} \tag{8}$$

where $N_i > 0$ is the gain of the control law. From (8), it is seen that the control law has an inherent saturation to ensure that the control values never become negative. In turn, this guarantees that the positivity constraint on the pressures delivered by the pumps is fulfilled (see the last remark at the end of the section). The use of saturated *proportional-integral* control laws to achieve *asymptotic* pressure regulation is more complicated and is not pursued here, although some results in the case of linear systems have appeared [28].

In what follows, the following terminology will be in use: a trajectory is *attracted* by a subset S of the state space if it is defined for all $t \geq 0$, and it belongs to S for all $t \geq T$, with $T > 0$ a finite time. Our control goal is the following.

Pressure Regulation Problem: Given system (3) with the outputs (7), any pair of compact sets of positive parameters $\mathcal{P}, \mathcal{V}$, any compact set of reference values $\mathcal{R}$ such that, for each $\hat{K}_v \in \mathcal{V}, \mathcal{R} \subseteq \mathrm{Image}(\hat{\mu}(\hat{K}_v, \cdot))$,[3] any arbitrarily large positive number q_M and compact set of initial conditions

$$\mathcal{Q} = \{q_f \in \mathbb{R}^n : |q_{fi}| \leq q_M, i = 1, \ldots, n\} \tag{9}$$

and any arbitrarily small positive number γ, find controllers of the form (8), such that, for any $(K_p, K_v) \in \mathcal{P} \times \mathcal{V}$ and $r \in \mathcal{R}$, every trajectory $q_f(t)$ of the closed-loop system (3), (7), and (8) with initial condition in $\mathcal{Q}$ is attracted by the set $\{q_f \in \mathbb{R}^n : |\hat{\mu}_i(\hat{K}_{vi}, q_{fi}) - r_i| \leq \gamma, i = 1, \ldots, n\}$.

In other words, the Pressure Regulation Problem is solved if the trajectories of the closed-loop system converge in finite time to a subset of the state space where the regulation errors $\epsilon_i := y_i - r_i, i = 1, \ldots, n$, are in magnitude smaller than γ.

Before stating the main result of the section, we introduce the error coordinates e defined as

$$e = q_f - \hat{\mu}^{-1}(\hat{K}_v, r) \tag{10}$$

where $\hat{\mu}^{-1}(\hat{K}_v, r) = (\hat{\mu}_1^{-1}(\hat{K}_{v1}, r_1) \ldots \hat{\mu}_n^{-1}(\hat{K}_{vn}, r_n))^T$. Observe that, since $\mathcal{R} \subseteq \mathrm{Image}(\hat{\mu}(\hat{K}_v, \cdot))$, for each $K_v \in \mathcal{V}$ and each $r \in \mathcal{R}$, there exists $q_f \in \mathbb{R}^n$ such that $r = \hat{\mu}(\hat{K}_v, q_f)$, i.e., $r_i = \hat{\mu}_i(\hat{K}_{vi}, q_{fi})$ for each $i = 1, 2, \ldots, n$. Since $\hat{\mu}_i(\hat{K}_{vi}, \cdot)$ is monotonically increasing and continuously differentiable, it is invertible and the inverse $\hat{\mu}_i^{-1}(\hat{K}_{vi}, r_i)$ is well-defined for any i. Hence, $\hat{\mu}^{-1}(\hat{K}_v, r)$ is defined on a domain which contains $\mathcal{R}$.

[3]By $\hat{K}_v \in \mathcal{V}$ it is meant the set of values taken by the subvector $\hat{K}_v$ of K_v when the latter ranges over $\mathcal{V}$. More in general, throughout the paper, given a vector x which lives in the set $\mathcal{X}$, by $\hat{x} \in \mathcal{X}$ it is meant the set of values taken by the subvector $\hat{x}$ of x when the latter ranges over $\mathcal{X}$.

Then, we derive a simple relation between the regulation error ϵ and the error coordinates e which is used in the forthcoming derivations.

Lemma 3: The relation between the error coordinates e and the regulated error ϵ is given by

$$\epsilon_i(e_i, \hat{K}_{vi}, r_i) = \hat{\mu}_i\left(\hat{K}_{vi}, e_i + \hat{\mu}_i^{-1}(\hat{K}_{vi}, r_i)\right) - r_i.$$

The function $\epsilon_i(e_i, \hat{K}_{vi}, r_i)$ is monotonically increasing and zero at $e_i = 0$, and moreover

$$\epsilon_i(e_i, \hat{K}_{vi}, r_i)e_i > 0, \qquad \text{for all } e_i \neq 0.$$

Proof: The relation between e_i and $\epsilon_i = y_i - r_i$ is obtained by replacing q_{fi} as a function of e_i in y_i. Observe that

$$\epsilon_i(0, \hat{K}_{vi}, r_i) = \hat{\mu}_i\left(\hat{K}_{vi}, \hat{\mu}_i^{-1}(\hat{K}_{vi}, r_i)\right) - r_i = r_i - r_i = 0.$$

Therefore if $e_i = 0$ then $\epsilon_i = 0$. Moreover

$$\frac{\partial \epsilon_i(e_i, \hat{K}_{vi}, r_i)}{\partial e_i} = \left.\frac{\partial \hat{\mu}_i(\hat{K}_{vi}, q_{fi})}{\partial q_{fi}}\right|_{q_{fi} = e_i + \hat{\mu}_i^{-1}(\hat{K}_{vi}, r_i)}.$$

From the definition of $\hat{\mu}_i$, we know that $(\partial \hat{\mu}_i(\hat{K}_{vi}, q_{fi}))/(\partial q_{fi}) > 0$ everywhere. As a result, $\epsilon_i(e_i, \hat{K}_{vi}, r_i)$ is monotonically increasing and $\epsilon_i(e_i, \hat{K}_{vi}, r_i) > 0$ (respectively, $\epsilon_i(e_i, \hat{K}_{vi}, r_i) < 0$) if and only if $e_i > 0$ (respectively, $e_i < 0$). Hence

$$\epsilon_i(e_i, \hat{K}_{vi}, r_i)e_i > 0, \qquad \text{for all } e_i \neq 0.$$

■

The following proposition is the main result of this section.

Proposition 2: For $i = 1, 2, \ldots, n$, there exist gains $N_i^* > 0$ such that, for all $N_i > N_i^*$, the controllers (8) solve the Pressure Regulation Problem.

Proof: We proceed defining a Lyapunov function and proving its derivative is negative on appropriate sets of the state space [31], [17].

In the error coordinates e defined in (10), the system (3) becomes

$$J\dot{e} = J\dot{q}_f = f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} + u. \tag{11}$$

Bearing in mind the definition of ϵ_i, in the new coordinates, the control law takes the form

$$u_i = \begin{cases} -N_i\epsilon_i(e_i, \hat{K}_{vi}, r_i), & \epsilon_i(e_i, \hat{K}_{vi}, r_i) \leq 0 \\ 0, & \epsilon_i(e_i, \hat{K}_{vi}, r_i) \geq 0 \end{cases}$$

or, equivalently, in view of Lemma 3

$$u_i = \begin{cases} -N_i\epsilon_i(e_i, \hat{K}_{vi}, r_i), & e_i \leq 0 \\ 0, & e_i \geq 0. \end{cases} \tag{12}$$

Consider the Lyapunov function $V(e) = e^T Je$. The derivative of the function $V(e)$ along the trajectories of (11) is given by

$$\dot{V}(e) = 2e^T\left(f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} + u\right). \tag{13}$$

Define the set of initial conditions in the e-coordinates as

$$\mathcal{E} = \{e \in \mathbb{R}^n : e = q_f - \hat{\mu}^{-1}(\hat{K}_v, r), \quad q_f \in \mathcal{Q}, r \in \mathcal{R}, \hat{K}_v \in \mathcal{V}\}$$

and let $\sigma > 0$ be a real number such that $\{e : V(e) \leq \sigma\} \supseteq \mathcal{E}$. Moreover, let $0 < \varrho < \sigma$ and $\gamma' > 0$ be such that

$$\begin{aligned}\{e \in \mathbb{R}^n : |e_i| \leq \gamma', i = 1, \cdots, n\} \\ \subseteq \{e \in \mathbb{R}^n : V(e) \leq \varrho\} \\ \subseteq \{e \in \mathbb{R}^n : |\epsilon_i(e_i, \hat{K}_{vi}, r_i)| \\ \leq \gamma, \forall \hat{K}_{vi} \in \mathcal{V}, \forall r_i \in \mathcal{R}, i = 1, \ldots, n\}\end{aligned}$$

and, finally, define $S = \{e : \varrho \leq V(e) \leq \sigma\}$. We let $M > 0$ be a constant such that $2e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} < M$ on S, for $(K_p, K_v) \in \mathcal{P} \times \mathcal{V}$, and $r \in \mathcal{R}$. We now investigate the sign of the derivative $\dot{V}(e)$ on different regions of the state space. The goal is to show that $\dot{V}(e) < 0$ for all $e \in S$.

Region 1: ($\mathcal{R}_1 = \{e \in S : e_i \leq 0, i = 1, \ldots, n\}$). Replacing $u_i, i = 1, \ldots, n$, with the controller expression (12) in the derivative of the Lyapunov function (13), the following is obtained:

$$\dot{V}(e) = 2e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} - \sum_{i=1}^{n} N_i e_i \epsilon_i(e_i, \hat{K}_{vi}, r_i).$$

By the definition of γ', any point e in $\mathcal{R}_1$ is such that $|e_{j(e)}| \geq \gamma'$ for at least an index $j(e) \in \{1, \ldots, n\}$, and therefore

$$\sum_{i=1}^{n} N_i e_i \epsilon_i(e_i, \hat{K}_{vi}, r_i) \geq N_{j(e)} \gamma' \epsilon_{j(e)}(\gamma', \hat{K}_{vj(e)}, r_j(e))$$

where $\epsilon_{j(e)}(\gamma', \hat{K}_{vj(e)}, r_j(e)) > 0$ because by Lemma 3 $\epsilon_{j(e)}(\cdot, \hat{K}_{vj(e)}, r_j(e))$ is positive for positive values of its argument. Then, choosing $N_i^{(1)}$ in such a way that $M - N_i^{(1)} \gamma' \epsilon_i(\gamma', \hat{K}_{vi}, r_i) < 0$ for all $i \in \{1, \cdots, n\}$, for all $K_v \in \mathcal{V}$, for all $r \in \mathcal{R}$, we have $\dot{V}(e) < 0$ for all $e \in \mathcal{R}_1$, for any $N_i \geq N_i^{(1)}$.

Region 2: ($\mathcal{R}_2 = \{e \in S : e_i \geq 0, i = 1, \ldots, n\}$). Due to the definition (12) of the controller, in this region $u = 0$. Moreover, since $\hat{\mu}_i^{-1}(\hat{K}_{vi}, r_i) > 0$ and $e_i \geq 0$ for $i = 1, \ldots, n$, the vector $q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)$ has all positive entries. Then, by Lemma 2, we have that $f_i(K_p, K_v, B^T q_f) < 0$ for all i. Hence, the derivative of the Lyapunov function (13) satisfies

$$\dot{V}(e) = 2e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} < 0$$

for all $e \in \mathcal{R}_2$.

Region 3: $\mathcal{R}_3 = S \setminus (\mathcal{R}_1 \cup \mathcal{R}_2)$. We consider the following partition of the set $\mathcal{R}_3$. Observe first that there exists $2^n - 2$ nonvoid intersections of $\mathcal{R}_3$ with the orthants of $\mathbb{R}^n$. Call these intersections $\mathcal{R}_{3\ell}$, with $\ell = 1, \ldots, 2^n - 2$, and consider the partition $\mathcal{R}_3 = \cup_{\ell=1}^{2^n-2} \mathcal{R}_{3\ell}$. Associated with each subregion $\mathcal{R}_{3\ell}$ there exists a unique set of indexes $\mathcal{L}_\ell \subset \{1, \ldots, n\}$ such that $e \in \mathcal{R}_{3\ell}$ if and only if $e_i \leq 0$ for each $i \in \mathcal{L}_\ell$ and $e_i \geq 0$ for each $i \in \bar{\mathcal{L}}_\ell$, with $\bar{\mathcal{L}}_\ell = \{1, \ldots, n\} \setminus \mathcal{L}_\ell$.

For a fixed $\ell \in \{1, \ldots, 2n - 2\}$, for $e \in \mathcal{R}_{3\ell}$, the derivative of the Lyapunov function computed along the trajectories of the closed-loop system writes as

$$\dot{V}(e) = 2e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} - \sum_{i \in \mathcal{L}_\ell} N_i e_i \epsilon_i(e_i, \hat{K}_{vi}, r_i).$$

Since $e_i \epsilon_i(e_i, \hat{K}_{vi}, r_i) > 0$ for all $e_i \neq 0$, it is also true that

$$\dot{V}(e) < 2e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)}.$$

From the analysis above, we know that $2e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} < 0$ for all $e \in \mathcal{R}_2$. In particular, the derivative is strictly negative for all e in the set $\{e \in S : e_i = 0, \forall i \in \mathcal{L}_\ell, e_i \geq 0, \forall i \in \bar{\mathcal{L}}_\ell\}$, which lies at the boundary between $\mathcal{R}_2$ and $\mathcal{R}_{3\ell}$. Since $\dot{V}(e)$ is a continuous function of its arguments, there must exist a sufficiently small value $\bar{e}_\ell > 0$ such that $\dot{V}(e)$ continues to be strictly negative on the subset $\mathcal{D}_{3\ell} = \{e \in \mathcal{R}_{3\ell} : e_i > -\bar{e}_\ell \quad \forall i \in \mathcal{L}_\ell, e_i \geq 0 \quad \forall i \in \bar{\mathcal{L}}_\ell\}$. Now, consider the remaining portion of $\mathcal{R}_{3\ell}$, namely the set of points $\mathcal{R}_{3\ell} \setminus \mathcal{D}_{3\ell}$ where $e_i \leq -\bar{e}_\ell$ for all $i \in \mathcal{L}_\ell$. Since $\epsilon_i(e_i, \hat{K}_{vi}, r_i)$ is a monotonically increasing function of e_i which is zero at zero, for all $e_i \leq -\bar{e}_\ell$, we have $e_i \epsilon_i(e_i, \hat{K}_{vi}, r_i) \geq -\bar{e}_\ell \epsilon_i(-\bar{e}_\ell, \hat{K}_{vi}, r_i) = \bar{e}_\ell \epsilon_i(\bar{e}_\ell, \hat{K}_{vi}, r_i) > 0$ (recall that $\epsilon_i(\bar{e}_\ell, \hat{K}_{vi}, r_i) > 0$). Let $N_{i\ell}^{(2)} > 0$ be such that $M - \sum_{i \in \mathcal{L}_\ell} N_{i\ell}^{(2)} \bar{e}_\ell \epsilon_i(\bar{e}_\ell, \hat{K}_{vi}, r_i) < 0$, for all $\hat{K}_{vi} \in \mathcal{V}$ and for all $r_i \in \mathcal{R}$. Then, for all $N_i \geq N_{i\ell}^{(2)}$ (with $i \in \mathcal{L}_\ell$), for all $e \in \mathcal{R}_{3\ell} \setminus \mathcal{D}_{3\ell}, \dot{V}(e) < 0$. This is true for all $\ell \in \{1, \ldots, 2n - 2\}$, and hence $\dot{V}(e) < 0$ on $\mathcal{R}_3$.

The thesis is finally inferred by letting, for $i = 1, \ldots, n, N_i^* = \max\{N_i^{(1)}, N_{i\ell}^{(2)}, \ell = 1, \ldots, 2n - 2\}$ and $N_i \geq N_i^*$. In fact, since $\dot{V}(e) < 0$ for all $e \in S$, then e converges in finite time to the level set $\{e \in \mathbb{R}^n : V(e) \leq \varrho\}$ which is contained in $\{e \in \mathbb{R}^n : |\epsilon_i(e_i, \hat{K}_{vi}, r_i)| \leq \gamma, \forall \hat{K}_{vi} \in \mathcal{P}, \forall r_i \in \mathcal{R}, i = 1, \ldots, n\}$. ■

Remark: The proof provides an estimate of the gains which guarantee practical regulation, namely $N_i^* = \max\{N_i^{(1)}, N_{i\ell}^{(2)}, \ell = 1, \ldots, 2n - 2\}$, where

$$N_i^{(1)} > \frac{M}{\gamma' \epsilon_{im}(\gamma')}, N_{i\ell}^{(2)} > \frac{M}{\bar{e}_\ell \epsilon_{im}(\bar{e}_\ell)}$$

with

$$\begin{aligned}\epsilon_{im}(c) &= \min_{\hat{K}_{vi} \in \mathcal{V}, r_i \in \mathcal{R}} \epsilon_i(c, \hat{K}_{vi}, r_i), \qquad c = \gamma', \bar{e}_\ell \\ \epsilon_i(c, \hat{K}_{vi}, r_i) &= \hat{\mu}_i(\hat{K}_{vi}, c + \hat{\mu}_i^{-1}(\hat{K}_{vi}, r_i)) - r_i.\end{aligned}$$

Observe, however, that the system we are dealing with is largely uncertain. As a matter of fact, in (3), not only are the parameters K_v, K_p, J uncertain but also the actual expression of the functions appearing in the vector field f are unknown, except for the fact that they satisfy the properties introduced in Section II-A. This implies that the quantities defining $N_i^{(1)}, N_{i\ell}^{(2)}$ are unknown, and they can be hardly helpful to provide a value for N_i^*. Nevertheless, they do provide the important indication that such gains exist and that the system under study has a gain stability margin which can be made arbitrarily large. In practice, the gains are tuned by a trial-and-error procedure which rely on the property established in the result above that increasing the gains eventually lead to the desired regulation goal.

Remark: The relation between the controller outputs and the pump pressures is described by (6). In (6), $\Delta h'_p$ is the vector of pressures delivered by the so-called booster pumps, which are in general used to help fulfilling constraints on the relative pressures across the network. Moreover, it is expected that the end-user pumps in general are too small to deliver the pressures necessary to obtain the desired flow. Therefore, both the end-user pumps and the booster pumps must provide the required control effort u. It is always possible to find a vector of nonnegative entries $\Delta h'_p$ such that each component of $u - F'\Delta h'_p = \Delta h^e_p$ is nonnegative provided that so is each component of u. For instance, one could choose $\Delta h'_p = (1)/(\|F'\|_\infty)\min\{u_1, \ldots, u_n\}$. In other words, *if one can solve the regulation problem using the positive control laws* (8), *then the actual control laws at the booster pumps $\Delta h'_p$ and at the end-user pumps Δh^e_p are positive as well.*

Recall that we have fixed (proof of Lemma 1) a reference direction for the pressure of each component, including the pumps. The pumps are installed in such a way that they deliver the required positive pressure consistently with the chosen reference direction.

IV. Pressure Regulation by Quantized Control

A. Motivation

In the previous section, we discussed a solution to the pressure regulation problem by proportional positive controllers. We also discussed that it is always possible to derive the actual control laws $\Delta h'_p$ and Δh^e_p as a function of u in such a way that each entry of both $\Delta h'_p$ and Δh^e_p is positive as well. There are other ways to derive $\Delta h'_p$ and Δh^e_p more efficiently, namely as functions of a subset of components of u. An example of such a more efficient distribution of the control action u to the pumps is given in Section V (a general treatment of methods to distribute the control action u goes beyond the scope of the paper).

Observe that u is the vector of control laws computed locally by each controller located at the end-users, and $\Delta h'_p$ and Δh^e_p are the actual control laws which the pumps in the network must deliver. Since controllers and pumps are distributed across the network and hence geographically separated, it is important to investigate a way in which the control laws (8) can actually be communicated to the pumps. In this section, we propose to use *quantized* control laws and prove that a quantized version of (8) achieves the same control objectives as the original control law.

By quantized control is meant a piece-wise constant control law which takes values in a finite set. The state space is partitioned into a finite number of regions, and a control value is assigned to each one of the regions. The transitions from one control value to another take place when the state crosses the boundaries of the regions. Since quantized control laws take values in a finite set, in principle these values can be transmitted over a finite bandwidth communication channel. Moreover, quantized control laws can be viewed as *event-based* control laws (the event being the crossing of the boundaries) whose design is based on the continuous-time model of the process. They do not require to derive sampled-data models of the process to control, and do not require equally spaced sampling (and transmission), requirements which would be very difficult to meet in the case study under investigation due to the complexity, the distributed nature and the uncertainty of the model. Quantized control for nonlinear systems has been investigated in a number of papers, among which we recall [13], [24], [15], [6], and [8]. Here, we extend the results of [8], where a quantized version of the so-called semiglobal backstepping lemma was proven, to the case in which *multiple positive* inputs are present. To the best of our knowledge, this is the first time a class of quantized controllers for a nonlinear multi-input industrial process is investigated.

B. Quantized Controllers

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the map

$$\psi(u) = \begin{cases} \psi_i, & \frac{\psi_i}{1+\delta} < u \le \frac{\psi_i}{1-\delta} \\ & 0 \le i \le j \\ 0, & u \le \frac{\psi_j}{1+\delta}. \end{cases} \tag{14}$$

In the definition above, j is a positive integer, ψ_0 is a positive real number, $\delta \in (0,1)$, and $\psi_i = \rho^i \psi_0$ for $i = 1, 2, \ldots, j$ with $\rho = (1-\delta)/(1+\delta)$. The parameters j, ψ_0, δ are to be designed. The map ψ is the classical *logarithmic* quantizer [13], with a few modifications. First, the output of ψ is zero for negative values of the argument u. This is because ψ is used below to quantize $r_i - y_i$ in the control input, and the latter is zero if $r_i - y_i \le 0$ (cf. (8) for the un-quantized case). Second, $\psi(u)$ is zero when the argument u approaches the origin. In this way, the *truncated* quantizer (14) has a finite number of quantization levels ($j+1$ quantization levels to be precise) and can be used in practical implementations, in contrast to the classical logarithmic quantizer which has an infinite number of quantization levels. Finally, we observe that other quantizers could be used, such as the uniform quantizers, and carry out a very similar analysis to the one presented below. For the sake of brevity, in the paper the analysis with uniform quantizers is not considered.

Consider now the quantized version of the control law (8), namely

$$u = N\Psi(-\epsilon) \tag{15}$$

where $N = \mathrm{diag}(N_1 \ldots, N_n)$ is a diagonal matrix of gains, $\Psi(-\epsilon) = (\psi(-\epsilon_1) \ldots \psi(-\epsilon_n))^T$, and $\epsilon_i = y_i - r_i$, and the resulting closed-loop system

$$J\dot{q}_f = f(K_p, K_v, B^T q_f) + N\Psi(-\epsilon) \tag{16}$$

where $\epsilon = \hat{\mu}(\hat{K}_v, q_f) - r$. Since $\Psi(-\epsilon)$ is a discontinuous function of the state variables, the closed-loop system (16) is a system with discontinuous right-hand side. For this system, the solutions are intended in the Krasowskii sense, a notion which is here briefly recalled.

Definition: A curve $\varphi : [0, +\infty) \to \mathbb{R}^n$ is a *Krasowskii solution* of a system of ordinary differential equations $\dot{x} = G(t, x)$, where $G : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, if it is absolutely continuous and for almost every $t \ge 0$ it satisfies the differential inclusion $\dot{x} \in K(G(t,x))$, where $K(G(t,x)) = \cap_{\delta>0}\overline{\mathrm{co}}\, G(t, B_\delta(x))$, where $\overline{\mathrm{co}}\, G$ is the convex closure of the set G, i.e., the smallest closed set containing the convex hull of G, and $B_\delta(x)$ is the open ball of radius δ centered at x.

Recalling [1, Theorem 1, Properties 2), 3), and 7)], we can state that the Krasowskii solutions of (16) are absolutely continuous functions which satisfy the differential inclusion

$$J\dot{q}_f \in f(K_p, K_v, B^T q_f) + Nv \tag{17}$$

where $v \in K(\Psi(-\epsilon)), K(\Psi(-\epsilon)) \subseteq \times_{i=1}^{n} K(\psi(-\epsilon_i))$ (here $\times$ denotes the Cartesian product), and [8]

$$K(\psi(-\epsilon_i)) \subseteq \begin{cases} \{-(1+\lambda_i\delta)\epsilon_i, & \lambda_i \in [-1,1]\} \\ & \frac{\psi_j}{1+\delta} < -\epsilon_i \le \frac{\psi_0}{1-\delta}, \\ \{-\lambda_i(1+\delta)\epsilon_i, & \lambda_i \in [0,1]\} \\ & 0 \le -\epsilon_i \le \frac{\psi_j}{1+\delta} \\ 0, & -\epsilon_i \le 0. \end{cases} \tag{18}$$

The result below proves to be analogous of Proposition 2, namely that the quantized controllers $N\Psi(-\epsilon)$ solve the Pressure Regulation Problem. This means in particular that every *Krasowskii solution* of (17) which starts in $\mathcal{Q}$ is attracted by the set $\{q_f \in \mathbb{R}^n : |\hat{\mu}_i(\hat{K}_{vi}, q_{fi}) - r_i| \le \gamma, i = 1, \cdots, n\}$.

Proposition 3: For any value of the quantization parameter $\delta \in (0,1)$, there exist gains $N_i^* > 0$ and parameters ψ_0, j of the quantizer such that, for all $N_i > N_i^*$, the quantized controllers (15) solve the Pressure Regulation Problem.

Proof: The proof uses the arguments of Proposition 2 above and [8, Proposition 1]. The symbols introduced in the proof of Proposition 2 are not repeated here.

As in Proposition 2, we adopt the error coordinates e defined in (10), so that the differential inclusion corresponding to the closed-loop system becomes

$$J\dot{e} \in f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} + Nv \tag{19}$$

where $v \in K(\Psi(-\epsilon))$. Similarly to Proposition 2, the proof of the thesis is based again on showing that $\dot{V}(e) < 0$ on S, with $V(e) = e^T J e$, but this time, using standard Lyapunov stability theory for differential inclusions (see [14], [6], and [8]), this must be true for all $v \in K(\Psi(-\epsilon))$. Since $K(\Psi(-\epsilon)) \subseteq \times_{i=1}^{n} K(\psi(-\epsilon_i))$ and in view of (18), we will investigate the sign of $\dot{V}(e) < 0$ when each component v_i of v, with $i = 1, 2, \ldots, n$, ranges in the sets on the right-hand side of (18).

As before, we let $M > 0$ be a constant such that $e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} < M$ on S, for $(K_p, K_v) \in \mathcal{P} \times \mathcal{V}$, and $r \in \mathcal{R}$.

Observe first that, for $e \in \mathcal{R}_2 = \{e \in S : e_i \ge 0, i = 1, \ldots, n\}$, $u = 0$ and so is v, and, as in Proposition 2, $\dot{V}(e) < 0$.

For $e \in \mathcal{R}_1 = \{e \in S : e_i \le 0, i = 1, \ldots, n\}$, we have that

$$\dot{V}(e) = e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} + \sum_{i=1}^{n} e_i N_i v_i.$$

Let us first design ψ_0 in such a way that the quantizers never undergo overflow as far as $e \in S$, i.e., the argument of each quantizer never exceeds the bound $\psi_0(1+\delta)^{-1}$. We let

$$\psi_0 \ge (1+\delta) \max_{e_i \in S, \hat{K}_{vi} \in \mathcal{V}, r_i \in \mathcal{R}} |\epsilon_i(e_i, \hat{K}_{vi}, r_i)|$$

for $i = 1, 2, \ldots, n$. Choose the integer j in the quantizer (14) in such a way that $\psi_j(1+\delta)^{-1} \le |\epsilon_i(-\gamma', \hat{K}_{vi}, r_i)|$ for any i. This amounts to choosing j in such a way that (recall that $\psi_j = \rho^j \psi_0$, with $0 < \rho < 1$)

$$\rho^j \le \frac{1}{\psi_0} \min_{\hat{K}_{vi} \in \mathcal{V}, r_i \in \mathcal{R}} |\epsilon_i(-\gamma', \hat{K}_{vi}, r_i)|$$

for $i = 1, 2, \ldots, n$. We recall from Proposition 2 that γ' is such that $\{e \in \mathbb{R}^n : |e_i| \le \gamma', i = 1, \cdots, n\} \subseteq \Gamma_\varrho$, and Γ_ϱ is the inner level set which defines S. Each term $N_i e_i v_i$ in the equality above is nonpositive, since $\epsilon_i(e_i, \hat{K}_{vi}, r_i) e_i > 0$ for all $e_i \neq 0$ and $v_i \in K(\Psi(-\epsilon_i))$. Moreover, for each $e \in \mathcal{R}_1$, there exists at least an index $j(e) \in \{1, 2, \ldots, n\}$ for which $e_{j(e)} \le -\gamma'$. As a result, recalling (18), we have that for each $e \in \mathcal{R}_1, v_{j(e)} = -(1 + \lambda_{j(e)}\delta)\epsilon_{j(e)}(e_{j(e)}, \hat{K}_{vj(e)}, r_{j(e)}))$ for some $\lambda_{j(e)} \in [-1, 1]$. Then

$$\begin{aligned}\dot{V}(e) \le\ & e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} \\ & - e_{j(e)}(1 + \lambda_{j(e)}\delta) N_{j(e)} \epsilon_{j(e)}(e_{j(e)}, \hat{K}_{vj(e)}, r_{j(e)})).\end{aligned}$$

Since

$$e_{j(e)}\epsilon_{j(e)}(e_{j(e)}, \hat{K}_{vj(e)}, r_{j(e)})) \ge \gamma' |\epsilon_{j(e)}(-\gamma', \hat{K}_{vj(e)}, r_{j(e)}))|$$

choosing $N_i^{(1)}$ in such a way that

$$M - \gamma' |\epsilon_i(-\gamma', \hat{K}_{vi}, r_i)|(1-\delta) N_i^{(1)} < 0$$

one guarantees that $\dot{V}(e) < 0$ for all $e \in \mathcal{R}_1$ and for all $v \in K(\Psi(-\epsilon))$.

Finally, we investigate $\dot{V}(e)$ for $e \in \mathcal{R}_3 = S \setminus (\mathcal{R}_1 \cup \mathcal{R}_2) = \cup_{\ell=1}^{2^n - 2} \mathcal{R}_{3\ell}$. As in Proposition 2, for each ℓ, $\dot{V}(e)$ is strictly negative on $\mathcal{D}_{3\ell} = \{e \in \mathcal{R}_{3\ell} : e_i > -\bar{e}_\ell\ \forall i \in \mathcal{L}_\ell, e_i \ge 0\ \forall i \in \bar{\mathcal{L}}_\ell\}$, with $\bar{e}_\ell > 0$. On $\mathcal{R}_{3\ell} \setminus \mathcal{D}_{3\ell}$, $e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)}$ is not guaranteed any further to be strictly negative, and one has to study the sign of

$$\dot{V}(e) = e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} + \sum_{i \in \mathcal{L}_\ell} e_i N_i v_i.$$

If one lets j be such that $\psi_j(1+\delta)^{-1} \le \min\{|\epsilon_i(-\gamma', \hat{K}_{vi}, r_i)|, |\epsilon_i(-\bar{e}_\ell, \hat{K}_{vi}, r_i)|\}$, i.e., if

$$\rho^j \le \frac{1}{\psi_0} \min_{\hat{K}_{vi} \in \mathcal{V}, r_i \in \mathcal{R}} \min\{|\epsilon_i(-\gamma', \hat{K}_{vi}, r_i)|, |\epsilon_i(-\bar{e}_\ell, \hat{K}_{vi}, r_i)|\}$$

for $i = 1, 2, \ldots, n$, then analogously to before

$$\begin{aligned}\dot{V}(e) =\ & e^T f(K_p, K_v, B^T q_f)|_{q_f = e + \hat{\mu}^{-1}(\hat{K}_v, r)} \\ & - \sum_{i \in \mathcal{L}_\ell} e_i(1 + \lambda_i\delta) N_i \epsilon_i(e_i, \hat{K}_{vi}, r_i).\end{aligned}$$

Let $N_{i\ell}^{(2)} > 0$ be such that $M - \sum_{i \in \mathcal{L}_\ell} \bar{e}_\ell N_{i\ell}^{(2)}(1 - \delta)|\epsilon_i(-\bar{e}_\ell, K_{vi}, r_i)| < 0$. Then, for all $N_i \ge N_{i\ell}^{(2)}$, for all $e \in \mathcal{R}_{3\ell} \setminus \mathcal{D}_{3\ell}, \dot{V}(e) < 0$. This is true for all $\ell \in \{1, \ldots, 2n-2\}$, and therefore $\dot{V}(e) < 0$ on $\mathcal{R}_3$.

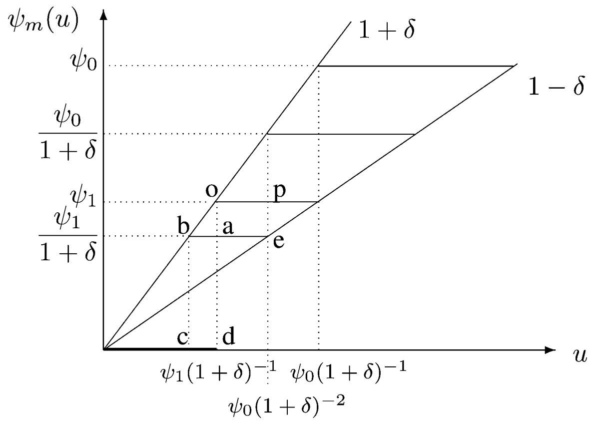

Fig. 4. Multivalued map $\psi_m(u)$ for $u > 0$, and with $j = 1$.

The thesis is inferred by letting, for $i = 1, \ldots, n, N_i^* = \max\{N_i^{(1)}, N_{i\ell}^{(2)}, \ell = 1, \ldots, 2n-2\}$ and $N_i \geq N_i^*$. ∎

Remark: From the proof above, one can observe that the gains $N_i^{(1)}, N_{i\ell}^{(2)}$ which define N_i^* in the quantized controllers are $(1-\delta)^{-1}$ times larger than the corresponding gains of the proportional controllers (cf. the Remark after the proof of Proposition 2 for an expression of these gains). In other words, the addition of quantizers introduces an uncertainty of magnitude $1-\delta$ in each input channel, and this can be counteracted by raising the quantized controllers' gains of a factor $(1-\delta)^{-1}$.

We cannot exclude that sliding modes may arise along those (switching) surfaces where $-(\hat{\mu}_i(\hat{K}_{vi}, q_{fi}) - r_i) = \psi_j(1+\delta)^{-1}$ for some i, j. This would give raise to chattering and it would jeopardize the possibility of transmitting the control values over a communication network, since a large bandwidth would be required. To this regard, we observe that it is always possible to replace the quantizers (14) with quantizers for which sliding modes are guaranteed to never occur. We follow the arguments of [8] and [15]. Let us introduce a new quantizer described by the following multivalued map:

$$\psi_m(u) = \begin{cases} \psi_i & \frac{\psi_i}{1+\delta} < u \leq \frac{\psi_i}{1-\delta}, \\ & 0 \leq i \leq j \\ \frac{\psi_i}{1+\delta} & \frac{\psi_i}{(1+\delta)^2} < u \leq \frac{\psi_i}{1-\delta^2}, \\ & 0 \leq i \leq j \\ 0 & u \leq \frac{\psi_j}{(1+\delta)^2}. \end{cases} \tag{20}$$

Fig. 4 gives a pictorial representation of the map in the case $j = 1$. Compared with the previous quantizer, in the quantizer (20), there are additional quantization levels equal to $\pm(\psi_i)/(1+\delta)$, $i = 0, 1, \ldots, j$. The figure helps to understand how the switching occurs with these quantizers. Suppose, for instance, that $\psi_m(u) = \psi_1$, u is decreasing and hits the point $\psi_1(1+\delta)^{-1}$ (in the figure this situation corresponds to point o). Then, a switching occurs and $\psi_m(u) = \psi_1(1+\delta)^{-1}$ (i.e., there is a jump from o to a in the Figure). If u decreases and becomes equal to $\psi_1(1+\delta)^{-2}$ (point b), then a new transition occurs (b $\rightarrow$ c). If, on the other hand, u increases until it reaches the value $\psi_0(1+\delta)^{-2}$ (point e), then a transition takes place from e to p.

From the above description, it should be clear that the new quantization levels and the new switching mechanism prevent the system to experience sliding modes and chattering. For the sake of simplicity, we shall refer to these quantizers as quantizers with hysteresis. One may then wonder whether Proposition 3 still holds. The answer is positive since the new quantization levels belong to the sets on the right-hand side of (18), and Proposition 3 was proven letting each component v_i of v range over these sets. Hence, Proposition 3 is still valid if we replace the quantizers (14) with the quantizers (20). The experimental results we present below are obtained using the quantizers with hystersis just introduced.

V. Experiments

Here, we present experimental results obtained using the proposed controllers on a specially designed setup. The setup corresponds to a "small" district heating system with four end-users with a network layout as the system shown in Fig. 5. Although this number is less than the number of end-users expected in real district heating systems by far, it makes it possible to build an operational setup in a laboratory, and it covers the main features of a real system. A photograph of the test setup is shown in Fig. 6.

The design of the piping of the test setup is aimed at emulating the dynamics of a real district heating system. However, due to physical constraints, the dynamics of the setup are approximately five to ten times faster than the dynamics expected in a real system.

The network comprises 29 components (valves, pumps and pipes) denoted by $c_1, \ldots, c_{29}$ and which correspond to the edges of the graph, and 26 nodes, denoted as $n_1, \ldots, n_{26}$. There are six pumps in the network. Pumps 1, 2, 4, and 5, labeled as $c_9, c_{27}, c_{23}, c_{19}$, are the end-users pumps, and they deliver the pressures $\Delta h_{p1}^e, \Delta h_{p2}^e, \Delta h_{p4}^e, \Delta h_{p5}^e$,, while Pumps 3 and 6, identified as components c_1 and c_5, are the booster pumps and deliver the pressures $\Delta h_{p3}', \Delta h_{p6}'$.

It is immediate to realize that there is a path between each pair of nodes, that is the graph is connected and Assumption 1 is satisfied. The end-user valves correspond to the components $c_{10}, c_{20}, c_{24}, c_{28}$. Each one of them is in series with a pipe and a pump. Moreover, if we remove from the graph all the edges which corresponds to the four end-user pipes, i.e., to $c_{11}, c_{21}, c_{25}, c_{29}$, we obtain a tree, that is a graph which does not have cycles. In other words, the end-user pipes $c_{11}, c_{21}, c_{25}, c_{29}$ are the chords of the graph, and Assumption 2 holds. Each chord identifies a fundamental loop, which is obtained by adding the chord to the tree. Hence, the fundamental loop associated to the chord c_{11} is given by the sequence of components $\{c_1, c_2, \ldots, c_{17}\}$. Similarly, the other fundamental loops are described by the sequences

$$\{c_1, c_2, c_{18}, c_{19}, c_{20}, c_{21}, c_{16}, c_{17}\}$$
$$\{c_1, c_2, c_3, c_{22}, c_{23}, c_{24}, c_{25}, c_{15}, c_{16}, c_{17}\}$$
$$\{c_1, c_2, c_3, c_4, c_5, c_6, c_{26}, c_{27}, c_{28}, c_{29}, c_{13}$$
$$c_{14}, c_{15}, c_{16}, c_{17}\}.$$

Each fundamental loop includes the component c_{17} which corresponds to the valve modeling the heat source. This implies that Assumption 3 also holds. Hence, the hydraulic network of Fig. 5 fulfills all the required Assumptions and both proportional and quantized controllers can be designed to guarantee semiglobal practical regulation.

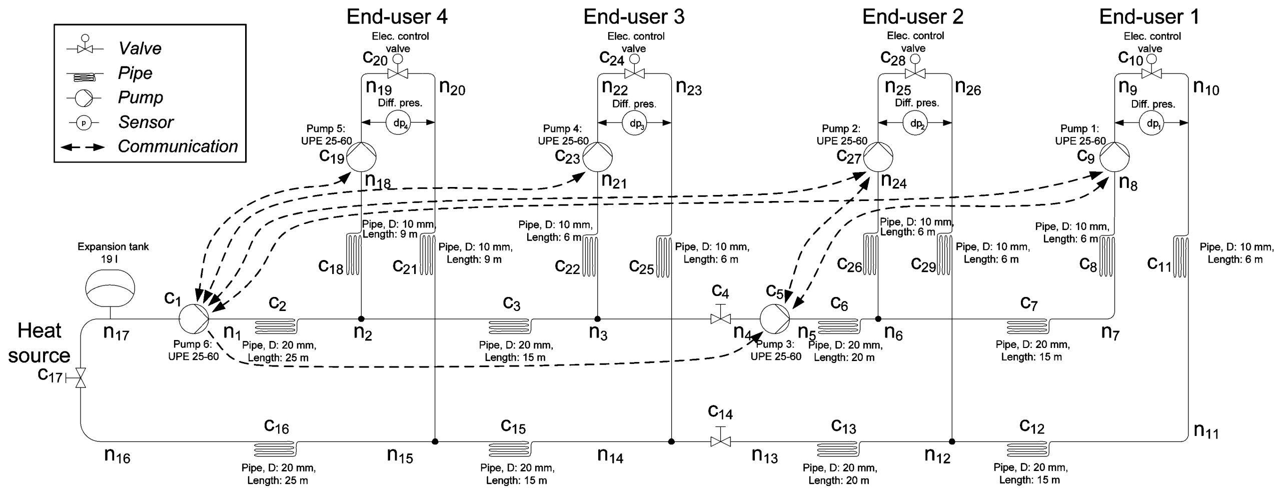

Fig. 5. Diagram of the hydraulic network of the test setup in Fig. 6. The system contains four end-user pumps and two booster pumps.

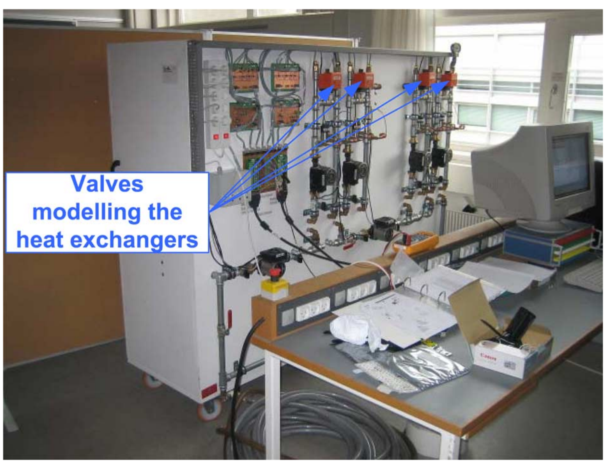

Fig. 6. Photograph of the test setup. The marked valves model the primary side of the heat exchanger of the end-users.

The control laws u_1, u_2, u_3, u_4 computed by the local controllers located at the end-user pumps $c_9, c_{27}, c_{23}, c_{19}$ are distributed to the pumps present in the network according to the rule

$$
\begin{aligned}
\Delta h'_{p6} &= 0.7 \min\{u_1, u_2, u_3, u_4\} \\
\Delta h'_{p3} &= 0.7 \min\{u_1, u_2\} - \Delta h'_{p6} \\
\Delta h^e_{p1} &= u_1 - \Delta h'_{p3} - \Delta h'_{p6} \\
\Delta h^e_{p2} &= u_2 - \Delta h'_{p3} - \Delta h'_{p6} \\
\Delta h^e_{p4} &= u_4 - \Delta h'_{p6} \\
\Delta h^e_{p5} &= u_4 - \Delta h'_{p6}
\end{aligned}
$$

where the first two expressions represent the pressures delivered by the booster pumps c_1 and c_5 and the last four corresponds to the pressures delivered by the end-user pumps $c_9, c_{27}, c_{23}, c_{19}$. It is straightforward to verify that the rule guarantees the six pumps to deliver positive pressures provided that u_1, u_2, u_3, u_4 are positive. We also remark that the rule defines a bidirectional communication graph among the pumps (see Fig. 5). As a matter of fact, it is clear from the first equality that the booster pump c_1 (which delivers the pressure $\Delta h'_{p6}$) must receive information from all of the controllers located at the end-user pumps $c_9, c_{27}, c_{23}, c_{19}$, while from the second equality it is understood that the end-user pumps c_9, c_{27} and the booster pump c_1 must transmit their delivered pressures to the booster pump c_5. It is interesting to observe that each pump transmits information only to pumps which are along its fundamental loop. These can be viewed as the "neighbors" of the pump.

To exemplify the performance of the controllers, a step response is tested, with the reference value changing from 0.2 [bar] to 0.45 [bar] and then back to 0.2 [bar].

The results of the test are illustrated in Fig. 7, where the top plot shows the controlled pressures at the end-users and the bottom plot shows the control inputs.

From the test results, it is immediately seen that there is a steady-state error between the measured pressures and the reference pressures. This is due to the fact that proportional controllers are used. Such steady-state errors can be reduced by adjusting the gains of the controllers. From the behavior of both the controlled pressures and the controller inputs, it is seen that the control system well behaves and that the steady state is achieved within a reasonably short period of time. The damped oscillation observed in the response is mainly due to the particular implementation of the controllers i.e., to be more specific, to a delay in the control hardware of the test setup. When considering the control of a real system, with dynamics five to ten times slower that the one of the test setup, we expect the effect to be less deleterious.

Second, results obtained with the quantized controllers given by Proposition 3 are shown. The design parameters of the quantizers (14) are chosen as $\psi_0 = 1, \delta = 0.25$, and $j = 3$. The gains of the controllers are set to $N_i = 1.5, i = 1, \ldots, 4$. They are determined by a trial-and-error procedure, starting from an initial value and then raising it until the desired regulation error is achieved. The theoretical results of the paper predict that such gains always exist.

To exemplify the performance of the quantized controllers, we carried out the same test as for the proportional controllers. The results of this test are shown in Fig. 8.

The experimental results confirm the theoretical analysis, namely that semiglobal practical regulation of the plant is guaranteed by proportional controllers. Moreover, the performance

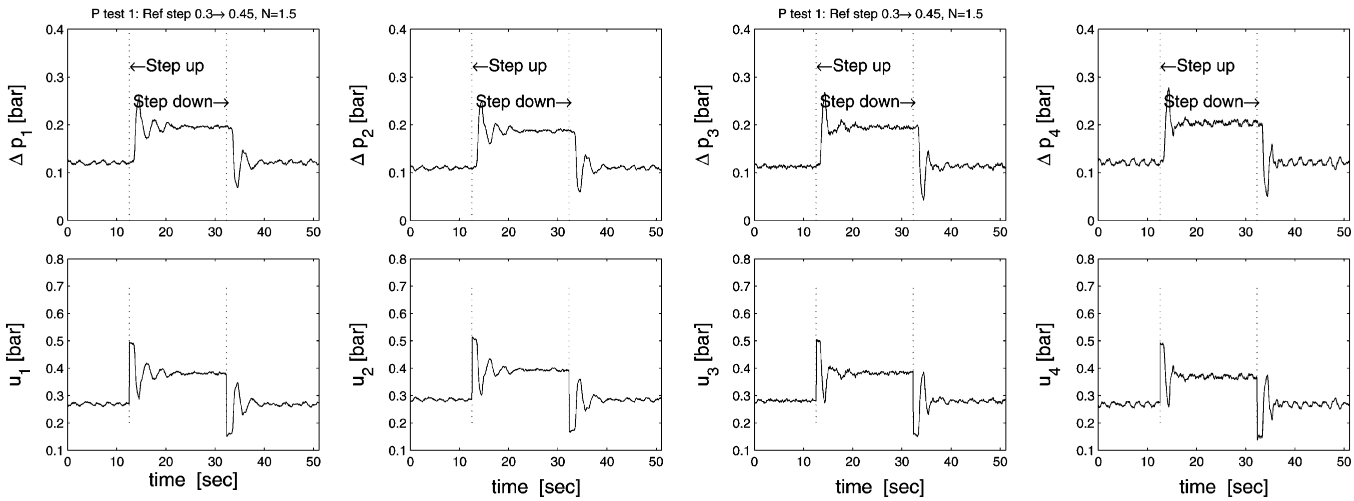

Fig. 7. Results obtained using the proposed proportional controllers. The top plot shows the controlled pressures and the bottom plot shows the control inputs.

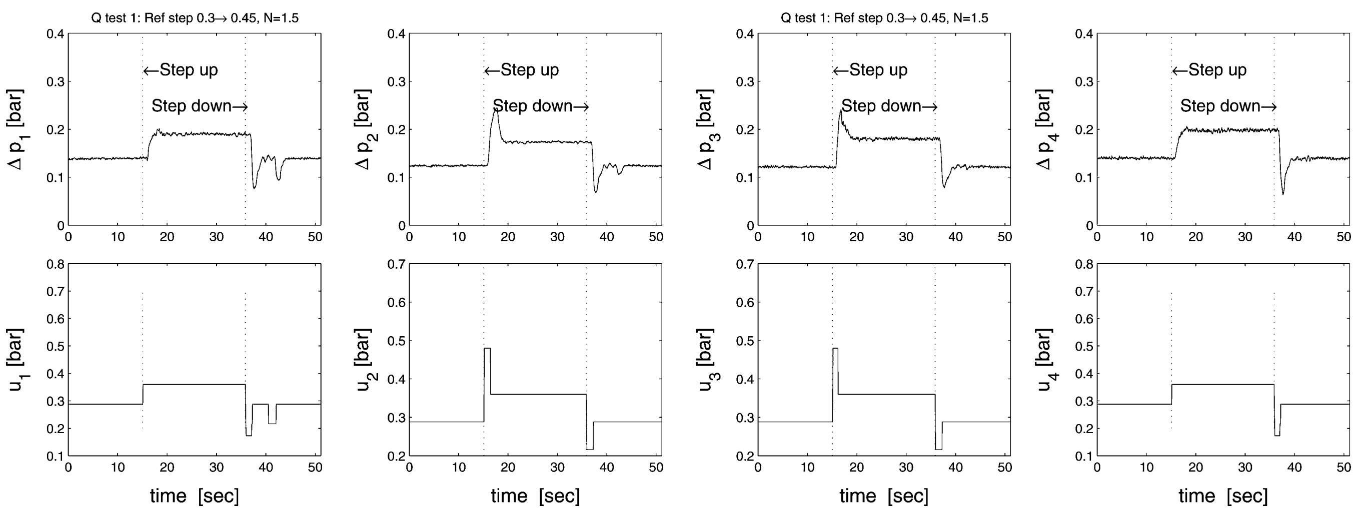

Fig. 8. Results obtained using the proposed quantized controllers. The top plot shows the controlled pressures and the bottom plot shows the quantized control inputs.

of the quantized controllers are comparable with those of the proportional controllers and this confirms the feasibility of the former as an effective industrial solution. The experiments emphasize that relatively large delays (as those introduced in these experiments by the hardware setup) can impose restrictions on the performance (oscillations) and on the accuracy of the controllers (large delays prevent from increasing the gains of the controllers and in turn from reducing the regulation error).

VI. Conclusion

The paper deals with the study of an industrial system distributed over a network. Positive proportional and quantized controllers have been proposed to practically regulate the pressure at the end-users and experimental validation of the results has been provided. The actual implementation of the quantized controller over an actual communication network in a urban environment is currently under investigation.

We plan to extend our findings to the case of proportional-integral controllers [18], [29], [30] and to include constraints on the sign of the flows as well [9]. Other research directions will focus on controller redesign when new end-users are added to the network, extension of the results to the case of open hydraulic networks [5], and robustification of the controllers to delays, the latter being a very important and challenging problem.

Finally, we point out the possibility to investigate the Pressure Regulation Problem with a different approach, in which each control law u_i renders the subsystem i input-to-state stable with respect to the state variables $q_{fj}, j \neq i$, affecting the subsystem, and in such a way that the coupling among the subsystems is weak in an appropriate sense. The approach rests on a small-gain theorem for networked nonlinear systems [7].

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