# *NETWORKED CONTROL FOR WATER DISTRIBUTION*

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# 2 Introduction

The aim of this project is to design and implement a networked control strategy for controlling pressure in a water distribution system. Improved pressure control in water distribution systems can contribute to saving water by decreasing water leakage and the number of pipe bursts.

Due to the fact that water distributed systems span over a large area, it is necessary to use a network to communicate with sensors and actuators. Networks adds complexity when designing a control system and certain strategies are implemented in order to overcome the effects of the network.

In this project, we shall design and implement a network control strategy for wAAUterWall test setup. To complete this project, the following steps will be taken. A mathematical model shall be derived to describe the system. A State space pressure controller shall be designed to control the system to minimise some cost function. A network will be simulated by running a control system on one computer and the wAAUterWall system on another computer; both connected to router for network communication. A control system shall be designed to minimise the effects of the network on the closed loop system.

# 3 System Components

The physical system is made of three main components; pipes, pumps and valves. These three components are modelled to have a relationship between differential pressure and water flow.

## 3.1 Pipe Model

The pipe model has the form

$$J\dot{q} = \Delta p - R|q|q - \Delta h$$

where J is the mass inertia of water, Δp is the differential pressure across the pipe, R represents the form and surface resistance and Δh is differential pressure due to a height difference.

## 3.2 Valve Model

There are two valve models, the first uses the pipe model and is done by assuming it has zero length and height difference. Thus, it has the form

$$0 = \Delta p - R|q|q$$

The other model is

$$q = k_v\sqrt{\Delta p}$$

where $k_v$ is specified by the valve's manufacturer.

## 3.3 Pump model

The pump model is given by

$$\Delta p = a_{h0}\omega^2 + a_{h1}q\omega - a_{h2}q^2$$

The $a_{h0}$, $a_{h1}$ and $a_{h2}$ parameters were provided but can be found experimentally.

# 4 System Diagram:

## 4.1 System Diagram of wAAUterWall:

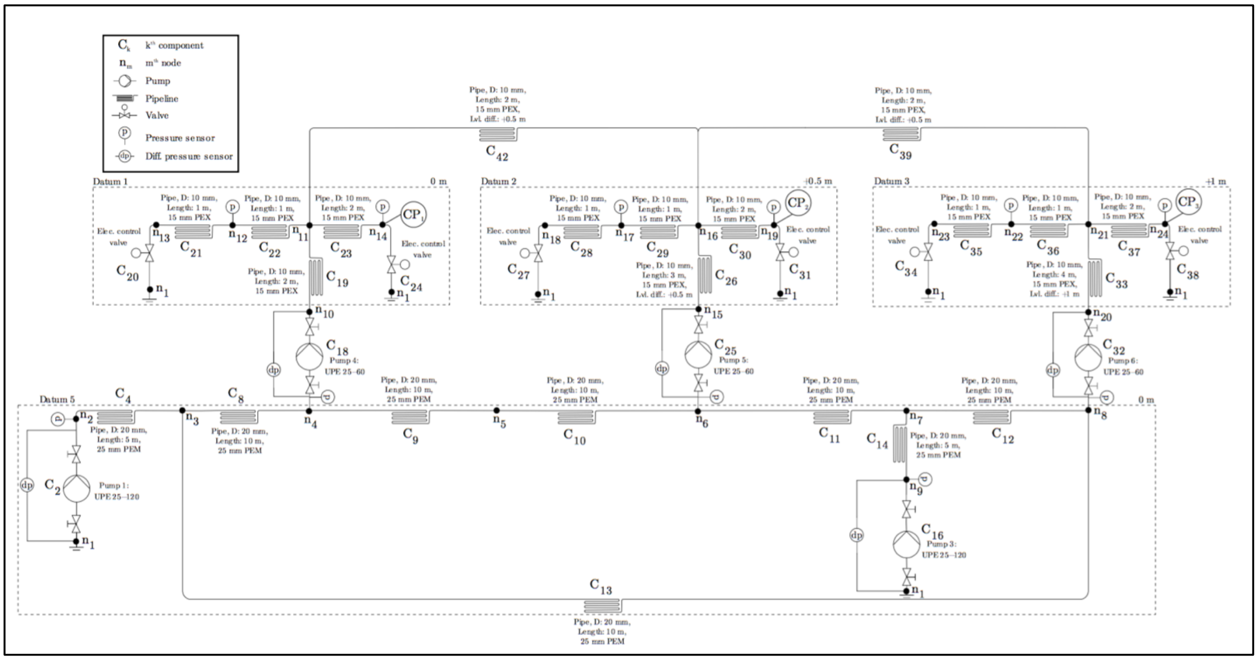


*Figure 1: wAAUterWall System Diagram*

## 4.2 Simplified Diagram for Analysis:

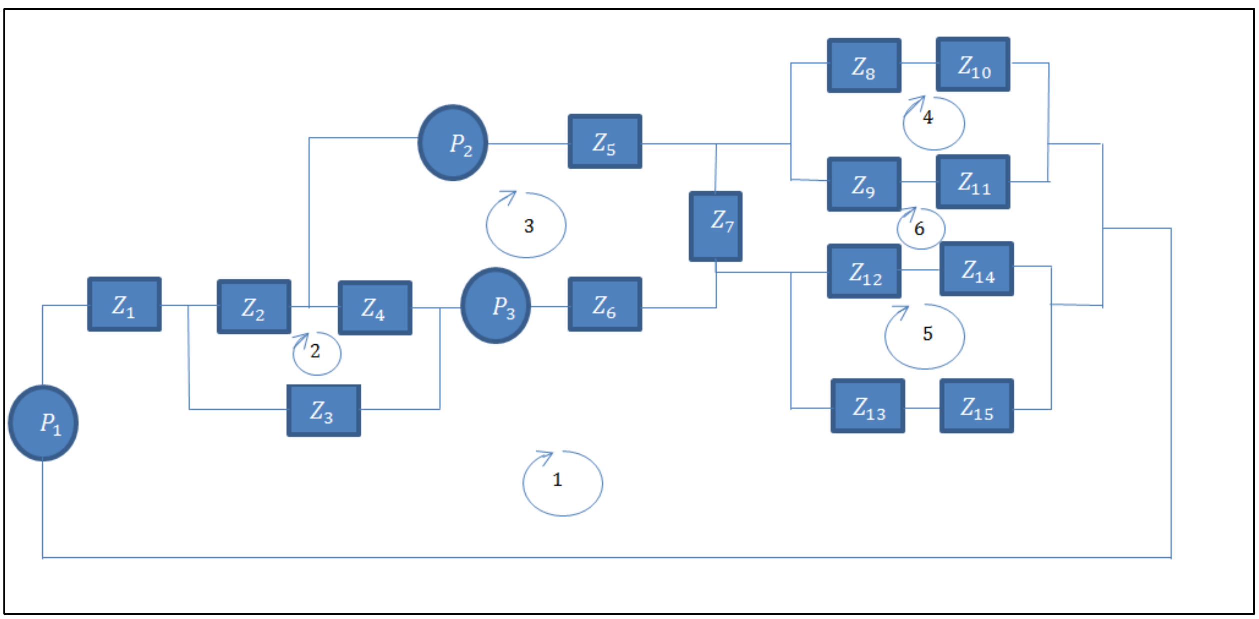



*Figure 2: Closed Loop System with two PMAs*

## 4.3 Equivalent components:

The equivalent components in reference to the figures above:

$Z_1 = C_4$

$Z_2 = C_8$

$Z_3 = C_{11} + C_{12} + C_{13}$

$Z_4 = C_9 + C_{10}$

$Z_5 = C_{19}$

$Z_6 = C_{26}$

$Z_7 = C_{42}$

$Z_8 = C_{21} + C_{22}$

$Z_9 = C_{23}$

$Z_{10} = C_{20}$

$Z_{11} = C_{24}$

$Z_{12} = C_{28} + C_{29}$

$Z_{13} = C_{30}$

$Z_{14} = C_{27}$

$Z_{15} = C_{31}$

## 4.4 Loop Equations:

The following equations were derived by applying Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL).

### 4.4.1 Loop 1:

$$V_1 - q_1 Z_1 - (q_1 - q_2)Z_3 + V_3 - (q_1 - q_3)Z_6 - (q_1 - q_5)Z_{13} - (q_1 - q_5)Z_{15} = 0$$

### 4.4.2 Loop 2:

$$-q_2 Z_2 - (q_2 - q_3)Z_4 + (q_1 - q_2)Z_3 = 0$$

### 4.4.3 Loop 3:

$$V_2 - q_3 Z_5 - (q_3 - q_6)Z_7 + (q_1 - q_3)Z_6 - V_3 + (q_2 - q_3)Z_4 = 0$$

### 4.4.4 Loop 4:

$$-q_4 Z_8 - q_4 Z_{10} - (q_4 - q_6)Z_{11} - (q_4 - q_6)Z_9 = 0$$

### 4.4.5 Loop 5:

$$-(q_5 - q_6)Z_{12} - (q_5 - q_6)Z_{14} + (q_1 - q_5)Z_{15} + (q_1 - q_5)Z_{13} = 0$$

### 4.4.6 Loop 6:

$$(q_3 - q_6)Z_7 + (q_4 - q_6)Z_9 + (q_4 - q_6)Z_{11} + (q_5 - q_6)Z_{14} + (q_5 - q_6)Z_{12} = 0$$

# 5 System Model:

## 5.1 Water Distribution Network Nonlinear Equations:

The KCL and KVL loops are translated to the water distribution network equations.

### 5.1.1 Equation 1

$$\dot{q}_1(J_4 + J_{11} + J_{12} + J_{13} + J_{26} + J_{30}) - \dot{q}_2(J_{11} + J_{12} + J_{13}) - \dot{q}_3 J_{26} - \dot{q}_4 J_{30} \\ = -R_4 q_1|q_1| - (R_{11} + R_{12} + R_{13})(q_1 - q_2)|q_1 - q_2| - R_{26}(q_1 - q_3)|q_1 - q_3| - (R_{30} \\ + R_{31})(q_1 - q_5)|q_1 - q_5| + \Delta p_1 + \Delta p_3$$

### 5.1.2 Equation 2

$$-\dot{q}_1(J_{11} + J_{12} + J_{13}) + \dot{q}_2(J_8 + J_9 + J_{10} + J_{11} + J_{12} + J_{13}) - \dot{q}_3(J_9 + J_{10}) \\ = -R_8 q_2|q_2| - (R_9 + R_{10})(q_2 - q_3)|q_2 - q_3| + (R_{11} + R_{12} + R_{13})(q_1 - q_2)|q_1 - q_2|$$

### 5.1.3 Equation 3

$$-\dot{q}_1 J_{26} + q_2'(J_9 + J_{10}) - \dot{q}_3(J_9 + J_{10} + J_{19} + J_{26} + J_{42}) - \dot{q}_6 J_{42} \\ = -R_{19} q_3|q_3| - R_{42}(q_3 - q_6)|q_3 - q_6| + R_{26}(q_1 - q_3)|q_1 - q_3| \\ + (R_9 + R_{10})(q_2 - q_3)|q_2 - q_3| + \varDelta p_2 - \varDelta p_3$$

### 5.1.4 Equation 4

$$\dot{q}_4(J_{21} + J_{22} + J_{23}) - \dot{q}_6 J_{23} = -(R_{20} + R_{21} + R_{22}) q_4|q_4| - (R_{23} + R_{24})(q_4 - q_6)|q_4 - q_6|$$

### 5.1.5 Equation 5

$$-\dot{q}_1 J_{30} + \dot{q}_5(J_{28} + J_{29} + J_{30}) - \dot{q}_6(J_{28} + J_{29}) \\ = -(R_{27} + R_{28} + R_{29})(q_5 - q_6)|q_5 - q_6| + (R_{30} + R_{31})(q_1 - q_5)|q_1 - q_5|$$

### 5.1.6 Equation 6

$$-\dot{q}_3 J_{42} - \dot{q}_4 J_{23} - q_5'(J_{28} + J_{29}) + \dot{q}_6(J_{23} + J_{28} + J_{29} + J_{42}) \\ = R_{42}(q_3 - q_6)|q_3 - q_6| + (R_{23} + R_{24})(q_4 - q_6)|q_4 - q_6| + (R_{27} + R_{28} + R_{29})(q_5 \\ - q_6)|q_5 - q_6|$$

## 5.2 Linearization using the Taylor Series Expansion:

### 5.2.1 Linearization method:

For a differential equation given in the form:

$$\dot{x} = f(x) + u$$

Linearizing at a given operating point:

$$\{x_0, u_0\}$$

Thus we can rewrite x and u in the small signal form:

$$x = x_0 + \tilde{x} \ \ and \ u = u_0 + \tilde{u}$$

We use the Taylor Series Expansion to approximate f(x) around an operating point and neglect the higher order terms of the Taylor Series Expansion, thus:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

At steady-state we have:

$$0 = f(x_0) + u_0$$

$$f(x_0) = -u_0$$

We now substitute the Taylor Series approximation back into the original differential equation:

$$\dot{x} \approx f(x_0) + f'(x_0)(x - x_0) + u$$

We can now substitute the small signal expressions as well as the equation from steady-state:

$$\dot{x} \approx -u_0 + f'(x_0)\tilde{x} + u_0 + \tilde{u}$$

Thus, the final expression is:

$$\dot{x} \approx f'(x_0)\tilde{x} + \tilde{u}$$

### 5.2.2 Generalized Taylor Series Expansions:

For

$$f(x) = x|x|$$

We can rewrite it as:

$$f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

We linearize at the operating point

$$\{x_0\}$$

Such that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

With the derivative of f(x):

$$f'(x) = 2\begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Since the piecewise function at x=0 is the same for both functions, we can rewrite it as:

$$f'(x) = 2|x|$$

Therefore:

$$f(x) \approx x_0|x_0| + 2|x_0|(x - x_0)$$

For

$$f(x, y) = (x - y)|x - y|$$

We can rewrite it as:

$$f(x) = \begin{cases} -(x - y)^2, & x < y \\ (x - y)^2, & x \geq y \end{cases}$$

We linearize at the operating point

$$\{x_0, y_0\}$$

Such that

$$f(x, y) \approx f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0, y_o} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{x_0, y_o} (y - y_0)$$

The partial derivatives of x and y are given below:

$$\frac{\partial f}{\partial x} = 2\begin{cases} -(x-y), & x < y \\ x-y, & x \geq y \end{cases}$$

$$\frac{\partial f}{\partial x} = 2|x-y|$$

$$\frac{\partial f}{\partial y} = -2\begin{cases} -(x-y), & x < y \\ x-y, & x \geq y \end{cases}$$

$$\frac{\partial f}{\partial y} = -2|x-y|$$

Therefore:

$$f(x,y) \approx f(x_0,y_0) + 2|x_0 - y_0|(x-x_0) - 2|x_0 - y_0|(y-y_0)$$

## 5.3 Linearized Equations:

The nonlinear equations for the system were linearized using the Taylor series. The linearized equations below describe the system's dynamic behaviour about a specific operating point.

**Note**: the linearized equations below are for the perturbation state.

The notation is such that *a,b* are constants from the above Taylor Series Expansion coefficients.

### 5.3.1 Equation 1:

$$\begin{aligned}\dot{q}_1(J_4+J_{11}+J_{12}+J_{13}+J_{26}+J_{30})-\dot{q}_2(J_{11}+J_{12}+J_{13})-\dot{q}_3J_{26}-\dot{q}_5J_{30}\\ =-R_4q_1|q_1|-(R_{11}+R_{12}+R_{13})(q_1-q_2)|q_1-q_2|-R_{26}(q_1-q_3)|q_1-q_3|-(R_{30}\\ +R_{31})(q_1-q_5)|q_1-q_5|+\Delta p_1+\Delta p_3\end{aligned}$$

Let:

$$J_4+J_{11}+J_{12}+J_{26}+J_{30}=J_{1-1}$$

$$J_{11}+J_{12}+J_{13}=J_{1-2}$$

$$R_{11}+R_{12}+R_{13}=R_{1-1}$$

$$R_{30}+R_{31}=R_{1-2}$$

So:

$$\begin{aligned}\dot{q}_1J_{1-1}-\dot{q}_2J_{1-2}-\dot{q}_3J_{26}-\dot{q}_5J_{30}\\ =-R_4q_1|q_1|-R_{1-1}(q_1-q_2)|q_1-q_2|-R_{26}(q_1-q_3)|q_1-q_3|\\ -R_{1-2}(q_1-q_5)|q_1-q_5|+\Delta p_1+\Delta p_3\end{aligned}$$

Using the Taylor Series method for linearization the following nonlinear terms are linearized to:

$$q_1|q_1|\to a_1q_1$$

$$(q_1-q_2)|q_1-q_2|\to a_{12}q_1+b_{12}q_2$$

$$(q_1-q_3)|q_1-q_3|\to a_{13}q_1+b_{13}q_3$$

$$(q_1-q_5)|q_1-q_5|\to a_{15}q_1+b_{15}q_5$$

Therefore, the linearized equation is given below:

$$\begin{aligned}\dot{q}_1J_{1-1}-\dot{q}_2J_{1-2}-\dot{q}_3J_{26}-\dot{q}_5J_{30}\\ =-R_4a_1q_1-R_{1-1}(a_{12}q_1+b_{12}q_2)-R_{26}(a_{13}q_1+b_{13}q_3)-R_{1-2}(a_{15}q_1+b_{15}q_5)\\ +\Delta P_1+\Delta P_3\end{aligned}$$

$$\dot{q}_1 J_{1-1} - \dot{q}_2 J_{1-2} - \dot{q}_3 J_{26} - \dot{q}_5 J_{30} = -(R_4 a_1 + R_{1-1} a_{12} + R_{26} a_{13} + R_{1-2} a_{15}) q_1 - R_{1-1} b_{12} q_2 - R_{26} b_{13} q_3 - R_{1-2} b_{15} q_5 + \Delta P_1 + \Delta P_3$$

With:

$$R_4 a_1 + R_{1-1} a_{12} + R_{26} a_{13} + R_{1-2} a_{15} = Rx_{11}$$

$$R_{1-1} b_{12} = Rx_{12}$$

$$R_{26} b_{13} = Rx_{13}$$

$$R_{1-2} b_{15} = Rx_{15}$$

Thus, the final linearized equation is:

$$\dot{q}_1 J_{1\_1} - \dot{q}_2 J_{1_2} - \dot{q}_3 J_{26} - \dot{q}_5 J_{30} = -Rx_{11} q_1 - Rx_{12} q_2 - Rx_{13} q_3 - Rx_{15} q_5 + \Delta P_1 + \Delta P_3$$

### 5.3.2 Equation 2:

$$-\dot{q}_1 (J_{11} + J_{12} + J_{13}) + \dot{q}_2 (J_8 + J_9 + J_{10} + J_{11} + J_{12} + J_{13}) - \dot{q}_3 (J_9 + J_{10}) = -R_8 q_2 |q_2| - (R_9 + R_{10})(q_2 - q_3)|q_2 - q_3| + (R_{11} + R_{12} + R_{13})(q_1 - q_2)|q_1 - q_2|$$

Let:

$$J_{11} + J_{12} + J_{13} = J_{2-1}$$

$$J_8 + J_9 + J_{10} + J_{11} + J_{12} + J_{13} = J_{2-2}$$

$$J_9 + J_{10} = J_{2-3}$$

$$R_9 + R_{10} = R_{2-1}$$

So:

$$-\dot{q}_1 J_{2-1} + \dot{q}_2 J_{2-2} - \dot{q}_3 J_{2-3} = -R_8 q_2 |q_2| - R_{2-1}(q_2 - q_3)|q_2 - q_3| + R_{1-2}(q_1 - q_2)|q_1 - q_2|$$

The linearized equation is:

$$-\dot{q}_1 J_{2-1} + \dot{q}_2 J_{2-2} - \dot{q}_3 J_{2-3} = -R_8 a_2 q_2 - R_{2-1}(a_{23} q_2 + b_{23} q_3) + R_{1-2}(a_{12} q_1 + a_{12} q_2)$$

$$-\dot{q}_1 J_{2-1} + \dot{q}_2 J_{2-2} - \dot{q}_3 J_{2-3} = R_{1-2} a_{12} q_1 + (R_{1-2} b_{12} - R_8 a_2 - R_{2-1} a_{23}) q_2 - R_{2-1} b_{23} q_3$$

With:

$$R_{1-2} a_{12} = Rx_{21}$$

$$R_{1-2} b_{12} - R_8 a_2 - R_{2-1} a_{23} = Rx_{22}$$

$$R_{2-1} b_{23} = Rx_{23}$$

Thus, the final linearized equation is:

$$-\dot{q}_1 J_{2-1} + \dot{q}_2 J_{2-2} - \dot{q}_3 J_{2-3} = Rx_{21} q_1 + Rx_{22} q_2 - Rx_{23} q_3$$

### 5.3.3 Equation 3:

$$-\dot{q}_1 J_{26} - \dot{q}_2 (J_9 + J_{10}) + \dot{q}_3 (J_9 + J_{10} + J_{19} + J_{26} + J_{42}) - \dot{q}_6 J_{42}$$
$$= -R_{19} q_3 |q_3| - R_{42}(q_3 - q_6)|q_3 - q_6| + R_{26}(q_1 - q_3)|q_1 - q_3|$$
$$+ (R_9 + R_{10})(q_2 - q_3)|q_2 - q_3| + \Delta p_2 - \Delta p_3$$

Let:

$$J_9 + J_{10} + J_{19} + J_{26} + J_{42} = J_{3-1}$$

So:

$$-\dot{q}_1 J_{26} - \dot{q}_2 J_{2_3} + \dot{q}_3 J_{3-1} - \dot{q}_6 J_{42}$$
$$= -R_{19} q_3 |q_3| - R_{42}(q_3 - q_6)|q_3 - q_6| + R_{26}(q_1 - q_3)|q_1 - q_3| + R_{2\_1}(q_2 - q_3)|q_2$$
$$- q_3| + \Delta p_2 - \Delta p_3$$

After linearization:

$$-\dot{q}_1 J_{26} - \dot{q}_2 J_{2-3} + \dot{q}_3 J_{3-1} - \dot{q}_6 J_{42}$$
$$= -R_{19} a_3 q_3 - R_{42}(a_{36} q_3 + b_{36} q_6) + R_{26}(a_{13} q_1 + b_{13} q_3) + R_{2\_1}(a_{23} q_2 + b_{23} q_3)$$
$$+ \Delta p_2 - \Delta p_3$$

$$-\dot{q}_1 J_{26} - \dot{q}_2 J_{2-3} + \dot{q}_3 J_{3-1} - \dot{q}_6 J_{42}$$
$$= R_{26} a_{13} q_1 + R_{2-1} a_{23} q_2 + (R_{26} b_{13} + R_{2-1} b_{23} - R_{19} a_3 - R_{42} a_{36}) q_3 - R_{42} b_{36} q_6$$
$$+ \Delta p_2 - \Delta p_3$$

With:

$$R_{26} a_{13} = Rx_{31}$$

$$R_{2-1} a_{23} = Rx_{32}$$

$$R_{26} b_{13} + R_{2-1} b_{23} - R_{19} a_3 - R_{42} a_{36} = Rx_{33}$$

$$R_{42} b_{36} = Rx_{36}$$

Thus, the final linearized equation is:

$$-\dot{q}_1 J_{26} - \dot{q}_2 J_{2-3} + \dot{q}_3 J_{3-1} - \dot{q}_6 J_{42} = Rx_{31} q_1 + Rx_{32} q_2 + Rx_{33} q_3 - Rx_{36} q_6 + \Delta p_2 - \Delta p_3$$

### 5.3.4 Equation 4:

$$\dot{q}_4 (J_{21} + J_{22} + J_{23}) - \dot{q}_6 J_{23} = -(R_{20} + R_{21} + R_{22})|q_4|(q_4) - (R_{23} + R_{24})|q_4 - q_6|(q_4 - q_6)$$

Let

$$J_{21} + J_{22} + J_{23} = J_{4-1}$$

$$R_{20} + R_{21} + R_{22} = R_{4-1}$$

$$R_{23} + R_{24} = R_{4-2}$$

So:

$$\dot{q}_4 J_{4-1} - \dot{q}_6 J_{23} = -R_{4-1}|q_4|(q_4) - R_{4-2}|q_4 - q_6|(q_4 - q_6)$$

After Linearization:

$$\dot{q}_4 J_{4-1} - \dot{q}_6 J_{23} = -R_{4-1} a_4 q_4 - R_{4-2}(a_{46} q_4 + b_{46} q_6)$$

$$\dot{q}_4 J_{4-1} - \dot{q}_6 J_{23} = -(R_{4-1} a_4 + R_{4-2} a_{46}) q_4 - R_{4-2} b_{46} q_6$$

With:

$$R_{4-1} a_4 + R_{4-2} a_{46} = Rx_{44}$$

$$R_{4-2} b_{46} = Rx_{46}$$

Thus, the final linearized equation is:

$$\dot{q}_4 J_{4-1} - \dot{q}_6 J_{23} = -Rx_{44} q_4 - Rx_{46} q_6$$

### 5.3.5 Equation 5:

$$\begin{aligned} -\dot{q}_1 J_{30} + \dot{q}_5 (J_{28} + J_{29} + J_{30}) - \dot{q}_6 (J_{28} + J_{29}) \\ = -(R_{27} + R_{28} + R_{29})(q_5 - q_6)|q_5 - q_6| + (R_{30} + R_{31})(q_1 - q_5)|q_1 - q_5| \end{aligned}$$

Let

$$J_{28} + J_{29} + J_{30} = J_{5-1}$$

$$J_{28} + J_{29} = J_{5-2}$$

$$R_{27} + R_{28} + R_{29} = R_{5-1}$$

$$R_{30} + R_{31} = R_{5\_2}$$

So:

$$-\dot{q}_1 J_{30} + \dot{q}_5 J_{5-1} - \dot{q}_6 J_{5-2} = -R_{5-1}(q_5 - q_6)|q_5 - q_6| + R_{1-2}(q_1 - q_5)|q_1 - q_5|$$

After Linearization:

$$-\dot{q}_1 J_{30} + \dot{q}_5 J_{5-1} - \dot{q}_6 J_{5-2} = -R_{5-1}(a_{56} q_5 + b_{56} q_6) + R_{1-2}(a_{15} q_1 + b_{15} q_5)$$

$$-\dot{q}_1 J_{30} + \dot{q}_5 J_{5-1} - \dot{q}_6 J_{5-2} = R_{1-2} a_{15} q_1 + (R_{1-2} b_{15} - R_{5-1} a_{56}) q_5 - R_{5-1} b_{56} q_6$$

With:

$$R_{1-2}a_{15} = Rx_{51}$$

$$R_{1-2}b_{15} - R_{5-1}a_{56} = Rx_{55}$$

$$R_{5-1}b_{56} = Rx_{56}$$

Thus, the final linearized equation is:

$$-\dot{q}_1 J_{30} + \dot{q}_5 J_{5-1} - \dot{q}_6 J_{5-2} = Rx_{51}q_1 + Rx_{55}q_5 - Rx_{56}q_6$$

### 5.3.6 Equation 6:

$$-\dot{q}_3 J_{42} - \dot{q}_4 J_{23} - \dot{q}_5(J_{28} + J_{29}) + \dot{q}_6(J_{23} + J_{28} + J_{29} + J_{42}) = R_{42}(q_3 - q_6)|q_3 - q_6| + (R_{23} + R_{24})(q_4 - q_6)|q_4 - q_6| + (R_{27} + R_{28} + R_{29})(q_5 - q_6)|q_5 - q_6|)$$

Let

$$J_{23} + J_{28} + J_{29} + J_{42} = J_{6-1}$$

So

$$-\dot{q}_3 J_{42} - \dot{q}_4 J_{23} - \dot{q}_5 J_{5-2} + \dot{q}_6 J_{6-1} = R_{42}(q_3 - q_6)|q_3 - q_6| + R_{4-2}(q_4 - q_6)|q_4 - q_6| + R_{5-1}(q_5 - q_6)|q_5 - q_6|)$$

After Linearization:

$$-\dot{q}_3 J_{42} - \dot{q}_4 J_{23} - \dot{q}_5 J_{5-2} + \dot{q}_6 J_{6-1} = R_{42}(a_{36}q_3 + b_{36}q_6) + R_{4-2}(a_{46}q_4 + b_{46}q_6) + R_{5-1}(a_{56}q_5 + b_{56}q_6)$$

$$-\dot{q}_3 J_{42} - \dot{q}_4 J_{23} - \dot{q}_5 J_{5-2} + \dot{q}_6 J_{6-1} = R_{42}a_{36}q_3 + R_{4-2}a_{46}q_4 + R_{5-1}a_{56}q_5 + (R_{42}b_{36} + R_{4-2}b_{46} + R_{5-1}a_{56})q_6$$

With:

$$R_{42}a_{36} = Rx_{63}$$

$$R_{4-2}a_{46} = Rx_{64}$$

$$R_{5-1}a_{56} = Rx_{65}$$

$$R_{42}b_{36} + R_{4-2}b_{46} + R_{5-1}a_{56} = Rx_{66}$$

Thus, the final linearized equation is:

$$-\dot{q}_3 J_{42} - \dot{q}_4 J_{23} - \dot{q}_5 J_{5-2} + \dot{q}_6 J_{6-1} = Rx_{63}q_3 + Rx_{64}q_4 + Rx_{65}q_5 + Rx_{66}q_6$$

## 5.4 Matrix form of equations:

$$\begin{bmatrix} J_{1-1} & -J_{1-2} & -J_{26} & 0 & -J_{30} & 0 \\ -J_{2-1} & J_{2-2} & -J_{2-3} & 0 & 0 & 0 \\ -J_{26} & -J_{2-3} & J_{3-1} & 0 & 0 & -J_{42} \\ 0 & 0 & 0 & J_{4-1} & 0 & -J_{23} \\ -J_{30} & 0 & 0 & 0 & J_{5-1} & -J_{5-2} \\ 0 & 0 & -J_{42} & -J_{23} & -J_{5-2} & J_{6-1} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$= \begin{bmatrix} -Rx_{11} & -Rx_{12} & -Rx_{13} & 0 & -Rx_{15} & 0 \\ Rx_{21} & Rx_{22} & -Rx_{23} & 0 & 0 & 0 \\ Rx_{31} & Rx_{32} & Rx_{33} & 0 & 0 & -Rx_{36} \\ 0 & 0 & 0 & -Rx_{44} & 0 & -Rx_{46} \\ Rx_{51} & 0 & 0 & 0 & Rx_{55} & -Rx_{56} \\ 0 & 0 & Rx_{63} & Rx_{64} & Rx_{65} & Rx_{66} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \end{bmatrix}$$

# 6 State Space Equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$$

## 6.1 System Equation:

Using the linearized equations above in the matrix form

$$\boldsymbol{J\dot{q}} = \boldsymbol{Rq} + \boldsymbol{G\Delta p}$$

This must be transformed into the state-space system equation form, such that:

$$\dot{\boldsymbol{q}} = \boldsymbol{J^{-1}}(\boldsymbol{Rq} + \boldsymbol{G\Delta p})$$

$$\dot{\boldsymbol{q}} = \boldsymbol{J^{-1}Rq} + \boldsymbol{J^{-1}G\Delta p}$$

Where

$$\boldsymbol{A} = \boldsymbol{J^{-1}R}$$

$$\boldsymbol{B} = \boldsymbol{J^{-1}G}$$

Thus in the form:

$$\dot{\boldsymbol{q}} = \boldsymbol{Aq} + \boldsymbol{B\Delta p}$$

## 6.2 Output Equation:

$$y_1 = R_{24}(q_4 - q_6)|q_4 - q_6|$$

Linearized equation:

$$y_1 = R_{24}(a_{46}q_4 + b_{46}q_6)$$

$$y_1 = R_{24}a_{46}q_4 + R_{24}b_{46}q_6$$

$$y_2 = R_{31}(q_1 - q_5)|q_1 - q_5|$$

Linearized equation:

$$y_2 = R_{31}(a_{15}q_1 + b_{15}q_5)$$

$$y_2 = R_{31}a_{15}q_1 + R_{31}b_{15}q_5$$

$$y = \begin{bmatrix} 0 & 0 & 0 & R_{24}a_{46} & 0 & R_{24}b_{46} \\ R_{31}a_{15} & 0 & 0 & 0 & R_{31}b_{15} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

Thus it is in the form:

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q}$$

# 7 Operating Points

The state-space model derived in Section 0 was a linear model of the system. Since the system is nonlinear, an operating point needs to be chosen in order to complete get numerical values for the Taylor Series' coefficients, thus completing the linear model.

This was done by selecting an operating point for which the system shall operate in a nearby neighborhood. The operating point of the system was carefully selected in order to allow the pumps to operate near the middle of its operating range. The input signal for the pumps is a voltage signal between 0V and 5V. The inputs were also chosen to ensure there is flow in the component $C_{42}$ (The pipe connecting the two PMAs). The inputs to the three pumps, $C_2$, $C_{18}$ and $C_{25}$ are shown in the table below. All Components $C_n$ are in reference to Figure 1.

| PUMP | INPUT VOLTAGE [V] (MAX 5V) | NORMALISED INPUT SIGNAL | DIFFERENTIAL PRESSURE [BAR] |
|---|---|---|---|
| **$C_2$** | 2.7 | 0.54 | 0.4153 |
| **$C_{18}$** | 2.6 | 0.52 | 0.4121 |
| **$C_{25}$** | 2.5 | 0.5 | 0.3616 |

*Table 1: Input Voltage Signals and Differential Pressures of Pumps*

The figure below shows the differential pressures measured across the pumps for the input signals given in Table 1 above.

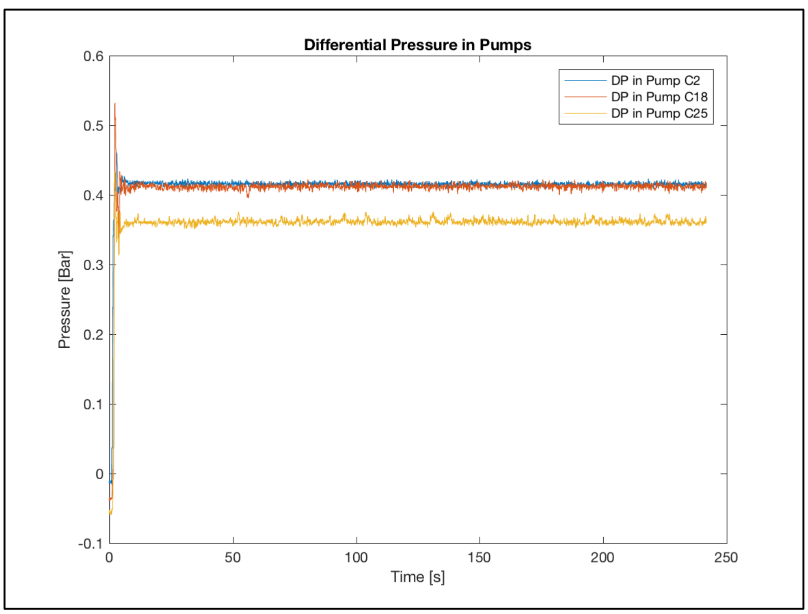



*Figure 3: Differential Pressure in the Pumps given the input signals from Table 1*

The output pressures at $n_{14}$ and $n_{19}$ are shown in the figure below. The nodes $n_n$ are in reference to Figure 1. These pressures are used to calculate when calculating the flows at the operating point.

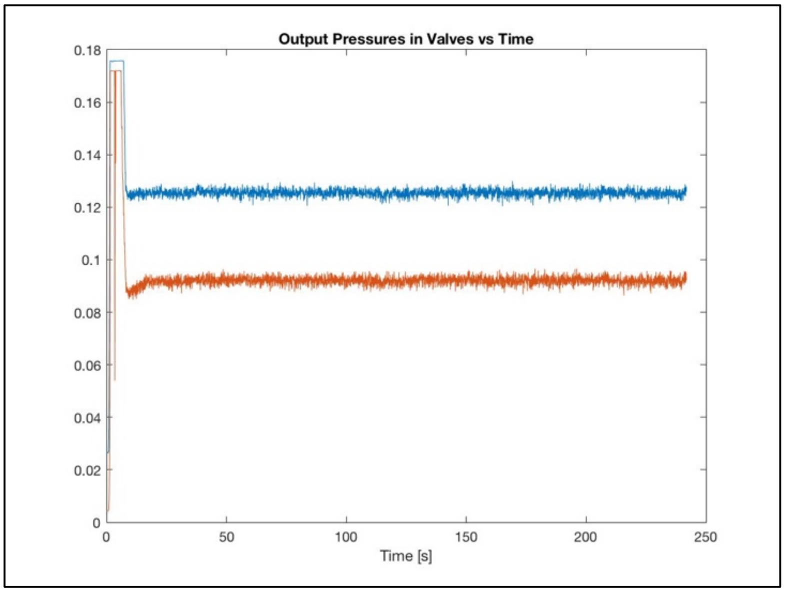



*Figure 4: wAUUterWall Output Pressures at Nodes $n_{14}$ and $n_{19}$*

## 7.1 Flow Calculations:

The pressures before $n_{14}$ and $n_{19}$ were measured as the outputs to the system. Pressures were also recorded at $n_{12}$ and $n_{17}$. The pressures before pumps $n_4$ and $n_6$ were also recorded. These pressures are shown in the table below. Using these data points, the flows through the loops were calculated using the following equations.

| NODE | PRESSURE [BAR] |
|---|---|
| **$n_4$** | 0.0889 |
| **$n_6$** | 0.0617 |
| **$n_{12}$** | 0.1389 |
| **$n_{14}$** | 0.1254 |
| **$n_{17}$** | 0.1037 |
| **$n_{19}$** | 0.0921 |

*Table 2: Measured Pressures at Nodes from System Diagram*

The flow through the open valve $C_{24}$ and $C_{31}$ are calculated as shown below:

$$q_6 - q_4 = \sqrt{p_{24}}$$

$$q_1 - q_5 = \sqrt{p_{31}}$$

Using the flow found above, the pressure at node $n_{11}$ was calculated using the following equation:

$$p_{n11} = R_{23}(q_6 - q_4)|q_6 - q_4| + p_{24}$$

Thus the flows were calculated as shown below:

$$q_4 = \sqrt{\frac{p_{n11} - p_{n12}}{R_{22}}}$$

$$q_6 = \sqrt{p_{24}} + q_4$$

The same process was used for the second PMA.

$$p_{n16} = R_{30}(q_1 - q_5)|q_1 - q_5| + p_{31}$$

$$q_5 - q_6 = \sqrt{\frac{p_{n16} - p_{n17}}{R_{29}}}$$

$$q_5 = \sqrt{\frac{p_{n16} - p_{n17}}{R_{29}}} + q_6$$

$$q_1 = \sqrt{p_{31}} + q_5$$

The flow through pipe $C_{42}$ was solved as shown below:

$$q_3 - q_6 = \sqrt{\frac{p_{n11} - p_{n16} - h}{R_{42}}}$$

$$q_3 = \sqrt{\frac{p_{n11} - p_{n16} - h}{R_{42}}} + q_6$$

The flow through pipes $C_9$ and $C_{10}$ were calculated as shown below:

$$q_2 - q_3 = \sqrt{\frac{p_{n4} - p_{n6}}{R_9 + R_{10}}}$$

$$q_2 = \sqrt{\frac{p_{n4} - p_{n6}}{R_9 + R_{10}}} + q_3$$

## 7.2 Calculated Flows at Operating Point:

In the table below, the calculated loop flows are shown.

| LOOP | FLOW [$m^3$/h] |
|---|---|
| $q_1$ | 1.2410 |
| $q_2$ | 0.4732 |
| $q_3$ | 0.0183 |
| $q_4$ | 0.2616 |
| $q_5$ | 0.9376 |

| | |
|---|---|
| **$q_6$** | 0.6157 |

*Table 3: Loop flows at the chosen operating point*

The output flows measured were:

$$q_6 - q_4 = 0.3542\ m^3/h$$

$$q_1 - q_5 = 0.3035\ m^3/h$$

From analysis of the system diagram Figure 1, flow through valve $C_{20}$ should be equivalent to the flow through $C_{24}$. Similarly, the flow through valve $C_{27}$ should be equivalent to the flow through $C_{31}$. This assumption is due to the pipe characteristics for pipes $C_{21}$+$C_{22}$ is the same as $C_{23}$ and similarly for $C_{28}$+$C_{29}$ and $C_{30}$. Thus, the overall output flow can be calculated as:

$$2(q_6 - q_4) + 2(q_1 - q_5) = 2(0.3542) + 2(0.3035) = 1.3154\ m^3/h$$

This gives a similar result to the input system flow q, thus it can be concluded as a reasonable value.

The flow $q_3$ is significantly smaller than the other loop flows. This is also acceptable as it can be seen from Figures 1 and 2 that the flow through the pipe connecting the PMAs, $C_{42}$, is given by the equation:

$$q_3 - q_6 = 0.0183 - 0.6157 = -0.5974\ m^3/h$$

This is in the direction from node $n_{11}$ to $n_{16}$. This is also reasonable as there is a 0.5m height difference between the two nodes, thus causing the flow to go from PMA 2 (at height 0.5m) to PMA 1 (at height 0m).

# 8 System Modelling:

The model of the system derived in Chapter 5 uses theoretical values for the pipe parameters and may not accurately represent the physical system. Simulating the physical system with the input pressures as described in Table 1 gives the following pressures in the outputs as shown in the figure below.

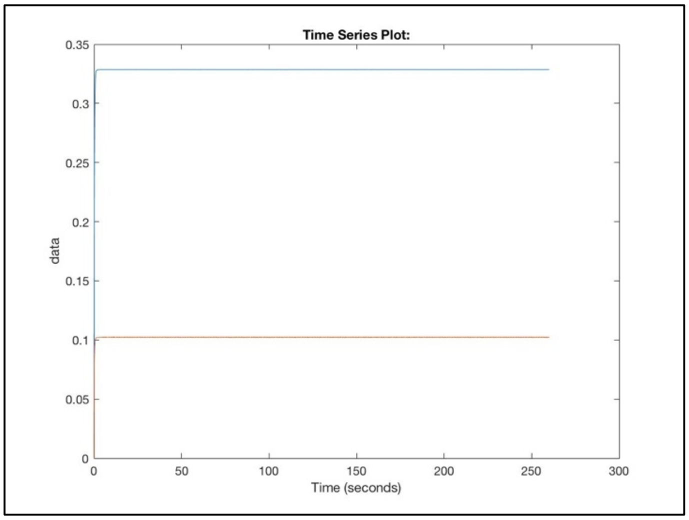



*Figure 5: Theoretical Outputs*

Comparing the theoretical outputs in Figure 5 and the physical outputs in Figure 4, it can be seen that there is a significant difference between the mathematical model and physical system. These differences can be due to a number of factors, including that the bending in the pipes was not taken into consideration. Also, the true form losses in the pipes are unknown.

To correctly model the system, a parameter estimation technique is employed to fit the model to the physical systems response.

The MATLAB System Identification Toolbox was used to estimate these parameters. They grey-box model estimation method was used.

The R parameters used to calculate the operating point flows, above, were fixed. The rest of the pipe R parameters were free to be changed.

Using the physical lab setup, the pump pressures were varied near the chosen operating point and the inputs and outputs of the system were recorded. These inputs and outputs of the system are shown below.

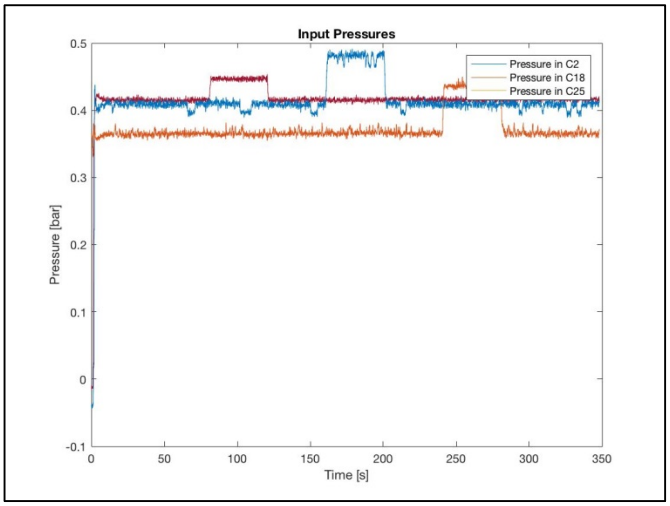



*Figure 6: Input Pressures for Parameter Estimation*

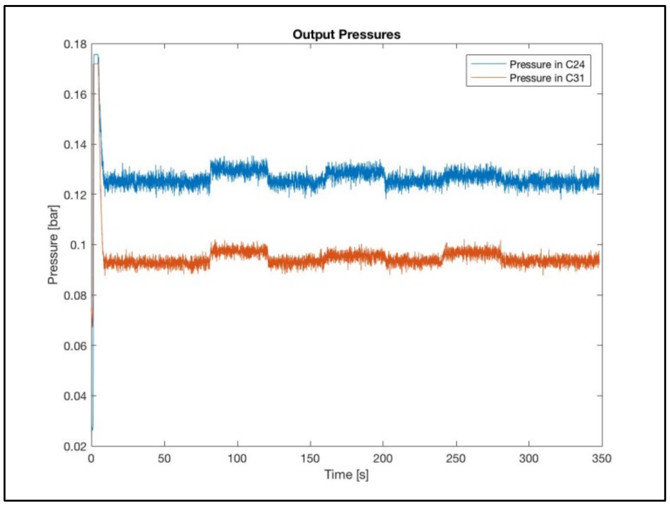



*Figure 7: Output Pressures for Parameter Estimation*

This data was processed such that only the steady-state values of the test were taken into consideration. Using the system equation shown below.

$$\boldsymbol{J\dot{q}} = \boldsymbol{Rq} + \boldsymbol{G\Delta p}$$

For the system to be in steady-state, the derivative $\dot{\boldsymbol{q}}$ will be $\boldsymbol{0}$. Thus we can rewrite the system equation as

$$\mathbf{0} = \boldsymbol{Rq} + \boldsymbol{G\Delta p}$$

This equation can be used to estimate the R parameters of the model such that model will fit the physical system.

The processed steady-state inputs and outputs of the system is shown below.

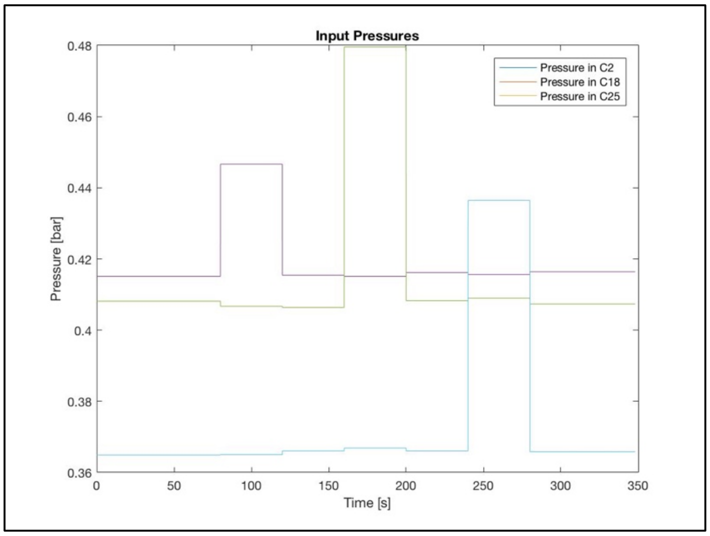



*Figure 8: Processed Input Pressures for Parameter Estimation*

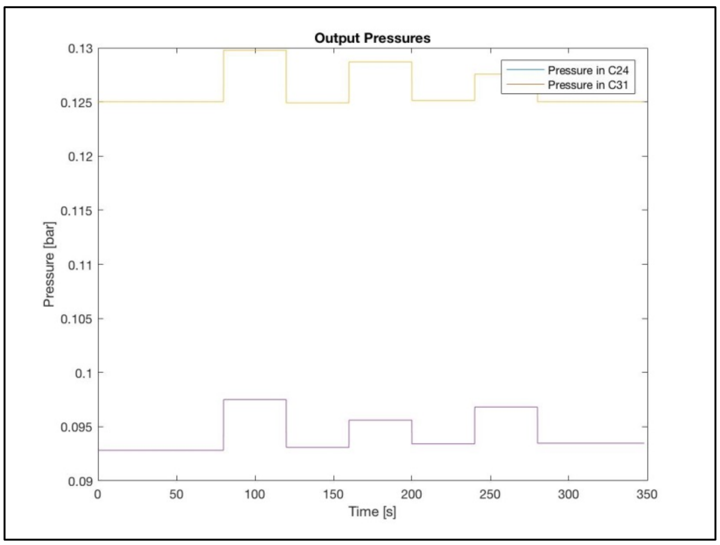



*Figure 9: Processed Outputs for Parameter Estimation*

This processed steady-state data was used to estimate the R parameters for the system.

Another test on using the lab setup was performed. The input data shown below was also given as an input to the updated system model.

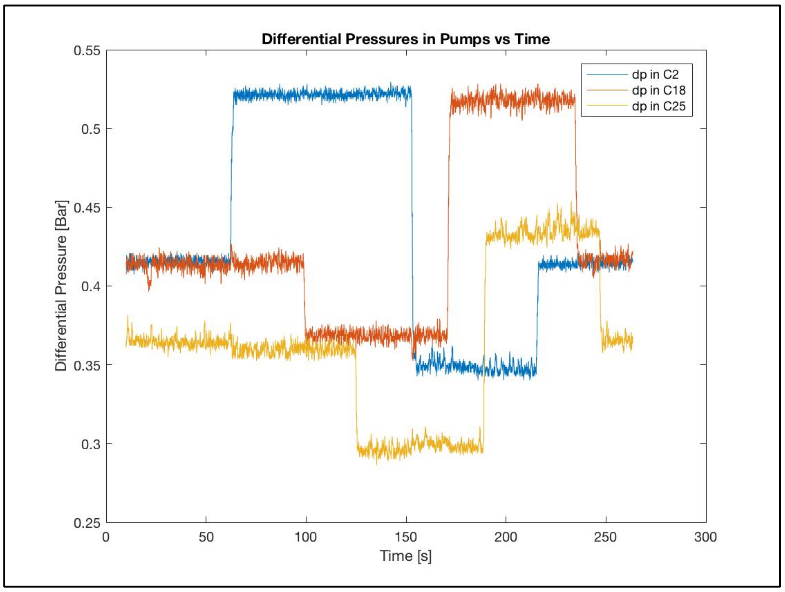



*Figure 10: Input Pressures for Model Verification*

The physical system's output versus the system model's output is shown in the figure below.

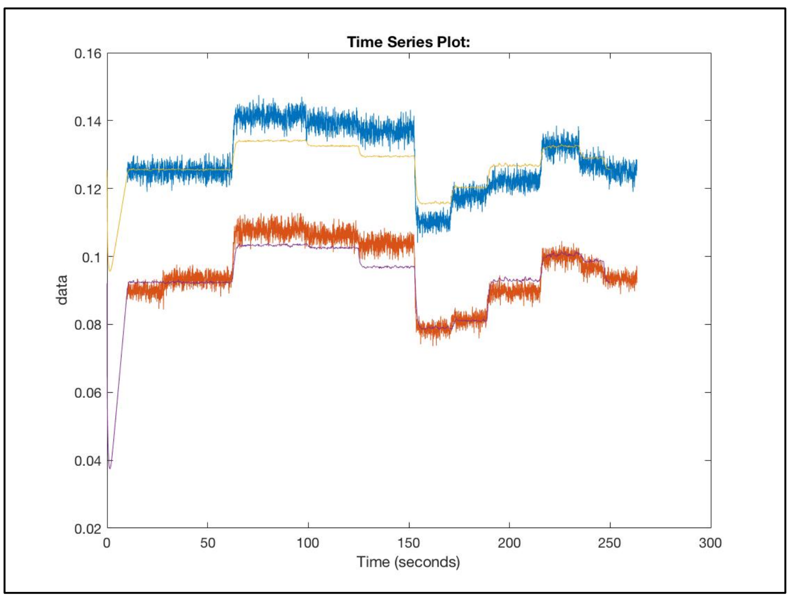



*Figure 11: Physical Output Pressure compared to Modelled Output Pressure*

In Figure 11, significant deviations between the linear model and the physical system can be seen between 50s and 150s. This deviation is due to inaccuracies between the linear model and nonlinear physical system. Smaller deviations can be seen between 190s and 220s are also due to these inaccuracies.

# 9 Networking

System used in this project is a network control system where LQR controller is connected as feedback from sensors to pumps. This introduced the delays from sensors to controlling part and also from controlling parts to actuators. In Water distribution network, delay is occurred in system by controllers, actuators and network controlling part exchanging data .Delay from controlling part of actuators are significantly high compare to others and others are negligible. Measured delay in pumps of the system is around 0.7 seconds:

$$d_T = d_A + d_S + d_C \sim d_A = 0.7\ second$$

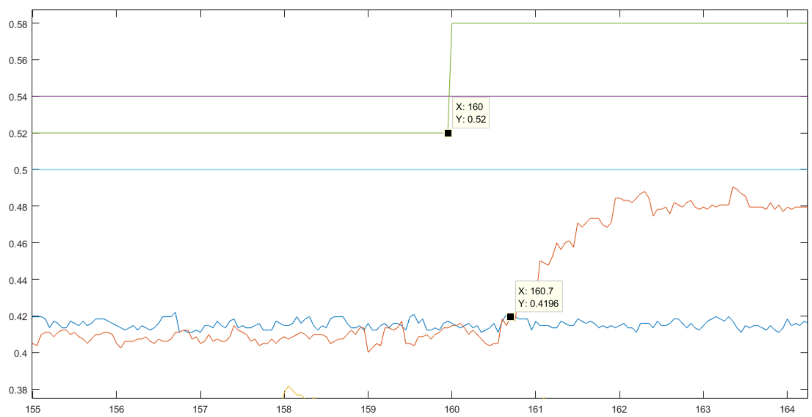



*Figure 12 time difference in change of voltage and change of pressure*

After that the delay was introduced to the system simulation using Pade approximation.

$$Z(t) = U(t - 0.7)$$

Where Z(t) is input without delay and U(t) is input with time delay. By taking these equation to Laplace domain:

$$Z(s) = U(s)e^{-0.7s}$$

By using Pade approximation for first order: (p=q=1)

$$e^{-T_d s} = \frac{1 - \frac{T_d}{2}s}{1 + \frac{T_d}{2}s}$$

$$Z(s) = \frac{1 - \frac{T_d}{2}s}{1 + \frac{T_d}{2}s}U(s)$$

After, this equation is taken to state space model:

$$Z(t) + \frac{T_d}{2} Z'(t) = U(t) - \frac{T_d}{2} U'(t)$$

$$Z(t) - U(t) = -\frac{T_d}{2} (U'(t) + Z'(t))$$

X(t) is taken as : $$X(t) = Z(t) + U(t)$$

So: $$X'(t) = Z'(t) + U'(t)$$

After implying the X(t) to equation:

$$X(t) - 2U(t) = -\frac{T_d}{2} X'(t)$$

So:

$$X'(t) = -\frac{1}{\frac{T_d}{2}} X(t) + \frac{2}{\frac{T_d}{2}} U(t)$$

$$Y(t) = X(t) - U(t)$$

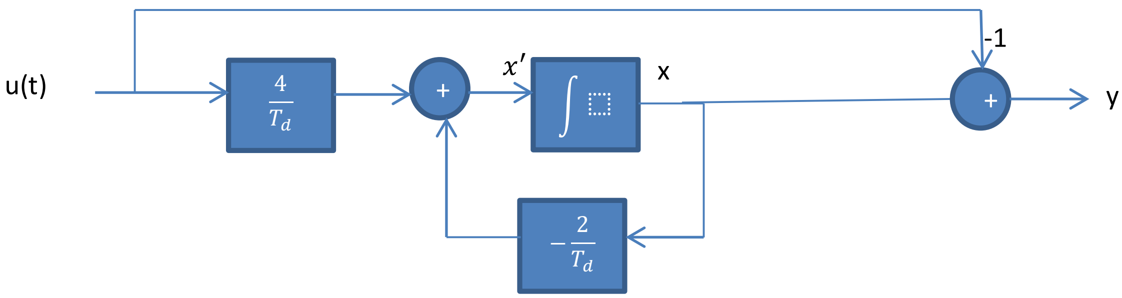
Therefor:

$$A = -\frac{1}{\frac{T_d}{2}} \ ; B = \frac{2}{\frac{T_d}{2}} \ ; C = 1 \ ; D = -1$$

We have three pumps that each of them affects the system separately. Therefor for three voltage inputs we have four orthogonal matrixes as below:

$$A = \begin{bmatrix} -\frac{2}{T_d} & 0 & 0 \\ 0 & -\frac{2}{T_d} & 0 \\ 0 & 0 & -\frac{2}{T_d} \end{bmatrix} B = \begin{bmatrix} \frac{4}{T_d} & 0 & 0 \\ 0 & \frac{4}{T_d} & 0 \\ 0 & 0 & \frac{4}{T_d} \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The block diagram of delay in each pump would be:




To get input pump model is implemented:

$$\Delta p = a_{h2}q^2 - a_{h1}qw - a_{h0}w^2$$

Where $a_{h2}$ and $a_{h1}$ are so small compare to $a_{h0}$ . So calculation can be estimated as:

$$\Delta p \sim -a_{h0}w^2$$

Where w (rotational speed ) is scaled from zero to 1 and it has linear proportion with voltage enforced to the pumps.

$$w \sim \frac{v_{input}}{5}$$

So updated equation would be:

$$\Delta p \sim -a_{h0}(\frac{v_{input}}{5})^2$$

Where

Where output is differential pressure with delay:

$$Z(t) = a_{h0}(X(t) - U(t))$$

In this case one extra state is gotten and other state space matrices should be updated:

B matrix for system is:

$$B = \begin{bmatrix} \mathrm{B11} & \mathrm{B12} & \mathrm{B13} \\ \mathrm{B21} & \mathrm{B22} & \mathrm{B23} \\ B31 & B32 & B33 \\ B41 & B42 & B43 \\ B51 & B52 & B53 \\ B61 & B62 & B63 \end{bmatrix}$$

And now instead of u(t) matrix matrix with delay is implied to the system. So:

$$\begin{bmatrix} \mathrm{B11} & \mathrm{B12} & \mathrm{B13} \\ \mathrm{B21} & \mathrm{B22} & \mathrm{B23} \\ B31 & B32 & B33 \\ B41 & B42 & B43 \\ B51 & B52 & B53 \\ B61 & B62 & B63 \end{bmatrix} \begin{bmatrix} (X1(t) - U1(t)) \\ (X2(t) - U2(t)) \\ (X3(t) - U3(t)) \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{B11} & \mathrm{B12} & \mathrm{B13} \\ \mathrm{B21} & \mathrm{B22} & \mathrm{B23} \\ B31 & B32 & B33 \\ B41 & B42 & B43 \\ B51 & B52 & B53 \\ B61 & B62 & B63 \end{bmatrix} \begin{bmatrix} X1(t) \\ X2(t) \\ X3(t) \end{bmatrix} - \begin{bmatrix} \mathrm{B11} & \mathrm{B12} & \mathrm{B13} \\ \mathrm{B21} & \mathrm{B22} & \mathrm{B23} \\ B31 & B32 & B33 \\ B41 & B42 & B43 \\ B51 & B52 & B53 \\ B61 & B62 & B63 \end{bmatrix} \begin{bmatrix} (U1(t) \\ U2(t) \\ U3(t) \end{bmatrix}$$

For delay matrixes three other state equation is to state space matrixes:

- $X'1(t) = -\frac{1}{\frac{T_d}{2}} X1(t) + \frac{2}{\frac{T_d}{2}} U1(t)$
- $X'2(t) = -\frac{1}{\frac{T_d}{2}} X2(t) + \frac{2}{\frac{T_d}{2}} U2(t)$
- $X'3(t) = -\frac{1}{\frac{T_d}{2}} X3(t) + \frac{2}{\frac{T_d}{2}} U3(t)$

Updated matrixes would be:

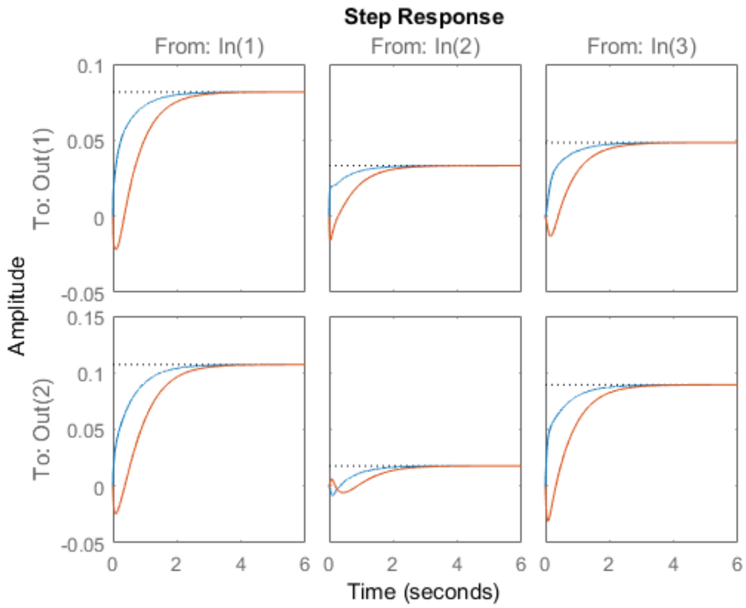
$$
\begin{bmatrix} q'1 \\ q'2 \\ q'3 \\ q'4 \\ q'5 \\ q'6 \\ x'1 \\ x'2 \\ x'3 \end{bmatrix}
=
\begin{bmatrix}
A11 & A12 & A13 & A14 & A15 & A16 & \mathrm{B11} & \mathrm{B12} & \mathrm{B13} \\
A21 & A22 & A23 & A24 & A25 & A26 & \mathrm{B21} & \mathrm{B22} & \mathrm{B23} \\
A31 & A32 & A33 & A34 & A35 & A36 & B31 & B32 & B33 \\
A41 & A42 & A43 & A44 & A45 & A46 & B41 & B42 & B43 \\
A5 & A52 & A53 & A54 & A55 & A56 & B51 & B52 & B53 \\
A61 & A62 & A63 & A64 & A65 & A66 & B61 & B62 & B63 \\
0 & 0 & 0 & 0 & 0 & 0 & -2/\mathrm{T\_d} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2/\mathrm{T\_d} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2/\mathrm{T\_d}
\end{bmatrix}
\begin{bmatrix} q1 \\ q2 \\ q3 \\ q4 \\ q5 \\ q6 \\ x1 \\ x2 \\ x3 \end{bmatrix}
$$

$$
+
\begin{bmatrix}
-B11 & -B12 & -B13 \\
-B21 & -B22 & -B23 \\
-B31 & -B32 & -B33 \\
-B41 & -B42 & -B43 \\
-B51 & -B52 & -B53 \\
-B61 & -B62 & -B63 \\
4/\mathrm{T\_d} & 0 & 0 \\
0 & 4/\mathrm{T\_d} & 0 \\
0 & 0 & 4/\mathrm{T\_d}
\end{bmatrix}
\begin{bmatrix} U1(t) \\ U2(t) \\ U3(t) \end{bmatrix}
$$

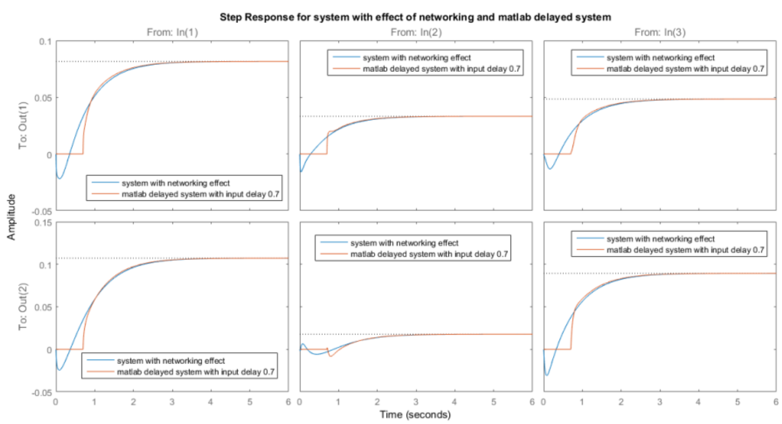
For C matrix ,it follows the rules of series systems so:

$$
C =
\begin{bmatrix}
C11 & C12 \\
C21 & C22 \\
C31 & C32 \\
C41 & C42 \\
C51 & C52 \\
C61 & C62 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
$$

The system with and without delay are simulated in MATLAB using code ss(A,B,C,D) and an step signal was enforced to them to check differences in responses andthe results are satisfactory.




The figure below shows comparison between input delayed system cone by matlab with 0.7 seconds and our delayed designed system




This comparison shows that delayed designed system works as it is supposed to.

# 10 Controllability and Observability

The controllability and observability matrices for the extended state-space system were made, such that the controllability matrix is given by the equation

$$\mathcal{C} = [B \; AB \dots A^{n-1}B]$$

and the observability matrix was calculated using

$$\mathcal{O} = [C \; CA \dots CA^{n-1}]^T$$

In the case of the extended state-space system, n=9. For the system to be both controllable and observable, the respective matrices must be full rank. It was seen that for the extended state-space system

$$rank(\mathcal{C}) = 5$$

and

$$rank(\mathcal{O}) = 4$$

The original state-space 6x6 system was then tested for controllability and observability to ensure the original system was controllable and observable. The matrices for the original system were both full rank.

The condition numbers of these matrices showed that the matrices were ill-conditioned and it is determined that physical system has a property of being almost uncontrollable and unobservable. Due to this property and numerical uncertainties, the extended system's controllability and observability matrices were not full rank.

To deal with the problem, two options were considered. The one option was to split the state-space system into a controllable part and an uncontrollable part and similarly for the observer. Using both the controllable and observable system, a controller can be designed for it. The second option was to use a model reduction technique to eliminate the uncontrollable states. Due to time constraints, the second option was chosen.

## 11 Model Reduction

The problem of the system being numerically unstable was researched and a solution to the problem was to reduce the order of the system to eliminate the states that were almost unstable. The system was reduced to a 6x6 system using a balanced reduction. The balanced reduction method introduced a D matrix into the system, but the values of the system were very small and were neglected. The step responses between the 9x9 delayed system and the 6x6 reduced system is shown below.

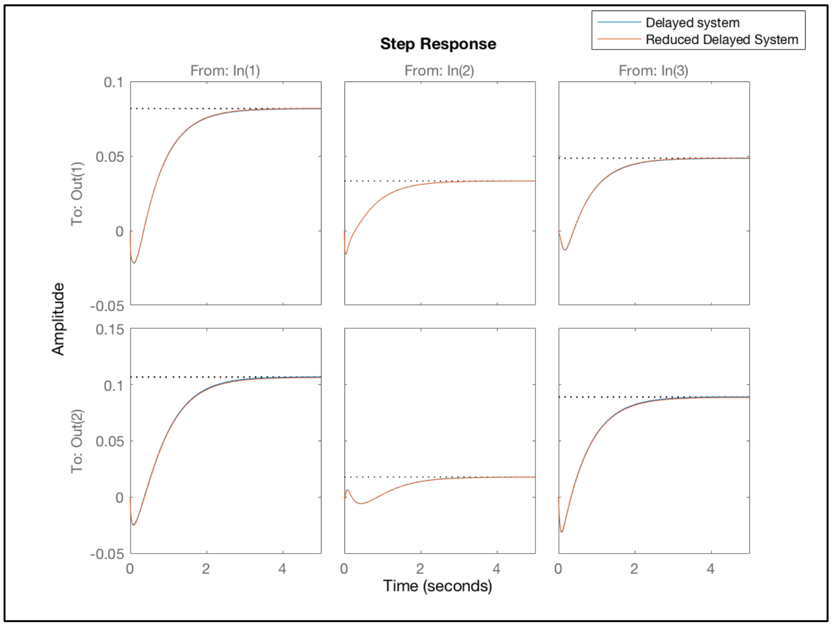



*Figure 13: Step Responses of the 9x9 Delayed System and the 6x6 Reduced Delayed System*

It can be seen in the Figure 13: Step Responses of the 9x9 Delayed System and the 6x6 Reduced Delayed System, the step response of both systems are very similar, thus the modelled dynamics of the physical system are still given by the reduced system.

The new 6x6 system was tested for controllability and observability to ensure the original system was controllable and observable. Both matrices were full rank and thus the numerically unstable states were removed while the dynamics of the system were still maintained.

# 12 Integral Control

It is desired for the system to be able to track a reference signal. This allows the operator to choose the suitable output pressures and the controller should be able to track it and thus have the desired outputs. To do this an additional integral term is added to the system. The input to this integrator is the error

$$e = \tilde{y}_r - \tilde{r}$$

where, $\tilde{r}$ , is the reference signal and, $\tilde{y}_r$, is the state-space output. This is used to extend the state-space model, in a similar manner to the delays. The error, $e$, is used as the two new states of the system.

$$\tilde{x}_i = \tilde{y}_r - \tilde{r}$$

The state-space model is then extended giving the new model

$$\begin{bmatrix} \dot{\tilde{x}}_r \\ \dot{\tilde{x}}_i \end{bmatrix} = \begin{bmatrix} A_r & 0 \\ C_r & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_r \\ \tilde{x}_i \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} \tilde{u}_r + \begin{bmatrix} 0 \\ -I \end{bmatrix} \tilde{r}$$

with

$$\dot{\tilde{x}}_I = \begin{bmatrix} \dot{\tilde{x}}_r \\ \dot{\tilde{x}}_i \end{bmatrix}; A_I = \begin{bmatrix} A_r & 0 \\ C_r & 0 \end{bmatrix}; B_I = \begin{bmatrix} B_r \\ 0 \end{bmatrix}; C_I = [C_r \ 0]$$

the new state-space model of the system is given as

$$\dot{\tilde{x}}_I = A_I \tilde{x}_I + B_I \tilde{u}_I + \begin{bmatrix} 0 \\ -I \end{bmatrix} \tilde{r}$$

$$\tilde{y}_I = C_I \tilde{x}_I$$

# 13 State Feedback Control

The chosen control method used was linear state feedback. For the given model

$$\dot{x} = Ax + Bu$$

the state feedback is in the form

$$u = Kx$$

where

$$K = [K_S\ K_I]$$

is the feedback gain. $K_s$ is the feedback gain of the system states and $K_I$ is the feedback gain of the integral states.

The feedback model for the full state-space system is given below

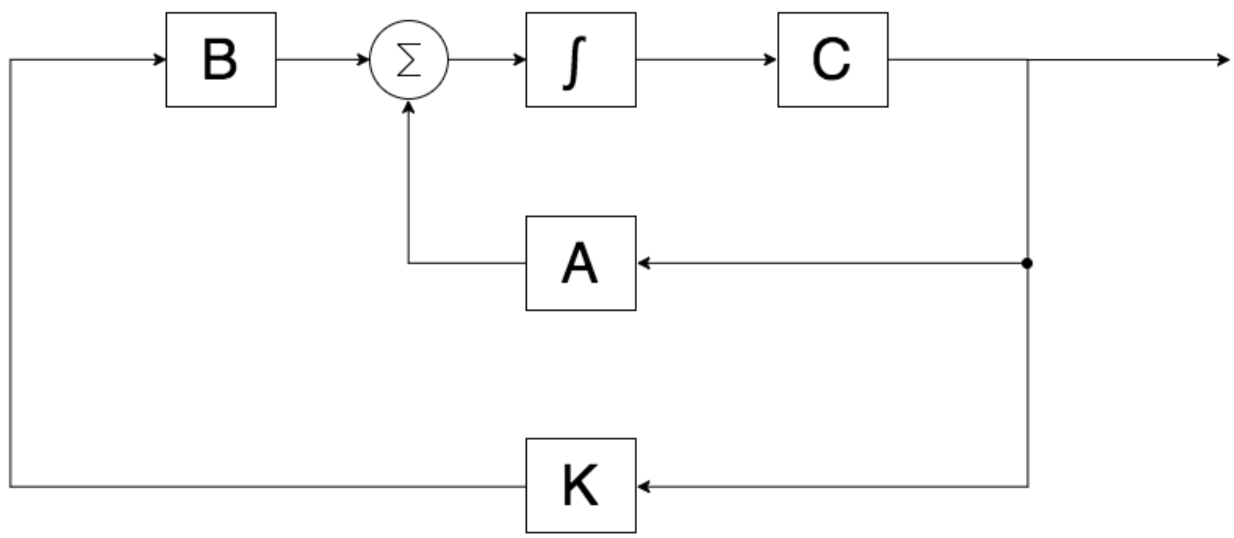



*Figure 14: Continuous-time State-Space Model with State Feedback*

Thus,

$$\dot{x} = Ax + BKx = (A + BK)x$$

In this equation, the eigenvalues of *A+BK* defines the poles of the system.

The method used to design the K matrix was Linear Quadratic Regulator (LQR)

## 13.1 LQR Control

The state feedback controller optimizes the cost function which is

$$\mathcal{J} = \int_0^{\infty} x^T Q_x x + u^T Q_u u\, dt$$

where $Q_x$ is a symmetric, positive semi-definite weight matrix and $Q_u$ is a symmetric, positive definite weight matrix.

For an LQR controller, the gain matrix, K, is given by

$$K = Q_u^{-1} B^T P$$

The P matrix satisfies the algebraic Ricatti equation

$$PA + A^T P - PBQ_u^{-1} B^T P + Q_x = 0$$

Bryson's Rule is used to choose the diagonal elements of the $Q_x$ and $Q_u$ matrices such that

$$Q_{x_{ii}} = \frac{1}{(maximum\ acceptable\ number\ of\ x_{ii})^2}$$

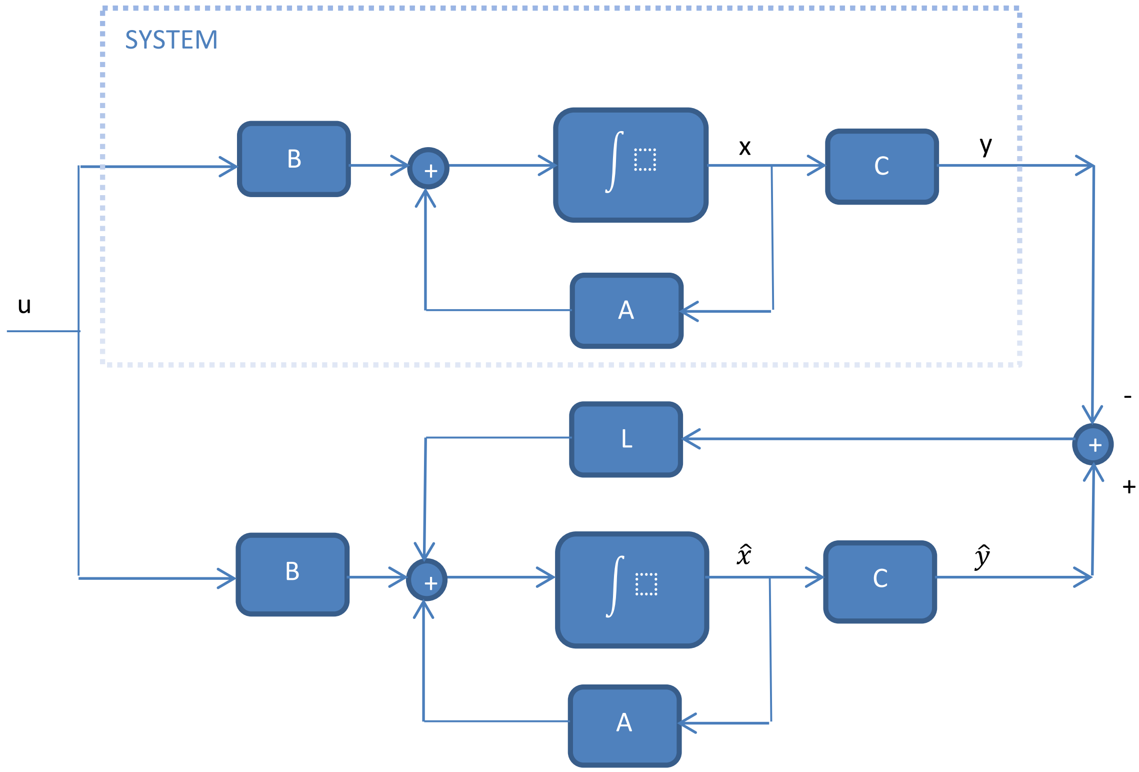
$$Q_{u_{ii}} = \frac{1}{(maximum\ acceptable\ number\ of\ u_{ii})^2}$$

The operating point for the flows and the inputs were used as the maximum acceptable numbers. The maximum acceptable number chosen was the values of the system's states and the inputs, at the operating point at which the system was linearized. The integral states, were given low weights as these states should not be penalized compared to the system's states.

Once the K matrix was calculated, it was split into $K_s$ and $K_I$.

## 13.2 Observer design

To design an observer for system, the system must be observable and by order reduction explained earlier the system is observable. Designing an observer is to get state feedback from the outputs and add it to the system before characteristic matrix (A). The observer gain is described by matrix L. In the system the two outputs are measured and there are six states so dimensions of L matrix is (2,6). Values of L matrix are taken in a way that eigenvalues of the closed loop matrix [$A - LC$] is stable and desirable. These eigenvalues can be arbitrarily placed to meet the given characteristics. Matlab code 'place' is used to get values for closed loop.


SYSTEM
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$\hat{y}$
+
A

# 14 Discretization

The model developed in the previous sections was done in the continuous time domain. The physical test setup uses MATLAB's real-time workshop which generates code from the Simulink model that runs on the microcontrollers controlling the test setup. This Simulink model used is a discrete-time system with a sampling time of T = 0.05s.

Initially, the continuous-time model was used, using a discrete-time integrator. This can be done if the poles of the system are slower than the sampling frequency.

When implementing the system, it was noted that the discrete-time integration method that gave an accurate response was the trapezoidal method. Unfortunately, this method cannot be used in real-time workshop. This is due to the trapezoidal creating an algebraic loop, thus code cannot be generated for it.

The solution was then to discretize the state-space model. The state-space model was discretized using the Zero-Order Hold method.

In discrete-time, the state-space system is given by the following equations

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

As mentioned in 0, the D matrix was neglected. The system was discretized using the MATLAB c2d command. The new Simulink model implemented is given below; a key difference between the continuous-time and discrete-time state-space models is the integrator used in continuous-time for the states is instead, a delay block in the discrete-time model.

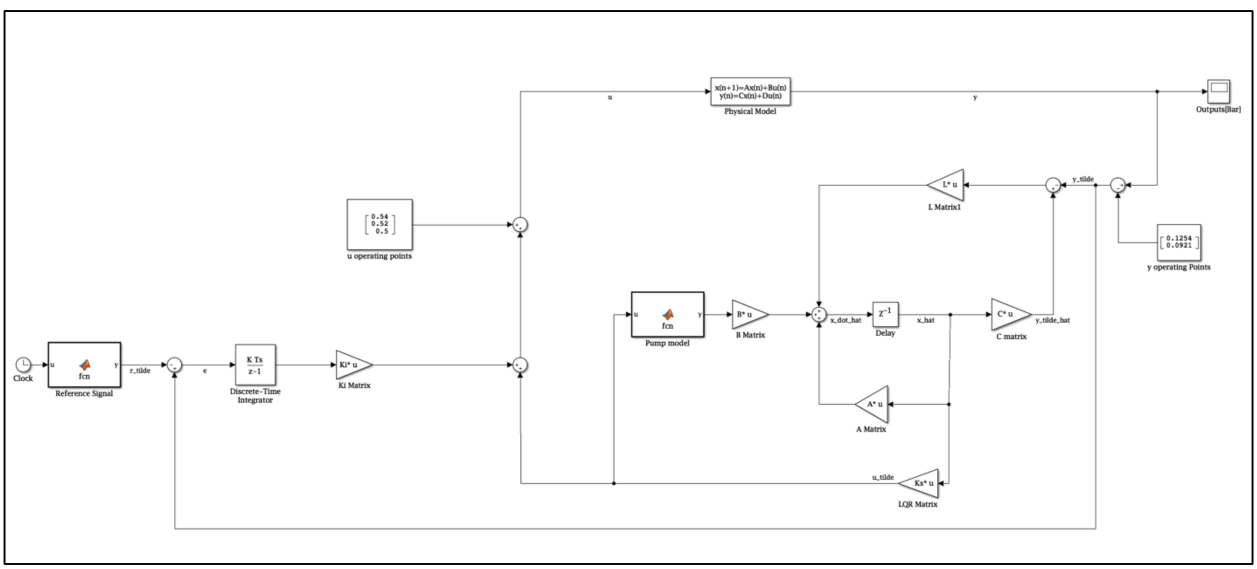


*Figure 15: Discrete-time State-Space Simulink Model*

# 15 Results

## 15.1 Simulated LQR Controller without Integral Control

The LQR controller, without integral control has been simulated in MATLAB Simulink and the following results have been obtained. The LQR weight matrices were tuned to penalize the inputs more than the states. The response due to an initial condition disturbance can be seen below.

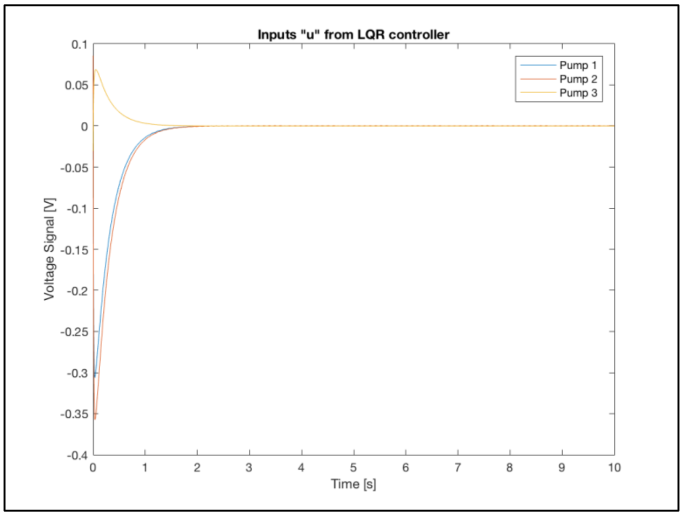


*Figure 16: Input signal to state-space model*

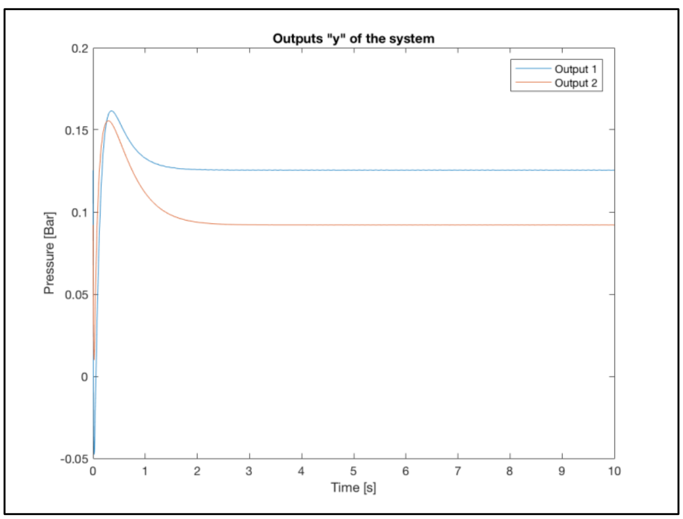


*Figure 17: Outputs of state-space model*

It can be seen that the system is able to drive the states to zero and maintain the operating point.

## 15.2 LQR Controller with Integral Controller

The LQR controller was implemented on the physical system. The reference to the system was stepped in order to see the effects of the controller. The output of the system can be seen in the figure below.

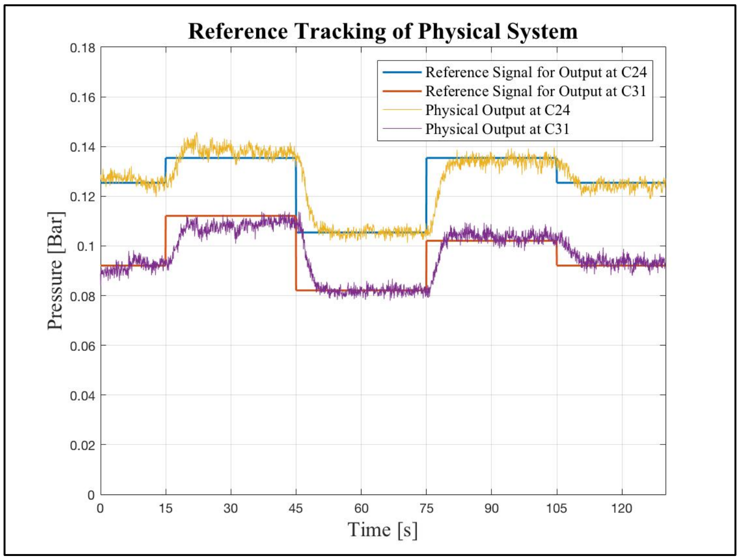


*Figure 18: System's Response to a Reference Signal*

It is seen that the system is able to track the reference signal for the outputs and is able to maintain the desired pressure. From the figure above, it is can be seen that from 15-45 seconds the output has a bigger tracking error than from 45-130 seconds. This is due to the reference being further away from the operating point. The means the state-space model is not as accurate at that point; thus it cannot track as well at that point.