
UiO : Matematisk institutt

Det matematisk-naturvitenskapelige fakultet


# On quantum channels and attractors

Masterpresentasjon


**Stian B. Søisdal**


**10. august 2020**

# Table of Contents

# Introduction to Attractors

# Introduction to Attractors

- Let $(X, d)$ be a complete metric space.
  and $f : X \to X$ be a strictly contractive map.

# Introduction to Attractors

- Let $(X, d)$ be a complete metric space.
  and $f : X \to X$ be a strictly contractive map.
- That is, for some $s \in [0, 1)$:

$$d\big(f(x), f(y)\big) \leq s d(x, y) \quad \forall x, y \in X.$$

# Introduction to Attractors

- Let $(X, d)$ be a complete metric space.
  and $f : X \to X$ be a strictly contractive map.
- That is, for some $s \in [0, 1)$:

$$d\big(f(x), f(y)\big) \leq sd(x, y) \quad \forall x, y \in X.$$

- Banach fixed-point theorem:
  the map $f$ has a unique fixed-point $x_0 \in X$.

# Introduction to Attractors

- Let $(X, d)$ be a complete metric space.
  and $f : X \to X$ be a strictly contractive map.
- That is, for some $s \in [0, 1)$:

$$d\big(f(x), f(y)\big) \leq sd(x, y) \quad \forall x, y \in X.$$

- Banach fixed-point theorem:
  the map $f$ has a unique fixed-point $x_0 \in X$.
- That is, $f(x_0) = x_0$ and

$$f^n(x) \to_{n\to\infty} x_0$$

  for all $x \in X$.

## Eksempel (1.1)

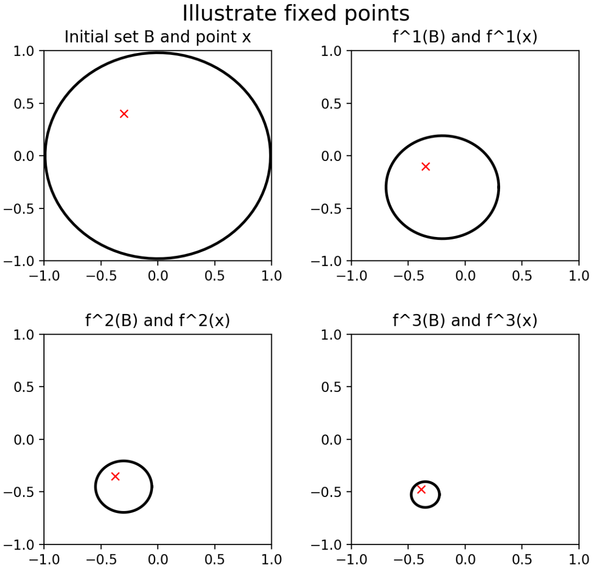
$X = \mathbb{R}^2$

$$f(x) = 0.5x - \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$

## Eksempel (1.1)

$$X = \mathbb{R}^2$$

$$f(x) = 0.5x - \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$



- Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of $X$.

- Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of $X$.
- There exists a complete metric $h$ on $\mathcal{K}(X)$ [1], called the hausdorf metric.

- Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of $X$.
- There exists a complete metric $h$ on $\mathcal{K}(X)$ [1], called the hausdorf metric.
- Let $f_1, f_2, \ldots f_N$ be strictly contractive maps on $(X, d)$. Then the map $F : \mathcal{K}(X) \to \mathcal{K}(X)$ given by

$$F(A) = f_1(A) \cup \ldots \cup f_N(A)$$

for $A \in \mathcal{K}(X)$, is a strictly contractive map on $(\mathcal{K}(X), h)$.

- Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of $X$.
- There exists a complete metric $h$ on $\mathcal{K}(X)$ [1], called the hausdorf metric.
- Let $f_1, f_2, \ldots f_N$ be strictly contractive maps on $(X, d)$. Then the map $F : \mathcal{K}(X) \to \mathcal{K}(X)$ given by

$$F(A) = f_1(A) \cup \ldots \cup f_N(A)$$

  for $A \in \mathcal{K}(X)$, is a strictly contractive map on $(\mathcal{K}(X), h)$.
- Then the map $F$ has a unique fixed-point $A_0 \in \mathcal{K}(X)$, so $F(A_0) = A_0$ and $F^n(A) \to_{n\to\infty} A_0$ for all $A \in \mathcal{K}(X)$.

- Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of $X$.
- There exists a complete metric $h$ on $\mathcal{K}(X)$ [1], called the hausdorf metric.
- Let $f_1, f_2, \dots f_N$ be strictly contractive maps on $(X, d)$. Then the map $F : \mathcal{K}(X) \to \mathcal{K}(X)$ given by

$$F(A) = f_1(A) \cup \dots \cup f_N(A)$$

  for $A \in \mathcal{K}(X)$, is a strictly contractive map on $(\mathcal{K}(X), h)$.
- Then the map $F$ has a unique fixed-point $A_0 \in \mathcal{K}(X)$, so $F(A_0) = A_0$ and $F^n(A) \to_{n\to\infty} A_0$ for all $A \in \mathcal{K}(X)$.
- Here the set $A_0$ is called the attractor to the iterated function system $(X; f_1, f_2, \dots, f_N)$.

## Eksempel (1.2)

$X = \mathbb{R}^2$

$$f_1(x) = 0.5x + \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}$$

$$f_2(x) = 0.3x + \begin{bmatrix} 0.5 \\ -0.3 \end{bmatrix}$$

$$f_3(x) = 0.4x + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$

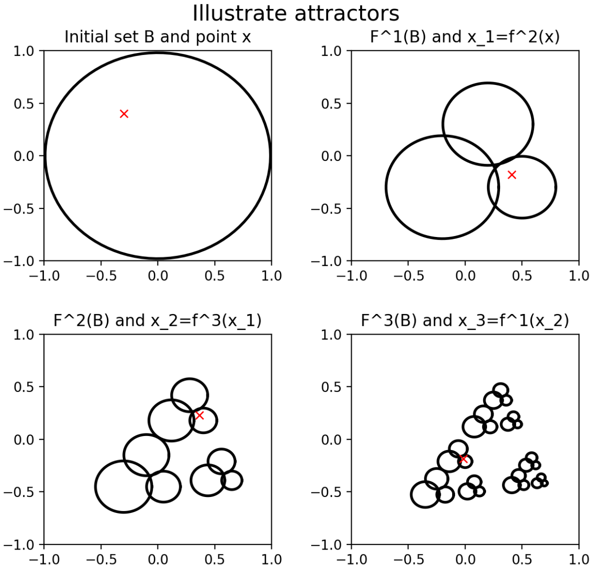
## Eksempel (1.2)

$$X = \mathbb{R}^2$$

$$f_1(x) = 0.5x + \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}$$

$$f_2(x) = 0.3x + \begin{bmatrix} 0.5 \\ -0.3 \end{bmatrix}$$

$$f_3(x) = 0.4x + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$



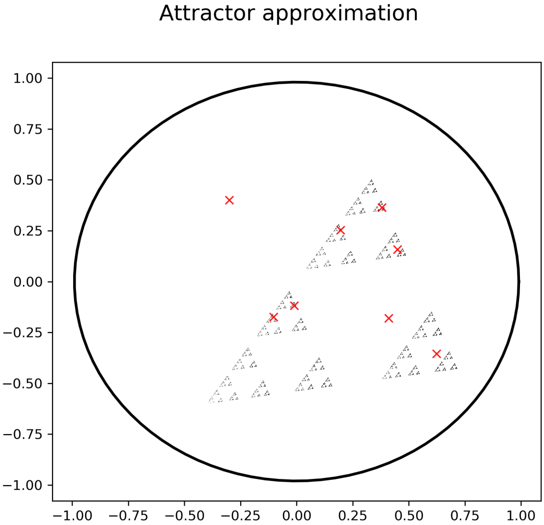
## Eksempel (1.2)

$X = \mathbb{R}^2$

$$f_1(x) = 0.5x + \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}$$

$$f_2(x) = 0.3x + \begin{bmatrix} 0.5 \\ -0.3 \end{bmatrix}$$

$$f_3(x) = 0.4x + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$



# Introduction to Quantum channels

# Introduction to Quantum channels

- When $\mathcal{A}$ is finite-dimensional $C^*$-algebra, $\mathcal{A} = \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_m}$ up to $*$-isomorphism.

# Introduction to Quantum channels

- When $\mathcal{A}$ is finite-dimensional $C^*$-algebra, $\mathcal{A} = \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_m}$ up to $*$-isomorphism.
- I will define the trace function by

$$Tr_{\mathcal{A}}(A) = \sum_{j=1}^{m} Tr(A_{n_j}), \quad A = (A_1, \ldots, A_j) \in \mathcal{A}.$$

# Introduction to Quantum channels

- When $\mathcal{A}$ is finite-dimensional $C^*$-algebra, $\mathcal{A} = \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_m}$ up to $*$-isomorphism.
- I will define the trace function by

$$Tr_{\mathcal{A}}(A) = \sum_{j=1}^{m} Tr(A_{n_j}), \quad A = (A_1, \ldots, A_j) \in \mathcal{A}.$$

- Will represent the states on $\mathcal{A}$ by the density operators

$$\mathcal{S}_{\mathcal{A}} = \{S \in \mathcal{A}^+ | Tr_{\mathcal{A}}(S) = 1\}$$

via $\varphi_S(A) = Tr_{\mathcal{A}}(AS)$ for each $S \in \mathcal{S}_{\mathcal{A}}$.

# Introduction to Quantum channels

- When $\mathcal{A}$ is finite-dimensional $C^*$-algebra, $\mathcal{A} = \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_m}$ up to $*$-isomorphism.
- I will define the trace function by

$$Tr_{\mathcal{A}}(A) = \sum_{j=1}^{m} Tr(A_{n_j}), \quad A = (A_1, \ldots, A_j) \in \mathcal{A}.$$

- Will represent the states on $\mathcal{A}$ by the density operators

$$\mathcal{S}_{\mathcal{A}} = \{S \in \mathcal{A}^+ | Tr_{\mathcal{A}}(S) = 1\}$$

via $\varphi_S(A) = Tr_{\mathcal{A}}(AS)$ for each $S \in \mathcal{S}_{\mathcal{A}}$.
- The quantum channels on $\mathcal{A}$ are the completely positive trace preserving linear ($CPT$) maps $\mathcal{A} \to \mathcal{A}$, sending $\mathcal{S}_{\mathcal{A}}$ into $\mathcal{S}_{\mathcal{A}}$.

# On quantum channels and attractors

# On quantum channels and attractors

- Will define the trace metric $d$ on $\mathcal{A}$ given by

$$d(A,B) = Tr_{\mathcal{A}}(|A-B|), \quad A,B \in \mathcal{A}.$$

# On quantum channels and attractors

- Will define the trace metric $d$ on $\mathcal{A}$ given by

$$d(A, B) = Tr_{\mathcal{A}}(|A - B|), \quad A, B \in \mathcal{A}.$$

- The associated metric to the trace norm

$$\|A\|_1 = Tr_{\mathcal{A}}(|A|), \quad A \in \mathcal{A}.$$

# On quantum channels and attractors

- Will define the trace metric $d$ on $\mathcal{A}$ given by

$$d(A,B) = Tr_{\mathcal{A}}(|A-B|), \quad A,B \in \mathcal{A}.$$

- The associated metric to the trace norm

$$\|A\|_1 = Tr_{\mathcal{A}}(|A|), \quad A \in \mathcal{A}.$$

- As $(\mathcal{A}, \|\cdot\|_1)$ is a Banach space [5],
  then $(\mathcal{S}_{\mathcal{A}}, d)$ ia a complete metric space. ($d$ restricted to $\mathcal{S}_{\mathcal{A}}$)

# On quantum channels and attractors

- Will define the trace metric $d$ on $\mathcal{A}$ given by

$$d(A,B) = Tr_{\mathcal{A}}(|A-B|), \quad A,B \in \mathcal{A}.$$

- The associated metric to the trace norm

$$\|A\|_1 = Tr_{\mathcal{A}}(|A|), \quad A \in \mathcal{A}.$$

- As $(\mathcal{A}, \|\cdot\|_1)$ is a Banach space [5], then $(\mathcal{S}_{\mathcal{A}}, d)$ ia a complete metric space. ($d$ restricted to $\mathcal{S}_{\mathcal{A}}$)
- So for quantum channels $Q_1, \ldots, Q_N$, restricted to $\mathcal{S}_{\mathcal{A}}$, that are strictly contractive gives an iterated function system

$$(\mathcal{S}_{\mathcal{A}}; Q_1, \ldots, Q_N)$$

with its attractor in the density operators.

# Examples of Attractor sets

# Examples of Attractor sets

- First when $n_j = 1$ for all $j \in \{1, N\}$, then $\mathcal{M}_{n_j} = \mathbb{C}$, and $\mathcal{A} = \mathbb{C}^m$.

# Examples of Attractor sets

- First when $n_j = 1$ for all $j \in \{1, N\}$,
  then $\mathcal{M}_{n_j} = \mathbb{C}$, and $\mathcal{A} = \mathbb{C}^m$. $\left(Tr(x) = \sum_j x_j,\ x \in \mathbb{C}^m\right)$

# Examples of Attractor sets

- First when $n_j = 1$ for all $j \in \{1, N\}$,
  then $\mathcal{M}_{n_j} = \mathbb{C}$, and $\mathcal{A} = \mathbb{C}^m$. $\left( Tr(x) = \sum_j x_j,\ x \in \mathbb{C}^m \right)$
- The density operators $\mathcal{S}_{\mathcal{A}} = \left\{ S \in \mathbb{R}_+^m \middle| Tr(S) = 1 \right\}$
  are the probability vectors $\mathbb{P}^m$.

# Examples of Attractor sets

- First when $n_j = 1$ for all $j \in \{1, N\}$, then $\mathcal{M}_{n_j} = \mathbb{C}$, and $\mathcal{A} = \mathbb{C}^m$. $\left(Tr(x) = \sum_j x_j,\ x \in \mathbb{C}^m\right)$
- The density operators $\mathcal{S}_{\mathcal{A}} = \left\{S \in \mathbb{R}_+^m \middle| Tr(S) = 1\right\}$ are the probability vectors $\mathbb{P}^m$.
- A quantum channel $Q$ on $\mathbb{P}^m$ is a stochastic matresis in $\mathbb{P}^{m,m}$.

# Examples of Attractor sets

- First when $n_j = 1$ for all $j \in \{1, N\}$, then $\mathcal{M}_{n_j} = \mathbb{C}$, and $\mathcal{A} = \mathbb{C}^m$. $\left(Tr(x) = \sum_j x_j,\ x \in \mathbb{C}^m\right)$
- The density operators $\mathcal{S}_{\mathcal{A}} = \left\{S \in \mathbb{R}_+^m \middle| Tr(S) = 1\right\}$ are the probability vectors $\mathbb{P}^m$.
- A quantum channel $Q$ on $\mathbb{P}^m$ is a stochastic matresis in $\mathbb{P}^{m,m}$.

**Teorem (3.1.1)**

*The quantum channels Q on $\mathbb{P}^m$ are strictly contractive when all entries of $Q^T Q$ are strictly positive. [4]*

# Examples of Attractor sets

- First when $n_j = 1$ for all $j \in \{1, N\}$, then $\mathcal{M}_{n_j} = \mathbb{C}$, and $\mathcal{A} = \mathbb{C}^m$. $\left(Tr(x) = \sum_j x_j,\ x \in \mathbb{C}^m\right)$
- The density operators $\mathcal{S}_{\mathcal{A}} = \left\{ S \in \mathbb{R}^m_+ \middle| Tr(S) = 1 \right\}$ are the probability vectors $\mathbb{P}^m$.
- A quantum channel $Q$ on $\mathbb{P}^m$ is a stochastic matresis in $\mathbb{P}^{m,m}$.

**Teorem (3.1.1)**

*The quantum channels Q on $\mathbb{P}^m$ are strictly contractive when all entries of $Q^T Q$ are strictly positive. [4]*

- On $\mathcal{S}_{\mathcal{A}} = \mathbb{P}^2$, the strictly contractive quantum channel are $\mathbb{P}^{2,2} \backslash \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.

## Eksempel (2.1)

$(\mathbb{P}^2; Q_1, Q_2)$

$$Q_1 = \begin{bmatrix} 1 & 2/3 \\ 0 & 1/3 \end{bmatrix}$$

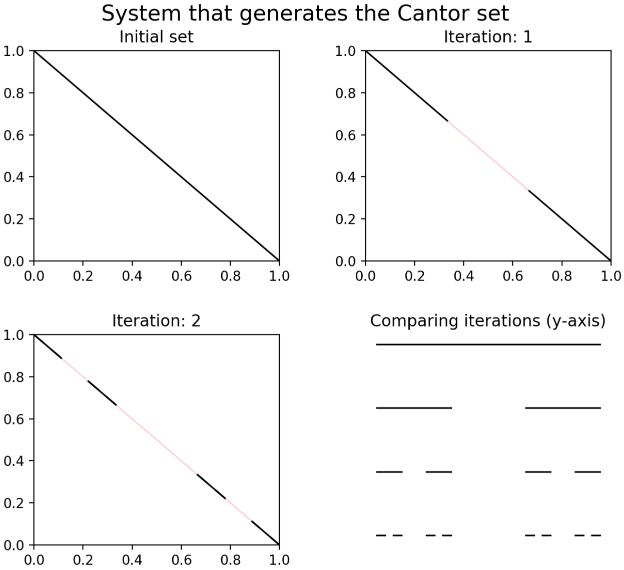
$$Q_2 = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1 \end{bmatrix}$$

## Eksempel (2.1)

$(\mathbb{P}^2; Q_1, Q_2)$

$$Q_1 = \begin{bmatrix} 1 & 2/3 \\ 0 & 1/3 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1 \end{bmatrix}$$



## Eksempel (2.2)

$(\mathbb{P}^2; Q_1, Q_2, Q_3)$

$$Q_1 = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.25 & 0 \\ 0.75 & 1 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$

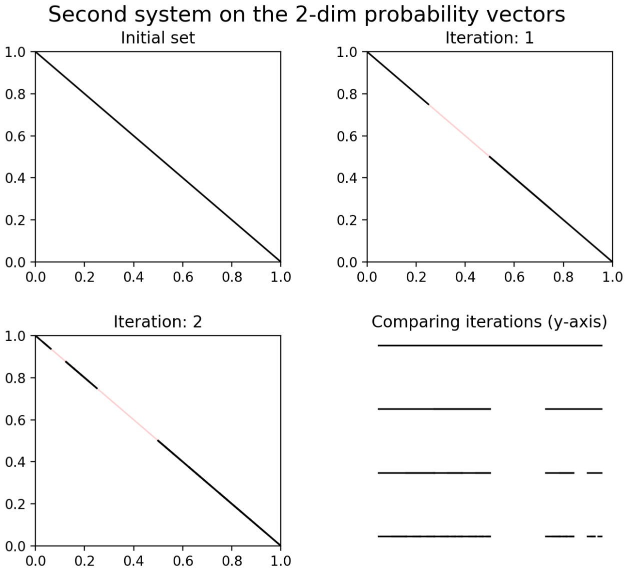
## Eksempel (2.2)

$$(\mathbb{P}^2; Q_1, Q_2, Q_3)$$

$$Q_1 = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.25 & 0 \\ 0.75 & 1 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$



## Eksempel (2.3)

$(\mathbb{P}^3; Q_1, Q_2, Q_3)$

$$Q_1 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix}$$

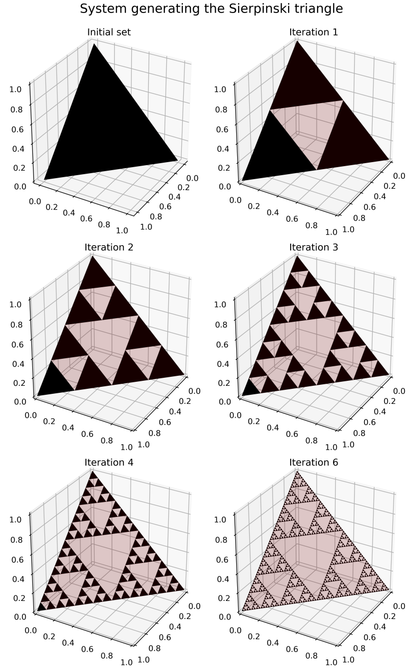
# Eksempel (2.3)

$(\mathbb{P}^3; Q_1, Q_2, Q_3)$

$$Q_1 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix}$$



## Eksempel (2.4)

$(\mathbb{P}^4; Q_1, Q_2, Q_3, Q_4)$

$$Q_1 = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$\vdots$

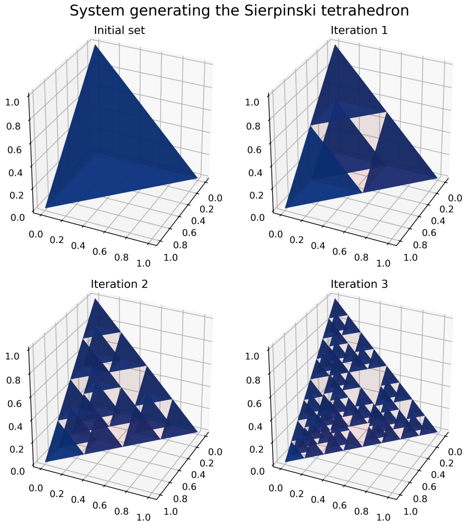
## Eksempel (2.4)

$(\mathbb{P}^4; Q_1, Q_2, Q_3, Q_4)$

$$Q_1 = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$\vdots$



# The 2 by 2 dimensional density matrises

# The 2 by 2 dimensional density matrises

- $\mathcal{A} = \mathcal{M}_2$, the 2 by 2 matrises
  $\mathcal{S}_{\mathcal{A}} = \{S \in \mathcal{M}_2^+ | Tr(S) = 1\}$, the density matrises.

# The 2 by 2 dimensional density matrises

- $\mathcal{A} = \mathcal{M}_2$, the 2 by 2 matrises
  $\mathcal{S}_{\mathcal{A}} = \{S \in \mathcal{M}_2^+ | Tr(S) = 1\}$, the density matrises.
- The Pauli basis:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

# The 2 by 2 dimensional density matrises

- $\mathcal{A} = \mathcal{M}_2$, the 2 by 2 matrises
  $\mathcal{S}_\mathcal{A} = \{S \in \mathcal{M}_2^+ | Tr(S) = 1\}$, the density matrises.
- The Pauli basis:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

- Identify $\mathcal{M}_2$ with $\mathbb{C}^4$ via
  $(r_0, r_1, r_2, r_3) \mapsto \frac{r_0}{2} I_2 + \frac{r_1}{2}\sigma_1 + \frac{r_2}{2}\sigma_2 + \frac{r_3}{2}\sigma_3$

# The 2 by 2 dimensional density matrises

- $\mathcal{A} = \mathcal{M}_2$, the 2 by 2 matrises
  $\mathcal{S}_{\mathcal{A}} = \{S \in \mathcal{M}_2^+ | Tr(S) = 1\}$, the density matrises.
- The Pauli basis:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

- Identify $\mathcal{M}_2$ with $\mathbb{C}^4$ via
  $(r_0, r_1, r_2, r_3) \mapsto \frac{r_0}{2} I_2 + \frac{r_1}{2}\sigma_1 + \frac{r_2}{2}\sigma_2 + \frac{r_3}{2}\sigma_3$
- For $S \in \mathcal{S}_{\mathcal{A}}$:
  trace one, so $r_0 = 1$,
  self-adjoint, so $(r_0, r_1, r_2, r_3) \in \mathbb{R}^4$,
  and positive, so $r_0^2 \geq r_1^2 + r_2^2 + r_3^2$.

# The 2 by 2 dimensional density matrises

- $\mathcal{A} = \mathcal{M}_2$, the 2 by 2 matrises
  $\mathcal{S}_{\mathcal{A}} = \{S \in \mathcal{M}_2^+ | Tr(S) = 1\}$, the density matrises.
- The Pauli basis:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

- Identify $\mathcal{M}_2$ with $\mathbb{C}^4$ via
  $(r_0, r_1, r_2, r_3) \mapsto \frac{r_0}{2} I_2 + \frac{r_1}{2}\sigma_1 + \frac{r_2}{2}\sigma_2 + \frac{r_3}{2}\sigma_3$
- For $S \in \mathcal{S}_{\mathcal{A}}$:
  trace one, so $r_0 = 1$,
  self-adjoint, so $(r_0, r_1, r_2, r_3) \in \mathbb{R}^4$,
  and positive, so $r_0^2 \geq r_1^2 + r_2^2 + r_3^2$.
- So $\mathcal{S}_{\mathcal{A}}$ can be identifyed with $\overline{B}_1(\mathbf{0}) \in \mathbb{R}^3$.

- A self-adjoint trace preserving linear map on $\mathbb{C}^4$ has the form $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t} & T \end{bmatrix}$, where $\mathbf{t} \in \mathbb{R}^3$ and $T \in \mathbb{R}^{3,3}$. ($\mathbf{r} \mapsto T\mathbf{r} + \mathbf{t}$ on $\mathbb{R}^3$)

- A self-adjoint trace preserving linear map on $\mathbb{C}^4$ has the form $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t} & T \end{bmatrix}$, where $\mathbf{t} \in \mathbb{R}^3$ and $T \in \mathbb{R}^{3,3}$. ($\mathbf{r} \mapsto T\mathbf{r} + \mathbf{t}$ on $\mathbb{R}^3$)
- It is positive when: $|\mathbf{t} + T\mathbf{r}| \leq 1$ for all $\mathbf{r} \in \mathbb{R}^3$ with $|\mathbf{r}| := \sqrt{r_1^2 + r_2^2 + r_3^2} \leq 1$.

- A self-adjoint trace preserving linear map on $\mathbb{C}^4$ has the form $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t} & T \end{bmatrix}$, where $\mathbf{t} \in \mathbb{R}^3$ and $T \in \mathbb{R}^{3,3}$. ($\mathbf{r} \mapsto T\mathbf{r} + \mathbf{t}$ on $\mathbb{R}^3$)
- It is positive when: $|\mathbf{t} + T\mathbf{r}| \leq 1$ for all $\mathbf{r} \in \mathbb{R}^3$ with $|\mathbf{r}| := \sqrt{r_1^2 + r_2^2 + r_3^2} \leq 1$.
- To check for quantum channels $Q$, need the form

$$Q = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & d_1 & 0 & 0 \\ t_2 & 0 & d_2 & 0 \\ t_3 & 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R_2^T \end{bmatrix}$$

## Teorem (4.1.1)

*For a positive trace-preserving map that corresponds to* $Q = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{t} & D \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R_2^T \end{bmatrix}$ *where* $|t_3| + |d_3| \leq 1$ *is satisfied, the map*
*is completely positive if and only if*

1. $(d_1 + d_2)^2 \leq (1 + d_3)^2 - t_3^2 - (t_1^2 + t_2^2)\left(\frac{1+d_3 \pm t_3}{1-d_3 \pm t_3}\right) \leq (1 + d_3)^2 - t_3^2$
2. $(d_1 - d_2)^2 \leq (1 - d_3)^2 - t_3^2 - (t_1^2 + t_2^2)\left(\frac{1-d_3 \pm t_3}{1+d_3 \pm t_3}\right) \leq (1 - d_3)^2 - t_3^2$
3. $[1 - |\mathbf{d}|^2 - |\mathbf{t}|^2]^2 \geq 4[d_1^2(t_1^2 + d_2^2) + d_2^2(t_2^2 + d_3^2) + d_3^2(t_3^2 + d_1^2) - 2d_1 d_2 d_3]$

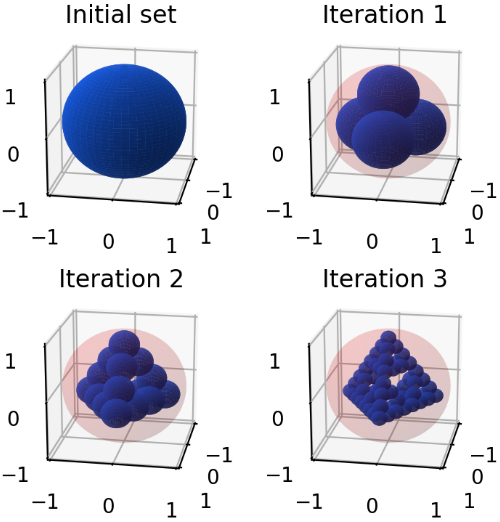
*hold. Here* 1. *and* 2. *are interpreted as* $t_1 = t_2 = 0$ *when* $|t_3| + |d_3| = 1$. *The CPT map is strictly contractive with respect to the trace norm when* $k < 1$*, where* $k = \max_i |d_i|$. *This* $k$ *is the contraction factor. [7]*

$(\mathcal{S}_{\mathcal{M}_2}; Q_1, Q_2, Q_3, Q_4)$

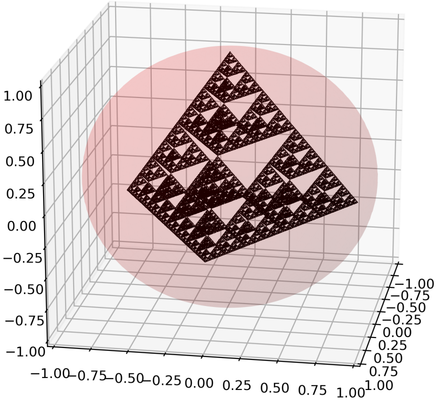
$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \sqrt{\frac{2}{9}} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

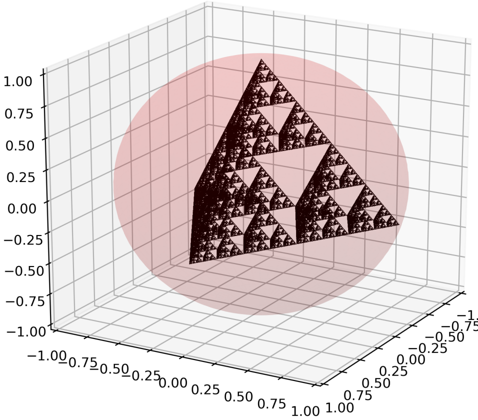
$$Q_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\sqrt{\frac{1}{18}} & \frac{1}{2} & 0 & 0 \\ \sqrt{\frac{1}{6}} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\sqrt{\frac{1}{18}} & \frac{1}{2} & 0 & 0 \\ -\sqrt{\frac{1}{6}} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

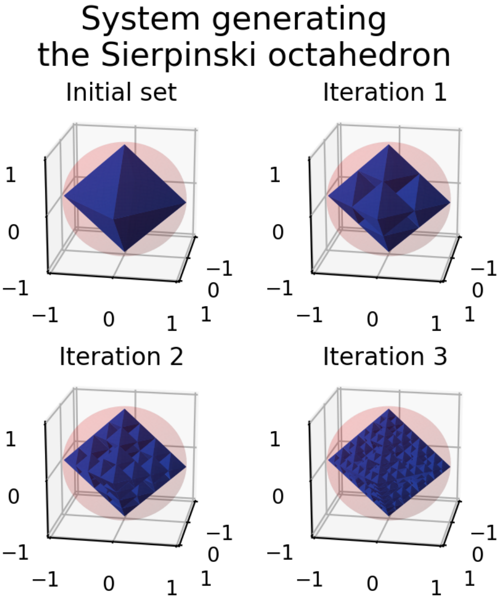
# System generating the Sierpinski tetrahedron



Random point approximation of the Sierpinski tetrahedron




System generating
the Sierpinski octahedron
Initial set
Iteration 1
Iteration 2
Iteration 3
1
0
−1
−1
0
1
−1
0
1

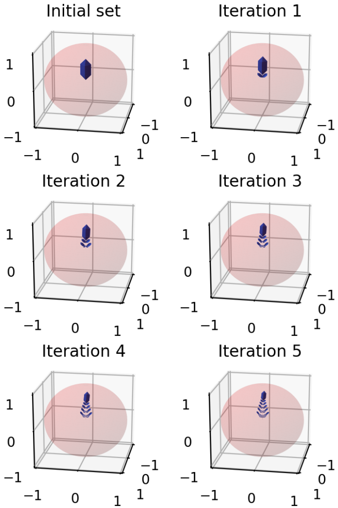
$(\mathcal{S}_{\mathcal{M}_2}; Q_1, Q_2, Q_3, Q_4)$

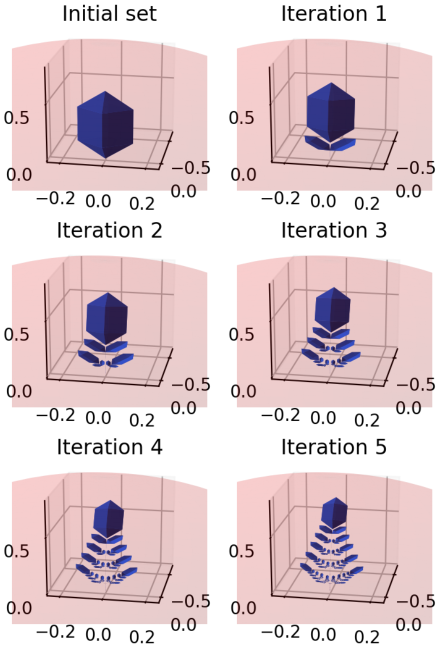
$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.18 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.85 & 0 & -0.1 \\ 0 & 0 & 0.85 & 0 \\ 0.16t & 0.1 & 0 & 0.85 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & -0.2 \\ 0.08t & 0 & 0.2 & 0.2 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & -0.2 & 0.2 \\ 0.08t & 0 & 0.2 & 0.2 \end{bmatrix}.$$

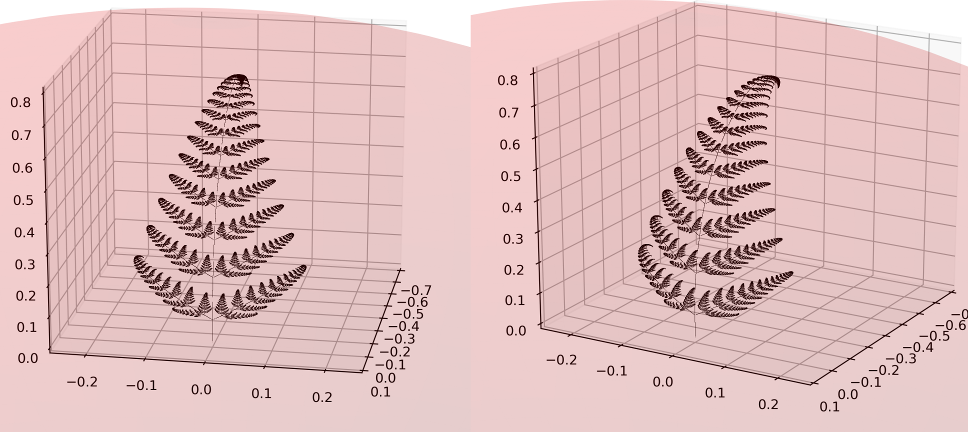
where $t = 0.900116$

# System generating a Fern attractor



Random point approximation
of the Fern attractor



# Conclusions and further research

# Conclusions and further research

- A iterative function systems of strictly contractive quantum channels has an attractor.

# Conclusions and further research

- A iterative function systems of strictly contractive quantum channels has an attractor.
- Recreated some of the more famous fractals as attractors

# Conclusions and further research

- A iterative function systems of strictly contractive quantum channels has an attractor.
- Recreated some of the more famous fractals as attractors
- Further research:
  $\mathcal{A} = \mathcal{M}_2 \oplus \mathbb{C}$

# Conclusions and further research

- A iterative function systems of strictly contractive quantum channels has an attractor.
- Recreated some of the more famous fractals as attractors
- Further research:
  $\mathcal{A} = \mathcal{M}_2 \oplus \mathbb{C}$
- Further research:
  Can any attractor (from affine maps) be placed in the density operators by change of coordinates?

# Conclusions and further research

- A iterative function systems of strictly contractive quantum channels has an attractor.
- Recreated some of the more famous fractals as attractors
- Further research:
  $\mathcal{A} = \mathcal{M}_2 \oplus \mathbb{C}$
- Further research:
  Can any attractor (from affine maps) be placed in the density operators by change of coordinates?
- Further research:
  Change trace function

# Conclusions and further research

- A iterative function systems of strictly contractive quantum channels has an attractor.
- Recreated some of the more famous fractals as attractors
- Further research:
  $\mathcal{A} = \mathcal{M}_2 \oplus \mathbb{C}$
- Further research:
  Can any attractor (from affine maps) be placed in the density operators by change of coordinates?
- Further research:
  Change trace function
- Further research:
  Fractal dimensions

# Referanser I

Barnsley, Michael F.
*Fractals everywhere*.
Academic Press Professional, 1993.

Erik Christopher Bedos
*exercises-sol-h19.pdf (Exercise 17)*, 2019
https://www.uio.no/studier/emner/matnat/math/MAT4460/h19/files/

Choi, Man-Duen
Completely positive linear maps on complex matrices
*Linear algebra and its applications*, 10:285–290, 1975.

# Referanser II

Kantrowitz, Robert and Neumann, Michael M and Ransford, Thomas J
Regularity, scrambling, and the steady state for stochastic matrices
*Contemp. Math*, 547:153–164, 2011

Pedersen, Gert K.
*Analysis now*.
Springer-Verlag, New York, 1989.

Raginsky, Maxim
Strictly contractive quantum channels and physically realizable quantum computers
*Physical Review A*, 65:032306, 2002

# Referanser III

Ruskai, Mary Beth and Szarek, Stanislaw and Werner, Elisabeth
An analysis of completely-positive trace-preserving maps on M2
*Linear Algebra and its Applications*, 347:159–187, 2002

Scherer, Wolfgang
*Mathematics of Quantum Computing: An Introduction*.
Springer Nature, 2019

UiO : **Matematisk institutt**

Det matematisk-naturvitenskapelige fakultet

# Stian B. Søisdal

# On quantum channels and attractors

Masterpresentasjon