بسمه تعالی
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تمرین سری **چهار**
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سوالات تئوری

۱. برای ماتریس زیر مقادیر منفرد (singular values) را بدست آورید(با راه حل کامل).

$$A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution. We compute AA^T and find $AA^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. The characteristic polynomial is

$$\begin{aligned} -\lambda^3 + 10\lambda^2 - 16\lambda &= -\lambda(\lambda^2 - 10\lambda + 16) \\ &= -\lambda(\lambda - 8)(\lambda - 2) \end{aligned}$$

So the eigenvalues of AA^T are $\lambda = 8, \lambda = 2, \lambda = 0$. Thus the singular values are $\sigma_1 = 2\sqrt{2}, \sigma_2 = \sqrt{2}$ (and $\sigma_3 = 0$).

To give the decomposition, we consider the diagonal matrix of singular values $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Next, we find an orthonormal set of eigenvectors for AA^T. For $\lambda = 8$, we find an eigenvector $(1, 2, 1)$ - normalizing gives $p_1 = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$. For $\lambda = 2$ we find $p_2 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, and finally for $\lambda = 0$ we get $p_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$.

This gives the matrix $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Finally, we have to find an orthogonal set of eigenvectors for $A^TA == \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

This can be done in two ways. We show both ways, starting with orthogonal diagonalization. We already know that the eigenvalues will be $\lambda = 8, \lambda = 2, \lambda = 0$. This gives eigenvectors $q_1 = (\frac{1}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}})$, $q_2 = (\frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}})$ and $q_3 = (\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2})$. Put these together to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{12}} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

For a quicker method, we calculate the columns of Q using those of P using the formula

$$p_i = \frac{1}{\sigma_i} A^T p_i.$$

Thus we calculate

$$p_1 = \frac{1}{\sigma_1} A^T p_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = q_1$$

and similarly for the other two columns.

Either way we can now we verify that we have $A = P\Sigma Q^T$.

۲. برای ماتریس زیر تجزیه SVD را بدست آورید(با راه حل کامل).

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution. We first compute

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We see immediately that the eigenvalues of AA^T are $\lambda_1 = \lambda_2 = 2$ (and hence that the eigenvalues of A^TA are 2 and 0, both with multiplicity 2), and thus the matrix A has singular value $\sigma_1 = \sigma_2 = \sqrt{2}$.

Next, an orthonormal basis of eigenvectors of AA^T is $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (You can choose any orthonormal basis for $\mathbb{R}^2$ here, but this one makes computation easiest.) Thus we set

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Lastly we have to find Q. We use the formula

$$q_1 = \frac{1}{\sigma_1} A^T p_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$q_2 = \frac{1}{\sigma_2} A^T p_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

We also need q_3 and q_4 but we can't compute those using the same formula, since we just ran our of p_i's. However, we know that the q_1, q_2, q_3, q_4 should be an orthonormal basis for $\mathbb{R}^4$, so we need to choose q_3 and q_4 in such a way that this indeed works out. We choose

$$q_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad q_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

giving

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

It is now easy to check that $A = P\Sigma Q^T$, where $\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$.

Note: we could also have diagonalized $A^T A$ to obtain Q, but we need to be careful, because if we choose the eigenvectors in the wrong way, we don't get $A = P\Sigma Q^T$; however, this can always be fixed by multiplying the eigenvectors by -1 as needed.

۳. الف) نشان دهید اگر A یک ماتریس مربعی باشد آنگاه $|\det A|$ حاصلضرب مقادیر تکین A است.

$$U : \text{orthogonal} \implies \det(I) = 1 = \det(U^T U) = \det(U)^2 \implies \det(U) = \pm 1$$

$$V : \text{orthogonal} \implies \det(I) = 1 = \det(V^T V) = \det(V)^2 \implies \det(V) = \pm 1$$

$$\implies$$

$$A = U\Sigma V^T \Rightarrow \det(A) = \det(\mathrm{U\Sigma V^T}) = \det(\mathrm{U})\det(\Sigma)\det(\mathrm{V})$$

$$\implies |\det(A)| = \det(\mathrm{U\Sigma V^T}) = |\det(\mathrm{U})|.|\det(\Sigma)|.|\det(\mathrm{V})|$$

$$\implies |\det(A)| = |\det(\Sigma)|$$

$$\xrightarrow{\Sigma\ is\ diagonal\ with\ singular\ values\ on\ diagonal} |det(A)| = |\det(\Sigma)| = \prod \sigma\mathrm{i}$$

ب) فرض کنید A ماتریسی مربعی و معکوس‌پذیر باشد، آنگاه SVD را برای A^{-1} بدست آورید.

$$A = U\Sigma V^T = U\Sigma V^{-1} \xrightarrow{entries\ on\ diagonal\ of\ \Sigma\ are\ nonezero}$$

$$A^{-1} = (U\Sigma V^{-1})^{-1} = V\Sigma^{-1}U^T$$