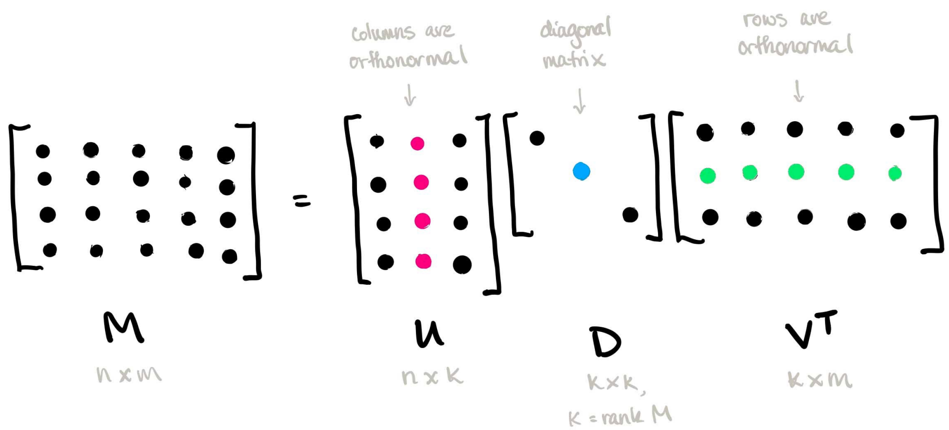
Linear Algebra

Fall 2021

Singular Value Decomposition (SVD)



Provided by:

Mohammad Hashemi

Outline

1. Definition of SVD
2. SVD calculation
3. SVD applications
4. Matrix approximation with SVD
5. Image compression with SVD 🧑🏻‍💻

Definition of SVD

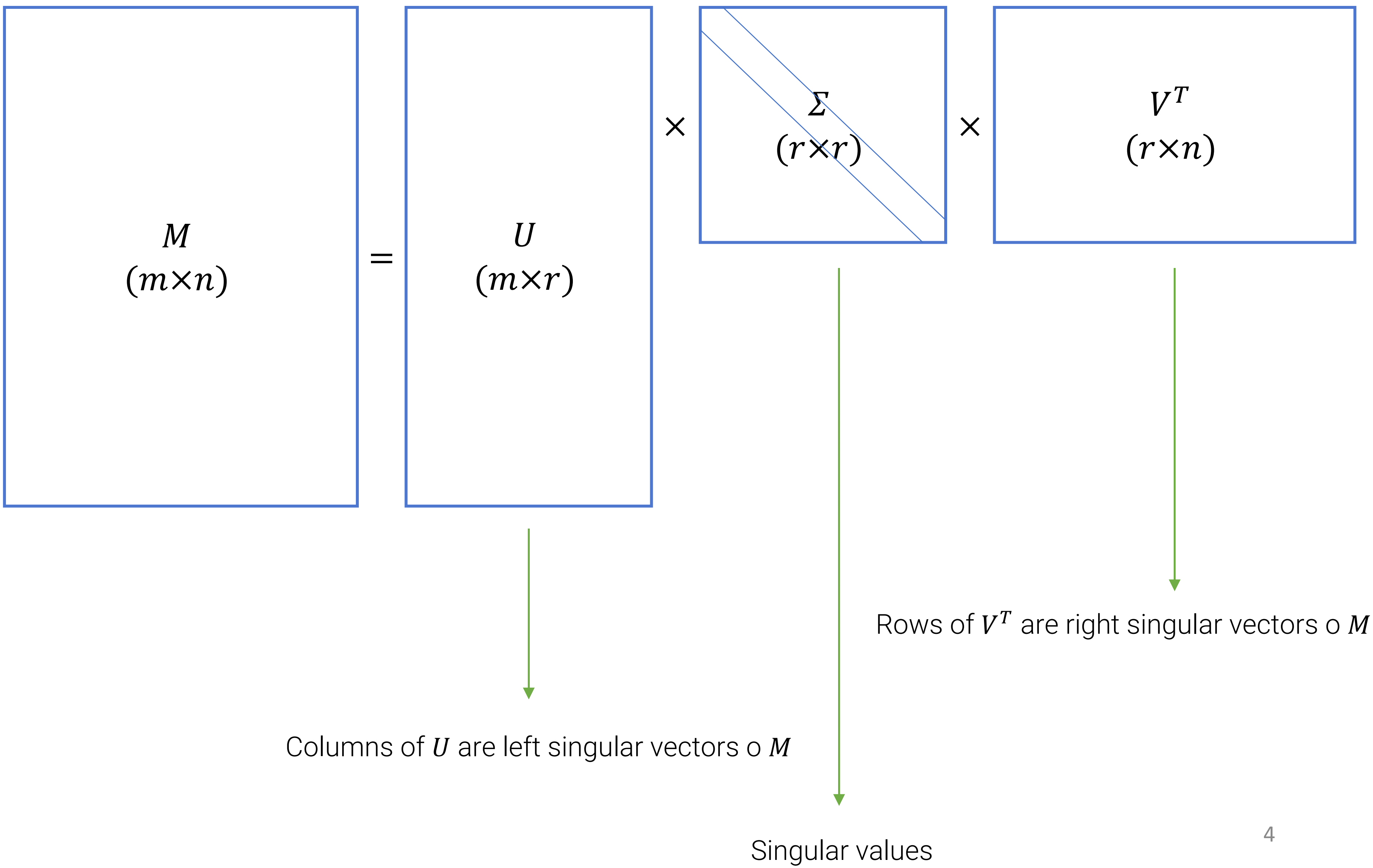
Let M be an $m \times n$ matrix, and let r be the rank of M. Then there exists a matrix factorization called Singular Value Decomposition (SVD) of M:

$$M = U\Sigma \mathrm{V}^{\mathrm{T}}$$

Where:

1. U is an $m \times r$ column-orthonormal matrix; that is, each of its columns is a unit vector and the dot product of any two columns is $\mathbf{0}$.
2. Σ is a diagonal matrix where the diagonal entries are called the singular values of M.
3. V^T is an $r \times n$ row-orthonormal matrix; that is, each of its rows is a unit vector and the dot product of any two rows is $\mathbf{0}$.

Definition of SVD

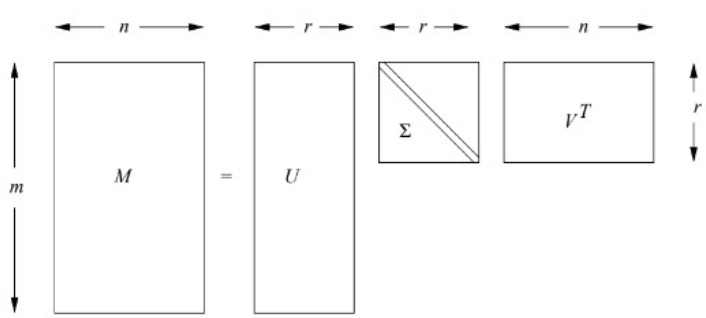


SVD calculation

The SVD of a matrix M has a strong connection of the eigenvectors of the matrix M^TM and MM^T.

Fact 1: The rows of V^T are the eigenvectors corresponding to the positive eigenvalues of M^TM.

Fact 2: The squares of the diagonal of Σ are the positive eigenvalues of M^TM.



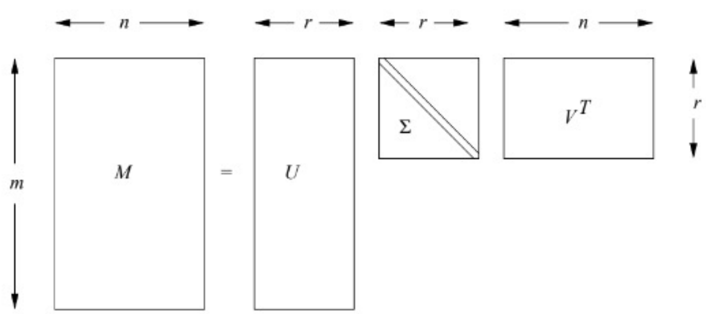
More specifically the i^{th} row of V^T is the eigenvector of M^TM whose corresponding eigenvalue if the square of the i^{th} entry of Σ.

https://www.math.ucsd.edu/_files/undergraduate/honors-program/honors-theses/2017-2018/Zecheng_Kuang_Honors_Thesis.pdf

SVD calculation

Fact 3: The columns of U are the eigenvectors corresponding to nonzero eigenvalues of MM^T.

Note: For any matrix M, M^TM and MM^T have the same nonzero eigenvalues.



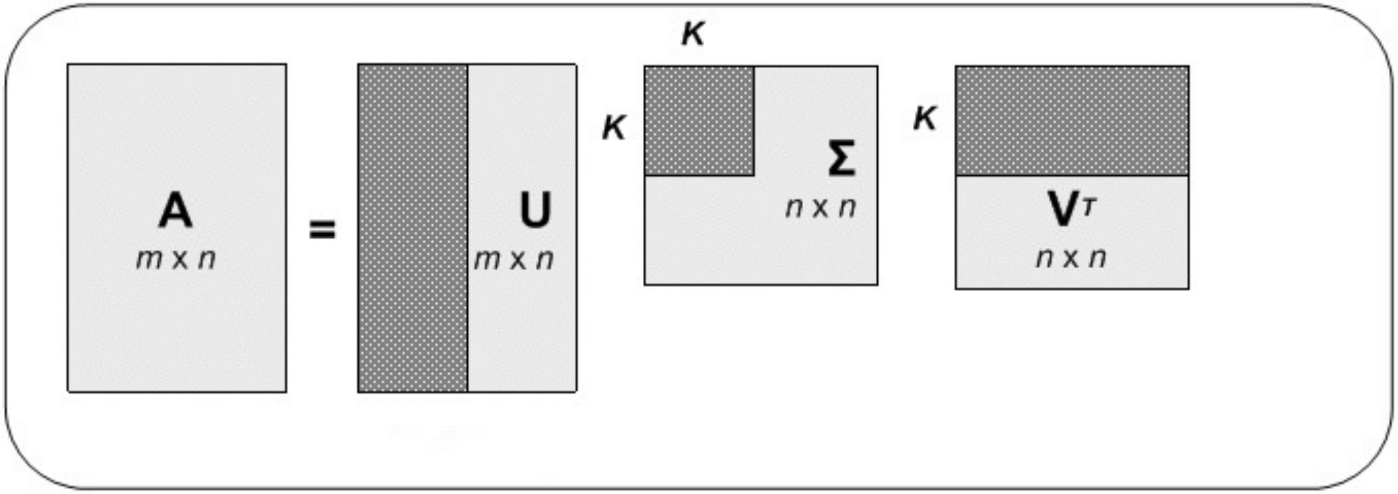
More specifically the i^{th} row of V^T is the eigenvector of M^TM whose corresponding eigenvalue if the square of the i^{th} entry of Σ.

https://www.math.ucsd.edu/_files/undergraduate/honors-program/honors-theses/2017-2018/Zecheng_Kuang_Honors_Thesis.pdf

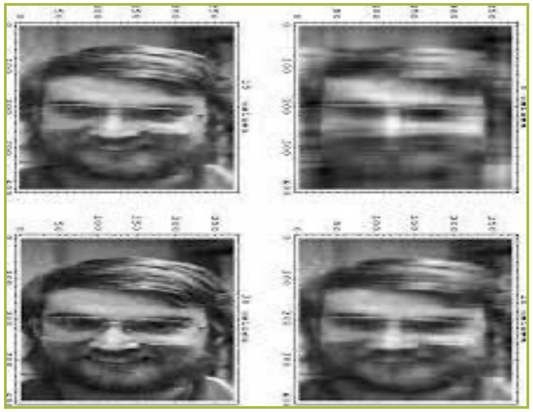
SVD calculation

Remark: Since $M^T M$ is an $n \times n$ matrix and MM^T is an $m \times m$ matrix, where n and m do not necessarily equal to the r. In fact, n and m are at least as large as r, which indicates that $M^T M$ and MM^T should have an additional $n - r$ and $m - r$ eigenpairs with eigenvalues of zeros.

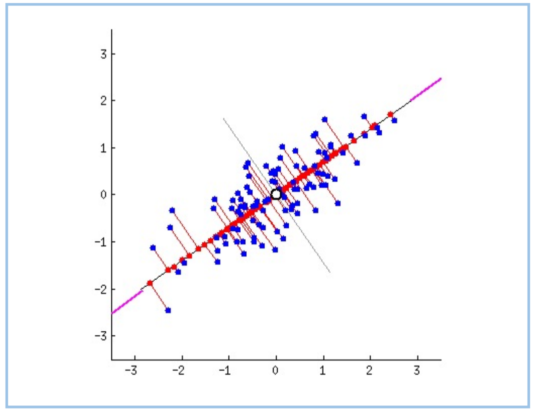
Full SVD | Reduced SVD



SVD applications

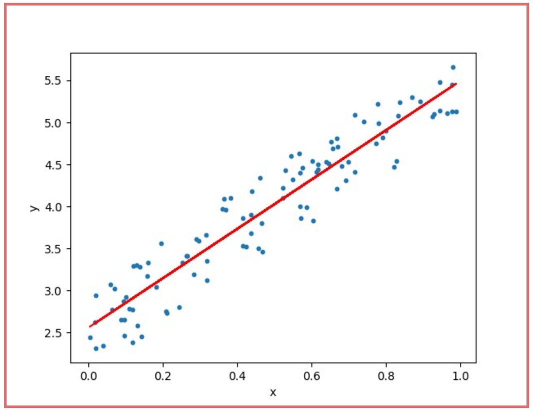


Data Reduction

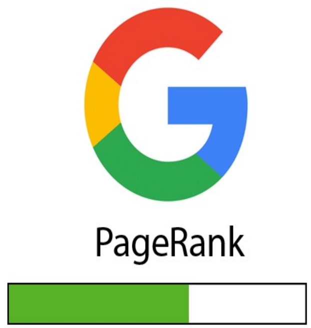


Principal Component Analysis

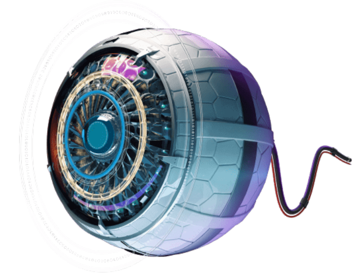
Recommender Systems



Solve $Ax = B$ for non-square A



Google PageRank



Computer Vision Tasks

Matrix Approximation

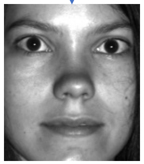
Assume we have X as a library of person faces with m faces. Then we can write the SVD form of X as:

$$X = \begin{bmatrix} | & | & & | \\ | & | & & | \\ | & | & & | \\ X_1 & X_2 & \dots & X_m \\ | & | & & | \\ | & | & & | \\ | & | & & | \end{bmatrix} = U\Sigma V^T = \begin{bmatrix} | & | & & | \\ | & | & & | \\ | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \\ | & | & & | \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sigma_m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & - & v_1^T & - & - \\ - & - & v_2^T & - & - \\ & & \cdot & & \\ & & \cdot & & \end{bmatrix}$$

X_1

X_2

...



X_m

$X_k \in \mathbb{R}^n \quad (n \gg m)$

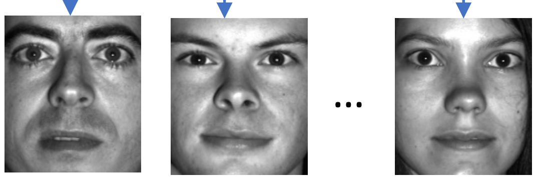
Fact 1: *U contains information about column-space of X.*

Fact 2: *V^T contains information about row-space of X.*

Matrix Approximation

$$X = \begin{bmatrix} | & | & & | \\ | & | & & | \\ | & | & & | \\ X_1 & X_2 & \dots & X_m \\ | & | & & | \\ | & | & & | \\ | & | & & | \end{bmatrix} = U\Sigma V^T = \begin{bmatrix} | & | & & | \\ | & | & & | \\ | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \\ | & | & & | \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sigma_m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & - & v_1^T & - & - \\ - & - & v_2^T & - & - \\ & & \cdot & & \\ & & \cdot & & \end{bmatrix}$$

m



X_1 X_2 X_m

$X_k \in \mathbb{R}^n$

$(n \gg m)$

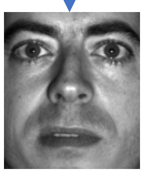
Important Note: *Since there are m columns in X, there are only m linearly independent columns in this n dimensional vector space that could be spanned by these vectors.*

Thus, only m columns of U matter for the representation of X.

Matrix Approximation

$$X = \begin{bmatrix} | & | & & | \\ | & | & & | \\ | & | & & | \\ X_1 & X_2 & \dots & X_m \\ | & | & & | \\ | & | & & | \\ | & | & & | \end{bmatrix} = U\Sigma V^T = \begin{bmatrix} | & | & & | \\ | & | & & | \\ | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \\ | & | & & | \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sigma_m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & - & v_1^T & - & - \\ - & - & v_2^T & - & - \\ & & \vdots & & \end{bmatrix}$$

m



X_1

X_2

...

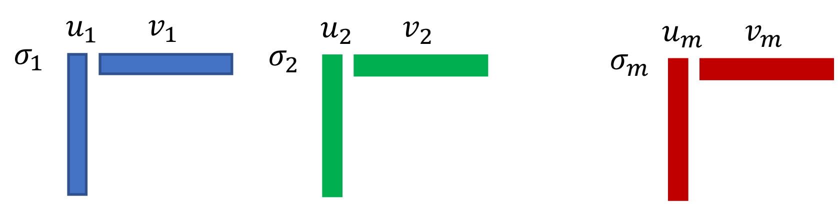
X_m

$X_k \in \mathbb{R}^n$

$(n \gg m)$

Expansion by rank-1 matrices:

$$X = \hat{U}\hat{\Sigma}V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_m u_m v_m^T$$



Economy SVD = Reduced SVD

Image compression

Let’s Do Some Code! 🧑🏻‍💻

https://github.com/SBU-CE/Linear-Algebra/04_Singular_Value_Decomposition

Thanks.

Mohammad Hashemi
mohammad99hashemi@gmail.com