

Perturbation Theory

Evan Saraivanov

November 28, 2022

Contents

1	Eulerian Perturbation Theory	2
1.1	Equations of Motion, The Fluid Approximation	2
1.1.1	The Boltzmann Equation	2
1.1.2	The Three Fluid Equations	3
1.2	Linear Solution	5
1.2.1	Equation of Motion in Fourier Space	5

Chapter 1

Eulerian Perturbation Theory

To begin, I need a few definitions. First, the *hubble length/time*, defined as $1/H_0$. It gives the age/size of the universe if inflation is perfectly linear/constant. Given a perfectly isotropic universe, this means the matter density decreases linearly with time. Because of gravitational instability, the small matter density fluctuations grow over time and a linear theory becomes insufficient at smaller redshifts. Additionally the non-linear perturbation theory fails at scales where non-perturbative effects dominate (would like examples).

Question 1. What is linear perturbation theory vs non-linear perturbation theory?

Donghui Jeong refers to the conditions where non-linear perturbation theory are valid as the *quasi-nonlinear regime* and it satisfies the following properties:

1. It's scale is smaller than the hubble scale. This allows the matter field to follow Newtonian Fluid EOM.
2. It's scale is larger than baryonic pressure. Both dark matter and baryonic matter are part of a pressureless matter.
3. Vorticity from non-linear effects is negligible.

In the subsequent sections, define

$$\delta(\tau, x) = \frac{\rho(\tau, x)}{\bar{\rho}(\tau)} - 1 \quad (1.1)$$

1.1 Equations of Motion, The Fluid Approximation

1.1.1 The Boltzmann Equation

We can write the number of particles located in a region of phase space as

$$N = f dx^3 \frac{dp^3}{(2\pi)^3} \quad (1.2)$$

We can look at the change in the number of particles over time in this region by looking at the distribution function f .

$$\frac{df}{dt} = \partial_t f + \dot{x} \nabla_x f + \dot{p} \nabla_p f = \partial_t f + \frac{p}{m} \nabla_x f + m a \nabla_p f = C[f] \quad (1.3)$$

Now we want to express this in a relativistic and expanding universe. Let $\lambda(t)$ be a monotonically increasing parameter. Then for any path through space the coordinate functions are parameterized by λ , giving $\dot{x} = x' \dot{\lambda}$, where primes denotes a derivative w.r.t λ and a dots denote the derivative w.r.t t . Plugging this in we get.

$$\frac{df}{dt} = \partial_t f + x' \dot{\lambda} \nabla_x f + p' \dot{\lambda} \nabla_p f \quad (1.4)$$

Then, we need to account for curvature. geodesic equation for the timelike components is

$$\begin{aligned} t'' &= -\Gamma_{\mu\nu}^0 (x^\mu)' (x^\nu)' \\ P^{0'} &= -\Gamma_{\mu\nu}^0 P^\mu P^\nu \end{aligned} \quad (1.5)$$

Furthermore, since $P^{0'} = \dot{P}^0 t' = P^0 \dot{P}^0$ we have

$$P^0 \dot{P}^0 = -\Gamma_{\mu\nu}^0 P^\mu P^\nu \quad (1.6)$$

Now, using $P^0 \dot{P}^0 = \frac{1}{2} \frac{d}{dt} E^2 = \frac{1}{2} \frac{d}{dt} (p^2 + m^2) = p\dot{p} = -Hp^2$ (last term from christoffel symbol), and that $P^0 = ma$, and that $\dot{p} = Hp = \nabla\phi$

$$\partial_t f + \frac{p^i}{ma^2} \partial_i f - \nabla\phi \partial_{p_i} f = C[f] \quad (1.7)$$

1.1.2 The Three Fluid Equations

The First Equation: The Continuity Equation

Our first fluid equation will come from the zero-th velocity moment of df/dt . We start by defining the density ρ ,

$$\begin{aligned} \int f(t, x, p) dp &= \rho(t, x) \\ \frac{d}{dt} \rho(t, x) &= \frac{d}{dt} \int f(t, x, p) dp \\ &= \int \frac{d}{dt} f dp \\ &= 0 \end{aligned} \quad (1.8)$$

on the other hand

$$\begin{aligned} \frac{d}{dt} \rho(t, x) &= \partial_t \rho + (\nabla_x \rho) \cdot \dot{x} \\ \Rightarrow \partial_t \rho + (\nabla_x \rho) \cdot v &= 0 \\ \Rightarrow \partial_t \rho + (\nabla_x \rho) \cdot v &= 0 \end{aligned} \quad (1.9)$$

Also

$$\bar{\rho}(t) = \int \rho(t, x) dx \Rightarrow \frac{d}{dt} \bar{\rho} = 0 \quad (1.10)$$

Hence

$$\partial_t (\rho - \bar{\rho}) + \frac{\bar{\rho}}{\rho} \nabla_x (\rho) \cdot v = 0 \quad (1.11)$$

$$\partial_t (\bar{\rho} \delta) + \bar{\rho} \nabla_x (1 + \delta) \cdot v = 0$$

$$\partial_t \delta + \nabla_x (1 + \delta) \cdot v = 0 \quad (1.12)$$

The Second Equation: The Euler Equation

Then next equation comes from the second velocity moment of the Vlasov equation (),

$$\begin{aligned} \int p^j f d^3 p &= \rho p^j \\ \partial_t \left(\int p^j f d^3 p \right) &= \partial_t (\rho p^j) \\ \int (\dot{p}^j f + p^j \partial_t f) d^3 p &= p^j \partial_t \rho + \rho \dot{p}^j \\ \int p^j \partial_t f d^3 p &= p^j \partial_t \rho \\ \int \frac{p^j p^i}{ma^2} \partial_{x,i} f - \partial_x^i \phi \partial_{p,i} f d^3 p &= p^j \partial_{x,i} \rho v^i \end{aligned} \quad (1.13)$$

Now we need to truncate the Hierarchy of moments. we have the second moment

$$\int p^i p^j f d^3 p = \rho p^i p^j + P^{ij} \quad (1.14)$$

Where we define P^{ij} as the pressure tensor. If we assume isotropic pressure we can close the hierarchy

$$P^{ij} = 0 \quad (1.15)$$

Now we can plug this into the second moment to get

$$\begin{aligned} \frac{p^j p^i}{ma^2} \nabla_{x,i} f - p^j \nabla \phi \nabla_{p,i} f d^3 p &= p^j \nabla_{x,i} \rho \frac{dp}{dt} \\ \frac{1}{ma^2} [p^i p^j \partial_{x,i} \rho - p g^{ij} \partial_{x,i} \rho] - \int p^j \nabla_{x,i} \phi \nabla_p^i f d^3 p &= v^j \partial_{x,i} \rho p^i + p^j \nabla_{x,i} \rho \frac{dp}{dt} \\ \frac{1}{ma^2} p^i p^j \partial_{x,i} \rho - \int p^j \nabla_{x,i} \phi \nabla_p^i f d^3 p &= p^j \nabla_{x,i} \rho \frac{dp}{dt} \end{aligned} \quad (1.16)$$

$$\begin{aligned} \int p^j \partial_t f d^2 p dp^j &= \iint \partial_t p^j f d^2 p dp^j \\ &= \iint \dot{p}^j f + p^j \partial_t f d^2 p dp^j \\ u = p^j, dv &= \int \partial_t f d^2 p dp^j, du = dp^j, v = \partial_t \rho \\ &= \dot{p}^j \rho + p^j \partial_t \rho - \int \partial_t \rho dp^j \\ &= \dot{p}^j \rho \end{aligned} \quad (1.17)$$

$$\begin{aligned} \iiint p^i \partial_{x,i} p^j f dp^i dp^j dp^k &= \iiint p^i f \partial_{x,i} p^j + p^i p^j \partial_{x,i} f dp^i dp^j dp^k \\ &= \iiint p^i f \partial_{x,i} p^j dp^i dp^j dp^k + \rho p^i p^j + P^{ij} \\ u = p^i, dv &= f \partial_{x,i} p^j dp^i, du = dp^i, v = \partial_{x,i} p^j \int f dp^i \\ &= p^i \partial_{x,i} p^j \rho - p^i \partial_{x,i} p^j \rho + \rho \partial_{x,i} (p^i p^j + P^{ij}) \\ &= \rho p^j \partial_{x,i} p^i + p^j \partial_{x,i} \rho p^i \end{aligned} \quad (1.18)$$

$$\begin{aligned} a^2 \iint \partial^i \phi \partial_{p,i} p^j f d^2 p dp^j &= a^2 \iint \partial^i \phi f \partial_{p,i} p^j d^2 p dp^j + \iint p^j \partial^i \phi \partial_{p,i} f d^2 p dp^j \\ &= a^2 \int \partial^j \phi f d^3 p + \partial^i \phi \iint p^j \partial_{p,i} f d^2 p dp^j \\ &= a^2 \rho \partial^j \phi \end{aligned} \quad (1.19)$$

Now if we put everything together and divide by ρ we get

$$\dot{p} + \frac{1}{ma^2} (p \cdot \nabla_x) p + \nabla_x \phi = 0 \quad (1.20)$$

Finally convert the time to conformal time ($t \rightarrow at$) and divide by m to get

$$\dot{v} + \mathcal{H}v + (v \cdot \nabla_x) v + \nabla_x \phi = 0 \quad (1.21)$$

The Third Equation: The Poisson Equation

Firstly, we are dealing with gravitationally interacting matter. Thus it is subject to the poisson equation relating the divergence of the gravitational field to the source of the gravitational field via Gauss's law.

$$\nabla^2 \phi = 4\pi G(\rho - \bar{\rho}) = 4\pi G a^2 \bar{\rho} \delta \quad (1.22)$$

1.2 Linear Solution

To summarize, the three fluid equations we have are

$$\partial_t \delta + \nabla_x (1 + \delta) \cdot v = 0 \quad (1.23)$$

$$\dot{v} + \mathcal{H}v + (v \cdot \nabla_x)v + \nabla_x \phi = 0 \quad (1.24)$$

$$\nabla^2 \phi = 4\pi G(\rho - \bar{\rho}) = 4\pi G a^2 \bar{\rho} \delta \quad (1.25)$$

We will make the following assumptions at large scales in this section:

1. Matter fluctuations are small compared to the homogenous contribution
2. Velocity vanishes on large scales

These assumptions allow the non-linear terms of the three fluid equations to vanish, so equations 1.23 and 1.24 become

$$\begin{aligned} \partial_t \delta + \nabla_x \cdot v &= 0 \\ \dot{v} + \mathcal{H}v &= -\nabla_x \phi \end{aligned} \quad (1.26)$$

Furthermore, the velocity can be decomposed into a divergence part $\theta = \nabla_x \cdot v$ and a vorticity part $w = \nabla_x \times v$

$$\partial_t \delta + \theta = 0 \quad (1.27)$$

From the 00 component Friedman equation,

$$\nabla_x^2 \phi = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta \quad (1.28)$$

This splits equation 1.24 into two parts,

$$\begin{aligned} \dot{\theta} + \mathcal{H}\theta + \frac{3}{2} \Omega_m \mathcal{H}^2 \delta &= 0 \\ \dot{w} + \mathcal{H}w &= 0 \end{aligned} \quad (1.29)$$

One of our assumptions was that vorticity is negligible, and the above equation demonstrates this because $w \propto e^{-a}$ thus at late times $w \rightarrow 0$ due to the expansion of the universe. Lets define the linear growth function $D_1(\tau)$ by $\delta(\tau, x) = D_1(\tau)\delta(0, x)$. The time derivative of divergence equation becomes

$$\ddot{D}_1 + \mathcal{H}\dot{D}_1 + \frac{3}{2} \Omega_m \mathcal{H}^2 D_1 = 0 \quad (1.30)$$

We have reduced eq 1.24 to a second order ODE in the linear regime, and thus it has two linearly independent solutions. Denote the fast solution as D_1^+ and the slow solution as D_1^- so that the general solution is

$$D_1 = D_1^+ A(x) + D_1^- B(x) \quad (1.31)$$

Now we can plug this solution into Eq. 1.27 we get $a\tau = t \Rightarrow dt = \dot{a}\tau + a d\tau$

$$\partial_\tau \delta = \dot{D}_1^+ A + \dot{D}_1^- B = -\theta(\tau, x) \quad (1.32)$$

1.2.1 Equation of Motion in Fourier Space

First lets find the Fourier transform of the continuity equation. The main difficulty here is that, moving the term $\nabla_x \delta \cdot v$ to the right hand side gives us a product of functions. This can be solved using the convolution theorem for the inverse Fourier transform. Furthermore, convolving any function with the delta function simply replaces the parameter in the integral. This means at the end, we also need to

convolve $\tilde{\theta}(\tau, k_1)$ with the dirac delta to get $\tilde{\theta}(\tau, k - k_2)$. Working out the algebra

$$\begin{aligned}
\mathcal{F}((\nabla_x \delta) \cdot v) &= \mathcal{F}(\nabla_x \delta) * \mathcal{F}(v) \\
&= ik\tilde{\delta}(\tau, k) * \tilde{v}(\tau, k) \\
&= \int ik\tilde{\delta}(\tau, k_2)\tilde{v}(\tau, k - k_2)d^3k_2 \\
&\quad k - k_2 = k_1 \\
&= \int -\frac{ik \cdot k_1 k_1}{k_1^2}\tilde{\delta}(\tau, k_2)\tilde{v}(\tau, k_1)d^3k_2 \\
&= \int \frac{k \cdot k_1}{k_1^2}\tilde{\delta}(\tau, k_2)\mathcal{F}(\nabla_x \cdot v)(k_1)d^3k_2 \\
&= \iint \delta_D(k - k_1 - k_2)\frac{k \cdot k_1}{k_1^2}\tilde{\delta}(\tau, k_2)\tilde{\theta}(\tau, k_1)d^3k_2d^3k_1
\end{aligned} \tag{1.33}$$

Thus the continuity equation becomes

$$\begin{aligned}
0 &= \int \partial_\tau \delta + \nabla_x(1 + \delta) \cdot v e^{-ik \cdot x} d^3x \\
\partial_\tau \tilde{\delta} + \tilde{\theta} &= - \iint \delta_D(k - k_1 - k_2) \frac{(k_1 + k_2) \cdot k_1}{k_1^2} \tilde{\delta}(\tau, k_1) \tilde{\theta}(\tau, k_2) \frac{d^3k_1}{(2\pi)^3} d^3k_2
\end{aligned} \tag{1.34}$$

Following the same methods, the Euler equation becomes

$$\partial_\tau \tilde{\theta} + \mathcal{H} \tilde{\theta} + \frac{3}{2} \Omega_m \mathcal{H}^2 \tilde{\delta} = - \iint \delta_D(k - k_1 - k_2) \frac{(k_1 + k_2)^2 k_1 \cdot k_2}{2k_1^2 k_2^2} \tilde{\theta}(\tau, k_1) \tilde{\theta}(\tau, k_2) \frac{d^3k_1}{(2\pi)^3} d^3k_2 \tag{1.35}$$

In Fourier space, the nonlinearities are represented as integrals over k -space while the linear terms remain as derivatives on the left. The non-linearities are now represented by couplings between different Fourier modes which are represented by the functions of k_1, k_2 in the integrands. For each fourier mode k , there are many combinations for k_1, k_2 such that $k = k_1 + k_2$. In equation 1.35 this manifests as a coupling between different divergences of the matter velocity and, as such, is in fact a requirement for translational invariance in the homogenous universe.

General Solution for an Einstein-de Sitter Cosmology

As a reminder, let me once more write the friedman equations.

$$\partial_\tau \mathcal{H} = \mathcal{H}^2 \left(\Omega_\Lambda - \frac{\Omega_m}{2} \right) \tag{1.36}$$

$$(\Omega_{\text{tot}} - 1) \mathcal{H}^2 = k \tag{1.37}$$

The Einstein-de Sitter cosmology has $\Omega_m = 1$ and $\Omega_\Lambda = 0$. Equation 1.36 becomes

$$\partial_\tau \mathcal{H} = -\frac{1}{2} \mathcal{H}^2 \tag{1.38}$$

Integrating w.r.t τ gives

$$\mathcal{H} = \frac{2}{\tau} \tag{1.39}$$

Now we apply the following perturbative expansion for $\tilde{\delta}$, which also gives the expression for $\tilde{\theta}$ via the linear continuity equation.

$$\begin{aligned}
\tilde{\delta} &= \sum_n a^n(\tau) \tilde{\delta}^{(n)}(k) \\
\tilde{\theta} &= \mathcal{H} \sum_n a^n(\tau) \tilde{\theta}^{(n)}(k)
\end{aligned} \tag{1.40}$$

Now plugging in the expansions to equation 1.34 and 1.35 gives

$$\begin{aligned}
n\tilde{\delta}^{(n)} + \tilde{\theta}^{(n)} &= A_n \\
\frac{3}{2}\mathcal{H}^2 a^n \tilde{\delta}^{(n)} + \mathcal{H}^2 a^n \tilde{\theta}^{(n)} + n\mathcal{H}^2 a^n \tilde{\theta}^{(n)} + \dot{\mathcal{H}} a^n \tilde{\theta}^{(n)} &= \tilde{B}_n \\
\frac{3}{2}\tilde{\delta}^{(n)} + \tilde{\theta}^{(n)} + n\tilde{\theta}^{(n)} - \frac{1}{2}\tilde{\theta}^{(n)} &= \frac{1}{\mathcal{H}^2 a^n} \tilde{B}_n \equiv \frac{1}{2}B_n \\
\Rightarrow \begin{pmatrix} n & 1 \\ 3 & 1+2n \end{pmatrix} \begin{pmatrix} \tilde{\delta}^{(n)} \\ \tilde{\theta}^{(n)} \end{pmatrix} &= \begin{pmatrix} A_n \\ B_n \end{pmatrix} \\
\Rightarrow \tilde{\delta}^{(n)} = \frac{(1+2n)A_n - B_n}{(2n+3)(n-1)}, \tilde{\theta}^{(n)} &= \frac{-3A_n + nB_n}{(2n+3)(n-1)}
\end{aligned} \tag{1.41}$$

And expand the k dependent functions in terms of the linear order density contrast field

$$\begin{aligned}
\tilde{\delta}^{(n)} &= \int \cdots \int \delta_D(k - q_1 - \cdots - q_n) F_n(q_1, \dots, q_n) \tilde{\delta}^{(1)}(q_1) \cdots \tilde{\delta}^{(1)}(q_n) d^3 q \cdots d^3 q_n \\
\tilde{\theta}^{(n)} &= \int \cdots \int \delta_D(k - q_1 - \cdots - q_n) G_n(q_1, \dots, q_n) \tilde{\delta}^{(1)}(q_1) \cdots \tilde{\delta}^{(1)}(q_n) d^3 q \cdots d^3 q_n
\end{aligned} \tag{1.42}$$

Writing A_n and B_n explicitly comes from the right hand side of the EOM. Keeping in mind the mode coupling, one must sum over each choice of modes for $\tilde{\theta}$ and $\tilde{\delta}$ such that the sum is n . Thus we have

$$\begin{aligned}
A_n &= - \iint \delta_D(k - k_1 - k_2) \frac{(k_1 + k_2) \cdot k_1}{k_1^2} \sum_{m=1}^{n-1} \tilde{\delta}^{(n-m)} \tilde{\theta}^{(m)} \frac{d^3 k_1}{(2\pi)^3} d^3 k_2 \\
B_n &= \iint \delta_D(k - k_1 - k_2) \frac{(k_1 + k_2) \cdot k_1}{k_1^2} \sum_{m=1}^{n-1} \tilde{\theta}^{(n-m)} \tilde{\theta}^{(m)} \frac{d^3 k_1}{(2\pi)^3} d^3 k_2
\end{aligned} \tag{1.43}$$

By simply plugging in the equations for A_n , B_n , $\tilde{\delta}$, and $\tilde{\theta}$ one finds the recursion relations for F_n and G_n

$$\begin{aligned}
F_n &= \\
G_n &=
\end{aligned} \tag{1.44}$$

By summing over all permutations of the q_i 's we obtain the symmetrized version of the functions F_n and G_n

$$\begin{aligned}
F_n^{(s)} &= \\
G_n^{(s)} &=
\end{aligned} \tag{1.45}$$