Perturbation Theory

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Chapter 1

Eulerian Perterbation Theory

To begin, I need a few definitions. First, the *hubble length/time*, defined as $1/H_0$. It gives the age/size of the universe if inflation is perfectly linear/constant. Given a perfectly isotropic universe, this means the matter density decreases linearly with time. Because of gravitational instability, the small matter density fluctuations grow over time and a linear theory becomes insufficient at smaller redshifts. Additionally the non-linear perturbation theory fails at scales where non-perturbative effects dominate (would like examples).

Question 1. What is linear perturbation theory vs non-linear perturbation theory?

Donghui Jeong refers to the conditions where non-linear perturbation theory are valid as the *quasi-nonlinear regime* and it satisfies the following properties:

1. It's scale is smaller than the hubble scale. This allows the matter field to follow Newtonian Fluid EOM.
2. It's scale is larger than baryonic pressure. Both dark matter and baryonic matter are part of a pressureless matter.
3. Vorticity from non-linear effects is negligable.

In the subsequent sections, define

$$\delta(\tau, x) = \frac{\rho(\tau, x)}{\bar{\rho}(\tau)} - 1 \tag{1.1}$$

1.1 Equations of Motion, The Fluid Approximation

1.1.1 The Boltzmann Equation

We can write the number of particles located in a region of phase space as

$$N = f dx^3 \frac{dp^3}{(2\pi)^3} \tag{1.2}$$

We can look at the change in the number of particles over time in this region by looking at the distribution function f.

$$\frac{df}{dt} = \partial_t f + \dot{x}\nabla_x f + \dot{p}\nabla_p f = \partial_t f + \frac{p}{m}\nabla_x f + ma\nabla_p f = C[f] \tag{1.3}$$

Now we want to express this in a relativistic and expanding universe. Let $\lambda(t)$ be a monatonically increasing parameter. Then for any path through space the coordinate functions are parameterized by λ, giving $\dot{x} = x'\dot{\lambda}$, where primes denotes a derivative w.r.t λ and a dots denote the derivative w.r.t t. Plugging this in we get.

$$\frac{df}{dt} = \partial_t f + x'\dot{\lambda}\nabla_x f + p'\dot{\lambda}\nabla_p f \tag{1.4}$$

Then, we need to account for curvature. geodesic equation for the timelike components is

$$\begin{aligned} t^{''} &= -\Gamma^0_{\mu\nu}(x^\mu)'(x^\nu)' \\ P^{0'} &= -\Gamma^0_{\mu\nu}P^\mu P^\nu \end{aligned} \tag{1.5}$$

Furthermore, since $P^{0'} = \dot{P}^0 t' = P^0\dot{P}^0$ we have

$$P^0\dot{P}^0 = -\Gamma^0_{\mu\nu}P^\mu P^\nu \tag{1.6}$$

Now, using $P^0\dot{P}^0 = \frac{1}{2}\frac{d}{dt}E^2 = \frac{1}{2}\frac{d}{dt}(p^2+m^2) = p\dot{p} = -Hp^2$ (last term from christoffel symbol), and that $P^0 = ma$, and that $\dot{p} = Hp = \nabla\phi$

$$\partial_t f + \frac{p^i}{ma^2}\partial_i f - \nabla\phi\partial_{p_i} f = C[f] \tag{1.7}$$

1.1.2 The Three Fluid Equations

The First Equation: The Continuity Equation

Our first fluid equation will come from the zero-th velocity moment of df/dt. We start by defining the density ρ,

$$\begin{aligned} \int f(t,x,p)dp &= \rho(t,x) \\ \frac{d}{dt}\rho(t,x) &= \frac{d}{dt}\int f(t,x,p)dp \\ &= \int \frac{d}{dt} f dp \\ &= 0 \end{aligned} \tag{1.8}$$

on the other hand

$$\begin{aligned} &\frac{d}{dt}\rho(t,x) = \partial_t\rho + (\nabla_x\rho)\cdot\dot{x} \\ &\Rightarrow \partial_t\rho + (\nabla_x\rho)\cdot v = 0 \\ &\Rightarrow \partial_t\rho + (\nabla_x\rho)\cdot v = 0 \end{aligned} \tag{1.9}$$

Also

$$\overline{\rho}(t) = \int \rho(t,x)dx \Rightarrow \frac{d}{dt}\overline{\rho} = 0 \tag{1.10}$$

Hence

$$\begin{aligned} \partial_t(\rho - \overline{\rho}) + \frac{\overline{\rho}}{\overline{\rho}}\nabla_x(\rho)\cdot v = 0 \\ \partial_t(\overline{\rho}\delta) + \overline{\rho}\nabla_x(1+\delta)\cdot v = 0 \end{aligned} \tag{1.11}$$

$$\partial_t\delta + \nabla_x(1+\delta)\cdot v = 0 \tag{1.12}$$

The Second Equation: The Euler Equation

Then next equation comes from the second velocity moment of the Vlasov equation (),

$$\begin{aligned} \int p^j f d^3p &= \rho p^j \\ \partial_t(\int p^j f d^3p) &= \partial_t(\rho p^j) \\ \int (\dot{p}^j f + p^j\partial_t f)d^3p &= p^j\partial_t\rho + \rho\dot{p} \\ \int p^j\partial_t f d^3p &= p^j\partial_t\rho \\ \int \frac{p^j p^i}{ma^2}\partial_{x,i}f - \partial^i_x\phi\partial_{p,i}f d^3p &= p^j\partial_{x,i}\rho v^i \end{aligned} \tag{1.13}$$

Now we need to truncate the Hierarchy of moments. we have the second moment

$$\int p^i p^j f d^3p = \rho p^i p^j + P^{ij} \tag{1.14}$$

Where we define P^{ij} as the pressure tensor. If we assume isotropic pressure we can close the higherarchy

$$P^{ij} = 0 \tag{1.15}$$

Now we can plug this into the second moment to get

$$\begin{gathered}\frac{p^j p^i}{ma^2}\nabla_{x,i} f - p^j \nabla\phi \nabla_{p,i} f d^3p = p^j \nabla_{x,i}\rho\frac{dp}{dt}\\ \frac{1}{ma^2}\left[p^i p^j \partial_{x,i}\rho - pg^{ij}\partial_{x,i}\rho\right] - \int p^j \nabla_{x.i}\phi\nabla^i_p f d^3p = v^j\partial_{x,i}\rho p^i + p^j\nabla_{x,i}\rho\frac{dp}{dt}\\ \frac{1}{ma^2}p^i p^j \partial_{x,i}\rho - \int p^j \nabla_{x.i}\phi\nabla^i_p f d^3p = p^j\nabla_{x,i}\rho\frac{dp}{dt}\end{gathered} \tag{1.16}$$

$$\begin{aligned}\int p^j \partial_t f d^2p dp^j &= \iint \partial_t p^j f d^2p dp^j\\ &= \iint \dot{p}^j f + p^j \partial_t f d^2p dp^j\\ u &= p^j, dv = \int \partial_t f d^2p dp^j, du = dp^j, v = \partial_t\rho\\ &= \dot{p}^j\rho + p^j\partial_t\rho - \int \partial_t\rho dp^j\\ &= \dot{p}^j\rho\end{aligned} \tag{1.17}$$

$$\begin{aligned}\iiint p^i\partial_{x,i}p^j f dp^i dp^j dp^k &= \iiint p^i f\partial_{x,i}p^j + p^i p^j\partial_{x,i} f dp^i dp^j dp^k\\ &= \iiint p^i f\partial_{x,i}p^j dp^i dp^j dp^k + \rho p^i p^j + P^{ij}\\ u &= p^i, dv = f\partial_{x,i}p^j dp^i, du = dp^i, v = \partial_{x,i}p^j\int f dp^i\\ &= p^i\partial_{x,i}p^j\rho - p^i\partial_{x,i}p^j\rho + \rho\partial_{x,i}(p^i p^j + P^{ij})\\ &= \rho p^j\partial_{x,i}p^i + p^j\partial_{x,i}\rho p^i\end{aligned} \tag{1.18}$$

$$\begin{aligned}a^2\iint \partial^i\phi\partial_{p,i}p^j f d^2p dp^j &= a^2\iint \partial^i\phi f\partial_{p,i}p^j d^2p dp^j + \iint p^j\partial^i\phi\partial_{p,i} f d^2p dp^j\\ &= a^2\int \partial^j\phi f d^3p + \partial^i\phi\iint p^j\partial_{p,i} f d^2p dp^j\\ &= a^2\rho\partial^j\phi\end{aligned} \tag{1.19}$$

Now if we put everything together and divide by ρ we get

$$\dot{p} + \frac{1}{ma^2}(p\cdot\nabla_x)p + \nabla_x\phi = 0 \tag{1.20}$$

FInally convert the time to conformal time ($t \to at$) and divide by m to get

$$\dot{v} + \mathcal{H}v + (v\cdot\nabla_x)v + \nabla_x\phi = 0 \tag{1.21}$$

The Third Equation: The Poisson Equation

Firstly, we are dealing with gravitationally interacting matter. Thus it is subject to the poisson equation relating the divergence of the gravitational field to the source of the gravitational field via Gauss's law.

$$\nabla^2\phi = 4\pi G(\rho - \overline{\rho}) = 4\pi G a^2\overline{\rho}\delta \tag{1.22}$$

1.2 Linear Solution

To summarize, the three fluid equations we have are

$$\partial_t \delta + \nabla_x (1+\delta) \cdot v = 0 \tag{1.23}$$

$$\dot{v} + \mathcal{H} v + (v \cdot \nabla_x) v + \nabla_x \phi = 0 \tag{1.24}$$

$$\nabla^2 \phi = 4\pi G(\rho - \overline{\rho}) = 4\pi G a^2 \overline{\rho} \delta \tag{1.25}$$

We will make the following assumptions at large scales in this section:

1. Matter fluctuations are small compared to the homogenous contribution
2. Velocity vanishes on large scales

These assumptions allow the non-linear terms of the three fluid equations to vanish, so equations 1.23 and 1.24 become

$$\begin{aligned} \partial_t \delta + \nabla_x \cdot v = 0 \\ \dot{v} + \mathcal{H} v = -\nabla_x \phi \end{aligned} \tag{1.26}$$

Furthermore, the velocity can be decomposed into a divergence part $\theta = \nabla_x \cdot v$ and a vorticity part $w = \nabla_x \times v$

$$\partial_t \delta + \theta = 0 \tag{1.27}$$

From the 00 component Friedman equation,

$$\nabla_x^2 \phi = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta \tag{1.28}$$

This splits equation 1.24 into two parts,

$$\begin{aligned} \dot{\theta} + \mathcal{H}\theta + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta = 0 \\ \dot{w} + \mathcal{H} w = 0 \end{aligned} \tag{1.29}$$

One of our assumptions was that vorticity is negligable, and the above equation demonstrates this because $w \propto e^{-a}$ thus at late times $w \to 0$ due to the expansion of the universe. Lets define the linear growth function $D_1(\tau)$ by $\delta(\tau, x) = D_1(\tau)\delta(0, x)$. The time derivative of divergence equation becomes

$$\ddot{D}_1 + \mathcal{H}\dot{D}_1 + \frac{3}{2}\Omega_m \mathcal{H}^2 D_1 = 0 \tag{1.30}$$

We have reduced eq 1.24 to a second order ODE in the linear regime, and thus it has two linearly independent solutions. Denote the fast solution as D_1^+ and the slow solution as D_1^- so that the general solution is

$$D_1 = D_1^+ A(x) + D_1^- B(x) \tag{1.31}$$

Now we can plug this solution into Eq. 1.27 we get $a\tau = t \Rightarrow dt = \dot{a}\tau + a d\tau$

$$\partial_\tau \delta = \dot{D}_1^+ A + \dot{D}_1^- B = -\theta(\tau, x) \tag{1.32}$$

1.2.1 Equation of Motion in Fourier Space

First lets find the Fourier transform of the continuity equation. The main difficulty here is that, moving the term $\nabla_x \delta \cdot v$ to the right hand side gives us a product of functions. This can be solved using the convolution theorem for the inverse Fourier transform. Furthermore, convoliving any function with the delta function simply replaces the parameter in the integral. This means at the end, we also need to convolve $\tilde{\theta}(\tau, k_1)$ with the dirac delta to get $\tilde{\theta}(\tau, k - k_2)$. Working out the algebra

$$\begin{aligned}
\mathcal{F}((\nabla_x \delta) \cdot v) &= \mathcal{F}(\nabla_x \delta) * \mathcal{F}(v) \\
&= ik\tilde{\delta}(\tau, k) * \tilde{v}(\tau, k) \\
&= \int ik\tilde{\delta}(\tau, k_2)\tilde{v}(\tau, k - k_2) d^3k_2 \\
& k - k_2 = k_1 \\
&= \int -\frac{ik \cdot k_1 k_1}{k_1^2}\tilde{\delta}(\tau, k_2)\tilde{v}(\tau, k_1) d^3k_2 \\
&= \int \frac{k \cdot k_1}{k_1^2}\tilde{\delta}(\tau, k_2)\mathcal{F}(\nabla_x \cdot v)(k_1) d^3k_2 \\
&= \iint \delta_D(k - k_1 - k_2)\frac{k \cdot k_1}{k_1^2}\tilde{\delta}(\tau, k_2)\tilde{\theta}(\tau, k_1) d^3k_2 d^3k_1
\end{aligned} \tag{1.33}$$

Thus the continuity equation becomes

$$\begin{aligned}
0 &= \int \partial_\tau \delta + \nabla_x(1+\delta) \cdot v e^{-ik \cdot x} d^3x \\
\partial_\tau \tilde{\delta} + \tilde{\theta} &= -\iint \delta_D(k - k_1 - k_2)\frac{(k_1 + k_2) \cdot k_1}{k_1^2}\tilde{\delta}(\tau, k_1)\tilde{\theta}(\tau, k_2)\frac{d^3k_1}{(2\pi)^3} d^3k_2
\end{aligned} \tag{1.34}$$

Following the same methods, the Euler equation becomes

$$\partial_\tau \tilde{\theta} + \mathcal{H}\tilde{\theta} + \frac{3}{2}\Omega_m \mathcal{H}^2 \tilde{\delta} = -\iint \delta_D(k - k_1 - k_2)\frac{(k_1 + k_2)^2 k_1 \cdot k_2}{2k_1^2 k_2^2}\tilde{\theta}(\tau, k_1)\tilde{\theta}(\tau, k_2)\frac{d^3k_1}{(2\pi)^3} d^3k_2 \tag{1.35}$$

In Fourier space, the nonlinearities are represented as integrals over k-space while the linear terms remain as derivatives on the left. The non-linearities are now represented by couplings between different Fourier modes which are represented by the functions of k_1, k_2 in the integrands. For each fourier mode k, there are many combinations for k_1, k_2 such that $k = k_1 + k_2$. In equation 1.35 this manifests as a coupling between different divergences of the matter velocity and, as such, is in fact a requirement for translational invariance in the homogenous universe.

General Solution for an Einstein-de Sitter Cosmology

As a reminder, let me once more write the friedman equations.

$$\partial_\tau \mathcal{H} = \mathcal{H}^2 \left(\Omega_\Lambda - \frac{\Omega_m}{2}\right) \tag{1.36}$$

$$(\Omega_{\text{tot}} - 1)\mathcal{H}^2 = k \tag{1.37}$$

The Einstein-de Sitter cosmology has $\Omega_m = 1$ and $\Omega_\Lambda = 0$. Equation 1.36 becomes

$$\partial_\tau \mathcal{H} = -\frac{1}{2}\mathcal{H}^2 \tag{1.38}$$

Integrating w.r.t τ gives

$$\mathcal{H} = \frac{2}{\tau} \tag{1.39}$$

Now we apply the following perturbative expansion for $\tilde{\delta}$, which also gives the expression for $\tilde{\theta}$ via the linear continuity equation.

$$\begin{aligned}
\tilde{\delta} &= \sum_n a^n(\tau)\tilde{\delta}^{(n)}(k) \\
\tilde{\theta} &= \mathcal{H}\sum_n a^n(\tau)\tilde{\theta}^{(n)}(k)
\end{aligned} \tag{1.40}$$

Now plugging in the expansions to equation 1.34 and 1.35 gives

$$\begin{aligned}n\tilde{\delta}^{(n)} + \tilde{\theta}^{(n)} &= A_n\\ \frac{3}{2}\mathcal{H}^2 a^n \tilde{\delta}^{(n)} + \mathcal{H}^2 a^n \tilde{\theta}^{(n)} + n\mathcal{H}^2 a^n \tilde{\theta}^{(n)} + \dot{\mathcal{H}} a^n \tilde{\theta}^{(n)} &= \tilde{B}_n\\ \frac{3}{2}\tilde{\delta}^{(n)} + \tilde{\theta}^{(n)} + n\tilde{\theta}^{(n)} - \frac{1}{2}\tilde{\theta}^{(n)} &= \frac{1}{\mathcal{H}^2 a^n}\tilde{B}_n \equiv \frac{1}{2}B_n\\ \Rightarrow \begin{pmatrix} n & 1\\ 3 & 1+2n\end{pmatrix}\begin{pmatrix}\tilde{\delta}^{(n)}\\ \tilde{\theta}^{(n)}\end{pmatrix} &= \begin{pmatrix}A_n\\ B_n\end{pmatrix}\\ \Rightarrow \tilde{\delta}^{(n)} = \frac{(1+2n)A_n - B_n}{(2n+3)(n-1)}, \tilde{\theta}^{(n)} &= \frac{-3A_n + nB_n}{(2n+3)(n-1)}\end{aligned} \tag{1.41}$$

And expand the k dependent functions in terms of the linear order density contrast field

$$\begin{aligned}\tilde{\delta}^{(n)} &= \int\cdots\int \delta_D(k - q_1 - \cdots - q_n) F_n(q_1,\ldots,q_n)\tilde{\delta}^{(1)}(q_1)\cdots\tilde{\delta}^{(1)}(q_n) d^3q\cdots d^3q_n\\ \tilde{\theta}^{(n)} &= \int\cdots\int \delta_D(k - q_1 - \cdots - q_n) G_n(q_1,\ldots,q_n)\tilde{\delta}^{(1)}(q_1)\cdots\tilde{\delta}^{(1)}(q_n) d^3q\cdots d^3q_n\end{aligned} \tag{1.42}$$

Writing A_n and B_n explicitely comes from the right hand side of the EOM. Keeping in mind the mode coupling, one must sum over each choice of modes for $\tilde{\theta}$ and $\tilde{\delta}$ such that the sum is n. Thus we have

$$\begin{aligned}A_n &= -\iint \delta_D(k - k_1 - k_2)\frac{(k_1+k_2)\cdot k_1}{k_1^2}\sum_{m=1}^{n-1}\tilde{\delta}^{(n-m)}\tilde{\theta}^{(m)}\frac{d^3k_1}{(2\pi)^3}d^3k_2\\ B_n &= \iint \delta_D(k - k_1 - k_2)\frac{(k_1+k_2)\cdot k_1}{k_1^2}\sum_{m=1}^{n-1}\tilde{\theta}^{(n-m)}\tilde{\theta}^{(m)}\frac{d^3k_1}{(2\pi)^3}d^3k_2\end{aligned} \tag{1.43}$$

By simply plugging in the equations for A_n, B_n, $\tilde{\delta}$, and $\tilde{\theta}$ one finds the recursion relations for F_n and G_n

$$\begin{aligned}F_n &=\\ G_n &=\end{aligned} \tag{1.44}$$

By summing over all permutations of the q_i's we obtain the symmetrized version of the functions F_n and G_n

$$\begin{aligned}F_n^{(s)} &=\\ G_n^{(s)} &=\end{aligned} \tag{1.45}$$

$$\begin{aligned}d\log(D_1) =& \frac{\dot{D}_1}{D_1}d\tau\\ \partial_\tau\delta =& \partial_\tau\sum_n D_n(\tau)\delta^{(n)}(k)\\ =& \sum_n \delta^{(n)}(k)\frac{d}{d\tau}D_n(\tau)\end{aligned} \tag{1.46}$$

General Non-Linear Solution

In general, the non linear solution has differing growth factors for each Fourier mode n. However, we still proceed with separation of variables. We also introduce a density-dependent function $f(\Omega_m, \Omega_\Lambda)$ that encompases the density-related components introduced by the Friedmann equations.

$$\begin{aligned}\delta(k,\tau) &= \sum_n D_n(\tau)\delta^{(n)}(k)\\ \theta(k,\tau) &= -\mathcal{H}(\tau)f(\Omega_m,\Omega_\Lambda)\sum_n E_n(\tau)\theta^{(n)}(k)\end{aligned} \tag{1.47}$$

One can then proceed as usual.

1.3 Non-Linear Growing and Decaying Modes

It should be noted that the above analysis only used the growing modes. To study decaying modes as well, we introduce a vecotr Ψ defined by

$$\Psi(k,z) = \left(\delta(k,z), -\frac{\theta(k,z)}{\mathcal{H}}\right) = (\Psi_1, \Psi_2) \tag{1.48}$$

(Note the minus sign on θ) where **it is assumed $\mathbf{\Omega_m = 1}$** and $z \equiv \ln(a)$. To ease notation (and follow my source :D), we introduce the notation such that repeated Fourier modes are integrated over. The equation of motions for Ψ can be concatenated to a single equation

$$\partial_z \Psi_a(k,z) + \Omega_{ab}\Psi_b(k,z) = \gamma_{abc}(k,k_1,k_2)\Psi_b(k_1,z)\Psi_c(k_2,z) \tag{1.49}$$

where the density tensor and 'coupling' tensor are determined by the original equations of motion as

$$\Omega_{ab} = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \tag{1.50}$$

$$\gamma_{121} = \delta_D(k-k_1-k_2)\alpha(k_1,k_2),\ \gamma_{222} = \delta_D(k-k_1-k_2)\beta(k_1,k_2) \tag{1.51}$$

The equivalent solution is

$$\Psi(k,z) = \sum_n F_n(z)\psi^{(n)}(k) \tag{1.52}$$

We can transform the equation of motion to an integral form by performing a Laplace transform from $z \mapsto \omega$

$$\begin{aligned}
\int_0^\infty e^{-z\omega}\partial_z\Psi_a(k,z) + \Omega_{ab}\Psi_b(k,z)dz &= \int_0^\infty e^{-z\omega}\gamma_{abc}(k,k_1,k_2)\Psi_b(k_1,z)\Psi_c(k_2,z)dz \\
\omega\Psi_a(k,\omega) - \Psi_a(k,0) + \Omega_{ab}\Psi_b(k,\omega) &= \gamma_{abc}(k,k_1,k_2)\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Psi_b(k_1,\sigma)\Psi_c(k_b,\omega-\sigma)d\sigma \\
\underbrace{\omega\Psi_a(k,\omega) + \Omega_{ab}\Psi_b(k,\omega)}_{\equiv\sigma^{-1}_{ab}(\omega)\Psi_b(k,z)} &= \Psi_a(k,0) + \gamma_{abc}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Psi_b(k_1,\sigma)\Psi_c(k_2,\omega-\sigma)d\sigma \\
\Psi_b(k,\omega) &= \sigma_{ab}(\omega)\Psi_a(k,0) + \sigma_{ab}\gamma_{abc}\int_{a-i\infty}^{a+i\infty}\Psi_b(k_1,\sigma)\Psi_c(k_2,\omega-\sigma)d\sigma
\end{aligned} \tag{1.53}$$

With some nice re-labelling of indices, we get

$$\Psi_a(k,\omega) = \sigma_{ab}(\omega)\Psi_b(k,0) + \sigma_{ab}\gamma_{bcd}\int_{a-i\infty}^{a+i\infty}\Psi_c(k_1,\sigma)\Psi_d(k_2,\omega-\sigma)d\sigma \tag{1.54}$$

And finally, performing the inverse Laplace transform gives

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z\omega}\Psi_a(k,\omega)d\omega &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z\omega}\sigma_{ab}(\omega)\Psi_b(k,0) + \sigma_{ab}\gamma_{bcd}\int_{a-i\infty}^{a+i\infty}\Psi_c(k_1,\sigma)\Psi_d(k_2,\omega-\sigma)d\sigma d\omega \\
\Psi_a(k,z) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z\omega}\sigma_{ab}(\omega)\Psi_b(k,0) + \int_0^z e^{z\omega}\sigma_{ab}(z-s)\gamma_{bcd}\Psi_c(k_1,s)\Psi_d(k_2,s)dsd\omega \\
\Psi_a(k,z) &= g_{ab}(z)\Psi_b(k,0) + \int_0^z g_{ab}(z-s)\gamma_{bcd}\Psi_c(k_1,s)\Psi_d(k_2,s)ds
\end{aligned} \tag{1.55}$$

where we have defined g_{ab} as the inverse Laplace transform of σ_{ab}.

Chapter 2

Lagrangian Perturbation Theory

Lagrangian perturbation theory (LPT) is based off Lagrangian coordinates. Suppose a coordinate in Eulerian space is given by $x(t)$. The corresponding Lagrangian coordinate is given by the initial position $q = x(0)$ and the *displacement field* $\psi(q, t)$,

$$x = q + \psi(q, t) \tag{2.1}$$

LPT has many benefits over the eulerian space. Firstly, we have only one field, the dispacement field, rather than two in EPT. Second, conservation of mass is automatically implied (this will be seen rigorously later in the chapter). The main concept to grasp is that Lagrangian coordinates are related to *flows*. Here, we attempt to match the two perturbation theories and emphasize the benefits of LPT.

2.1 Connecting Eulerian and Lagrangian Theories

Since the LPT picture and EPT picture are related by the displacement field, we can attempt to write the EPT equations in Lagrangian coordiantes.

First, lets examine the density, ρ. We are fortunate enough to be working with flows, thus the density in Lagrangian coordinates is constant! Lets denote the Jacobian from Eulerian coordinates to Lagrangian coordinates as J, so that the density is given by

$$\rho(x, t) = J^{-1}\rho_0 \tag{2.2}$$

and as such,

$$\delta(x, t) = J^{-1} - 1 \tag{2.3}$$

Second, lets examine the time derivatives. Rewrite equation 2.1 as

$$x = q + \psi(q, t) = f(q, t) \tag{2.4}$$

The time derivative of f is then

$$\frac{d}{dt}f(q, t) = \partial_t f(q, t) + v \cdot \nabla f(q, t) \tag{2.5}$$

where $v = \frac{dx}{dt}|_{t=0}$. I stress that, since the lagrangian coordinate q is given by $x(0)$, the derivatives do not necessarily vanish! This time derivative is called the *convective derivative along* v.

With these in hand, we can now write the Euler and Continuity equations to the Lagrangian coordinates. The continuity equation becomes

$$\partial_t J^{-1} + \nabla \cdot (J^{-1}v) = 0 \tag{2.6}$$

The Euler equation, on the other hand, can be solved by realizing that in lagrangian space,

$$\dot{f}(q, t) = v\,, \quad \ddot{f}(q, t) = -\nabla\phi \tag{2.7}$$

so that the Euler equation becomes

$$\partial_t v + (v \cdot \nabla) v = -\nabla\phi \Rightarrow \ddot{f} = \ddot{f} \tag{2.8}$$

The Euler equation is completely tautological!

Next, we have the poisson equation and the identity

$$\nabla \times (-\nabla\phi) = 0\,. \tag{2.9}$$

To convert these to the Lagrangian scheme, we need to define the *functional Jacobian* by

$$\mathcal{J}(A, B, C) = \frac{\partial(A, B, C)}{\partial(q_1, q_2, q_3)} = \epsilon_{ijk} A_{|i} B_{|j} C_{|k} \tag{2.10}$$

and the inverse transformation as $h = f^{-1}$. From the cofactor expansion, we can write

$$h_{k,j} = \frac{1}{2} J^{-1} \epsilon_{kab} \epsilon_{jcd} f_{a|c} f_{b|d} \tag{2.11}$$

Then derivatives of the gravitational field $g = -\nabla\phi$ can be represented by

$$g_{i,j} = g_{i|k} h_{k,j} = \frac{1}{2} J^{-1} \epsilon_{kab} \epsilon_{jcd} g_{i|k} f_{a|c} f_{b|d} = \frac{\epsilon_{jcd}}{2} J^{-1} \mathcal{J}(g_i, f_c, f_d) \tag{2.12}$$

So, the divergence is given by

$$g_{i,i} = \nabla \cdot g = \frac{1}{2} \epsilon_{abc} \mathcal{J}(\ddot{f}_a, f_b, f_c) J^{-1} \tag{2.13}$$

To compute the curl, on the other hand, we can use

$$g_{[i,j]} = g_{i,j} - g_{j,i} = -\frac{1}{2} (\nabla \times g)_k = \frac{1}{2} \epsilon_{pq[j} \mathcal{J}(\ddot{f}_{i]}, f_p, f_q) J^{-1} \tag{2.14}$$

Thus, the original constraint equations become

$$\begin{aligned} (\nabla \times g)_i =& \mathcal{J}(\ddot{f}_j, f_i, f_j) J^{-1} = 0 \\ \mathcal{J}(\ddot{f}_1, f_2, f_3) + \mathcal{J}(\ddot{f}_2, f_3, f_1) + \mathcal{J}(\ddot{f}_3, f_1, f_2) =& \Lambda \mathcal{J}(f_1, f_2, f_3) - 4\pi G \rho J \end{aligned} \tag{2.15}$$

Not that illuminating? Thats fine! The important thing here is to emphasize the ways LPT can simplify the system. We no longer have two fields, we have just one, the displacement field. The continuity equation becomes exactly solvable and the Euler equation becomes tautological. The Poisson equation and the curl-free condition restrains the jacobian that connects the Lagrangian and Eulerian coordinates.

Moving forward, one could directly attempt to perturb the Jacobian around a homogoneous background:

$$J = I + \epsilon J^{(1)} + \epsilon^2 J^{(2)} + \dots \tag{2.16}$$

However, at the one-loop level, there are IR divergences for $n \leq -1$ and UV divergences for $n \geq -1$. These will be addressed in the next section.

To make the switch to lagrangian perturbation theory, one must do a change of variables. Go from spatial/Eulerian coordinates to the *trajectory field* or *deformation field*.

$$x = f(X, t)\,; \quad X \equiv f(X, t_0). \tag{2.17}$$

the trajectories are thus entirely parameterized by time and their initial position $x_0 = X$. Also define the Lagrangian time derivative along the velocity as

$$\frac{d}{dt} f(X, t) \equiv \dot{f}(X, t) \equiv \partial_t f(X, t) + v \cdot \nabla f(X, t) \tag{2.18}$$

where $v = \dot{f}(X, y)$. The acceleration of this field is given as the double time derivative, $\ddot{f}(X, y) = g$. With these definitions, the Euler equation is automatically solved.

Now move to the continuity equation. the deformation tensor is the tensor of first derivatives of f

$$f^i_k = \partial_k f^i = \frac{\partial}{\partial X^k} f^i(X,t) \tag{2.19}$$

The volume of a deformed fluid element is given by the jacobian, the determinant of the deformation tensor,

$$\rho(X,t) = J^{-1}\rho_0 = J^{-1}\rho(X,t_0) \tag{2.20}$$

defining the inverse tranform (from lagrangian to eulerian coords),

$$X = h(x,t)\,, \quad h \equiv f^{-1} \tag{2.21}$$

$$h_{a,b} = \partial_b h^a = (J^{-1})^a_b \tag{2.22}$$

Define the functinal jacobian as

$$\mathcal{J}(A,B,C) = \frac{\partial(A,B,C)}{\partial(X_1,X_2,X_3)} = \epsilon_{ijk} A_{|i} B_{|j} C_{|k} \tag{2.23}$$

$\mathcal{J}$ is clearly antisymmetric under permutations i, j, k. Thus $\mathcal{J}$ is a differential 3 form. Nevertheless, this allows one to write $\nabla \cdot g$ as

$$g_{i,j} = g_{i|k} h_{k,j} = \frac{1}{2J}\epsilon_{k\ell m}\epsilon_{jpq} g_{j|k} f_{p|\ell} f_{q|m} = \frac{1}{2J}\mathcal{J}(g_i, f_p, f_q) \tag{2.24}$$

Hence

$$g_{[i,j]} = g_{i,j} - g_{j,i} = -\frac{1}{2}(\nabla \times g)_k = \frac{1}{2}\epsilon_{pq[j}\mathcal{J}(\ddot{f}_{i]}, f_p, f_q)J^{-1} \tag{2.25}$$

$$g_{i,i} = \nabla \cdot g = \frac{1}{2}\epsilon_{abc}\mathcal{J}(\ddot{f}_a, f_b, f_c)J^{-1} \tag{2.26}$$

Hence, from the original Eulerian equations, we have

$$\begin{aligned} (\nabla \times g)_i =& \mathcal{J}(\ddot{f}_j, f_i, f_j)J^{-1} = 0 \\ \mathcal{J}(\ddot{f}_1, f_2, f_3) + \mathcal{J}(\ddot{f}_2, f_3, f_1) + \mathcal{J}(\ddot{f}_3, f_1, f_2) =& \Lambda\mathcal{J}(f_1, f_2, f_3) - 4\pi G\rho J \end{aligned} \tag{2.27}$$

2.1.1 Lagrangian Dynamics

The continuity equation in index form is simply

$$\partial_t v_i + v_k v_{i,k} = g_i \tag{2.28}$$

Taking another spatial derivative

$$\begin{aligned} \partial_t v_{i,j} &= v_{k,j} v_{i,k} + v_k v_{i,kj} \\ \Rightarrow (v_{i,j})^\bullet &= g_{i,j} - v_{i,k} v_{k,j} \end{aligned} \tag{2.29}$$

decompose $v_{i,j}$ into a trace, symmetric traceless, and anti-symmetric tensors θ,σ_{ij}, and ω_{ij}, respectively.

$$v_{i,j} = \frac{1}{3}\theta\delta_{ij} + \sigma_{ij} + \omega_{ij} \tag{2.30}$$

This nicely simplifies the continuity equation. Plugging in each part separately ($\theta = v_{i,i}$, $\sigma_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) - \frac{1}{3}\theta\delta_{ij}$, and $\omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$).

$$\dot{\theta} = -\frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) + g_{i,i} \tag{2.31}$$

$$(\omega_{ij})^\bullet = -\frac{2}{3}\theta\omega_{ij} - \sigma_{ik}\omega_{kj} - \omega_{ik}\sigma_{kj} + g_{[i,j]} \tag{2.32}$$

$$(\sigma_{ij})^\bullet = -\frac{2}{3}\theta\sigma_{ij} - \sigma_{ik}\sigma_{kj} - \omega_{ik}\omega_{kj} + \frac{2}{3}(\sigma^2 - \omega^2)\delta_{ij} + g_{(i,j)} - \frac{1}{3}g_{k,k}\delta_{ij} \tag{2.33}$$

Now return to the Eulerian fields. The equations are

$$\dot{\rho} = -\rho\theta \tag{2.34}$$

$$\dot{\theta} = -\frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) + \Lambda - 4\pi G\rho \tag{2.35}$$

$$(\omega_i)^\bullet = -\frac{2}{3}\theta\omega_i + \sigma_{ij}\omega_j \tag{2.36}$$

$$(\sigma_{ij})^\bullet = -\frac{2}{3}\theta\sigma_{ij} - \sigma_{ik}\sigma_{kj} - \omega_{ik}\omega_{kj} + \frac{2}{3}(\sigma^2 - \omega^2)\delta_{ij} + E_{ij} \tag{2.37}$$

where $E_{ij} = g_{i,j} - g_{k,k}\delta_{ij}/3$ The first equation is the continuity equation. The second is the Raychaudhuri's equation, equivalent to the Poisson equation. The third is the Kelvin-Helmholtz vorticity transport equation. Equivalent to $\nabla \times g = 0$ given the Euler equation is true

2.1.2 Perturbations Around a Homogeneous Background

A perfectly homogenous space would mean the integral curves are trivial; there is no flow. Thus the Eulerian and Lagrangian coordinates are simply related by

$$x = X \tag{2.38}$$

Then, perturb the Lagrangian coordinates around this homogeneous background by the displacement field $\psi(X, t)$.

$$x = f(X, \tau) = X + \psi(X, \tau) \tag{2.39}$$

We then expand ψ perturbatively

$$\psi(X, t) = \sum_{i=0}^{\infty} \epsilon^i \psi^{(i)}(X, t) \tag{2.40}$$

The (linear) Euler equation says

$$\frac{d^2}{d\tau^2}x + \mathcal{H}\frac{d}{d\tau}x = -\nabla\phi \tag{2.41}$$

Pluging in $f(X, \tau)$ gives

$$\begin{aligned} \frac{d^2}{d\tau^2}(X + \psi(X,\tau)) + \mathcal{H}\frac{d}{d\tau}(X + \psi(X,\tau)) &= -\nabla\phi \\ \frac{d^2}{d\tau^2}\psi + \mathcal{H}\frac{d}{d\tau}\psi &= -\nabla\phi \\ \frac{d^2}{d\tau^2}\psi_{i,i} + \mathcal{H}\frac{d}{d\tau}\psi_{i,i} &= -\nabla^2\phi \\ \left[\frac{d^2}{d\tau^2}\psi_{i|j} + \mathcal{H}\frac{d}{d\tau}\psi_{i|j}\right] &= -\frac{3}{2}\Omega_m\mathcal{H}^2\delta J \\ \left[\frac{d^2}{d\tau^2}\psi_{i|j} + \mathcal{H}\frac{d}{d\tau}\psi_{i|j}\right] &= -\frac{3}{2}\Omega_m\mathcal{H}^2(J-1) \end{aligned} \tag{2.42}$$

Since in LPT, the density is fixed and the volume element evolves with the flow, where as in EPT the volume element is fixed and the density evolves. it holds that

$$\overline{\rho}(1+\delta)d^3x = \overline{\rho}d^3X \Rightarrow J\delta = 1 - J \Rightarrow 1 + \delta = \frac{1}{J} \tag{2.43}$$

Where J takes Eulerian coordinates to Lagrangian coordinates.

2.2 LPT Solutions

2.2.1 ZA Approximation

If we have a growth factor so that $\psi(X, \tau) = D_1(\tau)\psi(X)$, and vorticity vanishes, then

$$\partial_{q^i}\psi^i = -D(\tau)\delta(q) \tag{2.44}$$

The jacobian is diagonal, so $1 + \delta = \frac{1}{[1-\lambda_1 D_1][1-\lambda_2 D_1][1-\lambda_3 D_1]}$

Chapter 3

IR Resummation

In the previous chapter, we saw that the density in the Lagrangian picture is constant.