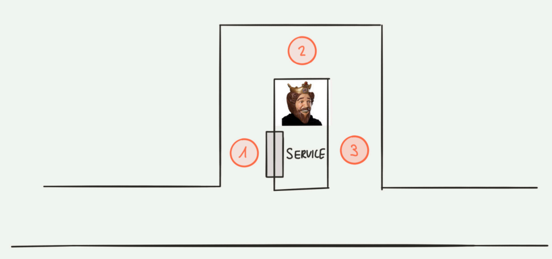
Modell



Mathematisches Modell

Grösse	Parameter
Anzahl Plätze in der Warteschlange	K
Ankommensrate $\frac{\text{Autos}}{\text{min}}$	λ
Servicerate $\frac{\text{Autos}}{\text{min}}$	μ
Mittlerer Umsatz pro Kunde	c_1
Servicekosten	$c_2 \cdot \mu$

$\lambda \cdot \Delta t$ (Δt klein)

$\mu \cdot \Delta t$ "

Def.: $P_k(t)$ Wkeit, dass zum Zeitpunkt t genau k Kunden

$$P_0(t+\Delta t) = P_1(t)\cdot \mu\Delta t \cdot (1-\lambda\Delta t) + P_0(t)\cdot(1-\lambda\Delta t) \qquad (k=0)$$

$$P_k(t+\Delta t) = P_{k-1}(t)\cdot\lambda\Delta t\cdot(1-\mu\Delta t) + P_k(t)\cdot(1-\lambda\Delta t)\cdot(1-\mu\Delta t) + P_{k+1}\cdot\mu\Delta t\cdot(1-\lambda\Delta t) \qquad (1\le k\le K)$$

$$P_K(t+\Delta t) = P_{K-1}(t)\cdot\lambda\cdot\Delta t\cdot(1-\mu\Delta t) + P_K(t)\cdot(1-\mu\Delta t) \qquad (k=K)$$

$$P_0(t+\Delta t) = \mu P_1(t)\cdot\Delta t + P_0(t) - \lambda P_0(t)\cdot\Delta t$$

$$P_0(t+\Delta t) - P_0(t) = \mu P_1(t)\Delta t - \lambda P_0(t)\Delta t$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = \mu P_1(t) - \lambda P_0(t)$$

$$P_k(t+\Delta t) = \lambda P_{k-1}(t)\Delta t - \lambda P_k(t)\Delta t - \mu P_k(t)\Delta t + P_k(t) + \mu P_{k+1}(t)\Delta t$$

$$\frac{P_k(t+\Delta t) - P_k(t)}{\Delta t} = \lambda P_{k-1}(t) - (\lambda+\mu)P_k(t) + \mu P_{k+1}(t)$$

$$P_k(t+\Delta t) = \lambda P_{k-1}(t)\Delta t + P_k(t) - \mu P_k(t)\Delta t$$

$$\frac{P_k(t+\Delta t) - P_k(t)}{\Delta t} = \lambda P_{k-1}(t) - \mu P_k(t)$$

Für $\Delta t \to 0$:

$$\boxed{\begin{aligned} P_0'(t) &= \mu P_1(t) - \lambda P_0(t) \\ P_k'(t) &= \lambda P_{k-1}(t) - (\lambda+\mu) P_k(t) + \mu P_{k+1}(t) \\ P_k'(t) &= \lambda P_{k-1}(t) - \mu P_k(t) \end{aligned}}$$

$t \to \infty$:

$$0 = \mu \pi_1 - \lambda \pi_0$$
$$0 = \lambda \pi_{k-1} - (\lambda+\mu)\pi_k + \mu \pi_{k+1}$$
$$0 = \lambda \pi_{k-1} - \mu \pi_k$$

$\pi_k := \lim_{t \to \infty} P_k(t)$

$\to \quad \lambda \pi_0 = \mu \pi_1$

$\frac{\lambda}{\mu} \cdot \pi_0 = \pi_1$

$$0 = \lambda \pi_0 - (\lambda+\mu)\pi_1 + \mu \pi_2$$
$$0 = \lambda \pi_0 - (\lambda+\mu) \cdot \frac{\lambda}{\mu} \pi_0 + \mu \pi_2$$
$$0 = \cancel{\lambda \pi_0} - \frac{\lambda^2}{\mu} \pi_0 - \cancel{\lambda \pi_0} + \mu \pi_2$$
$$\frac{\lambda^2}{\mu} \pi_0 = \mu \pi_2$$
$$\frac{\lambda^2}{\mu^2} \pi_0 = \pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \cdot \pi_0$$

$\to$ GF $q = \frac{\lambda}{\mu}$

Beweis mit Induktion : $k \to k+1$

gelte $\pi_k = \left(\frac{\lambda}{\mu}\right)^k \cdot \pi_0$

$$\pi_{k+1} = -\frac{\lambda}{\mu}\pi_{k-1} + \left(\frac{\lambda}{\mu}+1\right)\pi_k$$

$$= -\frac{\lambda}{\mu}\pi_{k-1} + \frac{\lambda}{\mu}\pi_k + \pi_k$$

$$= -\cancel{\left(\frac{\lambda}{\mu}\right)^k}\pi_0 + \left(\frac{\lambda}{\mu}\right)^{k+1}\pi_0 + \cancel{\left(\frac{\lambda}{\mu}\right)^k}\cdot\pi_0$$

$$\pi_{k+1} = \left(\frac{\lambda}{\mu}\right)^{k+1}\pi_0 \qquad \square$$

Berechnung von π_k: klar $\sum_{k=0}^{K}\pi_k = 1 = \pi_0 + \pi_1 + \dots + \pi_K$

$$\rightarrow \quad s_K = \pi_0 \cdot \frac{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\frac{\lambda}{\mu}} = 1$$

$$\Rightarrow \quad \pi_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}$$

$$\Rightarrow \quad \pi_k = \left(\frac{\lambda}{\mu}\right)^k \cdot \pi_0 = \underline{\underline{\left(\frac{\lambda}{\mu}\right)^k \cdot \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}}}$$

Spezialfall: $\left(\frac{\lambda}{\mu}=1\right)$ $\quad \pi_0 + \dots + \pi_K = 1$

$$\pi_k = \underline{\underline{\frac{1}{K+1}}}$$

Eintrittsrate: $\lambda \cdot (1 - \pi_K)$

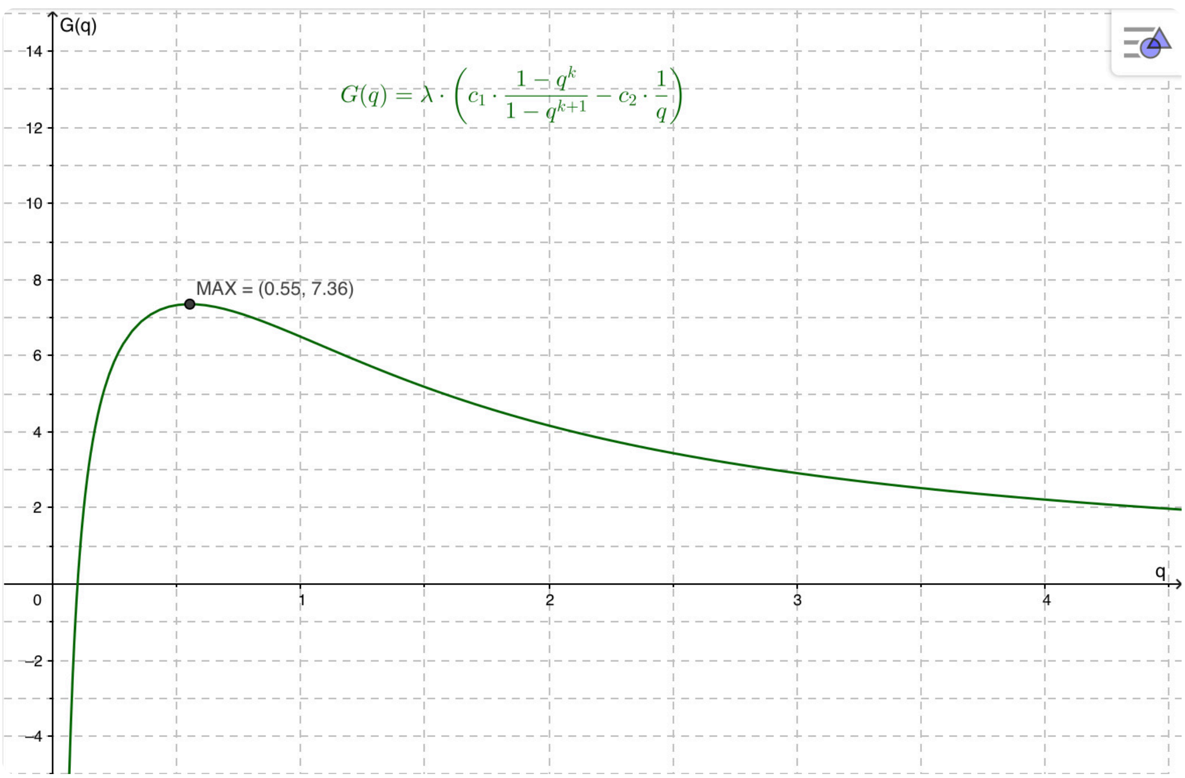
Gewinn pro Zeiteinheit: $G(q) = c_1 \cdot \lambda(1 - \pi_K) - c_2 \cdot \mu$

$q = \frac{\lambda}{\mu}$

$$= \lambda c_1 \left(1 - \frac{1-q}{1-q^{k+1}} \cdot q^K\right) - c_2 \cdot \frac{\mu\lambda}{\lambda}$$

$$= \lambda \left(c_1 \cdot \frac{1 - q^{k+1} - (1-q) q^K}{1 - q^{K+1}} - c_2 \cdot \frac{1}{q}\right)$$

$$G(q) = \lambda \left(c_1 \cdot \frac{1 - q^K}{1 - q^{K+1}} - c_2 \cdot \frac{1}{q}\right)$$



Für $\Delta t \to 0$:

$$P_0'(t) = \mu P_1(t) - \lambda P_0(t)$$

$$P_k'(t) = \lambda P_{k-1}(t) - (\lambda+\mu) P_k(t) + \mu P_{k+1}(t)$$

$$P_k'(t) = \lambda P_{k-1}(t) - \mu P_k(t)$$

$$\to \quad \begin{aligned} P_0'(t) &= -P_0(t) + 2\mu P_1(t) \\ P_1'(t) &= P_0(t) - 3P_1(t) + 2P_2(t) \\ P_2'(t) &= P_1(t) - 3P_2(t) + 2P_3(t) \\ P_3'(t) &= P_2(t) - 2P_3(t) \end{aligned}$$

$$\to \quad \begin{pmatrix} P_0' \\ P_1' \\ P_2' \\ P_3' \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

$$\chi_A = \det \begin{pmatrix} -1-\lambda & 2 & 0 & 0 \\ 1 & -3-\lambda & 2 & 0 \\ 0 & 1 & -3-\lambda & 2 \\ 0 & 0 & 1 & -2-\lambda \end{pmatrix}$$

$$\chi_A = (-1-\lambda) \cdot \begin{vmatrix} -3-\lambda & 2 & 0 \\ 1 & -3-\lambda & 2 \\ 0 & 1 & -2-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 1 & -3-\lambda & 2 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= (-1-\lambda) \cdot \left[(-3-\lambda) \cdot \begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 0 \\ 1 & -2-\lambda \end{vmatrix} \right]$$

$$+ 2 \cdot \left[\begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} \right]$$

$$= (-1-\lambda)\cdot\left[(-3-\lambda)\cdot\left((-3-\lambda)(-2-\lambda)-2\right) - 2(-2-\lambda)\right]$$

$$+ 2\cdot\left((-3-\lambda)(-2-\lambda) - 2\right)$$

$$= (-1-\lambda)\cdot\left[(-3-\lambda)\cdot\left(\lambda^2+5\lambda+6\right) - \overbrace{2 - 4 + 2\lambda}^{-6+2\lambda}\right]$$

$$+ 2\left[\left(\lambda^2+5\lambda+6\right) - 2\right]$$

$$= \left(\lambda^2+4\lambda+3\right)\left(\lambda^2+5\lambda+6\right) - \cancel{2}\lambda^2 + 4\lambda + 6$$

$$+ \cancel{2}\lambda^2 + 10\lambda + 8$$

$$= \lambda^4 + 9\lambda^3 + 29\lambda^2 + 53\lambda + 32$$

$$\chi_A = \det\begin{pmatrix} -1-\lambda & 2 & 0 & 0 \\ 1 & -3-\lambda & 2 & 0 \\ 0 & 1 & -3-\lambda & 2 \\ 0 & 0 & 1 & -2-\lambda \end{pmatrix}$$

$$\chi_A = (-1-\lambda)\cdot\begin{vmatrix} -3-\lambda & 2 & 0 \\ 1 & -3-\lambda & 2 \\ 0 & 1 & -2-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} 2 & 0 & 0 \\ 1 & -3-\lambda & 2 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\cdot\left[(-3-\lambda)\cdot\begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} 2 & 0 \\ 1 & -2-\lambda \end{vmatrix}\right]$$

$$- 2\cdot\begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\cdot\left[(-3-\lambda)\cdot\left((-3-\lambda)(-2-\lambda)-2\right) - 2(-2-\lambda)\right]$$

$$-2\cdot\left[(-3-\lambda)(-2-\lambda)-2\right]$$

$$= (-1-\lambda)(-3-\lambda)\cdot(\lambda^2+5\lambda+4) - 2(\lambda^2+3\lambda+2)$$

$$- 2(\lambda^2+5\lambda+4)$$

$$= (\lambda^2+4\lambda+3)(\lambda^2+5\lambda+4) - 4\lambda^2 - 16\lambda - 12$$

$$\chi_A = \lambda^4 + 9\lambda^3 + 23\lambda^2 + 15\lambda \overset{!}{=} 0$$

$\underline{\underline{\lambda_1 = 0}} \;\rightarrow\; \lambda^3 + 9\lambda^2 + 23\lambda + 15 = 0$

$\underline{\underline{\lambda_2 = -1}} \;\rightarrow\; (-1)^3 + 9\cdot(-1)^2 + 23\cdot(-1) + 15 = 0$ ✓

$$(\lambda^3+9\lambda^2+23\lambda+15) : (\lambda+1) = \lambda^2+8\lambda+15 = 0$$

$$\begin{array}{l} -(\lambda^3+\lambda^2) \\ \hline \;\;0 \quad 8\lambda^2 \\ \quad -(8\lambda^2+8\lambda) \\ \hline \end{array}$$

$$(\lambda+5)(\lambda+3) = 0$$

$\rightarrow\; \underline{\underline{\lambda_3 = -3}} \qquad \underline{\underline{\lambda_4 = -5}}$

$$\begin{array}{ll} 0 & 15\lambda \\ & \underline{-(15\lambda+15)} \\ & \qquad 0 \end{array}$$

$\lambda_1 = 0$:

$$\begin{aligned}
-1x + 2y + 0z + 0u &= 0 && \Leftrightarrow \quad x = 2y \quad y = \tfrac{1}{2}x \\
1x - 3y + 2z + 0u &= 0 && \Leftrightarrow \quad 2z = 3y - x = \tfrac{1}{2}x \\
0\cdot x + 1\cdot y - 3z + 2u &= 0 && \qquad z = \tfrac{1}{4}x \\
0\cdot x + 0\cdot y + 1\cdot z - 2u &= 0 && \Leftrightarrow \quad z = 2u \quad u = \tfrac{1}{2}z
\end{aligned}$$

$$\rightarrow \quad \vec{v_1} = \begin{pmatrix} 8 \\ 4 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$:

$$\begin{aligned}
0\cdot x + 2y &= 0 && \Leftrightarrow \quad y = 0 \\
x - 2y + 2z &= 0 && \Leftrightarrow \quad z = -\tfrac{1}{2}x \\
z - u & && \Leftrightarrow \quad u = z
\end{aligned}$$

$$\rightarrow \quad \vec{v_2} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

$\lambda_3 = -3$:

$$\begin{aligned}
2x + 2y &= 0 && \Leftrightarrow \quad y = -x \\
x + 0\cdot y + 2z &= 0 && \Leftrightarrow \quad z = -\tfrac{1}{2}x \\
z + u &= 0 && \Leftrightarrow \quad u = -z
\end{aligned}$$

$$\rightarrow \quad \vec{v_3} = \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda_4 = -5$:

$$\begin{aligned}
4x + 2y &= 0 && \Leftrightarrow \quad y = -2x \\
x + 2y + 2z &= 0 && \Leftrightarrow \quad z = \tfrac{3}{2}x \qquad \rightarrow \quad \vec{v_4} = \begin{pmatrix} 2 \\ -4 \\ 3 \\ -1 \end{pmatrix} \\
z + 3u &= 0 && \Leftrightarrow \quad u = -\tfrac{1}{3}z
\end{aligned}$$

$$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \\ 1 \end{pmatrix} \cdot p_0 \cdot e^{0} + \begin{pmatrix} 2 \\ 6 \\ -1 \\ -1 \end{pmatrix} \cdot p_1 \cdot e^{-t} + \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix} \cdot p_2 \cdot e^{-3t}$$

$$+ \begin{pmatrix} 2 \\ -4 \\ 3 \\ -1 \end{pmatrix} p_3 \cdot e^{-5t}$$

$$\rightarrow \quad P_0(t) = 8p_0 + 2p_1 e^{-t} + 2p_2 e^{-3t} + 2p_3 e^{-5t}$$

$$P_1(t) = 4p_0 \qquad - 2p_2 e^{-3t} - 4p_3 e^{-5t}$$

$$P_2(t) = 2p_0 - p_1 e^{-t} - p_2 e^{-3t} + 3p_3 e^{-5t}$$

$$P_3(t) = p_0 - p_1 e^{-t} + p_2 e^{-3t} - p_3 e^{-5t}$$

$$\rightarrow \quad 15 p_0 = 1 \qquad p_0 = \underline{\underline{\frac{1}{15}}}$$

Simulation: $P_0(0) = 0$, $P_1(0) = 1$, $P_2(0) = 0 = P_3(0)$

$$0 = \frac{8}{15} + 2p_1 + 2p_2 + 2p_3 \qquad (1)$$

$$1 = \frac{4}{15} \qquad - 2p_2 - 4p_3 \qquad (2)$$

$$0 = \frac{2}{15} - p_1 - p_2 + 3p_3 \qquad (3)$$

$$0 = \frac{1}{15} - p_1 + p_2 - p_3 \qquad (4)$$

$(1) + (2)$: $\quad 1 = \frac{12}{15} + 2p_1 - 2p_3 \qquad (1')$

$$(3)+(4) : 0 = \frac{3}{15} - 2p_1 + 2p_3 \qquad (2')$$

$$(2)+2\cdot(4) : 1 = \frac{6}{15} - 2p_1 - 6p_3 \qquad (3')$$

$$(1')+(3') : 2 = \frac{18}{15} - 8p_3$$

$$8p_3 = -\frac{12}{15}$$

$$p_3 = -\frac{3}{30} = \underline{\underline{-\frac{1}{10}}}$$

$$\rightarrow 0 = \frac{3}{15} - 2p_1 - \frac{3}{15}$$

$$p_1 = \underline{\underline{0}}$$

$$\rightarrow 1 = \frac{4}{15} - 2p_2 + \frac{6}{15}$$

$$2p_2 = -\frac{5}{15} = -\frac{1}{3}$$

$$p_2 = \underline{\underline{-\frac{1}{6}}}$$

$$\rightarrow \boxed{\begin{aligned} P_0(t) &= \frac{8}{15} - \frac{1}{3}e^{-3t} - \frac{1}{5}e^{-5t} \\ P_1(t) &= \frac{4}{15} + \frac{1}{3}e^{-3t} + \frac{2}{5}e^{-5t} \\ P_2(t) &= \frac{2}{15} + \frac{1}{6}e^{-3t} - \frac{3}{10}e^{-5t} \\ P_3(t) &= \frac{1}{15} - \frac{1}{6}e^{-3t} + \frac{1}{10}e^{-5t} \end{aligned}}$$