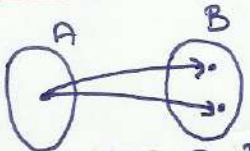
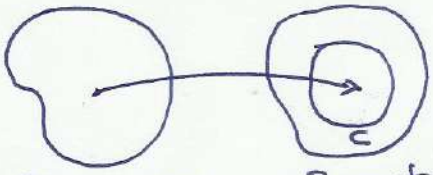
Functions

Definition: Let A and B be sets. A function from A into B, denoted by $f: A \to B$ is a rule that assigns to each element of A <u>Exactly One Element</u> of B.



NOT A FUNCTION

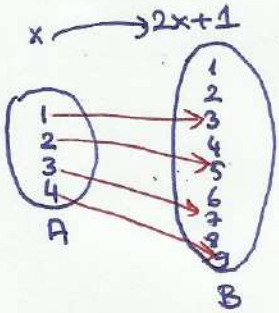


C: range of f
ran(f)

A: domain
dom(f)

B: codomain of f is the set B into which all of the output of the function is constrained to fall.

Example:



$A = \{1,2,3,4\}$ domain

$B = \{1,2,3,4,5,6,7,8,9\}$ codomain

$C = \{3,5,7,9\}$ range

Note: Most of the functions in this course will be real-valued functions of a real variable.

Examples: Find the domain and the range for each of the following functions.

- The domain can be determined from the rule defining the function
- To find the range of a function f:
  - Put $y = f(x)$
  - Solve x in terms of y
  - The range of f is the set of all real numbers y such that x can be solved.

1) $y = \boxed{f(x) = x^2+2}$ This function is defined for all real numbers x, so the domain of f is $\mathbb{R}$. $\mathrm{dom}(f) = \mathbb{R}$.

$y = x^2+2 \Rightarrow x^2 = y-2 \Rightarrow x = \pm\sqrt{y-2}$ Note that x can be solved if and only if (iff) $y-2 \geq 0$. Hence the range of f is $\{y \in \mathbb{R} : y-2 \geq 0\}$

$= \{y \in \mathbb{R} : y \geq 2\}$

$= [2, \infty)$

$\mathrm{dom}(f) = \mathbb{R}$

$\mathrm{ran}(f) = [2, \infty)$

2) $\boxed{y = g(x) = \dfrac{1}{x-2}}$ $g(x)$ is defined for all real numbers x except 2.

So $\mathrm{dom}(g) = \{x \in \mathbb{R} : x \neq 2\} = \mathbb{R} \setminus \{2\}$

$= (-\infty, 2) \cup (2, \infty)$

$y = \dfrac{1}{x-2} \Rightarrow x-2 = \dfrac{1}{y} \Rightarrow x = \dfrac{1}{y}+2$ x can be solved iff $y \neq 0$.

So $\mathrm{ran}(g) = \{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

3) $\boxed{y = h(x) = \sqrt{1+5x}}$ $h(x)$ is defined iff $1+5x \geq 0$.

So $\mathrm{dom}(h) = \{x \in \mathbb{R} : 1+5x \geq 0\} = \{x \in \mathbb{R} : x \geq -\frac{1}{5}\} = [-1/5, \infty)$

Firstly, note that $y = h(x)$ cannot be negative.

$\Downarrow$

$$\left.\begin{array}{l} y = \sqrt{1+5x} \quad y \geq 0 \\ y^2 = 1+5x \Rightarrow x = \dfrac{y^2-1}{5} \end{array}\right\}$$

x can always be solved for every $y \geq 0$.

$\mathrm{ran}(h) = \{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$

Example: Let $f(x) = \sqrt{x+7} - \sqrt{x^2+2x-15}$. Find the domain of f.

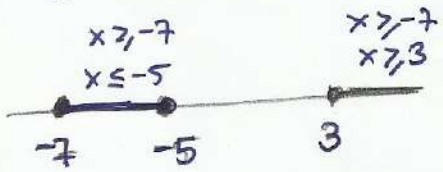
- Note that $f(x)$ is defined iff $x+7 \geq 0$ and $x^2+2x-15 \geq 0$.

$x+7 \geq 0 \Rightarrow x \geq -7$

$x^2+2x-15=(x+5)(x-3) \geq 0 \Rightarrow x \leq -5$ or $x \geq 3$

Note: For such kind of functions you can draw a table.

		-5		3	
$x-3$	$-$		$-$	0	$+$
$x+5$	$-$	0	$+$		$+$
$(x-3)(x+5)$	$+$	0	$-$	0	$+$
	$x \leq -5$				$x \geq 3$



To find the domain of $f(x)$ we have to consider both $\underbrace{x \geq -7}$ and $\underbrace{x \leq -5 \text{ or } x \geq 3}$

So $\text{dom}(f) = [-7,-5] \cup [3,\infty)$

Example: Let $f(x)=\dfrac{2x+1}{x^2+1}$. Find the range of f.

$y=f(x)=\dfrac{2x+1}{x^2+1} \Rightarrow yx^2+y=2x+1$

$x^2y-2x+(y-1)=0$ solve for x.

$\Delta = 4-4y(y-1) \Rightarrow x=\dfrac{2 \pm \sqrt{4-4y(y-1)}}{2y}$ if $y \neq 0$, $x=-\frac{1}{2}$ if $y=0$.

$=\dfrac{1 \pm \sqrt{1-y(y-1)}}{y}$ if $y \neq 0$

Remind: $ax^2+bx+c=0$

$\boxed{\Delta = b^2-4ac}$ $x=\dfrac{-b \pm \sqrt{\Delta}}{2a}$

Hence we see that x can be solved iff $1-y(y-1) \geq 0$ that is $y^2-y-1 \leq 0 \Rightarrow \text{ran}(f)=\{y \in \mathbb{R} : y^2-y-1 \leq 0\}$

Solve $y^2-y-1 \leq 0$. First find the zeros of $y^2-y-1=0$

$\Delta = 1+4=5 \quad y=\frac{1 \pm \sqrt{5}}{2}$

$y^2-y-1=\left(y-\frac{1+\sqrt{5}}{2}\right)\left(y-\frac{1-\sqrt{5}}{2}\right)$

		$\frac{1-\sqrt{5}}{2}$		$\frac{1+\sqrt{5}}{2}$	
$y-\frac{(1-\sqrt{5})}{2}$	$-$	0	$+$		$+$
$y-\frac{(1+\sqrt{5})}{2}$	$-$		$-$	0	$+$
y^2-y-1	$+$	0	$-$	0	$+$

$\text{ran}(f)=\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

Graphing a function: You get the graph of a function f by drawing all points whose coordinates are (x,y) where x must be in the domain of f and $y=f(x)$.

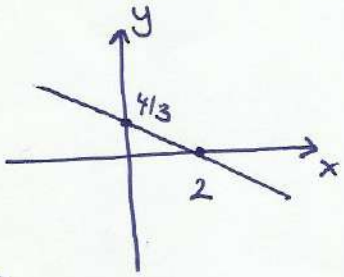
linear function: $y=f(x)=mx+n$, $m \neq 0$ and n are constants.

slope y-intercept

$dom(f)=\mathbb{R}$
$ran(f)=\mathbb{R}$

Its graph is a straight line.

Example: $2x+3y-4=0 \Rightarrow y=\frac{4-2x}{3}=-\frac{2}{3}x+\frac{4}{3}$



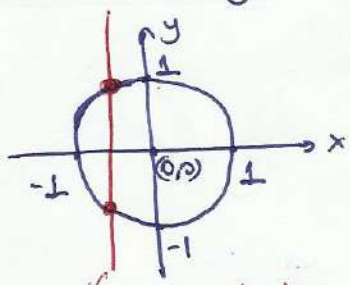
$x=0 \Rightarrow y=\frac{4}{3}$ The point $P(0,4/3)$: y-intercept

$y=0 \Rightarrow x=2$ The point $Q(2,0)$: x-intercept

The graph of this equation is the straight line passing through P and Q.

- **Note:** Not every curve you draw is the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so NO VERTICAL LINE CAN INTERSECT THE GRAPH OF A FUNCTION MORE THAN ONCE.

Example: $x^2+y^2=1$ equation for unit circle. center $(0,0)$
radius 1

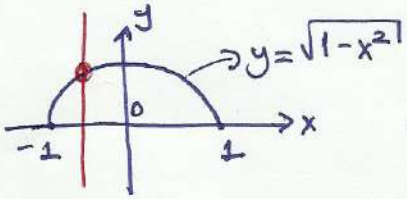


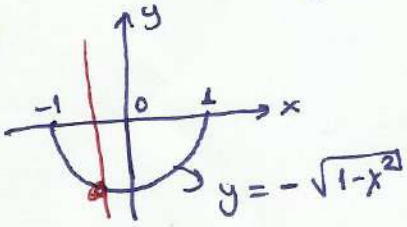
vertical line intersecting two different points.

(Imagine it as a perfect circle :))

The equation $x^2+y^2=1$ is not a function.

But if you take: $y^2=1-x^2$
$\Rightarrow y=\pm\sqrt{1-x^2}$





$y=\sqrt{1-x^2}$ and $y=-\sqrt{1-x^2}$ are function.

Quadratic function: $y=f(x)=ax^2+bx+c$ The graph of the quadratic function is called a parabola. $dom(f)=\mathbb{R}$

$a<0$ $\overset{\text{vertex}}{\wedge}$; $a>0$ $\underset{\text{vertex}}{\vee}$

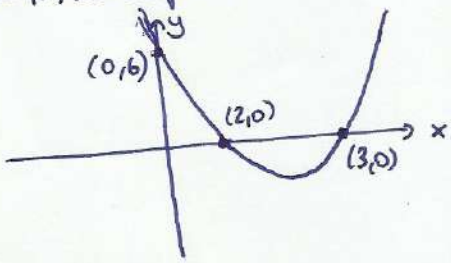
$ran(f) = [k,\infty)$ if $a>0$
$(-\infty,k)$ if $a<0$
k: y-coordinate of the vertex

Example: $y=x^2-5x+6=(x-2)(x-3)$

$x=2 \Rightarrow y=0$
$x=3 \Rightarrow y=0$) The points $(2,0)$ and $(3,0)$ are x-intercepts.

$x=0 \Rightarrow y=6$) The point $(0,6)$ is y-intercept.

since $a=1>0$, the parabola will open up.

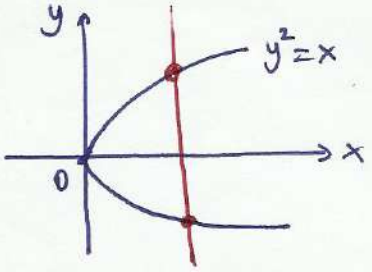


Example: $y^2=x$

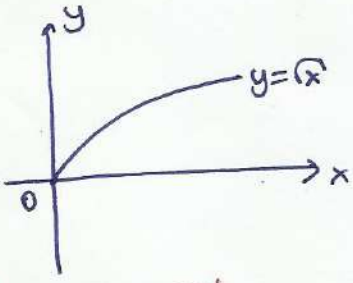
or $y=\pm\sqrt{x}$

It is clear that if (a,b) is a point belonging to parabola then $(a,-b)$ also belongs to the parabola since $b^2=(-b)^2$.

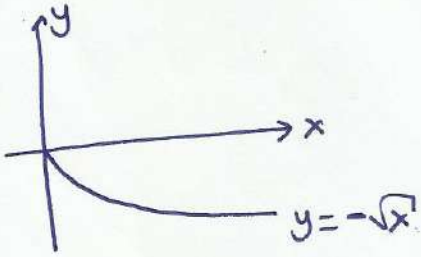
The graph passes from the origin $(0,0)$.



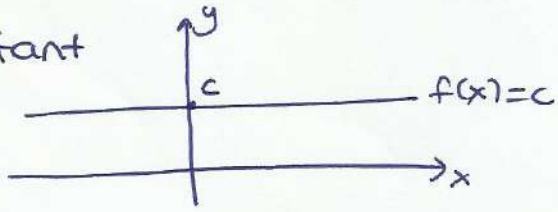
not a function of x. But if we reverse the roles of x and y then it is a function of y.



"square root function"



Constant function: $f(x)=c$, c constant



$dom(f)=\mathbb{R}$
$ran(f)=\{c\}$ a singleton

Polynomial function: $f(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ polynomial function of degree n.

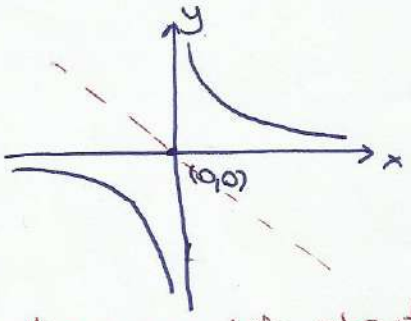
a_i , $i=0,1,\dots,n$ are constants
$a_n\neq 0$ $n\in\mathbb{N}$

* when $n=2 \Rightarrow$ quadratic function
$n=1 \Rightarrow$ linear function
$n=0 \Rightarrow$ constant function

Rational function: $f(x) = \frac{p(x)}{q(x)}$; $p(x)$ and $q(x)$ are polynomial functions.

Example: $y = f(x) = \frac{1}{x}$ $\quad dom(f) = \mathbb{R} \setminus \{0\}$

$y = 1/x \Rightarrow x = 1/y \quad \Rightarrow \quad ran(f) = \mathbb{R} \setminus \{0\}$



* symmetric about the origin

Exercise: $f(x) = 1/x^2$ $\quad dom(f) = ?$
$ran(f) = ?$
graph of f?

Combining functions f and g are given functions.

Sum: $(f+g)(x) = f(x) + g(x)$ $\quad dom(f+g) = dom(f) \cap dom(g)$

Difference: $(f-g)(x) = f(x) - g(x)$ $\quad dom(f-g) = dom(f) \cap dom(g)$

Product: $(fg)(x) = f(x)g(x)$ $\quad dom(fg) = dom(f) \cap dom(g)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\quad dom(f/g) = dom(f) \cap dom(g) \setminus \{g(x) \neq 0\}$

Composite functions: $(f \circ g)(x) = f(g(x))$. $\quad dom(f \circ g) = \{x \in dom(g) : g(x) \in dom(f)\}$

Examples: 1) Let $f(x) = x^2 + 1$ and $g(x) = x + 1$. Find $(f \circ g)(a^2)$ and $(g \circ f)(1)$.

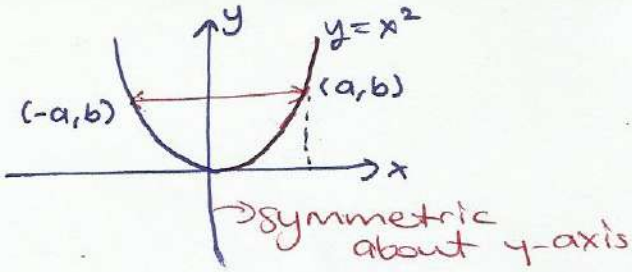
- $f \circ g(x) = f(g(x)) = (x+1)^2 + 1 \Rightarrow f \circ g(a^2) = (a^2+1)^2 + 1$
- $g \circ f(x) = g(f(x)) = (x^2+1) + 1 \Rightarrow g \circ f(1) = (1^2+1) + 1 = 3$.

Even functions and Odd Functions. $y = f(x)$

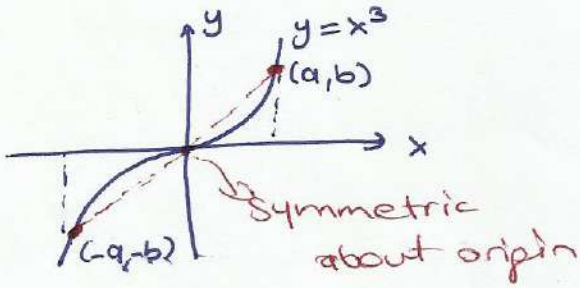
$f(-x) = f(x)$ even function (symmetric about the y-axis)

$f(-x) = -f(x)$ odd function (symmetric about the origin)

Examples: $f(x)=x^2$ $\longrightarrow$ even function
$f(-x)=(-x)^2=x^2$



$f(x)=x^3$ $\longrightarrow$ odd function
$f(-x)=(-x)^3=-x^3$

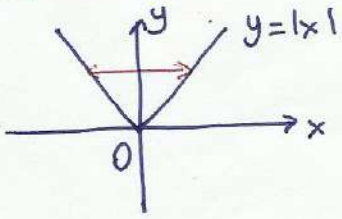


Piecewise Defined Functions different formulas on different parts of the domain of the function.

Example: Absolute Value Function

$$y=|x|=\begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0 \end{cases}$$

$dom(y)=\mathbb{R}=(-\infty,+\infty)$
$ran(y)=[0,\infty)$



"symmetric about y-axis"

$|a|$: distance from the point a to 0 on the real number line

$|a-b|$: distance between a and b

$$\boxed{\sqrt{a^2}=|a|}$$

Example: $f:[-2,6]\to\mathbb{R}$

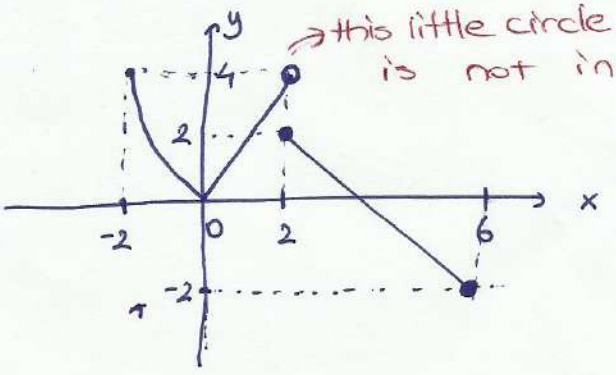
$$f(x)=\begin{cases} x^2 & \text{if } -2\le x<0 \\ 2x & \text{if } 0\le x<2 \\ 4-x & \text{if } 2\le x\le 6 \end{cases}$$

$f(-1)=?$

$f(1/2)=?$

Sketch the graph of f.

$f(-1)=(-1)^2=1$; $f(1/2)=2\cdot(1/2)=1$.

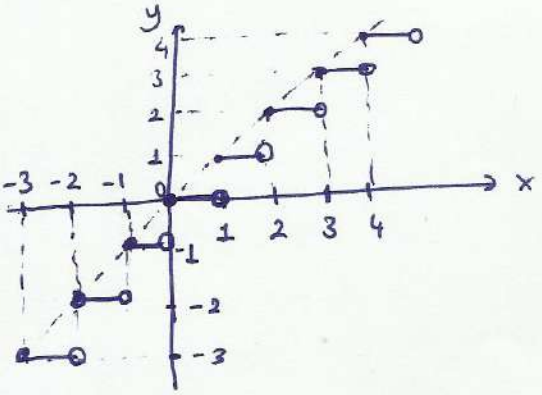


this little circle indicates that the point (2,4) is not included in the graph.

Example: The <u>greatest</u> integer function whose value at any number x is the greatest integer LESS THAN OR EQUAL to x. <u>Notation</u>: $\lfloor x \rfloor$ or $[x]$ or $[\![x]\!]$.

$\lfloor 2.4 \rfloor = 2$, $\lfloor 0 \rfloor = 0$, $\lfloor 1.9 \rfloor = 1$, $\lfloor -0.6 \rfloor = -1$, $\lfloor -1.4 \rfloor = -2$.

$\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$



Example: The <u>least</u> integer function whose value at any number x is the smallest integer GREATER THAN OR EQUAL to x.

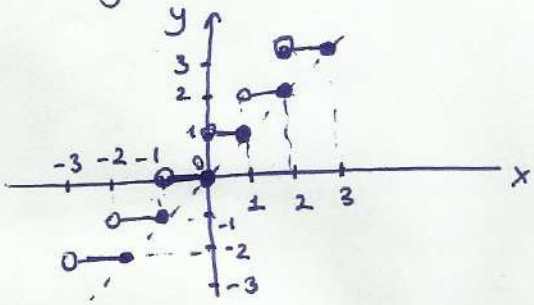
Notation: $\lceil x \rceil$

$\lceil 2.4 \rceil = 3$ $\lceil 2 \rceil = 2$

$\lceil 1.9 \rceil = 2$ $\lceil -2 \rceil = -2$

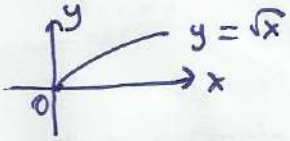
$\lceil -0.6 \rceil = 0$

$\lceil 1.4 \rceil = -1$



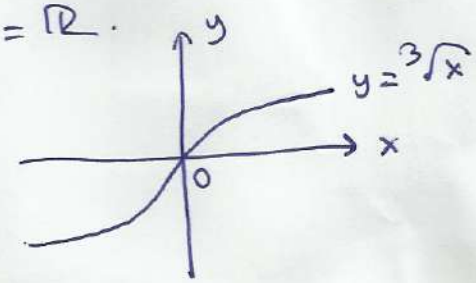
Root function: $f(x) = x^{1/n} = \sqrt[n]{x}$

For $n=2$ (EVEN), $f(x) = \sqrt{x}$ is the square root function and its domain is $\mathrm{dom}(f(x)) = [0, \infty)$.



For other <u>even</u> values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $\sqrt{x}$.

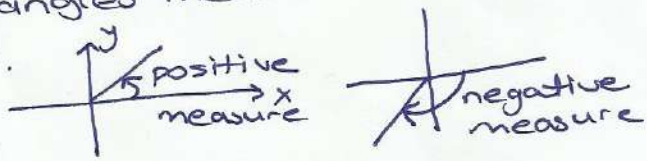
For $n=3$ (ODD), $f(x) = \sqrt[3]{x} = x^{1/3}$ (cube root function). Here $\mathrm{dom}(f) = \mathbb{R}$.



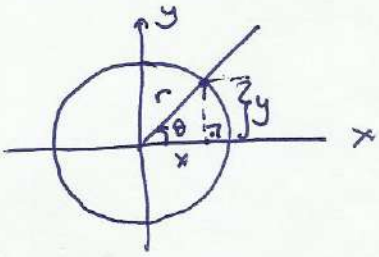
The graph of $y = \sqrt[n]{x}$ for n odd $(n>3)$ is similar to $y = \sqrt[3]{x}$

Trigonometric functions

- In calculus, it is best to use units called radians for the angles.
- π radians $= 180°$
- Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.



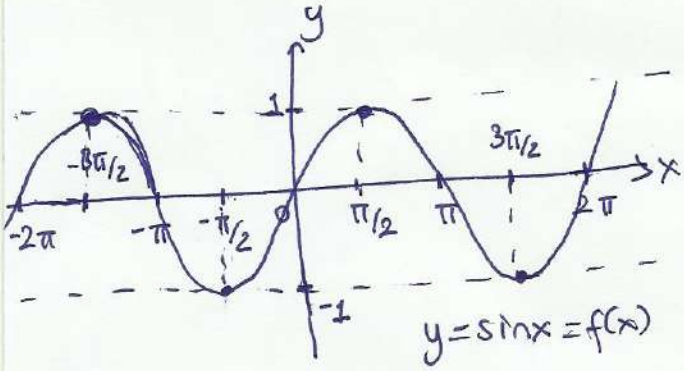
There are six basic trigonometric functions.



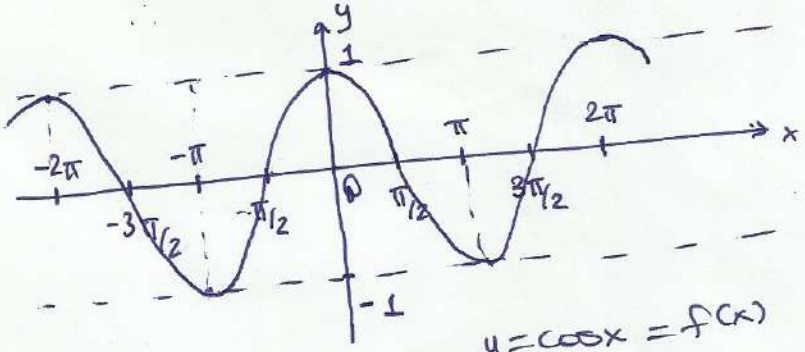
$\sin\theta = y/r$, $\cos\theta = x/r$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = y/x$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = x/y$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$, $\csc\theta = \operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$



$y = \sin x = f(x)$



$y = \cos x = f(x)$

$f(x) = \sin x$ ODD FUNCTION

$f(-x) = \sin(-x) = -\sin x$

$\mathrm{dom}(\sin) = \mathbb{R}$

$\mathrm{ran}(\sin) = [-1, 1]$

$-1 \le \sin x \le 1$

$-1 \le \cos x \le 1$

or

$|\sin x| \le 1$, $|\cos x| \le 1$

$f(x) = \cos x$ EVEN FUNCTION

$f(-x) = \cos(-x) = \cos x$

$\mathrm{dom}(\cos) = \mathbb{R}$

$\mathrm{ran}(\cos) = [-1, 1]$

$\sin x = 0$ when $x = n\pi$
n an integer

- sine and cosine functions are periodic with period 2π

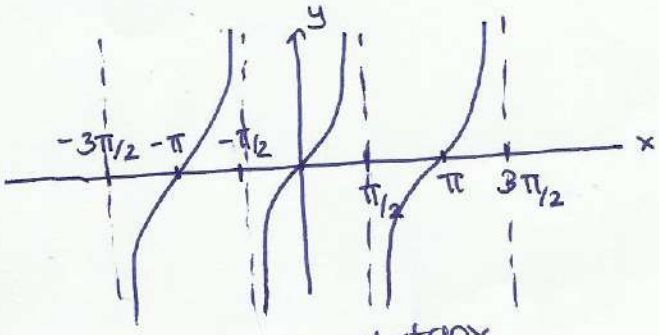
i.e. $\sin(x + 2\pi) = \sin x$, $\cos(x + 2\pi) = \cos x$.

- $\sin^2 x + \cos^2 x = 1$

$\tan x = \frac{\sin x}{\cos x}$ is undefined whenever $\cos x = 0$ i.e. when

$x = \pm \pi/2, \pm 3\pi/2, \ldots$ or $x = (2n+1)\frac{\pi}{2}$, n integer. $\text{ran}(\tan) = (-\infty, \infty)$

And it has period π; $\tan(x+\pi) = \tan x$ for all x.

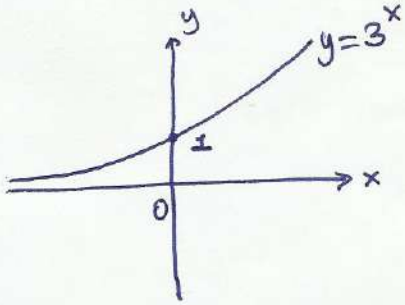


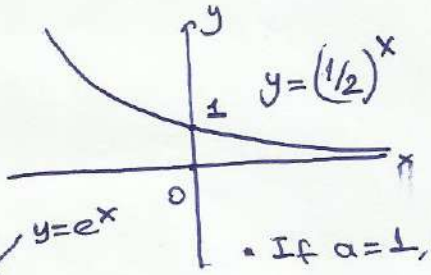
$y = \tan x$

There are several identities and properties for trigonometric functions but we will not give them here. We will recall them when we need them.

Exponential functions $f(x) = a^x$ a: positive constant

$\text{dom}(f) = \mathbb{R}$, $\text{ran}(f) = (0, \infty)$ $a \neq 1$

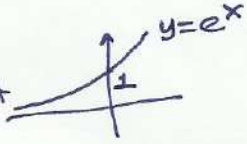




- $a^{x+y} = a^x a^y$
- $(a^x)^y = a^{xy}$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(ab)^x = a^x b^x$

$x, y \in \mathbb{R}$ a, b positive numbers

- If $a = 1$, then



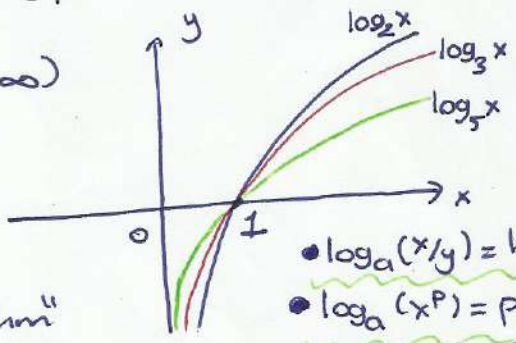
e: Euler Constant
$e \simeq 2.71828$

Logarithmic functions $f(x) = \log_a x$ the base a: positive constant $a \neq 1$

They are the <u>inverse</u> functions of the exponential functions.

$\text{dom}(f) = (0, \infty)$, $\text{ran}(f) = (-\infty, \infty)$

$x \in$



- $\log_a(xy) = \log_a(x) + \log_b(y)$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
- $\log_a(x^p) = p\log_a(x)$

Note:
- $\log_e x = \ln x$ "natural logarithm"
  notation

The trigonometric, inverse trigonometric, exponential and logarithmic functions are in the set of transcendental functions. (function that does not satisfy a polynomial equation)

Transformations of functions

vertical shift $\boxed{k>0}$, $f(x)$ a function

$f(x)+k$; shift the graph of f k units up $\uparrow k$ units

$f(x)-k$; " " down $\downarrow k$ units

horizontal shift

$f(x-k)$; " " right $\xrightarrow{}$ k units

$f(x+k)$; " " left $\xleftarrow{}$ k units

vertical streching

$kf(x)$; stretch the graph of f vertically by k

horizontal streching

$f(x/k)$; " " horizontally by k

vertical compressing

$f(x)/k$; compress the graph of f vertically by k

horizontal compresing

$f(kx)$; " " horizontally by k

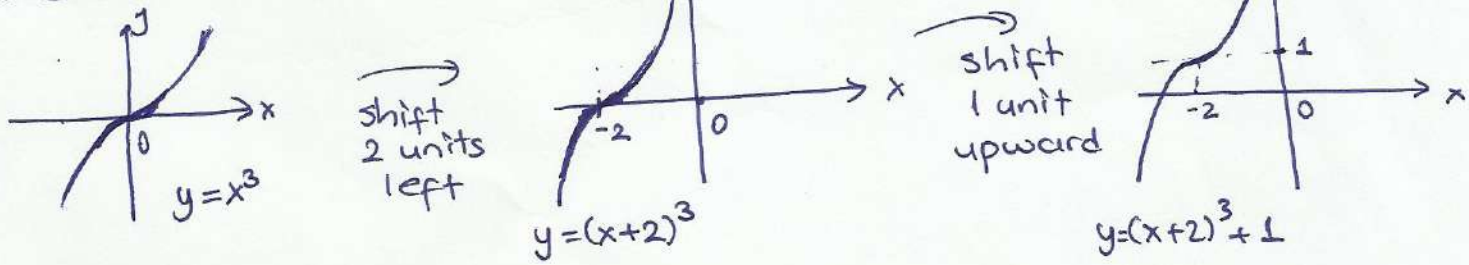
reflecting

$-f(x)$; reflect the graph of f about the x-axis

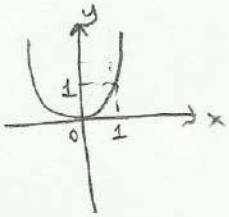
$f(-x)$; " " y-axis.

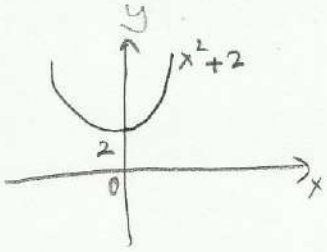
Example: $k(x)=(x+2)^3+1$. Sketch the graph of $k(x)$.

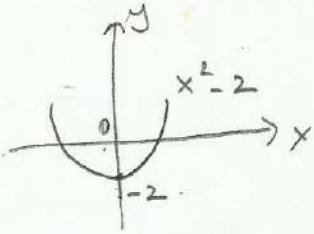
we know the graph of x^3.

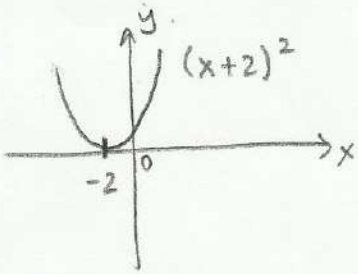


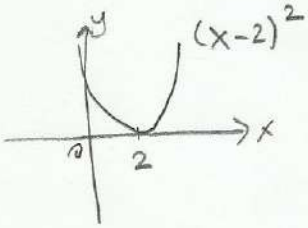
$f(x) = x^2$

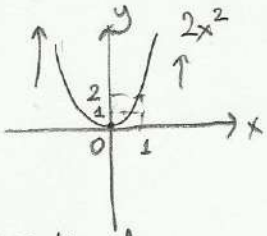




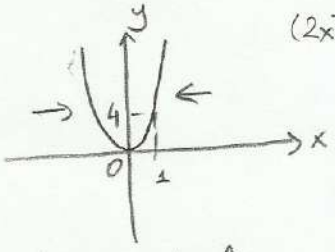




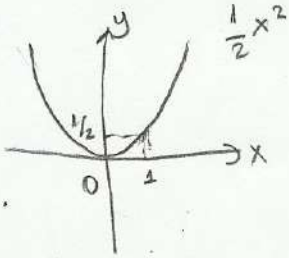




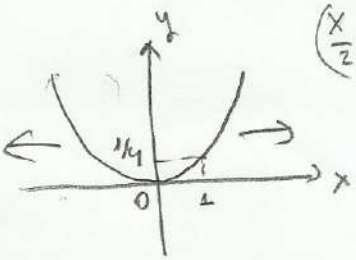
vertical
streching



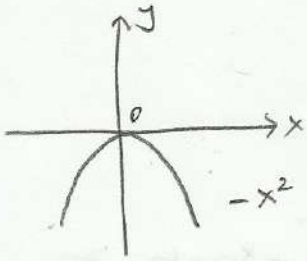
horizontal
compressing



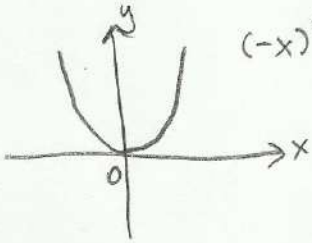
vertical
compressing



horizontal
streching

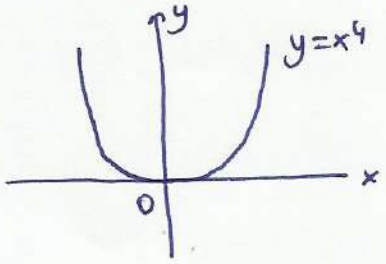


reflecting
about the
x-axis

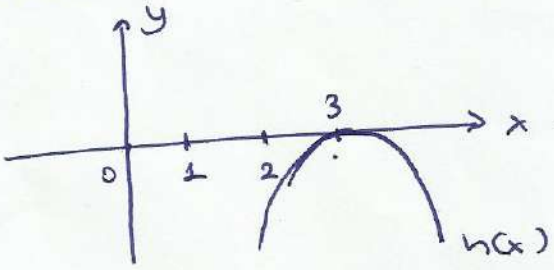


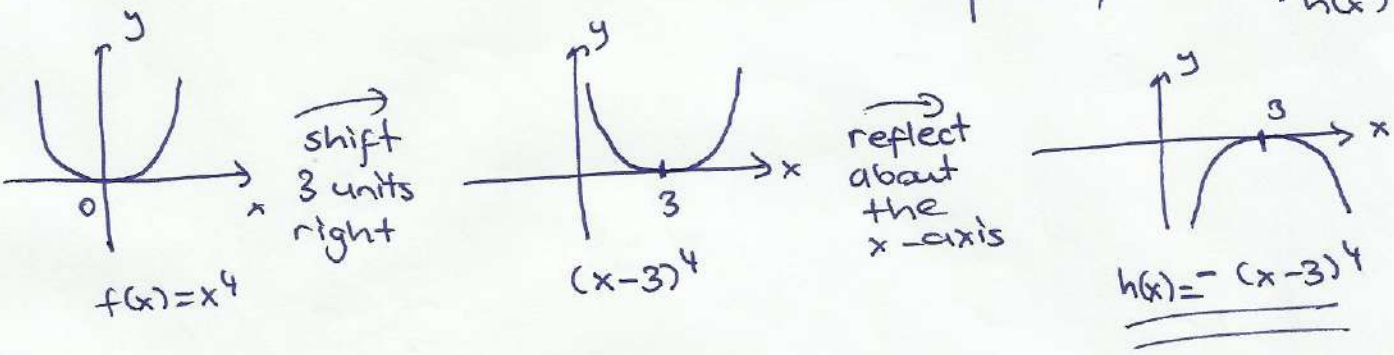
reflecting
about the
y-axis.

Example: The graph of $f(x)=x^4$ is shown below.



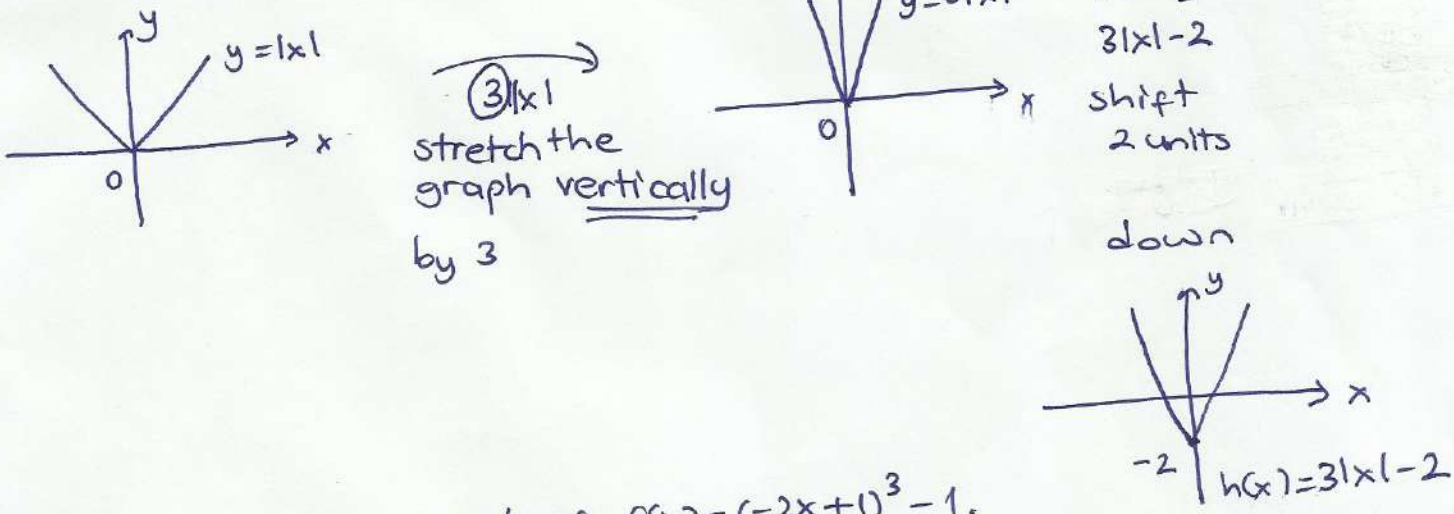
By using this information, write the equation of the function whose graph is a transformation of the graph of $f(x)=x^4$ and given as





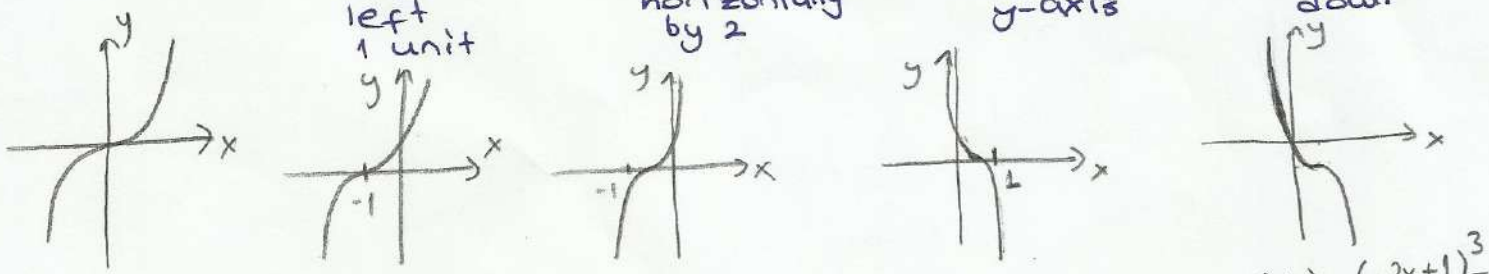
Example: Sketch the graph of $h(x)=3|x|-2$.

Start with $|x|$.



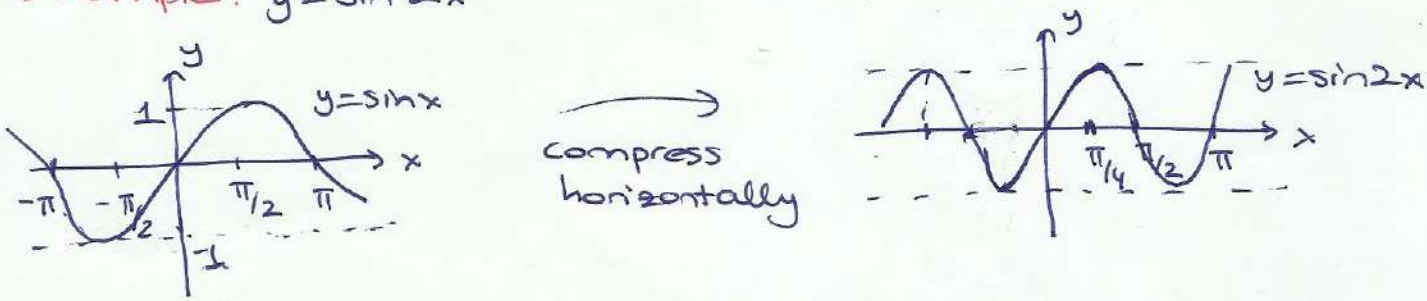
Example: Sketch the graph of $f(x)=(-2x+1)^3-1$.

start with x^3. $\xrightarrow{\text{shift left 1 unit}}$ $(x+1)^3$ $\xrightarrow{\text{compress horizontally by 2}}$ $(2x+1)^3$ $\xrightarrow{\text{reflect about y-axis}}$ $(-2x+1)^3$ $\xrightarrow{\text{shift 1 unit down}}$ $(-2x+1)^3-1$

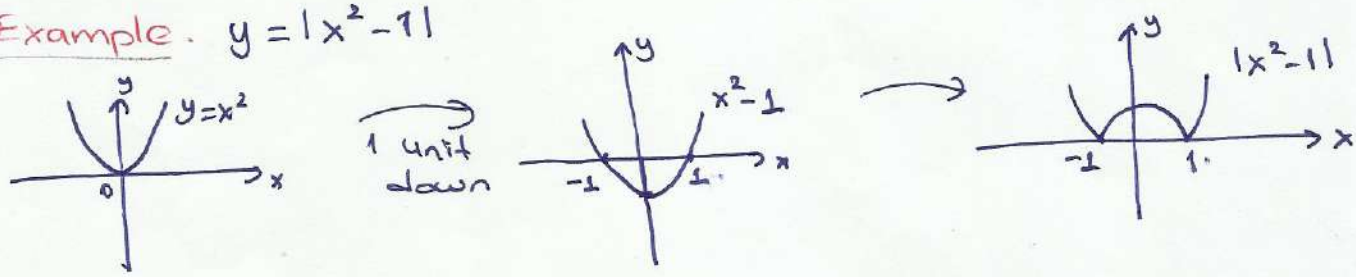


$f(x)=(-2x+1)^3-1$

Example. $y = \sin 2x$



Example. $y = |x^2 - 1|$



Inverse Fnctions

Example1) Let $f(x) = x^3$. Find x such that $f(x) = 8$.

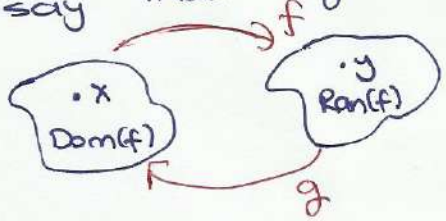
clearly, if $x^3 = 8 \Rightarrow x = 2$.

Example2) Let $f(x) = x^2$. Find x such that $f(x) = 9$.

clearly, if $x^2 = 9 \Rightarrow x = \pm 3$.

Assume that f and g have domains A and B, respectively and that $f(g(x))$ is defined for all $x \in B$ and $g(f(x))$ is defined for all $x \in A$.

If $f(g(x)) = x$ for all $x \in B$ and $g(f(x)) = x$ for all $x \in A$ we say that g is the inverse of f, written $g = f^{-1}$. $(f = g^{-1})$.



In Example 1), there is a function $g = f^{-1}$ s.t. $g(8) = f^{-1}(8) = 2$ but there is no such a g for f of Example 2). It is not clear that $g(9) = 3$ or $g(9) = -3$.

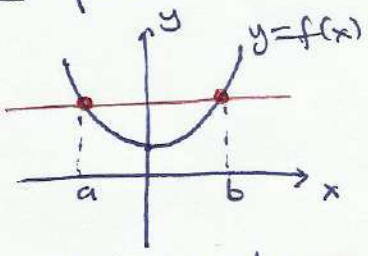
Definition: A function is called one-to-one when for every y in the range of f there is exactly one x in the domain of f for which $y = f(x)$.

Or, equivalently $f(a) = f(b) \Rightarrow a = b$.

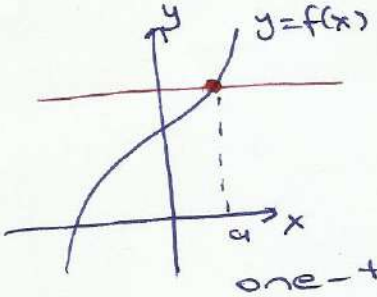
Example: Show that $f(x) = x^2$ is not one-to-one.

Note that $f(-3) = f(3)$ but $-3 \neq 3$.

Horizontal line test: A function f is one-to-one (1-1) if and only if every horizontal line intersects the graph of f in at most one point.



not one-to-one



one-to-one

Theorem: A function f has an <u>inverse</u> iff $\Longleftrightarrow$ it is <u>one-to-one</u>.

Example: 1) Find the inverse of $f(x) = x^3 - 5$.

To find the inverse of f, write $y = f(x)$ and solve for x.

$$y = x^3 - 5 \Rightarrow x^3 = y + 5 \Rightarrow \underset{f^{-1}(y)}{\underset{\parallel}{x}} = (y+5)^{1/3} \Rightarrow f^{-1}(x) = (x+5)^{1/3}$$

2) Find the inverse of $f(x) = 4x + 5$.

$$y = f(x) = 4x + 5 \Rightarrow x = f^{-1}(y) = \frac{y-5}{4} \Rightarrow f^{-1}(x) = \frac{x-5}{4}$$

3) Show that $f(x) = x^2 - 2x + 5$ is one-to-one on (a) $x \geq 1$. (b) $x \leq 1$

(a) $f(x) = x^2 - 2x + 5 = (x-1)^2 + 4$

If $x \geq 1 \Rightarrow x - 1 \geq 0$ and so $(x-1)^2$ are different for different $x-1$'s and hence for different x's.

consequently for $(x-1)^2+4$ are different for different x's.

Hence $f(x)$ is one-to-one on $x \geq 1$.

(b) Similar to (a).

Let us also find the inverse of $f(x)$ on $x \geq 1$.

$$y=(x-1)^2+4 \Rightarrow (x-1)^2=y-4 \Rightarrow \sqrt{(x-1)^2}=\sqrt{y-4} \Rightarrow |x-1|=\sqrt{y-4}$$

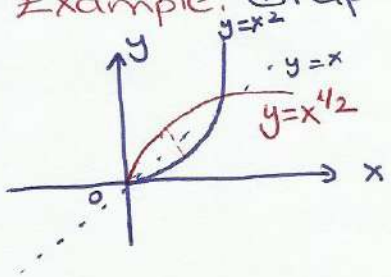
Since $x \geq 1 \Rightarrow x-1 \geq 1$ we have $x-1=\sqrt{y-4} \Rightarrow \underset{f^{-1}(y)}{\underset{\parallel}{x}}=1+\sqrt{y-4}$

Hence $f^{-1}(x)=1+\sqrt{x-4}$.

Remark: If (a,b) is a point on the graph of $y=f(x)$ and f has an inverse f^{-1}, then since $b=f(a)$ we have $f^{-1}(b)=a$ i.e. (b,a) is a point on the graph of $y=f^{-1}(x)$.

(b,a) is the reflection of the point (a,b) through the line $y=x$. Thus, given the graph of any one-to-one function, we can draw the graph of its inverse by simply reflecting the entire graph through the line $y=x$.

Example: Graph $f(x)=x^2$ and its inverse on $x \geq 0$.



Example: If $a>0$ and $a \neq 1$, the exponential function $f(x)=a^x$ is one-to-one by the horizontal line test. It therefore has an inverse function f^{-1} which is called the logarithmic function with base a as we mentioned before.

$$\boxed{\begin{array}{l} f^{-1}(x)=y \iff f(y)=x \\ \log_a x=y \iff a^y=x \end{array}}$$

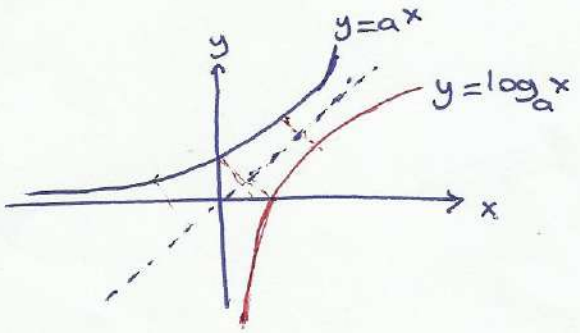
$f(x)=a^x$ and $f^{-1}(x)=\log_a x$

$\Rightarrow \quad f^{-1}(f(x))=x \quad$ i.e. $\log_a a^x = x \quad$ for every $x \in \mathbb{R}$

$f(f^{-1}(x))=x \quad$ i.e. $a^{\log_a x} = x \quad$ for every $x>0$.

$\boxed{a>1}$

<u>The graphs</u>



Particularly, $\ln x = y \Leftrightarrow e^y = x$

$\ln(e^x) = x \quad x \in \mathbb{R} \Rightarrow \ln e = 1$

$e^{\ln x} = x \quad x>0$

Example. Solve the equation $e^{3-5x}=8$.

Take natural logarithms of both sides of the equation

$$\ln(e^{3-5x}) = \ln 8$$

$$3-5x = \ln 8 \Rightarrow 3-\ln 8 = 5x$$

$$\Rightarrow x = \frac{3-\ln 8}{5}.$$

example. Show that for any positive number a $(a \neq 1)$, we have $\log_a x = \dfrac{\ln x}{\ln a}$.

Let $y=\log_a x \Rightarrow a^y = x \Rightarrow \ln(a^y) = \ln x \Rightarrow y \ln a = \ln x$

$$\Rightarrow y = \frac{\ln x}{\ln a}.$$

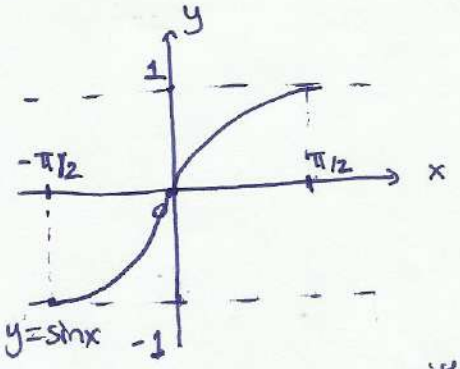
Example: (inverse trigonometric functions)

Note that $\sin x$ is not one-to-one on $\mathbb{R}$ and so it is not invertible. But if we <u>restrict its domain</u> to the interval $[-\pi/2, \pi/2]$, it will be one-to-one and so invertible there.

We define the inverse sine function by $y=\sin^{-1}x$ or $y=\arcsin x$ if and only if $\sin y = x$ and $-\pi/2 \le x \le \pi/2$.

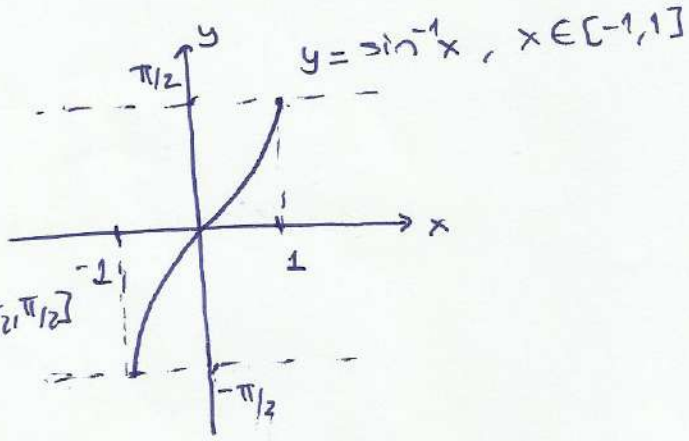
$$\sin(\sin^{-1}x) = x \quad \text{for all } x \in [-1,1]$$

$$\sin^{-1}(\sin x) = x \quad \text{for all } x \in [-\pi/2, \pi/2]$$



$x \in [-\pi/2, \pi/2]$

reflect the graph of $\sin x$, $x \in [-\pi/2, \pi/2]$ with respect to the line $y=x$.



example: $\sin^{-1}(\sqrt{3}/2) = ?$, $\sin^{-1}(-1/2) = ?$

- $\sin^{-1}(\sqrt{3}/2) = y \Rightarrow \sqrt{3}/2 = \sin y$, $y \in [-\pi/2, \pi/2]$, $y = ?$

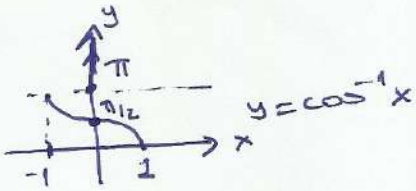
  $y = \pi/3 \in [-\pi/2, \pi/2]$ ✓

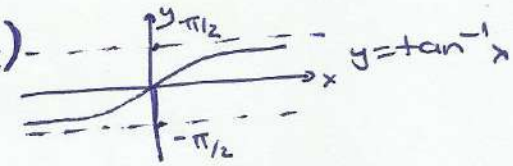
- $\sin^{-1}(-1/2) = y \Rightarrow -1/2 = \sin y$, $y \in [-\pi/2, \pi/2]$, $y = ?$

  $y = -\pi/6 \in [-\pi/2, \pi/2]$ ✓

Similarly, we define the other inverse trigonometric functions.

	Domain	Range
$y=\sin^{-1}(x)=\arcsin(x)$	$[-1,1]$	$[-\pi/2, \pi/2]$
$y=\cos^{-1}(x)=\arccos(x)$	$[-1,1]$	$[0,\pi]$
$y=\tan^{-1}(x)=\arctan(x)$	$(-\infty,\infty)$	$(-\pi/2, \pi/2)$
$y=\cot^{-1}(x)=\operatorname{arccot}(x)$	$(-\infty,\infty)$	$(0,\pi)$
$y=\operatorname{cosec}^{-1}(x)=\operatorname{arccosec}(x)$	$\lvert x\rvert \geq 1$	$(0,\pi/2] \cup (\pi, 3\pi/2]$
$y=\sec^{-1}(x)=\operatorname{arcsec}(x)$	$\lvert x\rvert \geq 1$	$[0,\pi/2) \cup [\pi, 3\pi/2)$

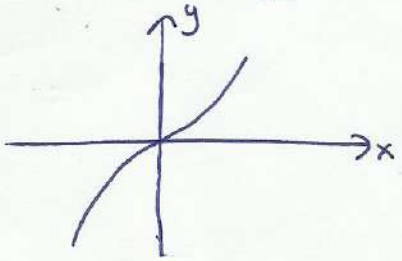




Hyperbolic functions:

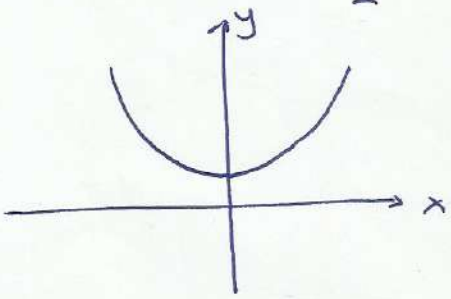
- Hyperbolic sine function

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$



- Hyperbolic cosine function.

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$



other hyperbolic functions are defined in the expected way.

For example, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

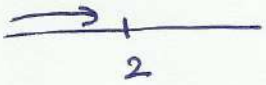
LIMIT

Let us start with an example.

Example: Let $f(x) = \frac{x^2 - 4}{x - 2}$. Examine the behavior of $f(x)$ as x approaches 2.

Let us compute some values of $f(x)$ for x close to 2.

x	$f(x) = (x^2-4)/(x-2)$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999



$f(x)$ is getting closer and closer to 4 as x approaches 2 from the left side.

The limit of $f(x)$ as x approaches 2 from LEFT is 4

$$\lim_{x \to 2^-} f(x) = 4$$

"left-hand limit"

x	$f(x) = (x^2-4)/(x-2)$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

The limit of $f(x)$ as x approaches 2 from RIGHT is 4

$$\lim_{x \to 2^+} f(x) = 4$$

"right-hand limit"

We call $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ "one-sided limits".

Since they are same we write

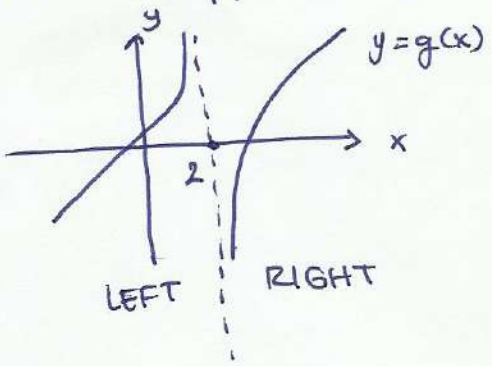
$$\lim_{x\to 2} f(x) = 4.$$

Definition: We write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (ON EITHER SIDE OF a) but not equal to a.

Indeed, since $x \neq 2$, $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

$$= \lim_{x\to 2} (x+2) = 4.$$

Example: Let $g(x) = \frac{x^2-5}{x-2}$. Examine the behavior of $g(x)$ as x approaches 2 by considering the graph of $g(x)$.

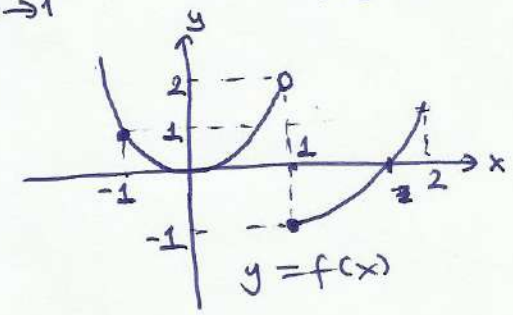


$\lim_{x\to 2^-} g(x) = +\infty$

$\lim_{x\to 2^+} g(x) = -\infty$

Since the one-sided limits of $g(x)$ are not equal to eachother, we say that the limit of $g(x)$ as x approaches 2 does not exist.

Example: Use the graph below to determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$, and $\lim_{x\to -1} f(x)$.



$\lim_{x\to 1^-} f(x) = 2$

$\lim_{x\to 1^+} f(x) = -1$

$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

Hence $\lim_{x\to 1} f(x)$ does not exist.

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = 1 \Rightarrow \lim_{x\to -1} f(x) = 1.$

Computation of Limits

Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, c is any constant, L and M are real numbers.

- $\lim_{x\to a} c = c$
- $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
- $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$

- Suppose that $\lim_{x\to a} f(x) = L$ and n is any positive integer. Then

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$

where for n even, we assume that $L > 0$.

Examples

1) $\lim_{x\to -3} \dfrac{3x+9}{x^2-9} = \lim_{x\to -3} \dfrac{3\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \lim_{x\to -3} \dfrac{3}{x-3} = -\dfrac{1}{2}$

2) $\lim_{x\to 3} \dfrac{3x+9}{x^2-9} = ?$ $\quad \lim_{x\to 3^-} \dfrac{3\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = -\infty$ $\quad$ limit does not exist.

$\lim_{x\to 3^+} \dfrac{3\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = +\infty$

3) $\lim_{x\to 0} \dfrac{x}{|x|} = ?$ Note that $\dfrac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

Hence $\lim_{x\to 0^+} \dfrac{x}{|x|} = 1$, $\lim_{x\to 0^-} \dfrac{x}{|x|} = -1$.

The left limit limit is not equal to the right limit so the limit does not exist.

4) $\lim_{x\to 2} (3x^2-5x+1) = \quad 3.(2)^2 - 5.2+1 = 3$

5) $\lim_{x\to 2} \sqrt[5]{3x^2-2x} = \sqrt[5]{\lim_{x\to 2} 3x^2-2x} = \sqrt[5]{3.4-4} = \sqrt[5]{8}$

6) $\lim_{x\to 0} \sin^{-1}\left(\frac{x+1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \pi/6$

$\sin^{-1}(1/2) = y \Rightarrow \frac{1}{2} = \sin(y)\ , y \in [-\pi/2, \pi/2] \Rightarrow y = \frac{\pi}{6}$

7) $\lim_{x\to 0} x\cot x \underset{\left(\frac{0}{0}\right)}{=?}$ We will evaluate this limit later.

8) $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = ?$

$$\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \cdot \frac{(\sqrt{x+2}+\sqrt{2})}{(\sqrt{x+2}+\sqrt{2})} = \lim_{x\to 0} \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+2}+\sqrt{2})} = \frac{1}{2\sqrt{2}}$$

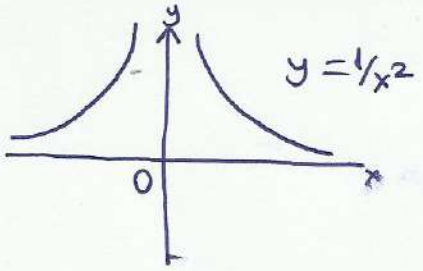
9) $\lim_{x\to 0} \frac{1}{x^2} = +\infty \quad \lim_{x\to 0^+} \frac{1}{x^2} = \lim_{x\to 0^-} \frac{1}{x^2} = +\infty.$

$\lim_{x\to 0} \left(-\frac{1}{x^2}\right) = -\infty$

Definition: The line $x=a$ is called a vertical asymptote of the curve of the curve $y=f(x)$ if at least one of the following statements is true:

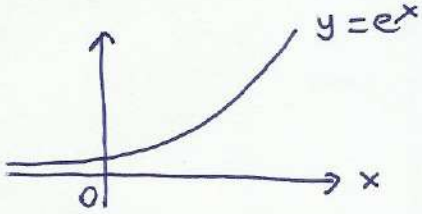
$\lim_{x\to a} f(x) = +\infty$, $\lim_{x\to a^+} f(x) = +\infty$, $\lim_{x\to a^-} f(x) = +\infty$

$\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = -\infty$



10) $\lim_{x \to \infty} x^3 + 10 = \infty$, $\lim_{x \to -\infty} x^3 + 10 = -\infty$

11) $\lim_{x \to \infty} e^x = \infty$



$\lim_{x \to -\infty} e^x = 0$

14) $y = \sin(1/x) = f(x)$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin(1/x)$

does not exist as $x \to 0$, since the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number.

12) $\lim_{x \to \infty} (x^2 - x) \neq \infty - \infty$ The limit laws cannot be applied to infinite limits i.e. you $\boxed{\text{cannot}}$ write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x$$

WRONG

$$\lim_{x \to \infty} (x^2 - x) = +\infty$$

13) $\lim_{x \to \infty} \dfrac{x^2 + x}{5 - 2x} = ?$

$$\lim_{x \to \infty} \frac{\cancel{x}(x+1)}{\cancel{x}\left(\frac{5}{x} - 2\right)} = -\infty$$

because $x + 1 \to \infty$ and $\frac{5}{x} - 2 \to -2$ as $x \to \infty$.

$\frac{5}{x} \to 0$

Sandwich Theorem: Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \quad \text{for some number } L.$$

Then $\lim_{x \to a} g(x) = L$.

Example 14) Evaluate $\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right]$.

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

multiply by x^2. $\quad -x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2$

$-x^2 \to 0$ as $x\to 0$, $\quad x^2 \to 0$ as $x \to 0$

$$f(x) \le g(x) \le h(x) \qquad f(x) = -x^2,\ h(x) = x^2$$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$. Hence by Sandwich theorem

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right] = 0.$$

15) Evaluate $\lim_{x\to\infty} \frac{\sin x}{x}$.

$$-1 \le \sin x \le 1$$

If $x>0$ $\quad -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$; If $x<0$ $\quad \frac{1}{x} \le \frac{\sin x}{x} \le -\frac{1}{x}$

$-\frac{1}{x} \to 0$, $\frac{1}{x} \to 0$, $\frac{1}{x} \to 0$, $-\frac{1}{x} \to 0$

By Sandwich theorem $\boxed{\lim_{x\to\infty} \frac{\sin x}{x} = 0.}$

16) Prove that $\lim_{x\to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

Since we have $-1 \le \sin(\pi/x) \le 1 \Rightarrow e^{-1} \le e^{\sin(\pi/x)} \le e^{1}$

$$\frac{\sqrt{x}}{e} \le \sqrt{x}\, e^{\sin(\pi/x)} \le \sqrt{x}\, e$$

$\frac{\sqrt{x}}{e} \to 0$ as $x\to 0$, $\quad \sqrt{x}\, e \to 0$ as $x \to 0$

Hence by Sandwich theorem

$$\lim_{x\to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0.$$

17) $$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1-\frac{1}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x^2}\right)} = 1$$

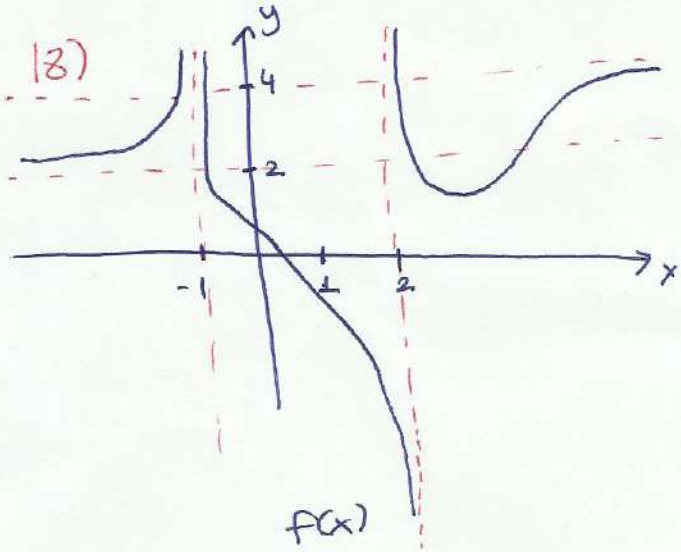
$\frac{1}{x^2} \to 0$

Definition: The line $y=L$ is called a horizontal asymptote of the curve $y=f(x)$ if either

$$\lim_{x\to\infty} f(x)=L \quad \text{or} \quad \lim_{x\to-\infty} f(x)=L.$$

In this example $y=1$ is a horizontal asymptote of the function $\frac{x^2-1}{x^2+1}$.

18)



Find the infinite limits, limits at infinity, and asymptotes for the function $f(x)$.

$$\lim_{x\to 2^+} f(x) = +\infty$$

$$\lim_{x\to 2^-} f(x) = -\infty$$

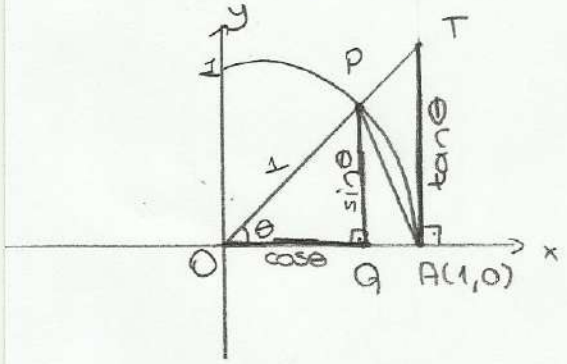
$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = +\infty$$

Hence both lines $x=-1$ and $x=2$ are vertical asymptotes of the function $f(x)$.

$$\lim_{x\to-\infty} f(x) = 2, \quad \lim_{x\to+\infty} f(x) = 4$$

Thus both $y=2$ and $y=4$ are horizontal asymptotes of $f(x)$.

Theorem: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ in radians)



To show that limit is 1, we have to show that right-hand limit and left hand limit are both 1.

Start with values of θ that are positive and less than $\pi/2$.

It is clear that

$$\text{Area } \Delta OAP < \text{Area sector } OAP < \text{Area } \Delta OAT$$

$$\begin{array}{ccc} \| & \| & \| \\ \frac{\sin\theta \times 1}{2} & \frac{1}{2}r^2\theta = \frac{1}{2}\theta & \frac{\tan\theta \times 1}{2} \end{array}$$

$$\Rightarrow \quad \frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2} = \frac{\sin\theta}{2\cos\theta}$$

Divide by the positive number $\frac{\sin\theta}{2}$, we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \Rightarrow \cos\theta < \frac{\sin\theta}{\theta} < 1$$

$\cos\theta \to 1$ as $\theta \to 0^+$, $\quad 1 \to 1$ as $\theta \to 0^+$

Hence by Sandwich Theorem $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$.

Since, $\sin\theta$ and θ are both odd functions $\frac{\sin\theta}{\theta}$ is an even function with a graph symmetric about the y-axis. This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-had limit.

$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \Rightarrow \underline{\underline{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}}$$

Examples

1) Question 7. at page 21

$$\lim_{x\to 0} x\cot x = \lim_{x\to 0} x\,\frac{\cos x}{\sin x} = \lim_{x\to 0}\frac{\overset{\to 1}{\cos x}}{\underset{\to 1}{\left(\frac{\sin x}{x}\right)}} = 1.$$

2) $$\lim_{x\to 0}\frac{\tan 2x}{5x} = \lim_{x\to 0}\frac{\sin 2x}{\cos 2x\cdot 5x}\cdot\frac{2}{2}$$

$$= \lim_{x\to 0}\overset{\to 1}{\frac{\sin 2x}{2x}}\cdot\frac{2}{5\cos 2x} = \frac{2}{5}$$

3) $$\lim_{x\to 0}\frac{5-6\cos x+\cos^2 x}{x\sin x} = \lim_{x\to 0}\frac{5-6\cos x+(1-\sin^2 x)}{x\sin x}$$

$$= \lim_{x\to 0}\frac{6(1-\cos x)}{x\sin x} - \lim_{x\to 0}\frac{\sin x}{x}$$

$$= \lim_{x\to 0}\frac{6\overbrace{(1-\cos x)}^{\sin^2 x}}{x\sin x}\,\frac{(1+\cos x)}{(1+\cos x)} - 1$$

$$= 6\lim_{x\to 0}\left(\underset{\to 1}{\frac{\sin x}{x}}\cdot\frac{1}{1+\cos x}\right) - 1 = 6\cdot 1\cdot\frac{1}{2} - 1 = 2$$

4) $$\lim_{x\to 3}\frac{\cos(\pi x/2)}{x-3} \underset{x=t+3}{=} \lim_{t\to 0}\frac{\cos\left(\frac{\pi}{2}t+\frac{3\pi}{2}\right)}{t}$$

$$= \lim_{t\to 0}\frac{\cos\left(\frac{\pi}{2}t\right)\overset{=0}{\cos\left(\frac{3\pi}{2}\right)} - \sin\left(\frac{\pi}{2}t\right)\overset{=-1}{\sin\left(\frac{3\pi}{2}\right)}}{t}$$

$$= \lim_{t\to 0}\underset{\to 1}{\frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}}\cdot\frac{\pi}{2} = \frac{\pi}{2}$$

CONTINUITY

A function f is continuous at $x=a$ when

(i) $f(a)$ is defined,

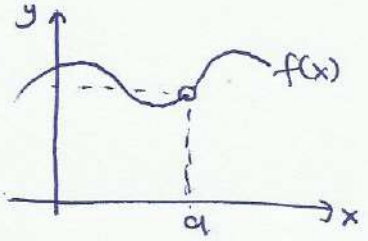
(ii) $\lim_{x\to a} f(x)$ exists, and

(iii) $\lim_{x\to a} f(x) = f(a)$.

Otherwise f is said to be discontinuous at $x=a$.

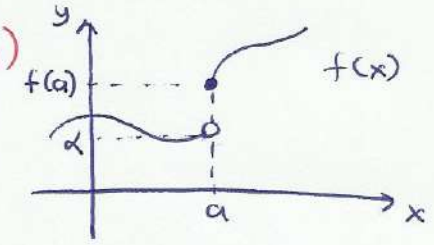
Examples: Some examples of functions that are discontinuous at $x=a$.

i)



The function $f(x)$ is not defined at $x=a$. "removable discontinuity"

ii)



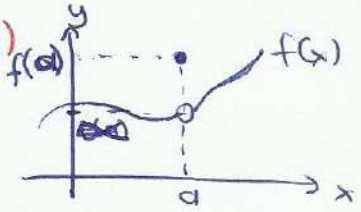
The function $f(x)$ is defined at $x=a$, but

$\lim_{x\to a^+} f(x) = f(a)$, $\lim_{x\to a^-} f(x) = \alpha$

$\alpha \neq f(a)$

$\Rightarrow \lim_{x\to a} f(x)$ does not exist. The graph has a JUMP at $x=a$ "Jump discontinuity"

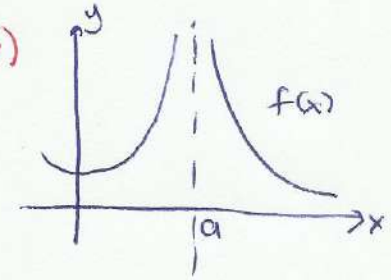
iii)



$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \alpha$ $\lim_{x\to a} f(x)$ exists.

But $\lim_{x\to a} f(x) \neq f(a)$. The graph has a hole at $x=a$ "removable discontinuity"

iv)



$\lim_{x\to a} f(x) = \infty$ and so $\lim_{x\to a} f(x) = f(a)$ never holds. The function BLOWS UP at $x=a$.

"infinite discontinuity"

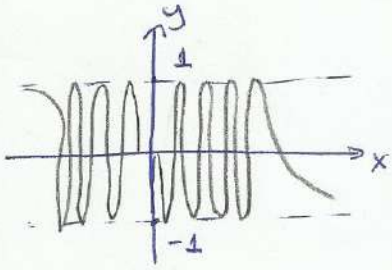
Definition: The point $x=a$ is called a removable discontinuity of a function f if one can remove the discontinuity by redefining the function at that point. Otherwise, it is called a nonremovable or an essential discontinuity of f. Clearly, a function has a removable discontinuity at $x=a$ if and only if $\lim_{x\to a} f(x)$ exists and is finite.

Example: Classify all the discontinuities of the following functions.

i) $f(x) = \dfrac{x^2+2x-3}{x-1}$ Notice that $f(1)$ is not defined so f is discontinuous at 2. This function would be continuous if it had $f(1)=4$. So $x=1$ is a removable discontinuity.

$$\lim_{x\to 1} \frac{(x+3)\cancel{(x-1)}}{\cancel{(x-1)}} = 4$$

ii) $f(x) = \sin\left(\frac{1}{x}\right)$



This function is discontinuous at the origin since it oscillates too much to have a limit as $x\to 0$.

Theorem: All polynomials are continuous everywhere. Additionally, $\sin x$, $\cos x$, $\tan^{-1}x$ and e^x are continuous everywhere, $\sqrt[n]{x}$ is continuous for all x, when n is odd and for $x>0$, when n is even. We also have $\ln x$ is continuous for $x>0$ and $\sin^{-1}x$ and $\cos^{-1}x$ are continuous for $-1<x<1$.

Theorem: Suppose that f and g are continuous at $x=a$. Then $f\pm g$, $f\cdot g$, $\frac{f}{g}$ (if $g(a)\neq 0$) are also continuous at $x=a$

Example: Find and classify all the discontinuities of $\dfrac{x^4-3x^2+2}{x^2-3x-4}$.

$$f(x)=\frac{x^4-3x^2+2}{x^2-3x-4}=\frac{(x^2-2)(x^2-1)}{(x-4)(x+1)}=\frac{(x^2-2)(x-1)(x+1)}{(x-4)(x+1)}$$

The function is not continuous at $\underbrace{x=-1}_{\text{removable discont.}}$ and $\underbrace{x=4}_{\text{infinite discont.}}$

Theorem: Suppose that $\lim_{x\to a} g(x)=L$ and f is continuous at L.

Then $\lim_{x\to a} f(g(x))=f(\lim_{x\to a} g(x))=f(L)$.

Example: Determine where $h(x)=\cos(x^2-6x+3)$ is continuous.

Here $g(x)=x^2-6x+3$. It is a polynomial and continuous everywhere.

$f(x)=\cos x$ is also continuous everywhere.

Hence $f(g(x))=\cos(x^2-6x+3)$ is continuous everywhere.

Remark: If f is continuous at every point on an open interval (a,b), we say that f is continuous on (a,b).

We say that f is continuous on the closed interval $[a,b]$ if f is continuous on (a,b) and

$$\lim_{x\to a^+} f(x)=f(a) \text{ and } \lim_{x\to b^-} f(x)=f(b).$$

If f is continuous on all of $(-\infty,\infty)$, we simply say that f is continuous.

Example: For what value of a is $f(x)=\begin{cases} x^2-1, & x<3 \\ 2ax, & x\geq 3 \end{cases}$

continuous at every x?

x^2-1 and $2ax$ are polynomials, they are continuous everywhere. But, what about the point $x=3$?

For continuity, we have to have $\lim_{x\to 3} f(x) = f(3)$.

$$\left.\begin{array}{l} \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-1) = 8 \\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2ax) = 6a = f(3) \end{array}\right\} \Rightarrow 6a = 8 \Rightarrow \underline{\underline{a = \frac{8}{6} = \frac{4}{3}}}$$

Example: where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2-1}$ continuous?

$\left.\begin{array}{l} \ln x \text{ is continuous for } x>0 \\ \tan^{-1} x \text{ is continous on } (-\infty,\infty) \end{array}\right\}$ $\ln x + \tan^{-1} x$ is continuous on $(0,\infty)$.

x^2-1 is a polynomial so it is continuous everywhere.

Hence $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2-1}$ is continuous on $(0,\infty)$ except the points that make the denominator zero i.e. $x = \pm 1$.

So f is continuous on $(0,1) \cup (1,\infty)$.

Example: Where is the function $f(x) = \ln(1+\cos x)$ continuous?

$\ln x$ is continuous for $x>0$.

$1+\cos x$ is continuous for any x.

Hence we must have $1+\cos x > 0 \Rightarrow \cos x > -1 \Rightarrow \cos x \neq -1$

Note that $\cos x = -1$ when $x = (2n+1)\pi$, $n \in \mathbb{Z}$. Thus $f(x)$ has discontinuities when $x = (2n+1)\pi$, $n \in \mathbb{Z}$ and is continuous on the intervals between these values.

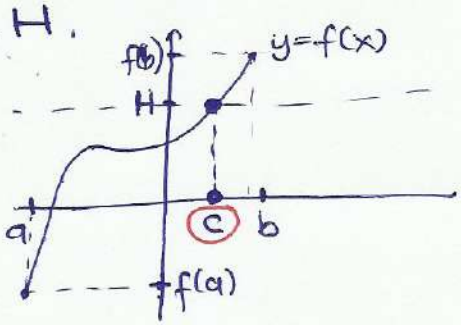
Example: Consider the function $f(x) = \begin{cases} \dfrac{\sqrt{1+x}-1}{x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$

where k is a constant. Determine the value of the constant k for which f is continuous at 0.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{1+x}+1)} = 1/2$$

Hence f is continuous at 0 when $k = f(0) = \lim_{x\to 0} f(x) = 1/2$.

The intermediate value theorem (IVT): Suppose that f is continuous on the closed interval $[a,b]$ and H is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a,b]$ for which $f(c) = H$.



* **Corollary:** Suppose that f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs. Then, there is at least one number $c \in (a,b)$ for which $f(c)=0$.

Example: Is there a continuous function f on $(0,\infty)$ such that $f(2x)f(x) < 0$ <u>for all $x>0$</u>? If 'yes', give an example; if 'no' prove that no such function exists.

No!

Let f be such a function. Then $f(2)f(1) < 0$. Hence $f(1)$ and $f(2)$ have opposite signs. Since f is continuous, we can apply intermediate value theorem to f on $[1,2]$ and conclude that there is a point c in $[1,2]$ such that $f(c)=0$.

Then $0 = f(2c)f(c) < 0$ gives a contradiction.

Example: Show that the function $f(x) = 2\tan x - 3x$ has at least one positive root.

$f(x) = 2\tan x - 3x$

$f(\frac{\pi}{3}) = 2\sqrt{3} - \pi > 0$ and $f(x)$ is continuous

$f(\frac{\pi}{4}) = 2 - \frac{3\pi}{4} < 2 - \frac{3.3}{4} = -\frac{1}{4} < 0$ on $[\frac{\pi}{4}, \frac{\pi}{3}]$

Hence by IVT, there is at least one c in $[\frac{\pi}{4}, \frac{\pi}{3}]$ such that $f(c)=0$.

Example: Is there a solution to $x^5-2x^3-2=0$ between $x=0$ and $x=2$?

$f(x)=x^5-2x^3-2$

$\boxed{\text{At } x=0}$ $\quad f(0)=-2<0$

$\boxed{\text{At } x=2}$ $\quad f(2)=32-16-2=14>0$

$f(x)$ function is a polynomial so it is continuous everywhere. Hence it is continuous on $[0,2]$. So by IVT, there is at least one c in $[0,2]$ such that $f(c)=c^5-2c^3-2=0$.

Continuous Extension

If $f(c)$ is not defined, but $\lim_{x\to c} f(x)=L$ exists, we can define a new function $F(x)$ by

$$F(x)=\begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x=c \end{cases}$$

The function $F(x)$ is continuous at $x=c$. It is called the continuous extension f to $x=c$.

Example: Find the continuous extension of $f(x)=\dfrac{x^2+2x-3}{x-1}$ to $x=1$.

$f(x)$ is not continuous at $x=1$ but $\lim_{x\to 1}\dfrac{x^2+2x-3}{x-1}=\lim_{x\to 1}\dfrac{(x+3)\cancel{(x-1)}}{\cancel{(x-1)}}$

$=4$

Hence $$F(x)=\begin{cases} \dfrac{x^2+2x-3}{x-1} & x\neq 1 \\ 4 & x=1 \end{cases}$$

is the continuous extension of f to $x=1$.

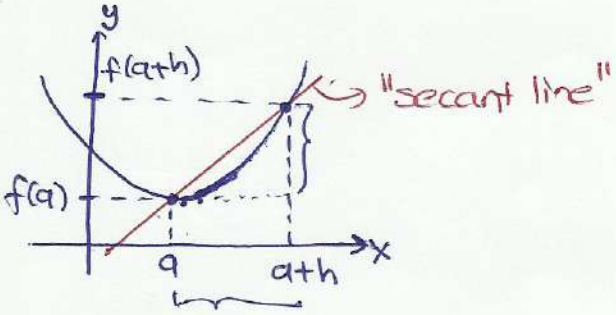
— o —

References

The notes are being prepared by the help of the following references.

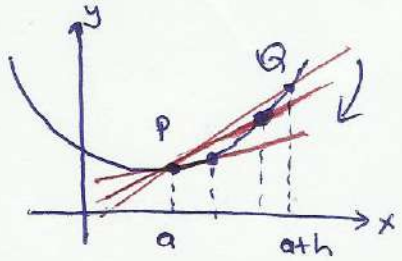
- George B. Thomas, Ross L. Finney, Calculus and Analytic Geometry, Ninth Edition
- James Stewart, Stewart Calculus-Early Transcendentals
- Lecture notes of Secil Gergün
- Pauls Online Notes

DIFFERENTIATION



The slope of the secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$ is given by

$$m_{sec} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$



In order to obtain an approximation to the tangent line, we come closer and closer toward to point P.
This makes h closer to 0.

We define the slope of the tangent line to be the limit of the slopes of the secant lines as h tends to 0, whenever this limit exists.

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$y = m_{tan}(x-a)+f(a) \quad \text{"equation of the tangent line"}$$

Example: Find an equation of the tangent line to $y = \underbrace{x^3+2x}_{f(x)}$ at $x=2$.

$$m_{tan} = \lim_{h \to 0} \frac{(2+h)^3+2(2+h)-12}{h} = \lim_{h \to 0} \frac{8+12h+6h^2+h^3+4+2h-12}{h}$$

$$= \lim_{h \to 0} 12+6h+h^2+2 = 14$$

So, the tangent line is $y = \underbrace{14}_{m_{tan}}(x-\underbrace{2}_{a})+\underbrace{12}_{f(a)}$

Example: Let $y=f(x)=|x|$. Find an equation of the tangent line at $a=-1, 0, 1$.

$\boxed{x=-1}$ $m_{tan} = \lim_{h\to 0} \frac{|-1+h|-1}{h} = \lim_{h\to 0} \frac{1-h-1}{h} = -1$

Tangent line: $y = -1(x-(-1)) + 1 = -x-1+1 = -x.$

!! $\boxed{x=0}$ $m_{tan} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{|h|}{h}$

$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$, $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$

$1 \neq -1$

$\Rightarrow$ Tangent line cannot be drawn to $y=f(x)$ at $x=0$.

$\boxed{x=1}$ $m_{tan} = \lim_{h\to 0} \frac{|1+h|-1}{h} = \lim_{h\to 0} \frac{1+h-1}{h} = \lim_{h\to 0} \frac{h}{h} = 1$

Tangent line: $y = 1.(x-1)+1 = x$

Remark: If $f(t)$ represents the position of an object relative to some fixed location at time t as it moves along a straight line, then the instantaneous velocity at time $t=a$ is given by

$$v(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

provided the limit exists.

The Derivative

The derivative of $f(x)$ is the function $f'(x)$ given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \text{ or } f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

let $h=a-x$

provided the limit exists. The process of computing a derivative is called differentiation.

f is diffentiable on an interval I if it is differentiable at every point in I.

Notation: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x)$, $\frac{d}{dx}$: differential operator

Example: Find $f'(x)$ if $f(x) = \frac{1}{x}$, $\neq 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-x-h}{x(x+h)h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{x(x+h)\cancel{h}} = -\frac{1}{x^2}$$

Example: Let $f(x) = \begin{cases} 2x^2+1 & \text{if } x<0 \\ 1 & \text{if } x=0 \\ 5x^3+1 & \text{if } x>0 \end{cases}$. Evaluate $f'(0)$ if it exists.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-1}{h}$$

$$\lim_{h\to 0^+} \frac{f(h)-1}{h} = \lim_{h\to 0^+} 5h^2 = 0, \quad \lim_{h\to 0^-} \frac{f(h)-1}{h} = \lim_{h\to 0^-} 2h = 0$$

So $f'(0) = 0$.

Example: Let $f(x) = |x^2-1|$. Evaluate $f'(0)$ and $f'(1)$ if they exist.

- $$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h^2-1|-1}{h} = \lim_{h\to 0} \frac{1-h^2-1}{h} = \lim_{h\to 0} \frac{-h^2}{h} = \lim_{h\to 0}(-h) = 0.$$

- $$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{|(1+h)^2-1|-0}{h}$$

$$\lim_{h\to 0^+} \frac{|(1+h)^2-1|}{h} = \lim_{h\to 0^+} \frac{(1+h)^2-1}{h} = \lim_{h\to 0^+} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0^+} 2+h = 2$$

$$\lim_{h\to 0^-} \frac{|(1+h)^2-1|}{h} = \lim_{h\to 0^-} \frac{1-(1+h)^2}{h} = \lim_{h\to 0^-} \frac{1-1-2h-h^2}{h} = \lim_{h\to 0^-} -2-h = -2$$

So $f'(1)$ does not exist.

Example: Let $f(x) = \begin{cases} 4 & \text{if } x<2 \\ 2x & \text{if } x\geq 2 \end{cases}$. Show that f is continuous but NOT differentiable at $x=2$.

$\lim_{x\to 2^-} f(x) = 4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x = 4$

$f(2) = 4$

since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$, $f(x)$ is continuous at $x=2$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \; ; \quad \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{2(2+h)-4}{h} = 2$$

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{4-4}{h} = 0.$$

So the limit does not exist. Hence $f'(2)$ does not exist.

$\Rightarrow f$ is NOT differentiable at $x=2$.

Theorem: If f is differentiable at a, then f is continuous at a.

Outline of the proof: We must show that $\lim_{x\to a} f(x) = f(a)$.

Since f is differentiable at a $\Rightarrow \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ exists.

$$f(x) - f(a) = \frac{f(x)-f(a)}{(x-a)} \cdot (x-a)$$

$$\Rightarrow \lim_{x\to a} \left(f(x)-f(a)\right) = \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} \cdot (x-a)\right] = \underbrace{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}_{\text{exists}} \cdot \underbrace{\lim_{x\to a} (x-a)}_{=0}$$

$$\Rightarrow \lim_{x\to a}\left(f(x)-f(a)\right) = 0 \quad \Rightarrow \lim_{x\to a} f(x) = f(a). \quad \square$$

Not differentiable

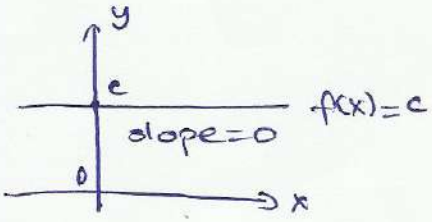
a corner

a discontinuity

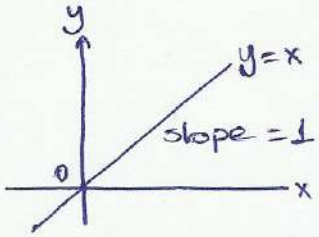
a vertical tangent

$\lim_{x\to a} |f'(x)| = \infty$

Differentiation Rules



$f(x)=c$ $\qquad \frac{d}{dx}c=0$, c constant



$\frac{d}{dx}(x)=1$

<u>power rule</u> $\frac{d}{dx}(x^n)=nx^{n-1}$ $\qquad$ n any real number

<u>constant multiple rule</u> $\frac{d}{dx}(cf(x))=c\frac{d}{dx}f(x)$ $\qquad$ c constant

<u>sum rule & difference rule</u> $\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$

Examples: 1) Find $f'(x)$.

i) $f(x)=2x^6+3\sqrt{x} \Rightarrow f(x)=2x^6+3x^{1/2}$

$f'(x)=2.6.x^5+3.\frac{1}{2}.x^{1/2-1}=12x^5+\frac{3}{2\sqrt{x}}$

ii) $f(x)=\frac{4x^2-3x+2\sqrt{x}}{x} \Rightarrow f(x)=4x-3+\frac{2}{\sqrt{x}}=4x-3+2x^{-1/2}$

$f'(x)=4-0+2.(-\frac{1}{2})x^{-1/2-1}=4-\frac{1}{x^{3/2}}$

iii) $f(x)=x\sqrt{x}+\frac{2}{x^2} \Rightarrow f(x)=x.x^{1/2}+2x^{-2}=x^{3/2}+2x^{-2}$

$f'(x)=\frac{3}{2}x^{3/2-1}-2.2.x^{-2-1}=\frac{3}{2}\sqrt{x}-\frac{4}{x^3}$

2) Find an equation of the tangent line to $f(x)=4-4x+\frac{2}{x}$ at $x=1$.

Firstly, find the slope of the tangent line.

$f'(x)=-4-\frac{2}{x^2}$, $f'(1)=-6=m$

Tangent line passes through the point $(x,y)=(1,\underbrace{4-4+\frac{2}{1}}_{f(1)})=(1,2)$

$\Rightarrow$ $y=m(x-a)+f(a)$
$=-6(x-1)+2$

3) At which point(s) the tangent line drawn to $y=2x^3$ is parallel to $y=6x-7$?

$\Downarrow$

slope of the tangent line and the slope of the line $y=6x-7$ will be equal.

$y=6x-7$ slope of this line is 6.

$f(x)=2x^3$, $f'(x)=6x^2$ $\quad f'(a)=6a^2=6 \Rightarrow a^2=1 \Rightarrow a=\pm 1$

So at points $(x,y)=(1,2)$ and $(x,y)=(-1,-2)$ the tangent line drawn to $y=2x^3$ is parallel to $y=6x-7$.

Higher order derivatives

Notation: $y'''=f'''(x)=\dfrac{d^3f}{dx^3}$, $y^{(4)}=f^{(4)}(x)=\dfrac{d^4f}{dx^4}$, ...

Example: If $f(x)=5x^4-2x^2+x+7$, compute as many derivatives as possible.

$f(x)=5x^4-2x^2+x+7$ $\qquad f^{(4)}(x)=120$

$f'(x)=20x^3-4x+1$ $\qquad f^{(5)}(x)=0.$

$f''(x)=60x^2-4$

$f'''(x)=120x$

- Product rule: f and g are differentiable at x.

$$\frac{d}{dx}\left(f(x)g(x)\right)=f'(x)g(x)+f(x)g'(x)$$

Example: $f(x)=(2x^4-3x+5)\left(x^2-\sqrt{x}+\dfrac{2}{x}\right)$, $f'(x)=?$

$$f'(x)=(2\cdot 4x^3-3)\left(x^2-x^{1/2}+2x^{-1}\right)+(2x^4-3x+5)\left(2x-\tfrac{1}{2}x^{-1/2}-2x^{-2}\right)$$

$$=8x^5-8x^{7/2}+16x^2-3x^2+3x^{1/2}-6x^{-1}+4x^5-x^{7/2}-4x^2$$

$$-6x^2+\tfrac{3}{2}x^{1/2}+6x^{-1}+10x-\tfrac{5}{2}x^{-1/2}-10x^{-2}$$

$$=12x^5+3x^2-9x^{7/2}+\tfrac{9}{2}x^{1/2}+10x-\tfrac{5}{2}x^{-1/2}-10x^{-2}$$

Example: Find an equation of the tangent line to

$f(x)=(2x^4-5x^2+2x)(x^3-x+1)$ at $x=0$.

$f'(x)=(8x^3-10x+2)(x^3-x+1)+(2x^4-5x^2+2x)(3x^2-1)$

$f'(0)=2$ slope of the tangent line

$y=2(x-0)+\underbrace{f(0)}_{=0}=2x$

- Quotient Rule f and g are differentiable at x and $g(x)\neq 0$.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$$

Example. Find $f'(x)$ if $f(x)=\dfrac{x^3-2x}{x^2+1}$.

$$f'(x)=\frac{(x^3-2x)'(x^2+1)-(x^3-2x)(x^2+1)'}{(x^2+1)^2}=\frac{(3x^2-2)(x^2+1)-(x^3-2x)(2x)}{(x^2+1)^2}$$

$$=\frac{x^4+5x^2-2}{(x^2+1)^2}$$

- Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then

$$\frac{d}{dx}\Big(f(g(x))\Big)=f'(g(x)).g'(x).$$

$$\left(y=f(u),\ u=g(x)\ \Rightarrow\ \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\right)$$

Example: i) $f(x)=\sqrt[5]{x^2+1}$, $f'(x)=?$

$f(x)=(x^2+1)^{1/5}$

$f'(x)=\frac{1}{5}(x^2+1)^{1/5-1}\cdot 2x$

$=\frac{2}{5}(x^2+1)^{-4/5}x$

ii) $f(x)=\dfrac{7x}{(x^3+5)^2}$, $f'(x)=?$

$$f'(x)=\frac{(7x)'\,(x^3+5)^2-7x\,[(x^3+5)^2]'}{(x^3+5)^4}=\frac{7(x^3+5)^2-7x\cdot 2(x^3+5)\,3x^2}{(x^3+5)^4}$$

$$=\frac{7}{(x^3+5)^2}-\frac{42x^3}{(x^3+5)^3}$$

iii) $f(x)=(x^2\sqrt{6x+2}+1)^{1/3}$, $f'(x)=?$

$$f'(x)=\frac{1}{3}\left(x^2\sqrt{6x+2}+1\right)^{-2/3}\cdot\left(2x\sqrt{6x+2}+x^2\cdot\frac{1}{2}(6x+2)^{-1/2}\cdot 6\right)$$

iv) $g(t)=(t^2+9)^3(t^2-2)^4$, $g'(\ )=?$

$$g'(t)=3(t^2+9)^2\cdot 2t\,(t^2-2)^4+(t^2+9)^3\,4(t^2-2)^3\,2t$$

v) Find $\dfrac{du}{dv}$ if $u=t+\dfrac{1}{t}$ and $t=1-\dfrac{1}{v}$.

$$\frac{du}{dv}=\frac{du}{dt}\cdot\frac{dt}{dv}\qquad \frac{du}{dt}=1-\frac{1}{t^2}\ ,\ \frac{dt}{dv}=+\frac{1}{v^2}$$

$$=\left(1-\frac{1}{t^2}\right)\frac{1}{v^2}=\left(1-\frac{1}{(1-\frac{1}{v})^2}\right)\cdot\frac{1}{v^2}=\frac{1-2v}{(v-1)^2v^2}$$

$t=1-\dfrac{1}{v}$

- Recall that we write $g(x)=f^{-1}(x)$ if $g(f(x))=x$, $x\in \operatorname{dom} f$
  $f(g(x))=x$, $x\in \operatorname{dom} g$

$$f(g(x))=x.$$

Use chain rule.

$$\frac{d}{dx}\left(f(g(x))\right)=\frac{d}{dx}x\ \Rightarrow\ f'(g(x))\cdot g'(x)=1$$

$$\Rightarrow\ \boxed{g'(x)=\frac{1}{f'(g(x))}}\quad (f'(g(x))\neq 0)$$

Theorem: If f is differentiable at all x and has an inverse function $g(x) = f^{-1}(x)$, then

$$g'(x) = \left[f^{-1}(x)\right]' = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))}$$

provided that $f'(g(x)) \neq 0$.

Example: Given that the function $f(x) = x^5 + 3x^3 + 2x + 1$ has an inverse function $g(x)$, compute $g'(7)$.

$$g'(7) = \frac{1}{f'(g(7))}$$

$f(1) = 7 \Rightarrow g(7) = 1$

$f'(x) = 5x^4 + 9x^2 + 2$

$f'(g(7)) = f'(1) = 5 + 9 + 2 = 16$

$\Rightarrow g'(7) = \frac{1}{16}$

Derivatives of Trigonometric Functions

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

Example: Find $f'(x)$.

i) $f(x) = 4\tan x - 5\operatorname{cosec} x \Rightarrow f'(x) = 4\sec^2 x + 5\operatorname{cosec} x \cot x$

ii) $f(x) = \cos(x^5) \Rightarrow f'(x) = 5x^4 \cdot -\sin(x^5) = -5x^4 \sin(x^5)$

iii) $f(x) = \sin^2(x^4) \Rightarrow f'(x) = 2\sin(x^4)\cos(x^4)\, 4x^3$
$= 8x^3 \sin(x^4)\cos(x^4)$

iv) $f(x) = \sin^2(x\sin x) \Rightarrow f'(x) = 2\sin(x\sin x)\cos(x\sin x)\left[\sin x + x\cos x\right]$

v) $f(x) = \dfrac{\sin^2(2x+1)}{\cos^3(\sqrt{x^2+x})}$

$$f'(x) = \frac{2\sin(2x+1)\cos(2x+1).2\cos^3(\sqrt{x^2+x}) - \sin^2(2x+1)3\cos^2(\sqrt{x^2+x})(-\sin(\sqrt{x^2+x})).\frac{1}{2}(x^2+x)^{-1/2}.(2x+1)}{\cos^6(\sqrt{x^2+x})}$$

$$= \frac{4\sin(2x+1)\cos(2x+1)}{\cos^3(\sqrt{x^2+x})} + \frac{3(2x+1)\sin^2(2x+1)\sin(\sqrt{x^2+x})}{2\sqrt{x^2+x}\,\cos^4(\sqrt{x^2+x})}$$

vi) Find $\dfrac{dy}{dx}$ if $y = \sin(u^2) + \dfrac{2}{u^3}$ and $u = x\cos(\cos x)$.

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{du} = 2u\cos(u^2) - 6u^{-4},$$

$$\frac{du}{dx} = \cos(\cos x) + x.(-\sin x)(-\sin(\cos x))$$

$$= \cos(\cos x) + x\sin x\sin(\cos x)$$

$$= (2u\cos(u^2) - 6u^{-4})[\cos(\cos x) + x\sin x.\sin(\cos x)]$$

$u = x\cos(\cos x)$

$$= [2x\cos(\cos x)\cos(x^2\cos^2(\cos x)) - 6x^{-4}\cos^{-4}(\cos x)][\cos(\cos x) + x\sin x\sin(\cos x)]$$

vii) Find $h'\left(\frac{1}{2}\right)$ if $h(s) = \cos(\pi s)\sin^2(\pi s)$.

$$h'(s) = \pi(-\sin(\pi s))\sin^2(\pi s) + \cos(\pi s)\,2\sin(\pi s)\cos(\pi s)\,\pi$$

$$h'\left(\tfrac{1}{2}\right) = -\pi\sin^3\left(\tfrac{\pi}{2}\right) + 2\pi\underbrace{\cos^2\left(\tfrac{\pi}{2}\right)}_{=0}\sin\left(\tfrac{\pi}{2}\right) = -\pi$$

Derivatives of Exponential and Logarithmic Functions

- For any constant $a>0$, $\frac{d}{dx}a^x = a^x \ln a$.

Particularly, if $a=e$ i.e. $\frac{d}{dx}e^x = e^x \ln e = e^x$.

Examples:

i) $f(x) = \pi^{x^2 \tan x}$, $f'(x) = ?$

$$f'(x) = \pi^{x^2\tan x} . \ln\pi . (x^2 \tan x)'$$

$$= \pi^{x^2\tan x} \ln\pi \, [2x\tan x + x^2 \sec^2 x]$$

ii) $f(x) = (x^3+4)^e + \pi^{\sin x}$, $f'(x) = ?$

$$f'(x) = e(x^3+4)^{e-1}(3x^2) + \pi^{\sin x} \ln\pi \cos x$$

iii) $f(x) = xe^{\frac{2}{x}} + e^{\sqrt{x}} + (x^5+5)^5 - e^{\pi x} + 4^{2x^2}$, $f'(x) = ?$

$$f'(x) = \left[e^{2/x} + x . e^{2/x} . \left(-\frac{2}{x^2}\right)\right] + \left[\frac{1}{2\sqrt{x}} e^{\sqrt{x}}\right] + \left[5(x^5+5)^4 . 5x^4\right] - \left[\pi e^{\pi x}\right] + \left[4x . 4^{2x^2} \ln 4\right]$$

- For $x>0$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$. $\left(\frac{d}{dx}\ln(u(x)) = \frac{u'(x)}{u(x)}\right)$

Examples:

i) $f(x) = \log_a x$, $f'(x) = ?$

$$\log_a x = \frac{\ln x}{\ln a} \quad , \quad f'(x) = \frac{1}{\ln a} . \frac{1}{x}$$

ii) $f(x) = \ln^2 x - \ln(2\ln x)$, $f'(x) = ?$

$$f'(x) = 2\ln x . \frac{1}{x} - \frac{(2\ln x)'}{2\ln x} = \frac{2}{x}\ln x - \frac{1}{x\ln x}$$

iii) $f(x) = \log_5(2^3 + e^{\pi} + e^{x^2})$, $f'(x) = ?$

$$\log_5(2^3+e^{\pi}+e^{x^2}) = \frac{\ln(2^3+e^{\pi}+e^{x^2})}{\ln 5} \quad , \quad f'(x) = \frac{1}{\ln 5} . \frac{2xe^{x^2}}{(2^3+e^{\pi}+e^{x^2})}$$

iv) $f(x) = (x^2+1)^{\sin x}$, $f'(x) = ?$

$f(x) = (x^2+1)^{\sin x} \Rightarrow \ln f(x) = \sin x \ln(x^2+1)$. Take derivative wrt x.

$$\frac{f'(x)}{f(x)} = \cos x \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}$$

$$\Rightarrow f'(x) = (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right]$$

v) $f(x) = \dfrac{(x+1)^4 (x+2)^2}{(x+5)^2 x^x 4^{x^2}}$, $f'(x) = ?$

$$f'(x) = \frac{\left[(x+1)^4(x+2)^2\right]' (x+5)^2 x^x 4^{x^2} - (x+1)^4(x+2)^2\left[(x+5)^2 x^x 4^{x^2}\right]'}{\left[(x+5)^2 x^x 4^{x^2}\right]^2}$$

$$= \frac{\left[4(x+1)^3(x+2)^2 + 2(x+1)^4(x+2)\right](x+5)^2 x^x 4^{x^2} - (x+1)^4(x+2)^2\left[2(x+5)x^x 4^{x^2} + (x+5)^2 x^x 2x 4^{x^2}\ln 4 + (x+5)^2 (x^x)' 4^{x^2}\right]}{\left[(x+5)^2 x^x 4^{x^2}\right]^2}$$

Let us find $(x^x)'$.

$g(x) = x^x \quad \ln g(x) = x\ln x \quad \Rightarrow \quad \dfrac{g'(x)}{g(x)} = \ln x + x \cdot \dfrac{1}{x}$

$g'(x) = (x^x)' = x^x(\ln x + 1)$

Hence $f'(x) = \dfrac{A}{B}$ where

$A = [4(x+1)^3(x+2)^2 + 2(x+1)^4(x+2)](x+5)^2 x^x 4^{x^2}$
$\quad - (x+1)^4(x+2)^2\left[2(x+5)x^x 4^{x^2} + (x+5)^2 x^x 4^{x^2} \times \ln 4 + (x+5)^2 x^x(\ln x+1)4^{x^2}\right]$

$B = \left[(x+5)^2 x^x 4^{x^2}\right]^2$.

vi) Express the number e as a limit.

We know that if $f(x)=\ln x$ then $f'(x)=\frac{1}{x}$. Thus $f'(1)=1$.

From the limit definition of a derivative we have

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\ln(1+h)-0}{h}=\lim_{h\to 0}\ln(1+h)^{1/h}=1$$

From the theorem

"If f is continuous at b and $\lim_{x\to a} g(x)=b$, then

$\lim_{x\to a} f(g(x))=f(b)$ i.e. $\lim_{x\to a} f(g(x))=f\left(\lim_{x\to a} g(x)\right)$", we have

$$e=e^{1}=e^{\lim_{h\to 0}\ln(1+h)^{1/h}}=\lim_{h\to 0} e^{\ln(1+h)^{1/h}}=\lim_{h\to 0}(1+h)^{1/h}$$

$$\Downarrow$$

$$\boxed{e=\lim_{h\to 0}(1+h)^{1/h}}$$

If we put $n=\frac{1}{h}$ then $n\to\infty$ as $h\to 0^{+}$ so we get an alternative expression for e which is

$$\boxed{e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}}$$

Derivatives of Inverse-Trigonometric Functions

The derivatives of six inverse trigonometric functions:

i) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $-1<x<1$

ii) $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, $-1<x<1$

iii) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $x \in \mathbb{R}$

iv) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$, $x \in \mathbb{R}$

v) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$, $|x|>1$

vi) $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$, $|x|>1$

Examples

i) $f(x) = \tan^{-1}(\ln x)\sin(\sqrt{x^3+1})$, $f'(x) = ?$ $(x>0)$

$$f'(x) = (\tan^{-1}(\ln x))'.\sin(\sqrt{x^3+1}) + \tan^{-1}(\ln x).[\sin(\sqrt{x^3+1})]'$$

$$= \frac{1}{x}.\frac{1}{1+(\ln x)^2}.\sin(\sqrt{x^3+1}) + \tan^{-1}(\ln x).\frac{1}{2\sqrt{x^3+1}}.(3x^2)\cos(\sqrt{x^3+1})$$

ii) $f(x) = (4x^5+7x)^{-1}\tan^{-1}6$, $f'(x) = ?$

$$f'(x) = -\tan^{-1}6\,(4x^5+7x)^{-2}(20x^4+7).$$

iii) $f(x) = \cos^3(\cot^{-1}(e+6\pi^2x))$, $f'(x) = ?$

$$f'(x) = 3\cos^2(\cot^{-1}(e+6\pi^2x)).-\sin(\cot^{-1}(e+6\pi^2x)).[\cot^{-1}(e+6\pi^2x)]'$$

$$= +18\pi^2\cos^2(\cot^{-1}(e+6\pi^2x))\sin(\cot^{-1}(e+6\pi^2x)).\frac{1}{1+(e+6\pi^2x)^2}$$

iv) $f(x) = (\sec^{-1}x)^2$, $f'(x) = ?$

$$f'(x) = 2\sec^{-1}x.\frac{1}{|x|\sqrt{x^2-1}}$$

Implicit Differentiation

The process of differentiating both sides of an equation with respect to x and then solving for $y'(x)$ is called implicit differentiation.

Example: Given $y^2 + 2e^{-xy} = 6$, find $y''(x)$. Then find the value of y'' at the point $(0,2)$.

$$y^2 + 2e^{-xy} = 6$$ Take derivative of both sides.

$$2yy' + 2(-y - xy')e^{-xy} = 0 \Rightarrow y'[2y - 2xe^{-xy}] = 2ye^{-xy}$$

$$\Rightarrow \boxed{y' = \frac{ye^{-xy}}{y - xe^{-xy}}}$$ Take derivative once again.

$$y'' = \frac{(y'e^{-xy} + y(-y-xy')e^{-xy})(y - xe^{-xy}) - ye^{-xy}(y' - e^{-xy} - x(-y-xy')e^{-xy})}{(y - xe^{-xy})^2}$$

$$= \frac{y'[\cancel{ye^{-xy}} - xe^{-2xy} - xy^2e^{-xy} + \cancel{x^2ye^{-2xy}} - \cancel{ye^{-xy}} - \cancel{x^2ye^{-2xy}}]}{(y - xe^{-xy})^2}$$

$$- y^3e^{-xy} + \cancel{xy^2e^{-2xy}} + ye^{-2xy} - \cancel{xy^2e^{-2xy}}$$

$$y''(x) = \frac{-\frac{xe^{-xy}(e^{-xy} + y^2)ye^{-xy}}{(y - xe^{-xy})} + ye^{-xy}(e^{-xy} - y^2)}{(y - xe^{-xy})^2}$$

at the point $(0,2)$

$$y''(0) = \frac{2(1-4)}{2^2} = -2.$$

Example:- Find $y'(x)$ for $x^2y^2-2x=4-4y$. Then find an equation of the tangent line at the point $(2,-2)$.

$x^2y^2-2x=4-4y$. Take derivative of both sides with respect to x.

$$2xy^2+2x^2yy'-2=-4y' \Rightarrow y'(2x^2y+4)=2-2xy^2$$

$$y'(x)=\frac{2-2xy^2}{2x^2y+4}$$

slope of the tangent line at $(2,-2)$.

$$y'(2)=\frac{2-2.2.4}{2.4.(-2)+4}=\frac{-14}{-12}=\frac{7}{6}$$

equation of the tangent line $\quad y(x)=y'(2)(x-a)+\underbrace{y(a)}_{-2}$

$$y=\frac{7}{6}(x-2)-2$$

Example: Find the equation of the tangent line to the curve $\sin(xy)+2xy=2\pi$ at $(1,\pi)$.

$\sin(xy)+2xy=2\pi$ Take derivative of both sides wrt x.

$$(y+xy')\cos(xy)+2y+2xy'=0.$$

$$\Rightarrow y'(x\cos(xy)+2x)=-y(2+\cos(xy))$$

$$y'=\frac{-y(2+\cos(xy))}{x(2+\cos(xy))}=-\frac{y}{x}$$

$$y'(1)=-\frac{\pi}{1}=-\pi$$

equation of the tangent line $\quad y(x)=-\pi(x-1)+\pi$.

Applications of Differentiation

• Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. Hence we use the tangent line at $(a, f(a))$ as an approximation to the curve $y=f(x)$ when x is near a.

$$y=f(a)+f'(a)(x-a) \quad \text{"an equation of tangent line"}$$

The approximation $f(x) \approx f(a)+f'(a)(x-a)$ is called the linear approximation of f at a.

$$L(x)=f(a)+f'(a)(x-a) \quad \text{"linearization of } f \text{ at } a\text{"}$$

Example: 1) Find the linearization of $f(x)=\sqrt{x+3}$ at $a=1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

$$f(x)=\sqrt{x+3} \Rightarrow f'(x)=\frac{1}{2\sqrt{x+3}}, \quad f'(1)=\frac{1}{4}$$

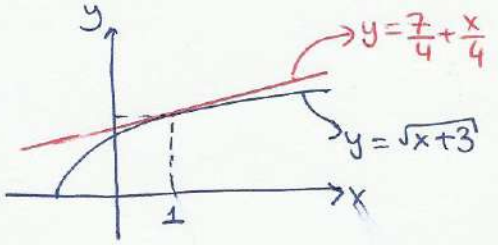
$$f(1)=2$$

Hence the linearization is $L(x)=2+\frac{1}{4}(x-1)=\frac{7}{4}+\frac{x}{4}$

$$\sqrt{x+3} \approx \frac{7}{4}+\frac{x}{4} \quad \text{when } x \text{ is near } 1.$$

$$\sqrt{3.98}=\sqrt{0.98+3} \approx \frac{7}{4}+\frac{0.98}{4}=1.995, \quad \sqrt{4.05}=\sqrt{1.05+3} \approx \frac{7}{4}+\frac{1.05}{4}$$

$$=2.0125$$



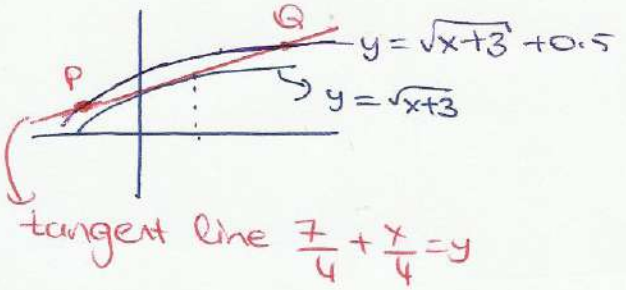
From the graph of the function it is clear that the approximations are overestimates because the tangent line lies above the curve.

ii) For what values of x is the linear approximation $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ accurate to within 0.5?

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right) \right| < 0.5$$

$$\Rightarrow \quad \sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5$$

Hence the linear approximation must lie between the curves obtained by shifting the curve $y = \sqrt{x+3}$ upward and downward by an amount 0.5.



we can estimate the x-coordinate of P and Q.

For instance, for Q

$$\sqrt{x+3} + \underbrace{\frac{1}{2}}_{0.5} = \frac{7}{4} + \frac{x}{4}$$

$$\sqrt{x+3} = \frac{5}{4} + \frac{x}{4}$$

$$\Rightarrow x+3 = \frac{25}{16} + \frac{5x}{8} + \frac{x^2}{16} \quad \Rightarrow \quad x^2 - 6x - 23 = 0$$

$$\Delta = 36 + 4 \cdot 23 = 128 \qquad x_{1,2} = \frac{6 \pm \sqrt{128}}{2}$$

$$\sqrt{128} \approx 11.3$$

Hence x-coordinate of Q is about 8.6. Similarly, x-coordinate of P can be found.

Thus the approximation is accurate to within 0.5 when

$$-2.6 < x < 8.6.$$

Example: Find the linearization of $\sin x$ at $a = 0$.

$f(x) = \sin x$, $f'(x) = \cos x$ $\quad f'(0) = 1$

$f(0) = 0$

$$L(x) = f(a) + f'(a)(x-a) = 0 + 1 \cdot (x - 0) = x$$

$\Rightarrow \sin x \approx x$ for x close to 0.

Example:- Use a linear approximation to approximate

i) $\sqrt[3]{8.02}$.

$f(x) = x^{1/3}$, $a=8$ $\quad f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f'(8) = \frac{1}{12}$

$f(8) = 2$

$L(x) = 2 + \frac{1}{12}(x-8) \Rightarrow x^{1/3} \approx 2 + \frac{1}{12}(x-8)$ for x close to 8.

Hence $\sqrt[3]{8.02} \approx 2 + \frac{1}{12}(8.02-8) \approx 2.001666667$.

(calculator gives $\sqrt[3]{8.02} \approx 2.001665280$).

ii) $\sin 46^\circ$

$f(x) = \sin x$, $a = 45^\circ = \frac{\pi}{4}$ $\quad f'(x) = \cos x$, $f'(a) = \frac{1}{\sqrt{2}}$

$f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ $\quad \Rightarrow L(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) \Rightarrow \sin x \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$

for x close to $\frac{\pi}{4}$

If $x = 46^\circ$, $x - a = 46^\circ - 45^\circ = 1^\circ = \frac{\pi}{180}$

$\Rightarrow \sin 46^\circ \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{\pi}{180} \approx 0.7194481230$

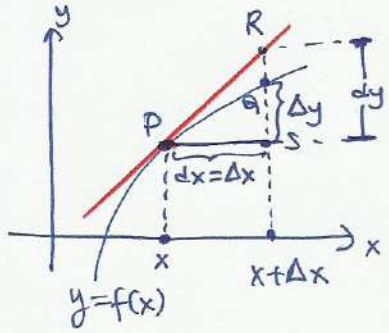
(calculator gives $\sin 46^\circ \approx 0.7193398005$)

Differentials

$$\underbrace{dy}_{\text{differential } dy} = f'(x)\, \underbrace{dx}_{\text{differential } dx}$$

differential dy — a dependent variable

differential dx — an independent variable



- $P = P(x, f(x))$, $Q = Q(x+\Delta x, f(x+\Delta x))$

The corresponding change in y is

$$\Delta y = f(x+\Delta x) - f(x),$$

- The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is

$$dy = f'(x)\Delta x = f'(x)dx.$$

dy: represents the amount that the tangent line rises or falls

Δy: represents the amount that the curve $y=f(x)$ rises or falls when x changes by an amount dx.

Example: Compare the values of Δy and dy if $y=f(x)=x^3+x^2-2x+1$ and x changes from 2 to 2.05.

- $x=2$, $x+\Delta x = 2.05 \Rightarrow \Delta x = 0.05 = dx$ (change in x)
- $f(2) = 2^3+2^2-2.2+1 = 9 = f(x)$
- $f(2.05) = 9.717625 = f(x+\Delta x)$
- $f'(x) = 3x^2+2x-2$, $f'(2) = 3.4+4-2 = 14$.

Hence $\Delta y = f(x+\Delta x) - f(x) = 9.717625 - 9 = 0.717625$.

$$dy = f'(x)dx \Rightarrow dy = 14 \times 0.05 = 0.7.$$

For more information and examples check Thomas'. Calculus and Stewart's Calculus.

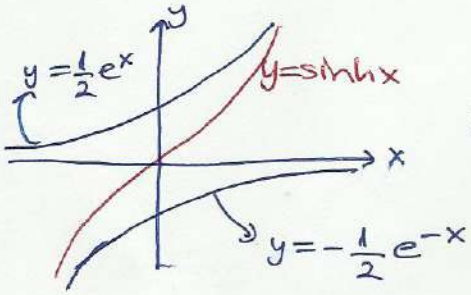
Hyperbolic functions and their derivatives

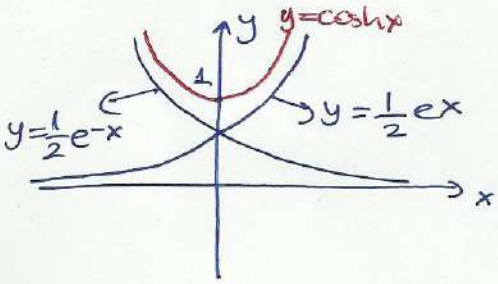
Hyperbolic functions are

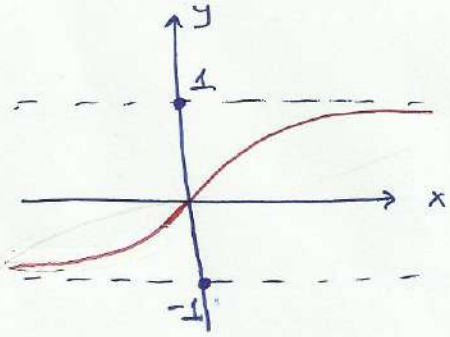
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{cosech} x = \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$







$\operatorname{dom}(\sinh) = \mathbb{R}$
$\operatorname{ran}(\sinh) = \mathbb{R}$

$\operatorname{dom}(\cosh) = \mathbb{R}$
$\operatorname{ran}(\cosh) = [1, \infty)$

$\operatorname{dom}(\tanh) = \mathbb{R}$
$\operatorname{ran}(\tanh) = (-1, 1)$

Identities: $\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x . \cosh y + \cosh x . \sinh y$$

$$\cosh(x+y) = \cosh x . \cosh y + \sinh x . \sinh y$$

Derivatives

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x . \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x . \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x .$$

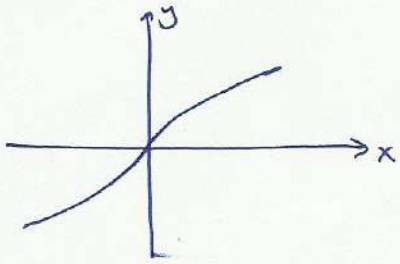
We can also define inverse hyperbolic functions.

$$y = \sinh^{-1} x \iff \sinh y = x$$

$$y = \cosh^{-1} x \iff \cosh y = x \text{ and } \boxed{y \geq 0.}\,*$$

*(cosh is not 1-to-1 but when restricted to domain $[0, \infty)$ it becomes 1-to-1.)

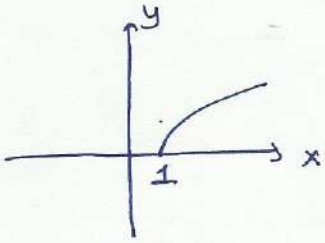
$y = \tanh^{-1} x \iff \tanh y = x.$



$y = \sinh^{-1} x$

$\mathrm{dom}(\sinh^{-1}) = \mathbb{R}$

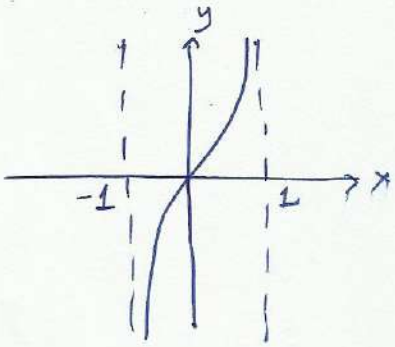
$\mathrm{ran}(\sinh^{-1}) = \mathbb{R}$



$y = \cosh^{-1} x$

$\mathrm{dom}(\cosh^{-1}) = [1, \infty)$

$\mathrm{ran}(\cosh^{-1}) = [0, \infty)$



$y = \tanh^{-1} x$

$\mathrm{dom}(\tanh^{-1}) = (-1, 1)$

$\mathrm{ran}(\tanh^{-1}) = \mathbb{R}$

Inverse hyperbolic functions can be expressed in terms of logarithms.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \quad x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad -1 < x < 1$$

Proof: $y = \sinh^{-1} x \Rightarrow x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$\Rightarrow e^y - 2x - e^{-y} = 0 \quad \Rightarrow e^{2y} - 2x e^y - 1 = 0.$ Let $e^y = t$

$t^2 - 2xt - 1 = 0$ quadratic equation

$\Delta = 4x^2 + 4$

$$t_{1,2} = \frac{2x \pm \sqrt{4x^2+4}}{2} = e^y \quad \Rightarrow \quad e^y = x \pm \sqrt{x^2+1}$$

Note that $e^y > 0$ but $x - \sqrt{x^2+1} < 0$. Thus the minus sign is admissible and we have

$e^y = x + \sqrt{x^2+1}$. Therefore $y = \ln(e^y) = \ln\left(x + \sqrt{x^2+1}\right)$

Derivatives of inverse hyperbolic functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\frac{d}{dx}(\mathrm{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\frac{d}{dx}(\mathrm{csch}^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2+1}}$	$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Indeterminate forms and L'Hôpital's Rule

Theorem (L'Hôpital's Rule) Suppose that f and g differentiable on the interval (a,b), except possibly at some fixed point $c \in (a,b)$ and that $g'(x) \neq 0$ on (a,b), except possibly at c. Suppose further that $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)} = L$ (or $\pm\infty$). Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Proof: For the special case in which $f(c)=g(c)=0$, f' and g' are continuous and $g'(c) \neq 0$.

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)} = \frac{\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim\limits_{x\to c} \frac{g(x)-g(c)}{x-c}} = \lim_{x\to c} \frac{f(x)-\overset{0}{f(c)}}{g(x)-\underset{0}{g(c)}} = \lim_{x\to c} \frac{f(x)}{g(x)}.$$

Note: L'Hôpital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is "$x\to a$" can be replaced by any of the symbols $x\to a^+$, $x\to a^-$, $x\to\infty$, or $x\to-\infty$.

Examples: Evaluate the following limits.

1) $\lim\limits_{x\to\infty} \frac{x^2+1}{x^3+4} \overset{\frac{\infty}{\infty}}{\underset{L.R}{=}} \lim\limits_{x\to\infty} \frac{2x}{3x^2} = \lim\limits_{x\to\infty} \frac{2}{3x} = 0.$

2) $\lim\limits_{x\to\infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty}}{\underset{L.R}{=}} \lim\limits_{x\to\infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty}}{\underset{L.R}{=}} \lim\limits_{x\to\infty} \frac{2}{e^x} = 0$

3) $\lim\limits_{x\to 0^+} \frac{\ln x}{\operatorname{cosec} x} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\cot x \cdot \operatorname{cosec} x} = \lim\limits_{x\to 0^+} -\frac{1}{x} \frac{\sin x}{\cos x} \cdot \sin x$

$$= -\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \frac{\sin x}{\cos x}$$

$$= -1 \cdot 0 = 0$$

4) $$\lim_{x\to 0} \frac{x-\sin x}{x^3} \underset{L.R}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{1-\cos x}{3x^2} \underset{L.R}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$$

5) $$\lim_{x\to 0} \frac{1-\cos(x^2)}{x^3\sin x} \underset{L.R}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{2x\sin(x^2)}{3x^2\sin x + x^3\cos x}$$

$$= \lim_{x\to 0} \frac{1}{3\underbrace{\frac{x^2}{\sin(x^2)}}_{1}\cdot\underbrace{\frac{\sin x}{2x}}_{1} + \underbrace{\frac{x^2}{\sin(x^2)}}_{1}\cdot\frac{\cos x}{2}} = \frac{1}{\frac{3}{2}+\frac{1}{2}} = \frac{1}{2}$$

6) $$\lim_{x\to 0} \frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x} \underset{L.R}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\cos x}$$

$$= \lim_{x\to 0} \frac{2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}{\cos x} \text{ does not exist}$$

Here L'Hôpital's theorem is not applicable.

$$\lim_{x\to 0} \frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x}\cdot \underbrace{\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)}_{\substack{\text{By sandwich theorem} \\ \parallel \\ 0}} = 1.0 = 0.$$

7) $$\lim_{x\to\infty} \frac{\ln x}{x^a}$$ a, constant

- If $a \leq 0$ $\quad \lim_{x\to\infty} \frac{\ln x}{x^a} \overset{\frac{\infty}{0}}{=} \infty$ not indeterminate

- If $a > 0$ $\quad \lim_{x\to\infty} \frac{\ln x}{x^a} \underset{L.R}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{ax^a} = 0.$

8) $\lim\limits_{x\to\infty} \dfrac{x^a}{e^x}$.

If $a \le 0$, $\lim\limits_{x\to\infty} \dfrac{x^a}{e^x} = \lim\limits_{x\to\infty} \dfrac{1}{x^{-a}e^x} = 0$.

If $a>0$, $\lim\limits_{x\to\infty} \dfrac{x^a}{e^x} \underset{L.R}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{x\to\infty} \dfrac{a x^{a-1}}{e^x}$ $\quad (=0$ if $a<1)$

(if $a>1$) we continue to apply L.R. $\underset{L.R}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{x\to\infty} \dfrac{a(a-1)x^{a-2}}{e^x}$ $\quad (=0$ if $a<2)$

$\vdots$

$\underset{L.R}{=} \lim\limits_{x\to\infty} a(a-1)\cdots(a-[\![a]\!]) \dfrac{x^{a-[\![a]\!]-1}}{e^x} = 0.$

(after $[\![a]\!]$ times application of L'Hôpital's rule)

Other Indeterminate Forms

$\infty-\infty$, $0\cdot\infty$, 0^0, 1^∞, and ∞^0 while evaluating a limit of this type, we first try to reduc it to one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so we can apply L'Hôpital's rule.

Note: $\frac{0}{\infty}$, $\frac{\infty}{0}$, 0^∞, 0^1, 1^0, ∞^1, and ∞^∞ are not indeterminate.

Examples: Evaluate the following limits.

1) $\lim\limits_{x\to 0} \left[\dfrac{1}{\ln(x+1)} - \dfrac{1}{x}\right] \overset{\infty-\infty}{=} \lim\limits_{x\to 0} \dfrac{x-\ln(x+1)}{x\ln(x+1)}$

$\underset{L.R.}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to 0} \dfrac{1-\frac{1}{x+1}}{\ln(x+1)+\frac{x}{x+1}}$

$= \lim\limits_{x\to 0} \dfrac{x}{(x+1)\ln(x+1)+x} \underset{L.R}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to 0} \dfrac{1}{\ln(x+1)+1+1} = \dfrac{1}{2}$

2) $\lim\limits_{x\to 0^+} (\sin x)^x \overset{0^0}{=} \lim\limits_{x\to 0^+} e^{\ln(\sin x)^x} = \lim\limits_{x\to 0^+} e^{x\ln(\sin x)} = \boxed{e^{\lim\limits_{x\to 0^+} x\ln(\sin x)}}$

$= e^0 = 1$ //

$\lim\limits_{x\to 0^+} x\ln(\sin x) \overset{0.\infty}{=} \lim\limits_{x\to 0^+} \dfrac{\ln(\sin x)}{\frac{1}{x}} \underset{L.R}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{x\to 0^+} \dfrac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}}$

$= \lim\limits_{x\to 0^+} -\dfrac{x}{\sin x} \cdot x\cos x$

$=0$

3) $\lim\limits_{x\to \frac{\pi}{2}} \tan x \cdot \ln(\sin x) \overset{\infty.0}{=} \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln(\sin x)}{\cot x} \underset{L.R.}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$

$= \lim\limits_{x\to \frac{\pi}{2}} -\sin x \cdot \cos x = 0$

4) $\lim\limits_{x\to 0} (1+x)^{x^2} \overset{1^0}{=} 1$

not undetermined

5) $\lim\limits_{x\to\infty} \left(1+\dfrac{a}{x}\right)^x \overset{1^\infty}{=} \lim\limits_{x\to\infty} e^{x\ln\left(1+\frac{a}{x}\right)} = \boxed{e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{a}{x}\right)}}$

$\lim\limits_{x\to\infty} x\ln\left(1+\dfrac{a}{x}\right) \overset{\infty.0}{=} \lim\limits_{x\to\infty} \dfrac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \underset{L.R.}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to\infty} \dfrac{\frac{1}{1+\frac{a}{x}} \cdot \frac{-a}{x^2}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to\infty} \dfrac{a}{1+\underbrace{\frac{a}{x}}_{\to 0}} = a$

$\Rightarrow \lim\limits_{x\to\infty} \left(1+\dfrac{a}{x}\right)^x = e^a$. $\left(\text{For } a=1,\ \lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x = e^1 = e\right)$

Maximum And Minimum Values

Definition: For a function f defined on a set S of real numbers and a number $c \in S$,

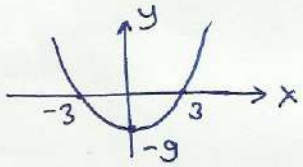
(i) $f(c)$ is the absolute maximum (or global maximum) of f on S if $f(c) \geq f(x)$ for all $x \in S$ and

(ii) $f(c)$ is the absolute minimum (or global minimum) of f on S if $f(c) \leq f(x)$ for all $x \in S$.

An absolute maximum or an absolute minimum is referred to as an absolute extremum. If a function has more than one extremum, we refer to these as extrema.

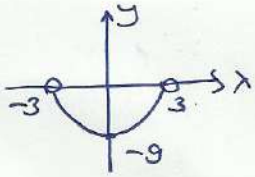
Examples 1) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval

(i) $(-\infty, \infty)$



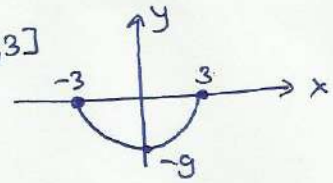
- f has no absolute maximum on $(-\infty, \infty)$.
- $f(0) = -9$ is the absolute minimum of f on $(-\infty, \infty)$

(ii) $(-3, 3)$



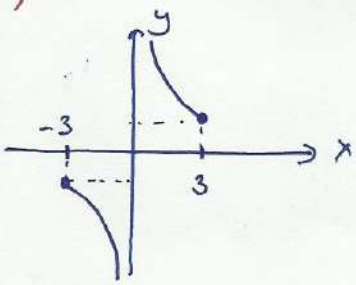
- f has no absolute maximum on $(-3, 3)$.
- $f(0) = -9$ is the absolute minimum of f on $(-3, 3)$.

(iii) $[-3, 3]$



- $f(3) = f(-3) = 0$ is the absolute maximum of f on $[-3, 3]$.
- $f(0) = -9$ is the absolute minimum of f on $[-3, 3]$.

2) Locate any absolute extrema of $f(x) = \frac{1}{x}$ on the interval $[-3, 3]$



f has no absolute maximum and no absolute minimum on $[-3, 3]$.

The Extreme Value Theorem: A continuous function f defined on a closed, bounded interval $[a,b]$ attains both an absolute maximum and an absolute minimum on that interval.

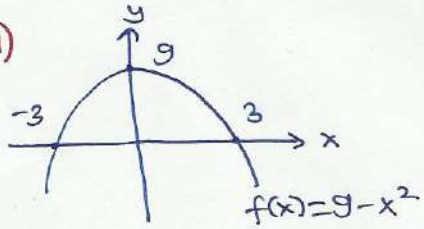
Definition: (i) $f(c)$ is a local (relative) maximum of f if $f(c) \geq f(x)$ for all x in some open interval containing c.

(ii) $f(c)$ is a local (relative) minimum of f if $f(c) \leq f(x)$ for all x in some open interval containing c.

Example: The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos(2n\pi) = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x. Similarly, $\cos(2n+1)\pi = -1$ is its (local and absolute) minimum value, where n is any integer.

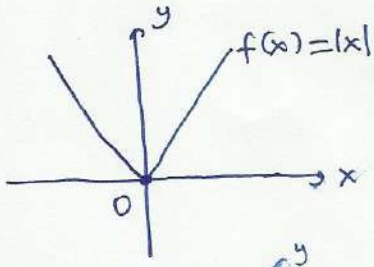
Example: Locate any local extrema for $f(x)$ and describe the behavior of the derivative at the local extremum if a) $f(x) = 9 - x^2$, b) $f(x) = |x|$.

a)



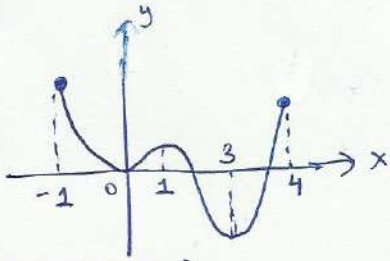
$f(0) = 9$ is a local (and absolute) maximum of f. No local (or absolute) min. Note that $f'(0) = 0$.

b)



$f(0) = 0$ is a local (and absolute) minimum of f. No local (or absolute) max. Note that $f'(0)$ does not exist.

Example:



$f(x) = 3x^4 - 16x^3 + 18x^2$, $-1 \leq x \leq 4$

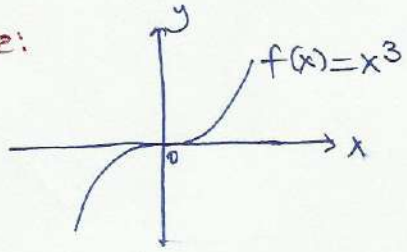
$f(1) = 5$ local max.
$f(0) = 0$ local min
$f(3) = -27$ local and absolute min.
$f(-1) = 37$ absolute max. but not a local max. because it occurs at an endpoint

$f(4) = 32$ neither a local nor an absolute max.

Definition: A number c in the $\boxed{\text{domain}}$ of a function f is called a critical number (point) of f if $f'(c)=0$ or $f'(c)$ is undefined.

Fermat's Theorem: Suppose that f has a local maximum or minimum at c. Then c must be a critical point.

Example:



$f(x)=x^3$

$f'(x)=3x^2 \Rightarrow f'(0)=0.$

But f has no maximum or minimum at 0. Indeed, $f'(0)=0$ simply means that the curve $y=x^3$ has a horizontal tangent at $(0,0)$.

Hence even when $f'(c)=0$ there need not be a maximum or minimum at c. That is, the converse of Fermat's Theorem is false in general.

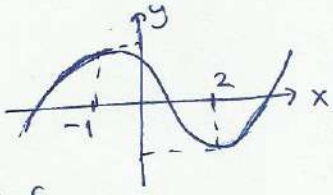
Example: Find the critical numbers an local extrema of

a) $f(x)=2x^3-3x^2-12x+5$ b) $f(x)=(3x+1)^{\frac{2}{3}}$

a) $f(x)=2x^3-3x^2-12x+5 \Rightarrow f'(x)=6x^2-6x-12=6(x^2-x-2)$
$=6(x-2)(x+1)$

$f'(x)=0$ when $x=-1$ and $x=2$.

$x=-1$ and $x=2$ are critical points.

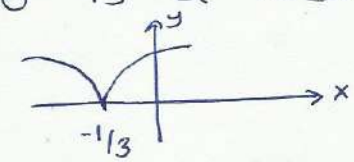


$f(-1)=-2-3+12+5=12$ local maximum of f

$f(2)=16-12-24+5=-15$ local minimum of f

b) $f(x)=(3x+1)^{2/3} \Rightarrow f'(x)=\frac{2}{3}(3x+1)^{-1/3}=\frac{2}{3(3x+1)^{1/3}}$

At $x=-1/3$, $f'(x)$ is undefined. Hence $x=-1/3$ is a critical point of $f(x)$. $f(-\frac{1}{3})=0$ is a local minimum of f.



Example: Find all the critical numbers of

a) $f(x) = \dfrac{2x^2}{x+2}$ b) $f(x) = x^2 - 2|x|$.

a) $f(x) = \dfrac{2x^2}{x+2} \Rightarrow f'(x) = \dfrac{4x(x+2) - 2x^2}{(x+2)^2} = \dfrac{2x(x+4)}{(x+2)^2}$

$f(x) = 0$ for $x = 0$ and $x = -4$.

$x = 0$ and $x = -4$ are critical points of f.

<u>Note</u>: !! $f'(-2)$ is undefined but $-2 \notin dom(f)$.

b) $f(x) = x^2 - 2|x|$.

$$f(x) = \begin{cases} x^2 - 2x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x^2 + 2x & \text{if } x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x - 2 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ 2x + 2 & \text{if } x < 0 \end{cases}$$

Since $f'(1) = f'(-1) = 0$ and $f'(0)$ is undefined; $x = 1$, $x = -1$, and $x = 0$ are critical points of f.

Theorem: Suppose that f is continuous on the closed, bounded interval $[a,b]$. Then, the absolute extrema of f must occur at an endpoint (a or b) or at a critical point.

Remark: This theorem gives a procedure to obtain the absolute extrema of a continuous function on a closed, bounded interval $[a,b]$

1) Find all the critical points in (a,b) and compute the values of f at these points.
2) Compute function values at the endpoints of the interval.
3) The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example: Find the absolute extrema of the given function on the given interval.

1) $f(x) = x^3 - 9x^2 + 15x$ on $[0,4]$.

$f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x-5)(x-1)$

$f'(x) = 0$ for $x=5$ and $x=1$ but $x=5$ does not belong to $[0,4]$.

$x=1$ is the critical point of f on $[0,4]$.

end points $x=0$, $x=4$.

$f(0) = 0$

$f(4) = -20$ (abs. min)

$f(1) = 7$ (abs. max)

2) $f(x) = |x^2 - 3x + 2|$ on $[-10,10]$. $x^2 - 3x + 2 = (x-2)(x-1)$

$$f(x) = \begin{cases} x^2 - 3x + 2 & \text{if } x \le 1 \\ -x^2 + 3x - 2 & \text{if } 1 \le x \le 2 \\ x^2 - 3x + 2 & \text{if } x \geq 2 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x - 3 & \text{if } x < 1 \\ \text{undefined} & \text{if } x = 1 \\ -2x + 3 & \text{if } 1 < x < 2 \\ \text{undefined} & \text{if } x = 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$$

$x=1$, $x=\frac{3}{2}$, $x=2$ are critical points of f.

$x=-10$, $x=10$ are endpoints.

$f(-10) = 132$ (abs. max.)

$f(1) = 0$ (abs. min.)

$f(3/2) = 1/4$

$f(2) = 0$ (abs. min)

$f(10) = 72$

3) $f(x) = \sqrt{5-4x}$ on $[-1,1]$.

$f(x) = \sqrt{5-4x} \Rightarrow f'(x) = \dfrac{-4}{2\sqrt{5-4x}} = \dfrac{-2}{\sqrt{5-4x}}$

$x = \frac{5}{4}$ is the only critical point but it does not belong to $[-1,1]$.

$f(-1) = 3$ (abs. max.)

$f(1) = 1$ (abs. min.)

Rolle's Theorem: Let f be a function that satisfies the followings:

1) f is continuous on the closed, bounded interval $[a,b]$

2) f is differentiable on the open interval (a,b).

3) $f(a)=f(b)$.

Then there is a number c in (a,b) such that $f'(c)=0$.

Example: Prove that the equation $x^3+x-1=0$ has <u>exactly</u> one real root.

Firstly, we will use Intermediate Value Theorem to show that there is at least one root.

Let $f(x)=x^3+x-1$. This function is a polynomial and it is continuous everywhere, and $f(1)=1>0$ & $f(0)=-1<0$. Hence the Intermediate Value Theorem states that there is at least one number $c \in (0,1)$ such that $f(c)=0$.

To show that the equation has <u>no other real root</u>, we will use Rolle's Theorem. Suppose that it has two roots a and b. Then $f(a)=0=f(b)$ and since f is a polynomial, it is continuous on any interval $[a,b]$, and differentiable on (a,b). Hence by Rolle's Theorem, there is a number $c \in (a,b)$ such that $f'(c)=0$. But $f'(x)=3x^2+1 \geq 1$ for all x, that is $f'(x)$ can never be 0. This is a contradiction. Therefore the equation cannot have two real roots.

The Mean Value Theorem: Let f be a function that satisfies the followings:

1) f is continuous on the closed, bounded interval $[a,b]$.

2) f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Example: Let $f(x)=x^3-x$ be a function defined on $[0,2]$.

Find the number c that satisfies the conclusion of the Mean Value Theorem.

- $f(x)=x^3-x$ is a polynomial so it is continuous and differentiable for all x, so it is continuous on $[0,2]$ and differentiable on $(0,2)$

Therefore by the Mean Value Theorem, there is a number $c \in (0,2)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(0)}{2-0}=\frac{6-0}{2-0}=3$$

$f'(x)=3x^2-1 \Rightarrow f'(c)=3c^2-1=3 \Rightarrow c=\pm\frac{2}{\sqrt{3}}$. But c must lie in $(0,2)$, so $c=2/\sqrt{3}$.

Example: If $f(1)=10$ and $f'(x)\geq 2$ for $1\leq x\leq 4$, how small can $f(4)$ possibly be?

We are given in the question that f is differentiable (and therefore continuous) for $1\leq x\leq 4$. Hence we can apply the Mean Value Theorem on $[1,4]$. There exists a number c such that

$$f'(c)=\frac{f(4)-f(1)}{4-1}=\frac{f(4)-10}{3}, \quad c\in(1,4)$$

Since $f'(x)\geq 2$ for $1\leq x\leq 4$ we also have $f'(c)\geq 2$, $c\in(1,4)$

$$f'(c)=\frac{f(4)-10}{3}\geq 2 \Rightarrow f(4)-10\geq 6 \Rightarrow f(4)\geq 16.$$

Hence the smallest possible value for $f(4)=16$.

Example: Prove that if $f'(x)=0$ for all x in an interval (a,b), then f is constant on (a,b).

We are given in the question that f is differentiable (and therefore continuous) on (a,b).

Let x_1 and x_2 be any two numbers in (a,b) with $x_1<x_2$. Since f is differentiable on (a,b), it must be differentiable on (x_1,x_2), and continuous on $[x_1,x_2]$.

By the Mean Value Theorem, there is a number $c \in (x_1, x_2)$ so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad c \in (x_1, x_2)$$

Since $f'(x) = 0$ for all $x \in (a,b)$, $f'(c) = 0$.

Thus $f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$. Therefore f has the same value at any two numbers x_1 and x_2 in (a,b). This yields that f is constant on (a,b).

By using this example, we can prove that:

- If $f'(x) = g'(x)$ for all x in an interval (a,b), then $f-g$ is constant on (a,b); that is, $f(x) = g(x) + c$ where c is a constant.

  (To prove, it is enough to define the function $F(x) = f(x) - g(x)$. Then $F'(x) = f'(x) - g'(x) = 0$ for all x in (a,b). By the above example, F is constant i.e. $f(x) - g(x)$ is constant.)

Example: Prove the identity $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

$f(x) = \tan^{-1} x + \cot^{-1} x$.

$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$ for all values of x.

Hence $f(x) = c$, a constant. To determine the value of c, we put $x = 1$.

$$c = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

$\Rightarrow \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

Increasing and Decreasing Functions

- A function f is (strictly) increasing on an interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) < f(x_2)$.
- A function f is (strictly) decreasing on an interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) > f(x_2)$.

Theorem: Suppose that f is differentiable on an interval I.

(i) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.

(ii) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.

Example: Find the intervals where $f(x)$ is increasing or decreasing.

1) $f(x) = x^2(x-1) \Rightarrow f'(x) = 3x^2 - 2x = x(3x-2)$

$f'(x) = 0$ for $x = 0$ and $x = \frac{2}{3}$

x		0		$2/3$	
f'	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

f is increasing on $(-\infty, 0)$ and $(2/3, \infty)$.

f is decreasing on $(0, 2/3)$.

2) $f(x) = 2x^3 + 9x^2 - 24x - 10 \Rightarrow f'(x) = 6x^2 + 18x - 24 = 6(x^2 + 3x - 4)$

$= 6(x+4)(x-1)$

$f'(x) = 0$ for $x = -4$ and $x = 1$

x		-4		1	
f'	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

f is increasing on $(-\infty; 4)$ and $(1, \infty)$.

f is decreasing on $(-4, 1)$.

First Derivative Test: Suppose that c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local maximum at c.

b) If f' changes from negative to positive at c, then f has a local minimum at c.

c) If f' does not change sign at c, then f has no local maximum or minimum at c.

Example: In each part find and classify the critical points of $f(x)$.

i) $f(x) = \frac{2}{3}x^3 - 2x + 1 \Rightarrow f'(x) = 2x^2 - 2 = 2(x^2-1) = 2(x-1)(x+1)$

$f'(x) = 0$ for $\underbrace{x=1 \text{ and } x=-1}_{\text{critical points}}$

x		-1		1	
f'	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

f is increasing on $(-\infty,-1)$ and $(1,\infty)$.

f is decreasing on $(-1,1)$

$f(-1) = 7/3$ local max. of f

$f(1) = -1/3$ local min. of f.

ii) $f(x) = x^{5/3} - 3x^{2/3} \Rightarrow f'(x) = \frac{5}{3}x^{2/3} - 2x^{-1/3} = x^{-1/3}\frac{(5x-6)}{3}$

$= \frac{5x-6}{3x^{1/3}}$

$f'(x) = 0$ for $x = 6/5$

$f'(x) =$ undefined for $x = 0$

$\Big\}$ $x = 6/5$ and $x = 0$ are critical points.

x		0		$6/5$	
f'	$+$	$\|\|$	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

$f(0) = 0$ local max. of f

$f(6/5) \approx -2.03$ local min. of f.

f is increasing on $(-\infty,0)$ and $(6/5,\infty)$

f is decreasing on $(0, 6/5)$.

iii) $g(x) = x + 2\sin x \quad , \quad 0 \le x \le 2\pi$

$g'(x) = 1 + 2\cos x \quad , \quad g'(x) = 0$ when $\cos x = -\frac{1}{2}$.

Hence $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$; $\frac{2\pi}{3} \in [0, 2\pi]$, $\frac{4\pi}{3} \in [0, 2\pi]$.

Since $g(x)$ is differentiable everywhere i.e. there is no point that makes $g'(x)$ undefined, the only critical numbers are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

x	0		$2\pi/3$		$4\pi/3$		2π
$g'(x)$		$+$	0	$-$	0	$+$	
$g(x)$		$\nearrow$		$\searrow$		$\nearrow$	
			local max at $2\pi/3$		local min at $4\pi/3$		

$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$ local max. value

$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$ local min. value

Concavity and the Second Derivative Test

For a function f that is differentiable on an interval I, the graph of f is

(i) concave up on I if f' is increasing on I or

(ii) concave down on I if f' is decreasing on I.

Theorem: Suppose that f'' exists on an interval I.

(i) If $f''(x) > 0$ on I, then the graph of f is concave up on I.

(ii) If $f''(x) < 0$ on I, " concave down on I

Definition: A point $P = (c, f(c)), c \in (a,b)$ on a curve $y = f(x)$ is called an <u>inflection point</u> if f is continuous on the interval (a,b) and the curve changes from concave up to concave down or from concave down to concave up at P. (the point that the concavity changes)

- Note that, if $P = (c, f(c))$ is an inflection point, then either $f''(c) = 0$ or $f''(c)$ is undefined. But beware that not all points where $f''(x)$ is zero or undefined correspond to inflection points.

Example: Determine where the graph of $f(x)=2x^3+9x^2-24x-10$ is concave up and concave down, and draw a graph showing all significant features of the function.

$$f(x)=2x^3+9x^2-24x-10$$

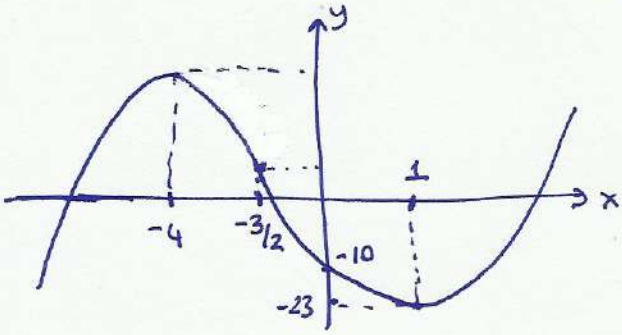
$$f'(x)=6x^2+18x-24=6(x^2+3x-4)=6(x+4)(x-1)$$

$$f''(x)=12x+18=6(2x+3)$$

$f'(x)=0$ when $x=-4$, $x=1$ (critical points)

$f''(x)=0$ when $x=-\frac{3}{2}$ (a candidate to be an inflection point)

x		-4		$-3/2$		1	
f'	$+$ ↗	0	$-$ ↘		$-$ ↘	0	$+$ ↗
f''	$-$ down		$-$ down	0	$+$ up		$+$ up
f	⌒		⌣		⌣		⌒
		local max		inflection pt.		local min	



Example: Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x)>0$ on $(-\infty,1)$, $f'(x)<0$ on $(1,\infty)$.

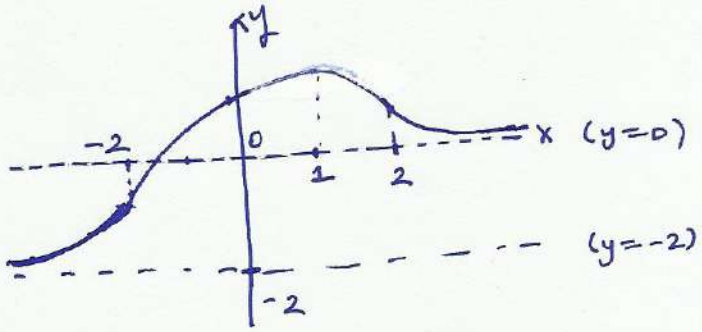
(ii) $f''(x)>0$ on $(-\infty,-2)$ and $(2,\infty)$, $f''(x)<0$ on $(-2,2)$

(iii) $\lim_{x\to-\infty} f(x)=-2$, $\lim_{x\to\infty} f(x)=0$.

By using these conditions we can draw a table.

x		-2		1		2	
f'	$+$ ↗		$+$ ↗		$-$ ↘		$-$ ↘
f''	up $+$		$-$ down		$-$ down		$+$ up
	⌣		⌒		⌣		⌣
		inf. pt		local max.		inf. pt	

- horizontal asymptotes $y=-2$ and $y=0$



Example: Determine where the graph of $f(x)=x^4-6x^2+1$ is concave up and concave down, find any inflection points and draw a graph showing all significant features.

$f(x)=x^4-6x^2+1$

$f'(x)=4x^3-12x=4x(x^2-3)=4x(x-\sqrt{3})(x+\sqrt{3})$

$f''(x)=12x^2-12=12(x^2-1)=12(x-1)(x+1)$

$f'(x)=0$ when $x=0$, $x=\sqrt{3}$, $x=-\sqrt{3}$. (critical points)

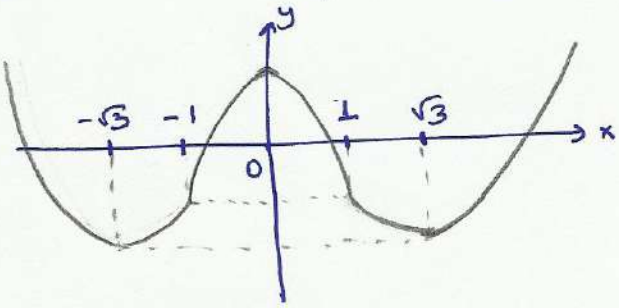
$f''(x)=0$ when $x=-1$, $x=1$.

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$ ↘	0	$+$ ↗		$+$ ↗	0	$-$ ↘		$-$ ↘	0	$+$ ↗
f''	$+$ up		$+$ up	0	$-$ down		$-$ down	0	$+$ up		$+$ up
f	∪		∪		∩		∩		∪		∪
		local min		inf. pt		local max		inf. pt		local min	

concave up on $(-\infty,-1)$ and $(1,\infty)$.

concave down on $(-1,1)$.

inflection points $(-1,-4)$, $(1,-4)$.



The Second Derivative Test: Suppose f'' is continuous near c.

(i) If $f'(c)=0$ and $f''(c)>0$, then f has a local min. at c.

(ii) If $f'(c)=0$ and $f''(c)<0$, then f has a local max. at c.

Note: If $f''(c)=0$ or $f''(c)$ is undefined, the Second Derivative Test yields no conclusion. That is, $f(c)$ may be a local max., a local min. or neither.

Example: Use the Second Derivative Test to find the local extrema of $f(x)=x^4-8x^2+10$.

$f(x)=x^4-8x^2+10$

$f'(x)=4x^3-16x=4x(x^2-4)=4x(x-2)(x+2)$.

$f'(x)=0$ when $x=0$, $x=-2$, $x=2$. (critical points).

$f''(x)=12x^2-16$

$f''(0) = -16 < 0 \Rightarrow f(0) = 10$ is a local max. of f

$f''(-2) = 32 > 0 \Rightarrow f(-2) = -6$ is a local min. of f

$f''(2) = 32 > 0 \Rightarrow f(2) = -6$ is a local min. of f.

Example: Use the Second Derivative Test to try to classify any local extrema i) for $f(x) = x^3$.

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$

$f'(x) = 0$ when $x = 0$. $x = 0$ is the only critical point of f.

$f''(0) = 0$ the second derivative test fails.

x		0	
f'	+	0	+
f	↗		↗

f has no local max., no local min.

ii) for $f(x) = x^4$.

$f(x) = x^4 \Rightarrow f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2$

$f'(x) = 0$ when $x = 0$. $x = 0$ is the only critical point of f.

$f''(0) = 0$ the second derivative test fails.

x		0	
f'	−	0	+
f	↘		↗
		local min	

f has local min. at $x = 0$.

Curve Sketching

1) Determine the DOMAIN of f.
2) Locate x- and y- INTERCEPTS, if any
3) Determine the ASYMPTOTES of f.
4) Determine where f is INCREASING and DECREASING, find any LOCAL EXTREMA.
5) Determine where the graph is CONCAVE UP and CONCAVE DOWN and locate any INFLECTION POINTS.

Example: Draw a graph of $f(x) = \frac{x^2+25}{x}$, showing all significant features.

- $Dom(f) = \mathbb{R} - \{0\}$
- No x- or y- intercepts.
- oblique (slant) asymptote: when the degree of the numerator is exactly one more than the degree of the denominator, the graph of the rational function will have an oblique (slant) asymptote.

$\lim_{x \to 0^+} x + \frac{25}{x} = +\infty$

$\lim_{x \to 0^-} x + \frac{25}{x} = -\infty$

$x = 0$ vertical asymptote

One can find the oblique asymptote by diving the denominator into the numerator. $f(x) = \frac{x^2+25}{x} = x + \frac{25}{x} \to 0$ as $x \to \pm\infty$

$y = x$ is the oblique asymptote.

$f(x)$ and its oblique asymptote $y=x$ are close to eachother when $|x|$ is large. Thus the graph of the rational function $f(x)$ is asymptotic to the graph of the polynomial $y=x$ as $x \to \pm\infty$.

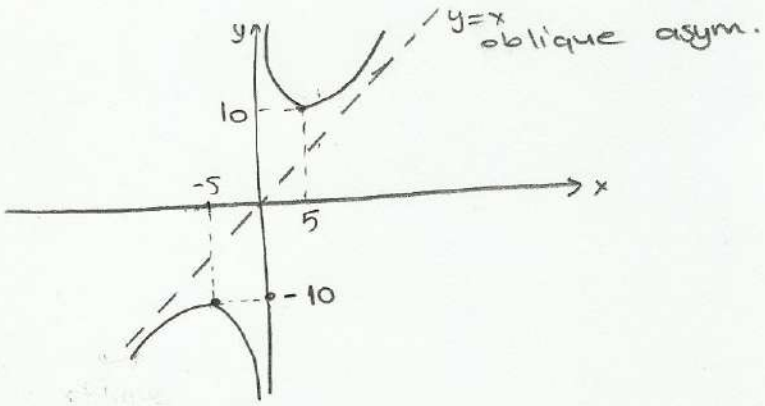
- $f'(x) = 1 - \frac{25}{x^2} = \frac{x^2-25}{x^2} = \frac{(x-5)(x+5)}{x^2}$

$f'(x) = 0$ when $x = -5$ and $x = 5$) critical points

$f'(x) =$ undefined when $x = 0$ (not in dom(f))

- $f''(x) = \frac{50}{x^3}$ $f''(x) =$ undefined when $x = 0$.

x		-5		0		5	
f'	+ ↗	0	− ↘	‖	− ↘	0	+ ↗
f''	− down		− down	‖	+ up		+ up
f	⌒		⌒	‖	⌣		⌣



Example: Sketch the graph of $f(x)=\dfrac{x^2}{x^2-4}$, showing all significant features.

- $\mathrm{dom}(f)=\mathbb{R}\setminus\{\pm 2\}$.

- $y=0 \Rightarrow x=0$ (x - intercept)
  $x=0 \Rightarrow y=0$ (y - intercept)
  } the graph passes through the origin.

- $\lim\limits_{x\to 2^+}\dfrac{x^2}{x^2-4}=+\infty$, $\lim\limits_{x\to 2^-}\dfrac{x^2}{x^2-4}=-\infty$

  $\lim\limits_{x\to -2^+}\dfrac{x^2}{x^2-4}=-\infty$, $\lim\limits_{x\to -2^-}\dfrac{x^2}{x^2-4}=+\infty$

  } $x=-2$ and $x=2$ are vertical asymtotes.

- $\lim\limits_{x\to\pm\infty}\dfrac{x^2}{x^2-4}=1 \Rightarrow y=1$ is a horizontal asymtote.

- $f'(x)=\dfrac{2x(x^2-4)-2x.x^2}{(x-2)^2(x+2)^2}=\dfrac{-8x}{(x-2)^2(x+2)^2}$

  $f'(x)=0$ when $x=0$) critical point

  $f'(x)=$ undefined when $x=2$, $x=-2$ but they are not in $\mathrm{dom}(f)$.

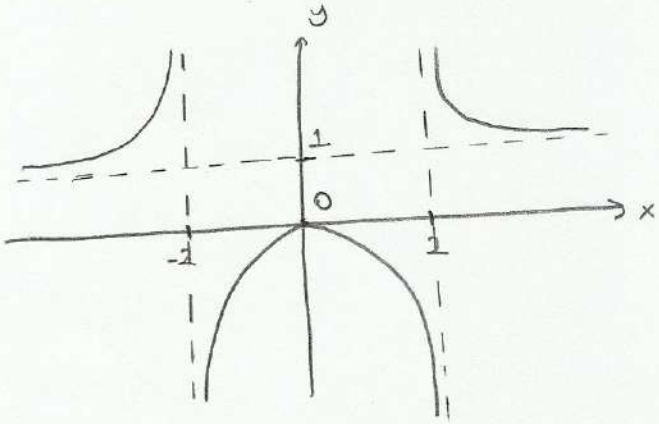
- $f''(x)=\left[\dfrac{-8x}{(x^2-4)^2}\right]'=\dfrac{-8.(x^2-4)^2+8x\,2(x^2-4)2x}{(x^2-4)^4}=\dfrac{-8(x^2-4)+32x^2}{(x^2-4)^3}=\dfrac{24x^2+32}{(x^2-4)^3}$

  $=\dfrac{8(3x^2+4)}{(x^2-2)^3(x^2+2)^3}$

  $f''(x)$ is never zero for real x.

  $f''(x)=$ undefined for $x=\pm 2$.

x		-2		0		2	
f'	$+\nearrow$	‖	$+\nearrow$	0	$-\searrow$	‖	$-\searrow$
f''	$+$ up	‖	$-$ down		$-$ down	‖	$+$ up
f	)	‖	)	local max	)	‖	)



Example: Sketch the graph of $g(x)=\frac{x^2-4}{x-1}$, showing all significant features.

- $\mathrm{dom}(g)=\mathbb{R}-\{1\}$.

- $y=0 \Rightarrow x=\pm 2$ (x-intercepts)

  $x=0 \Rightarrow y=4$ (y-intercept)

- $\lim_{x\to 1^+}\frac{x^2-4}{x-1}=-\infty$, $\lim_{x\to 1^-}\frac{x^2-4}{x-1}=+\infty \Rightarrow x=1$ is a vertical asymp.

$$\begin{array}{r|l} x^2-4 & x-1 \\ \cline{2-2} -x^2+x & x+1 \\ \hline x-4 & \\ -x\pm 1 & \\ \hline -3 & \end{array} \Rightarrow x^2-4=x+1-\underbrace{\frac{3}{x-1}}_{\to 0 \text{ as } x\to\pm\infty}$$

$\Rightarrow y=x+1$ is a oblique (slant) asymp.

- $g'(x)=\frac{2x(x-1)-(x^2-4)}{(x-1)^2}=\frac{x^2-2x+4}{(x-1)^2}=\frac{(x-1)^2+3}{(x-1)^2}$

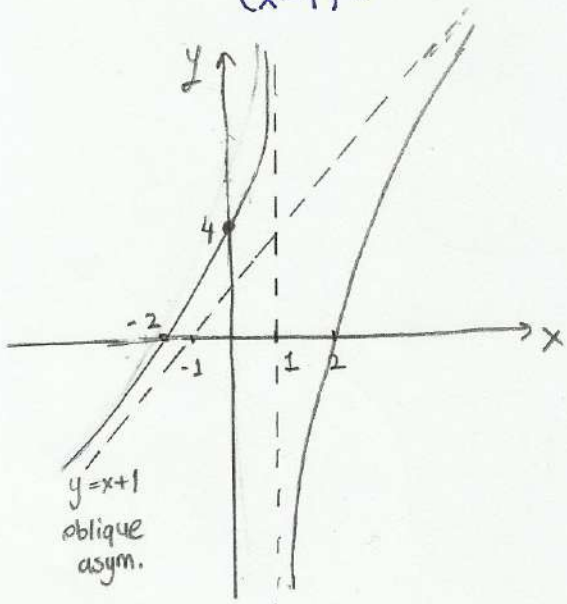
  $g'(x)$ is never zero for real x since $\Delta=(-2)^2-4.1.4=-12<0$.
  $g'(x)=$ undefined for $x=1$.

- $g''(x)=\frac{(2x-2)(x-1)^2-2(x-1)(x^2-2x+4)}{(x-1)^4}=\frac{2(x-1)^2-2(x^2-2x+4)}{(x-1)^3}$

$$=\frac{2x^2-4x+2-2x^2+4x-8}{(x-1)^3}=\frac{-6}{(x-1)^3}$$

$g''(x)$ is undefined for $x=1$.

x		1	
g'	+ ↗	‖	+ ↗
g''	+ up	‖	− down
g	⌒	‖	⌒



Example: Draw the graph of $y = e^{1/x}$, showing all significant features

$y = f(x) = e^{1/x}$

. $\text{dom}(f) = \mathbb{R} - \{0\}$

- There is no x-intercept and y-intercept.

- $\lim_{x \to 0^+} e^{1/x} = +\infty$, $\lim_{x \to 0^-} e^{1/x} = 0$ $\quad x = 0$ is vertical asymp.

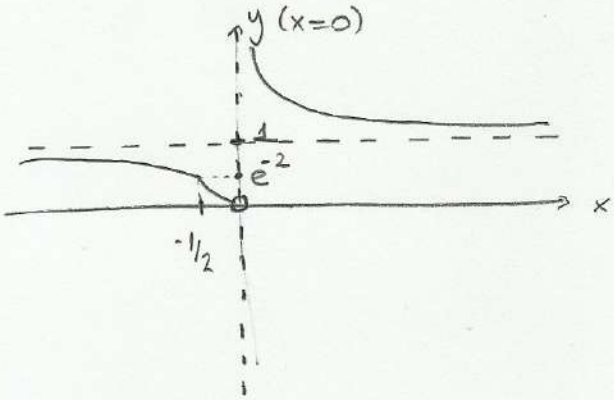
$\lim_{x \to \pm\infty} e^{1/x} = 1$ $\quad y = 1$ is horizontal asymp.

- $f'(x) = -\dfrac{e^{1/x}}{x^2}$ $\quad f'(x) =$ undefined for $x = 0$

- $f''(x) = -\left(\dfrac{-1/x^2 \, e^{1/x} \cdot x^2 - 2x e^{1/x}}{x^4} \right) = \dfrac{e^{1/x}(1+2x)}{x^4}$

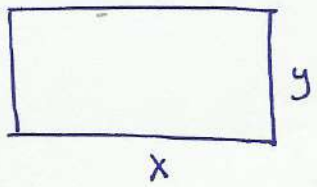
$f''(x) = 0$ for $x = -1/2$, $f''(x) =$ undefined for $x = 0$.

x		$-1/2$		0	
f'	$-$ ↘		$-$ ↘	‖	$-$ ↘
f''	$-$ down	0	$+$ up	‖	$+$ up
f		inflection pt.		‖	



Optimization Problems

Example: You have 40 feet of fencing with which to enclose a rectangular space for a garden. Find the largest area that can be enclosed with this much fencing and the dimensions of the corresponding garden.



$2(x+y)=40 \Rightarrow \boxed{y=20-x}$

Area $= A(x,y) = xy$. Substitute $y=20-x$.

$\Rightarrow A(x) = x(20-x) \quad 0 \le x \le 20$

The problem asks us to find the absolute maximum of $A(x)$ on $[0,20]$. $A'(x) = 20-2x = 0 \Rightarrow x=10$ critical point.

$\left.\begin{matrix} x=0 \\ x=20 \end{matrix}\right)$ end points.

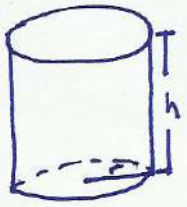
$A(0)=0$

$A(20)=0$

$A(10)=100$ abs. max.

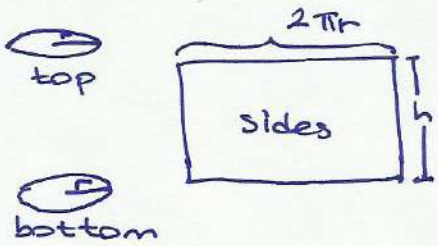
The largest possible area obtained in this way is $100\,ft^2$ and coresponding dimensions or $10ft \times 10ft$.

Example: A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.



r: radius
h: height

In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides).



A: area $\quad A = \text{top} + \text{bottom} + \text{sides}$

$= \pi r^2 + \pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r h$

The volume is given as $1\,L = 1000\,cm^3$. Thus

$\pi r^2 h = 1000 \Rightarrow h = 1000/(\pi r^2)$. So

$Area = A(r) = 2\pi r^2 + \dfrac{2000}{r}, \; r>0 \quad dom(A) = (0,\infty)$

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

$A'(r)=0$ when $\pi r^3 = 500$ so the only critical number is $r = \left(\frac{500}{\pi}\right)^{1/3}$.

$$A'(r) = \frac{4(\pi r^3 - 500)}{r^2}$$

r		$\left(\frac{500}{\pi}\right)^{1/3}$	
A'	$-$	0	$+$
A	$\searrow$	abs. min.	$\nearrow$

The value of h corresponding to $r = \left(\frac{500}{\pi}\right)^{1/3}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \frac{(500)^{2/3}}{\pi^{2/3}}} = \frac{2 \cdot 500}{\pi^{1/3}(500)^{2/3}} = 2\left(\frac{500}{\pi}\right)^{1/3}$$

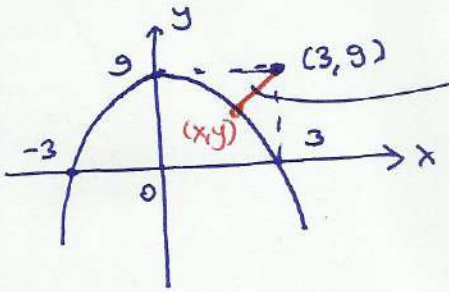
First Derivative Test for Absolute Extreme Values:

Suppose that c is a critical number of a continuous function f defined <u>on an interval</u>

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$ then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example: Find the point on the parabola $y = 9 - x^2$ closest to the point $(3, 9)$.



distance $d(x,y) = \sqrt{(3-x)^2 + (9-y)^2}$

Since $y = 9 - x^2$, we have

$$d(x,y) = d(x) = \sqrt{(3-x)^2 + x^4}.$$

It is enough to maximize $D(x) = (3-x)^2 + x^4$ on $[0,3]$. clearly, the closest point lies in the first quadrant.

$D'(x) = -6+2x+4x^3 = 2(x-1)(2x^2+2x+3) = 0 \Rightarrow \underbrace{x=1}$ critical point.

$\left.\begin{matrix} x=0 \\ x=3 \end{matrix}\right)$ end-points.

$D(0)=9$

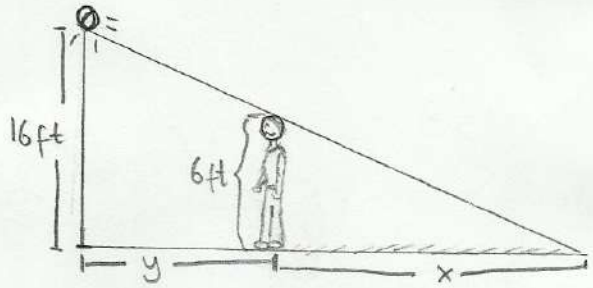
$D(1)=5$ $\Rightarrow$ abs. min. Hence $(1,8)$ is the closest point and the minimum distance is $d(1)=\sqrt{5}$.

$D(3)=81$

Related Rates:

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity.

Example: A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. How fast is the length of his shadow changing?



x: length of shadow at time t

y: distance from the lamp at time t.

$\frac{dy}{dt} = \overset{!}{-}5\text{ft/sec}$ (y is decreasing)

From the triangle we have $\frac{x}{x+y} = \frac{6}{16} \Rightarrow 16x = 6x+6y$

$10x = 6y$

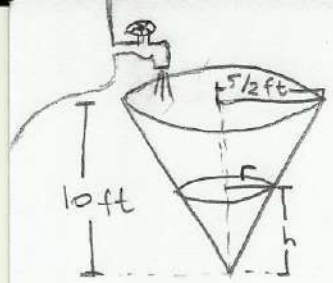
$\Rightarrow \boxed{5x = 3y}$

$5x = 3y$

Take derivative with respect to t.

$5\frac{dx}{dt} = 3\frac{dy}{dt} \Rightarrow \frac{dx}{dt} = \frac{3}{5}\cdot(-5) = -3\text{ ft/sec}$ //

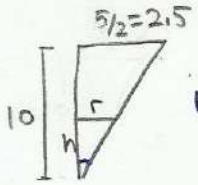
Example: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base of 5 ft. How fast is the water level rising when the water is 6 ft deep?



Volume of a cone $V = \frac{\pi r^2 h}{3}$

It is given that $\frac{dV}{dt} = 9 ft^3/min$.

The question is $\frac{dh}{dt} = ?$ when $h = 6$ ft.



we have $\frac{r}{h} = \frac{2.5}{10} \Rightarrow r = \frac{h}{4}$

$$V(h) = \pi \frac{h^2}{16} \cdot \frac{h}{3} = \frac{V h^3}{48}$$

$$9 = \frac{dV}{dt} = \frac{\pi}{48} 3h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{144}{\pi h^2}$$

when $h = 6$ ft $\quad \frac{dh}{dt} = \frac{144}{\pi 36} = \frac{4}{\pi}$ ft/min

Antiderivatives

we call F an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example: i) Find an antiderivative of $f(x) = \sin x$.

$F'(x) = \sin x \qquad F(x) = -\cos x$ since $F'(x) = -\frac{d}{dx}\cos x = \sin x$

ii) Find an antiderivative of $f(x) = x^3$.

$F'(x) = x^3 \qquad F(x) = \frac{x^4}{4}$ since $F'(x) = \frac{1}{4}\frac{d}{dx}x^4 = \frac{1}{4} \cdot 4x^3 = x^3$.

Theorem: Suppose that F and G are both antiderivatives of f on an interval I. Then $G(x) = F(x) + c$ for some constant c.

Definition: Let F be any antiderivative of f. The indefinite integral of $f(x)$ is defined by

$$\int \underbrace{f(x)}_{\text{integrand}} dx = F(x) + C$$

integration constant (it is arbitrary)

dx identifies x as the variable of integration.

Example. 1) $\int 3x^2 dx = x^3 + c$ since $\frac{d}{dx} x^3 = 3x^2$

2) $\int t^7 dt = \frac{t^8}{8} + c$ since $\frac{d}{dt}\left(\frac{t^8}{8}\right) = t^7$.

Power Rule: For any $\boxed{r \neq -1}$!! $\int x^r dx = \frac{x^{r+1}}{r+1} + c$

Example: 1) $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} = -\frac{1}{2x^2}$

2) $\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{x^{1/2+1}}{1/2+1} = \frac{2x^{3/2}}{3}$

By reversing any derivative formula we get a corresponding integration formula. Examples:

1) $\int x^r dx = \frac{x^{r+1}}{r+1} + c, r \neq -1$

2) $\int \cos x dx = \sin x + c$

3) $\int \sec x \tan x dx = \sec x + c$

4) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

5) $\int e^{-x} dx = -e^{-x} + c$

6) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

7) $\int \sin x dx = -\cos x + c$

8) $\int \sec^2 x dx = \tan x + c$

9) $\int \operatorname{cosec}^2 x dx = -\cot x + c$

10) $\int e^x dx = e^x + c$

11) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

12) $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c$

Theorem: Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then, for any constants, a and b

$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx.$$

Example: Evaluate the following integrals.

1) $\int (5\cos x + 4x^9) dx = 5\int \cos x dx + 4\int x^9 dx$

$= 5\sin x + 4 \cdot \frac{x^{10}}{10} + c$

2) $\int (u-6)(u^2+5)\,du \overset{!}{\neq} \left(\int (u-6)\,du\right)\left(\int (u^2+5)\,du\right)$

$$\int (u-6)(u^2+5)\,du = \int (u^3+5u-6u^2-30)\,du = \frac{u^4}{4}+\frac{5u^2}{2}-\frac{6u^3}{3}-30u+c$$

3) $\int \left(7e^x - \frac{2}{1+x^2}\right)dx = 7\int e^x\,dx - 2\int \frac{1}{1+x^2}\,dx$

$$= 7e^x - 2\tan^{-1}x + c$$

4) $\int t x^3\,dt = x^3 \frac{t^2}{2}$

Theorem: For $x \neq 0$, $\frac{d}{dx}\ln|x| = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \ln|x| + c$

More generally, if $f(x) \neq 0$ and f is differentiable, we have

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)} \Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

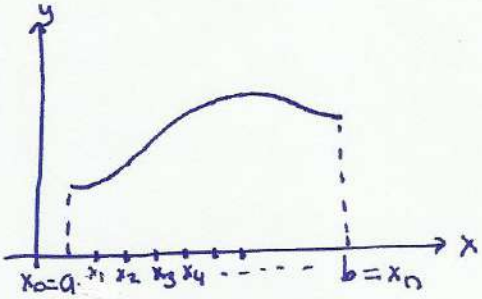
Example: $\int \frac{\sec^2 x}{\tan x}\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|\tan x| + c$

$f(x) = \tan x$

Area

Problem: Estimate the area below the graph of $y=f(x)$ and above the x-axis for $a \le x \le b$.

For now, assume that $f(x) \geq 0$ and f is continuous on $[a,b]$.



Divide the interval $[a,b]$ into n equal pieces. (regular partition of $[a,b]$)

The width of each subinterval in the partition is then $\frac{b-a}{n} = \Delta x$

$x_1 = x_0 + \Delta x$

$x_2 = x_1 + \Delta x$

$\vdots$

$x_i = x_{i-1} + \Delta x = x_0 + i\Delta x$, $i = 1, 2, \ldots, n$

On each subinterval $[x_{i-1}, x_i]$, construct a rectangle of height $f(x_i)$. The area under the curve A is roughly the same as the sum of the areas of these n rectangles.

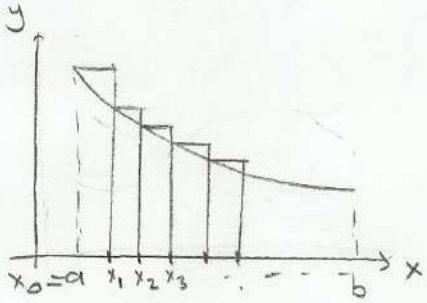
$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x = A_n$$

Indeed, we can pick any points $c_1, c_2, \ldots, c_n$ where $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. The sum $\sum_{i=1}^{n} f(\underbrace{c_i}_{\text{evaluation points}})\Delta x$ is called Riemann sum.

For a continuous, nonnegative function f, the area under the curve $y = f(x)$ is the limit of the Riemann sums:

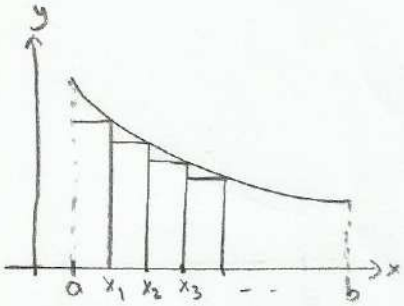
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} A_n$$

For any continuous function f, the above limit is the same for any choice of $c_i \in [x_{i-1}, x_i]$. The most common choices for the evaluation points c_i are x_i (the right end point), x_{i-1} (the left endpoint), and $\frac{1}{2}(x_{i-1} + x_i)$ (the midpoint)



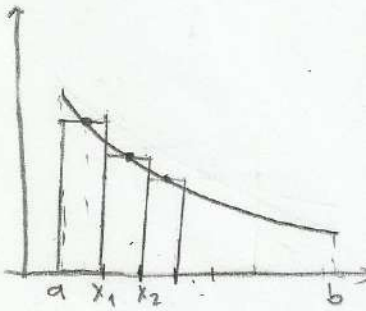
$c_i = x_{i-1}$ (left endpoint)

$U = \sum_{i=1}^{n} f(c_i)\Delta x$ (upper Riemann sum)



$c_i = x_i$ (right endpoint)

$L = \sum_{i=1}^{n} f(c_i)\Delta x$ (lower Riemann sum)



$c_i = \frac{1}{2}(x_{i-1} + x_i)$ (the mid point)

Example: Find the area under the curve $y = f(x) = 2x - 2x^2$ on the interval $[0, 1]$.

Consider the partition of $[0,1]$ into n subintervals.

$\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$.

$$\Rightarrow A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\left(2\left(\frac{i}{n}\right) - 2\left(\frac{i}{n}\right)^2\right]\cdot\frac{1}{n}$$

$$= \frac{2}{n^2}\underbrace{\sum_{i=1}^{n} i} - \frac{2}{n^3}\underbrace{\sum_{i=1}^{n} i^2}$$

$$= \frac{2}{n^2}\cdot\frac{n(n+1)}{2} - \frac{2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)}{n} - \frac{(n+1)(2n+1)}{3n^2}$$

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{n+1}{n} - \lim_{n\to\infty}\frac{2n^2+3n+1}{3n^2} = 1-\frac{2}{3} = \frac{1}{3} = A \quad //$$

The Definite Integral

For any function f defined on $[a,b]$, the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x,$$

whenever the limit exists and is the same for any choice of evaluation points $c_1, c_2, \ldots, c_n$. When the limit exists, we say that f is integrable on $[a,b]$.

• Remark on the next page.

Theorem: If f is continuous on $[a,b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$, that is the definite integral $\int_a^b f(x)\,dx$ exists.

Theorem: If f and g are integrable on $[a,b]$, then the following are true:

a) For any constants c and d,

$$\int_a^b [cf(x) + dg(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$$

b) For any c in $[a,b]$,

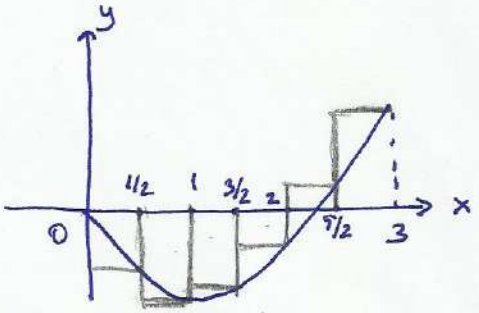
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

c) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ d) $\int_a^a f(x)\,dx = 0$ (if $f(a)$ is defined)

Remark: Suppose that $f(x) \geq 0$ on the interval $[a,b]$ and A_1 is the area bounded between $y=f(x)$ and the x-axis for $a \leq x \leq b$. Further, suppose that $f(x) \leq 0$ on the interval $[b,c]$ and A_2 is the bounded area between the curve $y=f(x)$ and the x-axis for $b \leq x \leq c$. The signed area between $y=f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 - A_2$, and the total area between $y=f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 + A_2$.

Example: i) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the evaluation points to be right endpoints and $a=0, b=3$, and $n=6$.

ii) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.



i) With $n=6$, the interval width is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

$$x_1 = 1/2,\ x_2 = 1,\ x_3 = 1 + \frac{1}{2} = \frac{3}{2},\ x_4 = \frac{3}{2} + \frac{1}{2} = 2,\ x_5 = 2 + \frac{1}{2} = \frac{5}{2},\ x_6 = 3$$

So the Riemann sum is

$$A_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x = f(\tfrac{1}{2}) \cdot \tfrac{1}{2} + f(1) \cdot \tfrac{1}{2} + f(\tfrac{3}{2}) \cdot \tfrac{1}{2} + f(2) \cdot \tfrac{1}{2} + f(\tfrac{5}{2}) \cdot \tfrac{1}{2} + f(3) \cdot \tfrac{1}{2}$$

$$= -3.9375$$

* Note that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles.

ii) We can evaluate $\int_0^3 (x^3 - 6x)\,dx$ by using the Riemann sum with n-subintervals (Exercise) or

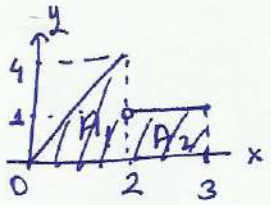
$$\int_0^3 (x^3 - 6x)\,dx = \left(\frac{x^4}{4} - \frac{6x^2}{2}\right)\Bigg|_0^3 = \frac{81}{4} - 27 = -6.75 = -\frac{81}{4}$$

This comes from the fundamental theorem of calculus (Part I).* (FTC1)

If f is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ then $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

Example: Evaluate $\int_0^3 f(x)dx$ where $f(x)$ is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \le 2 \\ 1, & \text{if } x > 2 \end{cases}.$$



$$\int_0^3 f(x)dx = A_1 + A_2 = \frac{2.4}{2} + 1.1 = 5$$

or $$\int_0^3 f(x)dx = \int_0^2 2x\,dx + \int_2^3 1.dx = x^2\Big|_0^2 + x\Big|_2^3 = (4-0) + (3-2) = 5.$$

Theorem: Suppose that $g(x) \le f(x)$ for all $x \in [a,b]$ and that f and g are integrable on $[a,b]$. Then,

$$\int_a^b g(x)dx \le \int_a^b f(x)\,dx.$$

Definition: The average value of a function f on the interval $[a,b]$ is defined by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example: Find the average value of the function $f(x) = x^2 - 2x$ on $[0,2]$.

$$f_{ave} = \frac{1}{2-0}\int_0^2 (x^2-2x)dx = \frac{\left(\frac{x^3}{3} - \frac{2.x^2}{2}\right)\Big|_0^2}{2} = \frac{\frac{8}{3} - 4}{2} = -\frac{2}{3}$$

Theorem: If M and m are the maximum and minimum values of f on $[a,b]$, respectively, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

or $$m \le \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{f_{ave}} \le M$$

Example: Use above theorem to estimate $\int_0^1 e^{-x^2}dx$.

$f(x)=e^{-x^2}$ is a decreasing function on $[0,1]$ since $f'(x)=-2xe^{-x^2}\leq 0$ on $[0,1]$. Its absolute maximum value is $M=f(0)=1$ and its absolute minimum value is $m=f(1)=e^{-1}$.

Thus by the theorem

$$e^{-1}(1-0)\leq \int_0^1 e^{-x^2}dx \leq 1.(1-0)$$

$$\Rightarrow e^{-1}\leq \int_0^1 e^{-x^2}dx \leq 1$$

Example: Show that $\frac{1}{10}\leq \int_2^3 \frac{1}{x^2+1}dx \leq \frac{1}{5}$.

$f(x)=\frac{1}{x^2+1}$ is a decreasing function on $[2,3]$.

Its absolute maximum value is $M=\underbrace{\frac{1}{2^2+1}}_{f(2)}=\frac{1}{5}$ and its absolute minimum value is $m=\underbrace{\frac{1}{3^2+1}}_{f(3)}=\frac{1}{10}$. Thus by the theorem

$$m(3-2)=\frac{1}{10}\leq \int_2^3 \frac{1}{x^2+1}dx \leq \frac{1}{5}=M(3-2)$$

Theorem: (Integral Mean Value Theorem) If f is continuous on $[a,b]$, then there is a number $c\in(a,b)$ for which

$$f(c)=\frac{1}{b-a}\int_a^b f(x)dx.$$

The Fundamental Theorem of Calculus (Part II) (FTC2) If f is continuous on $[a,b]$ and $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$ on $[a,b]$.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)=f(x).$$

Remark: The general form of the Fundamental Theorem of Calculus is: If

$$F(x)=\int_{\alpha(x)}^{\beta(x)} f(t)\,dt \quad \text{then} \quad F'(x)=f(\beta(x))\beta'(x)-f(\alpha(x))\alpha'(x).$$

Example: Compute $F'(x)$ if

1) $F(x)=\int_2^x \frac{t^{5/2}}{\sqrt{t^2+41}}\,dt$. By FTC2, $F'(x)=\frac{x^{5/2}}{\sqrt{x^2+41}}$

2) $F(x)=\int_2^{x^2}\sqrt{t^2+1}\,dt$ $\quad \sqrt{(x^2)^2+1}\cdot 2x = 2x\sqrt{x^4+1} = F'(x)=\frac{d}{dx}\left(\int_2^{x^2}\sqrt{t^2+1}\,dt\right)$ by FTC2

3) $F(x)=\int_{\sin x}^{x^2}(1+t)\,dt$, By FTC2, $F'(x)=\frac{d}{dx}\left(\int_{\sin x}^{x^2}(1+t)\,dt\right)$

$$=(1+x^2)\cdot 2x-(1+\sin x)\cos x$$

Example: For the function $F(x)=\int_4^{x^2}\ln(t^3+4)\,dt$ find an equation of the tangent line at $x=2$.

$$F'(x)=\frac{d}{dx}\left[\int_4^{x^2}\ln(t^3+4)\,dt\right]=\ln(x^6+4)\,2x$$

$F'(2)=4\ln(68)$

$F(2)=\int_4^4 \ln(t^3+4)\,dt=0$

equation of the tangent line $y=F'(2)(x-2)+F(2)$

$$y=4\ln(68)(x-2)$$

Example. Find

1) $\displaystyle\int_0^{\pi/2}\left[\frac{d}{dx}(\sin^5 x)\right]dx = \sin^5 x\Big|_0^{\pi/2} = 1$

2) $\displaystyle\frac{d}{dx}\Big(\underbrace{\int_a^c f(x)\,dx}_{\text{constant}}\Big) = 0$

3) $\displaystyle\frac{d}{da}\Big(\int_a^b f(x)dx\Big) = -\frac{d}{da}\Big(\int_b^a f(x)\,dx\Big) = -f(a)$ by FTC2

4) $\displaystyle\frac{d}{dt}\Big(\underbrace{\int_0^1 (2+\sec t)^{17}\,dt}_{\text{constant}}\Big) = 0.$

Integration Techniques

- Integration By Substitution

1) $\displaystyle\int (5x+1)^{1/2}dx = \int u^{1/2}\cdot\frac{1}{5}du = \frac{1}{5}\,\frac{u^{1/2+1}}{1/2+1} + c = \frac{2}{15}u^{3/2} + c$

$5x+1 = u$
$5dx = du$
$\Rightarrow dx = \frac{1}{5}du$

$\displaystyle = \frac{2}{15}(5x+1)^{3/2} + c$

2) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \int -\frac{du}{u} = -\ln|u| + c = -\ln|\cos x| + c$

$\displaystyle = \ln|\sec x| + c$

$\cos x = u$
$-\sin x\,dx = du$

3) $\displaystyle\int_0^a \frac{dx}{a^2+x^2} = \int_0^a \frac{dx}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)} = \int_0^1 \frac{a\,du}{a^2(1+u^2)} = \frac{1}{a}\Big[\arctan u\Big]_0^1$

$\displaystyle = \frac{1}{a}\left(\arctan 1 - \arctan 0\right)$

$\displaystyle = \frac{1}{a}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4a}$

a: constant

$\frac{x}{a} = u \Rightarrow$ $x=a \Rightarrow u=1$, $x=0 \Rightarrow u=0$

$dx = a\,du$

You also have to change the limits of the integration

4) $\displaystyle\int \frac{e^{4/x}}{x^2}dx = -\int e^u\frac{du}{4} = -\frac{1}{4}e^u + c = -\frac{1}{4}e^{4/x} + c$

$\frac{4}{x} = u$
$-\frac{4}{x^2}dx = du$

5) $\int \frac{18\tan^2 x \sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{6\,du}{u^2} = -6 \cdot \frac{1}{u} + c$

$= \frac{-6}{2+\tan^3 x} + c$

$2+\tan^3 x = u$

$3\tan^2 x \sec^2 x\,dx = du$

6) $\int 4\cos^2(3\theta)\,d\theta = 4\int \frac{\cos(6\theta)+1}{2}\,d\theta = 2\int (\cos u)\frac{du}{6} + 2\int d\theta$

$\cos(2x) = 2\cos^2 x - 1$

$6\theta = u$

$6\,d\theta = du$

$= \frac{\sin u}{3} + 2\theta + c$

$\cos(6\theta) = 2\cos^2(3\theta) - 1$

$= \frac{\sin(6\theta)}{3} + 2\theta + c$

$\Rightarrow \cos^2(3\theta) = \frac{\cos(6\theta)+1}{2}$

7) $\int \frac{(\tan^{-1}x)^2}{1+x^2}\,dx = \int u^2\,du = \frac{u^3}{3} + c = \frac{(\tan^{-1}x)^3}{3} + c$

$\tan^{-1}x = u$

$\frac{1}{1+x^2}\,dx = du$

8) $\int \sec x\,dx = ?$ $\quad \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{du}{u}$

$= \ln|u| + c$

$\sec x + \tan x = u$

$(\sec x \tan x + \sec^2 x)\,dx = du$ $\quad = \ln|\sec x + \tan x| + c$

9) $\int \sin x \cos x\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{\sin^2 x}{2} + c$

$u = \sin x$

$du = \cos x\,dx$

10) $\int_0^{\sqrt{\pi}/2} x\sin^3(x^2)\cos(x^2)\,dx = \int_0^{\sqrt{2}/2} u^3\,\frac{du}{2} = \frac{u^4}{8}\Big|_0^{\sqrt{2}/2}$

$\sin(x^2) = u$

$2x\cos(x^2)\,dx = du$

$= \frac{1}{4 \cdot 8} = \frac{1}{32}$

limits of integration

$x = 0 \Rightarrow u = \sin(0) = 0$

$x = \frac{\sqrt{\pi}}{2} \Rightarrow u = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

11) $\displaystyle\int_0^1 \frac{x+1}{(x^2+2x+6)^2}\,dx = \int_6^9 \frac{du}{2u^2} = \frac{-1}{2u}\Big|_6^9 = -\frac{1}{2}\left(\frac{1}{9}-\frac{1}{6}\right) = \frac{1}{36}$

$x^2+2x+6 = u$ $\quad$ $x=0 \Rightarrow u=6$

$x=1 \Rightarrow u=9$

$(2x+2)\,dx = du$

$\Rightarrow (x+1)\,dx = du/2$

• Integration of rational functions using partial fractions

To write a rational function $\frac{P(x)}{Q(x)}$ as a sum of partial fractions:

(1) The degree of $P(x)$ must be less than the degree of $Q(x)$

(2) Factor out the Polynomial $Q(x)$.

- For each linear factor, assign the sum of m partial fractions to this factor where m is the exponent of that linear factor as follows:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

- For each irreducible quadratic factor, assign the sum of n partial fractions to this factor where n is the exponent of that quadratic factor as

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}$$

Examples: 1) $\displaystyle\int \frac{x-1}{x^2+x}\,dx = ?$

$$\frac{x-1}{x^2+x} = \frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow Ax + A + Bx = x-1$$

$$(A+B)x + A = x-1$$

$$A+B=1 \quad A=-1 \Rightarrow B=2$$

$$\int \frac{x-1}{x^2+x}\,dx = \int\left(\frac{-1}{x} + \frac{2}{x+1}\right)dx = -\ln|x| + 2\ln|x+1| + c$$

$$= \ln\left|\frac{(x+1)^2}{x}\right| + c$$

2) $\int \frac{2x-3}{x^3+x}\,dx = ?$

$$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$ (x^2+1 is an irreducible polynomial)

$$Ax^2+A+Bx^2+Cx = 2x-3 \Rightarrow A+B=0$$
$$C=2$$
$$A=-3 \Rightarrow B=3$$

$$\int \frac{2x-3}{x^3+x}\,dx = \int \frac{-3}{x}\,dx + \int \frac{3x+2}{x^2+1}\,dx = \int -\frac{3}{x}\,dx + \int \frac{3x\,dx}{x^2+1} + \int \frac{2}{x^2+1}\,dx$$

$x^2+1=u$
$2x\,dx = du$

$$= -3\ln|x| + \frac{3}{2}\int \frac{du}{u} + 2\arctan x + c$$

$$= -3\ln|x| + \frac{3}{2}\ln|x^2+1| + 2\arctan x + c$$

3) $\int \frac{x-1}{(x+1)^3}\,dx = ?$

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \Rightarrow A(x+1)^2 + B(x+1) + C = x-1$$

$A=0$
$2A+B=1 \Rightarrow B=1$
$B+C=-1 \Rightarrow C=-2$

$$\int \frac{x-1}{(x+1)^3}\,dx = \int \frac{1}{(x+1)^2}\,dx + \int \frac{-2\,dx}{(x+1)^3}$$

$$= -\frac{1}{x+1} + \frac{1}{(x+1)^2} + c$$

4) $\int \frac{4x^3-3x+5}{x^2-2x}\,dx = ?$

$$\begin{array}{l|l} 4x^3-3x+5 & x^2-2x \\ \cline{2-2} -4x^3 \pm 8x^2 & 4x+8 \\ \hline 8x^2-3x+5 & \\ -\ 8x^2 \pm 16x & \\ \hline 13x+5 & \end{array}$$

$$\int \frac{4x^3-3x+5}{x^2-2x}\,dx = \int \left(4x+8 + \frac{13x+5}{x^2-2x}\right)dx = 2x^2+8x + \int \frac{13x+5}{x^2-2x}\,dx.$$

$$\frac{13x+5}{x^2-2x} = \frac{13x+5}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$Ax - 2A + Bx = 13x + 5$

$A + B = 13$, $A = -5/2$

$B = 31/2$

$$\int \frac{4x^3-3x+5}{x^2-2x}\,dx = 2x^2+8x + \int \frac{-5/2}{x}\,dx + \int \frac{31/2}{x-2}\,dx$$

$$= 2x^2+8x - \frac{5}{2}\ln|x| + \frac{31}{2}\ln|x-2| + c.$$

- Integration By Parts

Remember that if u and v are differentiable functions of x, then u.v is also differentiable and

$$\frac{d}{dx}(u.v) = u\frac{d}{dx}v + v\frac{d}{dx}u \quad \text{or} \quad d(uv) = u\,dv + v\,du$$

Take the integral of both sides, we have $\int d(uv) = \int u\,dv + \int v\,du$.

Note that $\int d(uv) = uv$. Therefore,

$$uv = \int u\,dv + \int v\,du \Rightarrow \boxed{\int u\,dv = uv - \int v\,du.}$$

Examples: 1) $\int \underbrace{x}_{u}\,\underbrace{e^{2x}\,dx}_{dv} = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$

$x = u \qquad e^{2x}\,dx = dv$

$dx = du \qquad \frac{e^{2x}}{2} = v$

2) $\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = x\ln x - \int dx = x\ln x - x + c$

$\ln x = u \qquad dx = dv$

$\frac{1}{x}dx = du \qquad x = v$

3) $\int \underbrace{\arcsin(3x)}_{u}\,\underbrace{dx}_{dv} = x\arcsin(3x) - \int \frac{3x}{\sqrt{1-9x^2}}\,dx$

$1-9x^2 = t$

$-18x\,dx = du$

$\arcsin(3x) = u \qquad dx = dv \qquad = x\arcsin(3x) + \frac{1}{6}\int \frac{du}{\sqrt{u}}$

$\frac{3}{\sqrt{1-(3x)^2}}dx = du \qquad x = v \qquad = x\arcsin(3x) + \frac{1}{3}\sqrt{\arcsin(3x)} + c$

3) $\int e^x \sin x\,dx = ?$

$$I = \int \underbrace{e^x}_{u} \underbrace{\sin x\,dx}_{dv} = -\cos x\, e^x + \int \underbrace{e^x}_{u} \underbrace{\cos x\,dx}_{dv}$$

$e^x = u \quad \cos x\,dx = dv$
$e^x dx = du \quad \sin x = v$

$e^x = u \quad \sin x\,dx = dv$
$e^x dx = du \quad -\cos x = v$

$$= -\cos x\, e^x + \sin x\, e^x - \underbrace{\int e^x \sin x\,dx}_{I} + c$$

$$\Rightarrow 2I = -\cos x\, e^x + \sin x\, e^x + c \Rightarrow I = \int e^x \sin x\,dx = \frac{1}{2}\left(-\cos x\, e^x + \sin x\, e^x\right)$$

4) $\int_1^2 x^3 \ln x\,dx = ?$

- Trigonometric Integrals

$$\int \sin^m x \cos^n x\,dx \quad , m,n \geq 0.$$

1) <u>m is odd</u> , $m = 2k+1$.

$$\sin^m x = \sin^{2k+1}(x) = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x,$$

$$\int \sin^m x \cos^n x\,dx = \int (1-\cos^2 x)^k \cos^n x \sin x\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\int (1-u^2)^k u^n\,du$$

2) <u>m is even and n is odd</u>: Write $n = 2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ and repeat what is done in case 1) changing the places of sine and cosine.

3) <u>m is even and n is even</u>: Substitute

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad , \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Examples: 1) $\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cos x\,dx$

$m = 2$ even
$n = 3$ odd

$$= \int \sin^2 x (1-\sin^2 x) \cos x\,dx$$

$\sin x = u$
$\cos x\,dx = du$

$$= \int u^2 (1-u^2)\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

2) $\int \sin^5 2x \, dx = \int \sin^4 2x \sin 2x \, dx$

$$= \int (1-\cos^2 2x)^2 \sin 2x \, dx$$

$u = \cos 2x$

$du = -2\sin 2x \, dx$

$$= \int (1-u^2)^2 \left(-\frac{du}{2}\right) = -\frac{1}{2}\int (1-2u^2+u^4)\, du$$

$$= -\frac{1}{2}\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + c$$

$$= -\frac{1}{2}\left[\cos 2x - \frac{2\cos^3 2x}{3} + \frac{1}{5}\cos^5 2x\right] + c$$

Products of Sines and Cosines

$\int \sin(mx)\sin(nx)\,dx$, $\int \sin(mx)\cos(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$

One can use the identities:

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin(m-n)x + \cos(m+n)x\right]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right].$$

- Trigonometric Substitution

This method is for transforming complicated integrals involving a^2-x^2, a^2+x^2, and x^2-a^2.

For a^2+x^2, use $x = a\tan\theta$

For a^2-x^2, use $x = a\sin\theta$

For x^2-a^2, use $x = a\sec\theta$.

Note: We want any substitution to be reversible so that we can go back to original variable. We should remember that the functions in these substitutions have inverses only for selected values of θ.

$x = a\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$, $-\pi/2 < \theta < \pi/2$

$x = a\sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$, $-\pi/2 \le \theta \le \pi/2$

$x = a\sec\theta \Rightarrow \theta = \sec^{-1}(x/a)$, $0 \le \theta < \pi/2$ if $x/a \ge 1$
$\pi/2 < \theta \le \pi$ if $x/a \le -1$

Examples: 1) $\displaystyle\int \frac{dx}{\sqrt{9+x^2}} = ?$

$x = 3\tan\theta \,,\ -\pi/2 < \theta < \pi/2$

$dx = 3\sec^2\theta\, d\theta$

$9 + x^2 = 9 + 9\tan^2\theta = 9(1+\tan^2\theta)$
$= 9\sec^2\theta$

$$\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9\sec^2\theta}}$$

$$= \int \frac{3\sec^2\theta}{|3\sec\theta|}\, d\theta$$

$$\underset{-\pi/2<\theta<\pi/2}{=} \int \frac{\sec^2\theta}{\sec\theta}\, d\theta$$

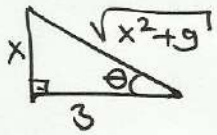
$$= \int \sec\theta \frac{(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\, d\theta = \int \frac{du}{u} = \ln|\tan\theta + \sec\theta| + c$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2+9}}{3}\right| + c$$

$u = \sec\theta + \tan\theta$

$du = \sec\theta\tan\theta + \sec^2\theta$

$\tan\theta = \frac{x}{3}$



$\Rightarrow \sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{x^2+9}}{3}$

2) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\, dx = ?$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

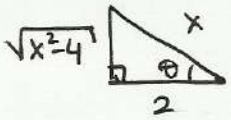
$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta$

$$\int \frac{dx}{x^2\sqrt{x^2-4}} \overset{*}{=} \int \frac{2\sec\theta\tan\theta\, d\theta}{4\sec^2\theta \cdot 2\tan\theta} = \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta + c$$

$\sec\theta = \frac{x}{2} \Rightarrow \cos\theta = \frac{2}{x}$

$$= \frac{1}{4}\frac{\sqrt{x^2-4}}{x} + c$$



$\Rightarrow \sin\theta = \frac{\sqrt{x^2-4}}{x}$

* To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $\frac{x}{a} \geq 1$. This will place θ in $[0, \frac{\pi}{2})$ and make $\tan\theta \geq 0$. We will then have $\sqrt{x^2-a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, provided that $a > 0$.

3) $\int \frac{1}{\sqrt{2x-x^2}}dx = ?$

$$\sqrt{2x-x^2} = \sqrt{-(x^2-2x)}$$
$$= \sqrt{-(x-1)^2+1}$$
$$= \sqrt{1-(x-1)^2}$$

Let $x-1=u \Rightarrow dx=du$

$$\int \frac{du}{\sqrt{1-u^2}} = \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} = \int d\theta = \theta + c = \sin^{-1}(u) + c$$
$$= \sin^{-1}(x-1)+c.$$

$u=\sin\theta$, $\theta \in [-\pi/2, \pi/2]$ $\qquad u=\sin\theta$

$du = \cos\theta d\theta$ $\qquad \Rightarrow \theta = \sin^{-1} u$

Improper Integrals

When we study the definite integrals, we require two things: the domain of integration is finite and the function is finite on this domain.

If the integration interval is infinite or if the integrand has an infinite discontinuity in this interval then the integral is called an improper integral.

If the limits exist we evaluate them with the following definitions:

(1) If f is continuous on $[a,\infty)$, then
$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

(2) If f is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

Improper integral of type I.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called <u>convergent</u> if the corresponding limit exists and is a finite number, and <u>divergent</u> if the limit does not exist or is not a finite number.

If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^\infty f(x)dx \quad \left(\text{any real number } a \text{ can be used}\right).$$

3) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx.$$

if this limit exists (as a finite number).

4) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$$

if this limit exists (as a finite number).

improper integral of type II

- If f has a discontinuity at α where $a<\alpha<b$, and both $\int_a^\alpha f(x)dx$ and $\int_\alpha^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^\alpha f(x)dx + \int_\alpha^b f(x)dx.$$

Examples: Determine whether the following integrals are convergent or divergent.

1) $$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left[-\ln x\cdot\frac{1}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}dx\right]$$

improper integral of type I

$u=\ln x \quad dv=\frac{dx}{x^2}$

$du=\frac{1}{x}dx \quad v=-\frac{1}{x}$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \overset{\to 0}{\frac{1}{b}} + 1\right]$$

$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b}\right)+1 \overset{L.R}{\underset{\frac{\infty}{\infty}}{=}} \lim_{b\to\infty}\frac{-1/b}{1}+1$$

$$=0+1=1$$

Hence the integral converges to 1.

2) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \tan^{-1}x \Big|_{a}^{0} = \lim_{a\to-\infty} (\tan^{-1}(0) - \tan^{-1}(a))$$

$$= 0 - (-\pi/2) = \pi/2 .$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1}x \Big|_{0}^{b} = \lim_{b\to\infty} (\tan^{-1}(b) - \tan^{-1}(0))$$

$$= \pi/2 - 0 = \pi/2 .$$

Hence $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi/2 + \pi/2 = \pi$. $\boxed{\text{Convergent}}$

3) $$\int_{0}^{\infty} \sin x \, dx = \lim_{b\to\infty} \int_{0}^{b} \sin x \, dx = \lim_{b\to\infty} -\cos x \Big|_{0}^{b} = \lim_{b\to\infty} (1-\cos b)$$

limit does not exist.

The integral is divergent.

4) $$\int_{-\infty}^{\infty} x \, dx = \int_{-\infty}^{0} x \, dx + \int_{0}^{\infty} x \, dx$$

$$\int_{0}^{\infty} x dx = \lim_{b\to\infty} \int_{0}^{b} x dx = \lim_{b\to\infty} \frac{b^2}{2} = +\infty \quad \boxed{\text{divergent}}$$

$$\int_{-\infty}^{0} x dx = \lim_{a\to-\infty} \int_{a}^{0} x dx = \lim_{a\to-\infty} -\frac{a^2}{2} = -\infty \quad \boxed{\text{divergent}}$$

divergent + divergent $\Downarrow$ Hence $\int_{-\infty}^{\infty} x dx$ is also divergent.

5) For what values of p is the integral $\int_{1}^{\infty} \frac{1}{x^p} dx$ convergent?

If $p=1$, then $$\int_{1}^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x} dx = \lim_{b\to\infty} \ln|x| \Big|_{1}^{b} = \lim_{b\to\infty} \ln(b) - 0$$

$$= +\infty \quad \boxed{\text{divergent}}$$

If $p \neq 1$ $\displaystyle\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b$

$$= \lim_{b\to\infty} \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \begin{cases} \dfrac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1 \end{cases}$$

Hence $\displaystyle\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p>1 \quad \boxed{\text{convergent}} \\ \infty, & p\le 1 \quad \boxed{\text{divergent}} \end{cases}$

6) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{c\to 1^-} \int_0^c \frac{dx}{\sqrt{1-x^2}} = \lim_{c\to 1^-} \arcsin x \Big|_0^c$

improper integral of type II

$f(x) = \dfrac{1}{\sqrt{1-x^2}}$ is continuous on $[0,1)$, but it is NOT continuous at $x=1$.

$$= \lim_{c\to 1^-} \left(\arcsin(c) - \arcsin(0)\right)$$

$$= \pi/2 - 0 = \pi/2. \quad \boxed{\text{convergent}}$$

7) $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^-} \int_0^c \frac{dx}{(x-1)^{2/3}} + \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$

$$= 3 + 3\sqrt[3]{2} \quad \boxed{\text{convergent}}$$

$f(x) = \dfrac{1}{(x-1)^{2/3}}$ is continuous on $[0,1)\cup(1,3]$ but it is discontinuous at $x=1$.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^-} \int_0^c \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^-} 3(x-1)^{1/3}\Big|_0^c = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} 3(x-1)^{1/3}\Big|_c^3 = 3\sqrt[3]{2}$$

8) For what values of p is the integral $\int_a^b \frac{dx}{(b-x)^p}$ convergent?

- If $p=1$, $\int_a^b \frac{dx}{b-x} = \lim_{c\to b^-} \int_a^c \frac{dx}{b-x} = \lim_{c\to b^-} -\ln|b-x| \Big|_a^c$

$f(x)=\frac{1}{b-x}$ is not continuous at $x=b$.

$$= \lim_{c\to b^-} -\ln|b-c| + \ln|b-a|$$

$$= +\infty$$

- If $p\neq 1$, $\int_a^b \frac{dx}{(b-x)^p} = \lim_{c\to b^-} \int_a^c \frac{dx}{(b-x)^p} = \lim_{c\to b^-} \frac{(b-x)^{-p+1}}{p-1} \Big|_a^c$

$f(x)=\frac{1}{(b-x)^p}$ is not continuous at $x=b$ if $p>1$

$$= \lim_{c\to b^-} \frac{(b-c)^{-p+1}}{p-1} - \frac{(b-a)^{-p+1}}{p-1}$$

$$= \begin{cases} \frac{(b-a)^{-p+1}}{1-p} & \text{if } p<1 \\ +\infty & \text{if } p>1 \end{cases}$$

Hence $\int_a^b \frac{dx}{(b-x)^p} = \begin{cases} \frac{(b-a)^{-p+1}}{1-p} & \text{if } p<1 \quad \boxed{\text{convergent}} \\ +\infty & \text{if } p\geq 1 \quad \boxed{\text{divergent}} \end{cases}$

Tests for convergence and divergence

Direct Comparison Test: Let f and g be continuous on $[a,\infty)$ and suppose that $\underline{\underline{0\leq f(x)\leq g(x)}}$ for all $x\geq a$.

Then (1) $\int_a^\infty g(x)dx$ converges $\Rightarrow \int_a^\infty f(x)dx$ converges.

(2) $\int_a^\infty f(x)dx$ diverges $\Rightarrow \int_a^\infty g(x)dx$ diverges.

Similar test can be given for the improper integrals of type II.

Example: Investigate the convergence of $\int_1^\infty e^{-x^2}dx$.

The integral $\int_1^\infty e^{-x^2}dx$ cannot be evaluated directly.

By using the fact $e^{-x^2} \le e^{-x}$ for $x \geq 1$ we get

$$\int_1^\infty e^{-x^2}dx \le \int_1^\infty e^{-x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty}(e^{-1}-e^{-b})$$

$$= \frac{1}{e} < \infty$$

Hence by direct comparison test $\int_1^\infty e^{-x^2}dx$ converges.

Example: Is $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ convergent or divergent?

$$0 \le \sin^2 x \le 1 \quad \Rightarrow \quad \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$$

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b}+1 = 1 \text{ convergent.}$$

Hence by direct comparison test $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ is convergent.

Example: Investigate the convergence of $\int_1^\infty \frac{\ln x}{\sqrt{x}}dx$ and $\int_0^1 \frac{\ln x}{\sqrt{x}}dx$

For $x \geq e$ we have $\ln x \geq 1$. So $\frac{\ln x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$ for $x \geq e$.

Thus $$\int_1^\infty \frac{\ln x}{\sqrt{x}}dx = \int_1^e \frac{\ln x}{\sqrt{x}}dx + \int_e^\infty \frac{\ln x}{\sqrt{x}}dx \geq \underbrace{\int_1^e \frac{\ln x}{\sqrt{x}}dx}_{\text{a constant}} + \underbrace{\int_e^\infty x^{-1/2}dx}_{+\infty} = +\infty$$

since $p = -\frac{1}{2} < 1$ this integral is divergent. (check page 98)

So by direct comparison test

$\int_1^\infty \frac{\ln x}{\sqrt{x}}dx$ is divergent.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx \underset{u=1/x}{=} \int_\infty^1 \frac{\ln(1/u)}{\sqrt{1/u}}\cdot -\frac{1}{u^2}\,du = \int_1^\infty \frac{-\ln u}{u^{3/2}}\,du = -\int_1^\infty \frac{\ln u}{u^{3/2}}\,du$$

$\frac{1}{u} = x \Rightarrow -\frac{1}{u^2}\,du = dx$

$x=0 \Rightarrow u=\infty$

$x=1 \Rightarrow u=1$

$\ln u \le u^{1/4}$ for $u \geq u_0$ for some u_0

(In fact for $u \geq 256$)

and so $\frac{\ln u}{u^{3/2}} \le \frac{u^{1/4}}{u^{3/2}} = \frac{1}{u^{5/4}}$

As $\int_{u_0}^\infty \frac{du}{u^{5/4}}$ converges ($p=\frac{5}{4}>1$ page 98), we get that $\int_1^\infty \frac{\ln u}{u^{3/2}}\,du$ converges $\Rightarrow \int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ converges.

Example: $\int_1^\infty \frac{\sin x}{x\sqrt{x}}\,dx$ Investigate convergence of this integral.

$f(x) = \frac{\sin x}{x\sqrt{x}}$ To use the direct comparison test we must have $f(x) \geq 0$ on the integration interval.

Hence we will consider $|f(x)|$.

$$|f(x)| = \left|\frac{\sin x}{x\sqrt{x}}\right| \le \frac{1}{x\sqrt{x}} = g(x) \quad \forall x \in [1,\infty)$$

$$\int_0^1 g(x)\,dx = \int_1^\infty \frac{dx}{x\sqrt{x}} = \int_1^\infty \frac{dx}{x^{3/2}} < \infty \text{ convergent}$$

$p=\frac{3}{2}>1$

Hence by direct comparison test $\int_1^\infty |f(x)|\,dx = \int_1^\infty \frac{|\sin x|}{x\sqrt{x}}\,dx$ is convergent.

Definition: If $\int |f(x)|\,dx$ is convergent, then we say that the improper integral $\int f(x)\,dx$ is absolutely convergent. If the improper integral $\int f(x)\,dx$ is convergent while the improper integral $\int |f(x)|\,dx$ is divergent, we say that it is conditionally convergent.

If the improper integral $\int |f(x)|dx$ is convergent, then the integral $\int f(x)dx$ is also convergent. Thus in our example,

$\int_1^\infty \frac{\sin x}{x\sqrt{x}}\,dx$ is also convergent.

Example: Investigate convergence of $\int_1^\infty \frac{x^2}{\sqrt{1+x^8}}\,dx$.

$$0 \le f(x) = \frac{x^2}{\sqrt{1+x^8}} = \frac{x^2}{x^4\sqrt{\frac{1}{x^8}+1}} = \frac{1}{x^2\sqrt{\frac{1}{x^8}+1}} \le \frac{1}{x^2} = g(x)$$

$\int_1^\infty \frac{1}{x^2}\,dx < \infty$ convergent. Hence by the direct comparison test
$p=2>1$

$\int_1^\infty \frac{x^2}{\sqrt{1+x^8}}\,dx$ is also convergent.

Limit Comparison Test: For nonnegative functions $f(x)$ and $g(x)$ which are continuous on $[a,\infty)$ we have if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ for some $0<L<\infty$, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

Similarly, for improper integrals of type II, we give this test:

Let $f(x)$ and $g(x)$ be nonnegative and continuous functions on $(a,b]$, but not continuous at $x=a$. If $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ for some $0<L<\infty$, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

Example: Investigate the convergence of $\int_0^2 x\sqrt{\frac{x+1}{2-x}}\,dx$ by using the limit comparison test.

$f(x) = x\sqrt{\frac{x+1}{2-x}}$ is not continuous at $x=2$. This integral is an improper integral of type II.

$f(x) = x\sqrt{\frac{x+1}{2-x}} \geq 0$, $\forall x \in [0,2)$. Choose $g(x) = \frac{1}{\sqrt{2-x}} > 0$.

$$\lim_{x\to 2^-} \frac{f(x)}{g(x)} = \lim_{x\to 2^-} x\sqrt{x+1} = 2\sqrt{3} \neq 0, \infty$$

$$\int_0^2 g(x)\,dx = \int_0^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{c\to 2^-} \int_0^c \frac{1}{\sqrt{2-x}}\,dx = \lim_{c\to 2^-} -2\sqrt{2-x}\Big|_0^c$$

$$= \lim_{c\to 2^-} \left[-2\sqrt{2-c} + 2\sqrt{2}\right] = 2\sqrt{2}$$

convergent

Hence by limit comparison test $\int_0^2 f(x)\,dx = \int_0^2 x\sqrt{\frac{x+1}{2-x}}\,dx$ is also convergent.

Example: Investigate the convergence of $\int_0^1 \frac{\sin x}{x^3}\,dx$.

$f(x) = \frac{\sin x}{x^3}$, $g(x) = \frac{1}{x^2}$. Both of these functions are nonnegative and continuous on $(0,1]$, but not continuous at $x=0$.

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\frac{\sin x}{x^3}}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \neq 0, \infty$$

and $$\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^2}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x^2}\,dx = \lim_{c\to 0^+} -\frac{1}{x}\Big|_c^1 = \lim_{c\to 0^+} \left[-1 + \frac{1}{c}\right]$$

$= +\infty$

divergent.

Hence by limit comparison test

$\int_0^1 \frac{\sin x}{x^3}\,dx$ is also divergent.

Example: Investigate the convergence of $\displaystyle\int_1^\infty \frac{\sqrt[5]{x^3-2x}}{\sqrt[3]{x^5+4x+1}}\,dx$.

improper integral of type I.

$f(x)=\dfrac{\sqrt[5]{x^3-2x}}{\sqrt[3]{x^5+4x+1}}$ compare with $g(x)=\dfrac{x^{3/5}}{x^{5/3}}=\dfrac{1}{x^{16/15}}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\dfrac{(x^3-2x)^{1/5}}{(x^5+4x+1)^{1/3}}}{x^{-16/15}}=\lim_{x\to\infty}\frac{x^{3/5}\left(1-2/x^2\right)^{1/5}}{x^{5/3}\left(1+4\cdot\frac{1}{x^4}+\frac{1}{x^5}\right)^{1/3}}\cdot x^{16/15}$$

$$=\lim_{x\to\infty}\frac{\left(1-2/x^2\right)^{1/5}}{\left(1+4/x^4+1/x^5\right)^{1/3}}=1\neq 0,\infty$$

Thus $\displaystyle\int_1^\infty f(x)dx$ and $\displaystyle\int_1^\infty g(x)dx$ both converge or both diverge.

Since $\frac{16}{15}>1$, we know that $\displaystyle\int_1^\infty g(x)dx=\int_1^\infty \frac{dx}{x^{16/15}}$ converges.

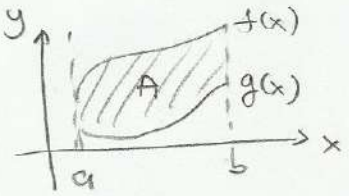
Hence $\displaystyle\int_1^\infty \frac{\sqrt[5]{x^3-2x}}{\sqrt[3]{x^5+4x+1}}dx$ also converges by limit comparison test.

Applications of Integration:

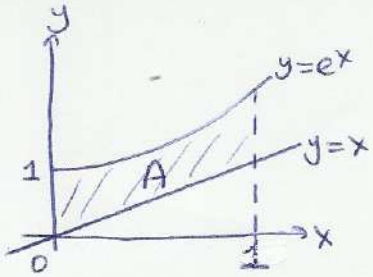
- Areas between curves

The area A of the region bounded by the curves $y=f(x)$, $y=g(x)$, and the lines $x=a$, $x=b$ where f and g are continuous and $f(x)\geq g(x)$ for all x in $[a,b]$ is

$$A=\int_a^b [f(x)-g(x)]\,dx.$$



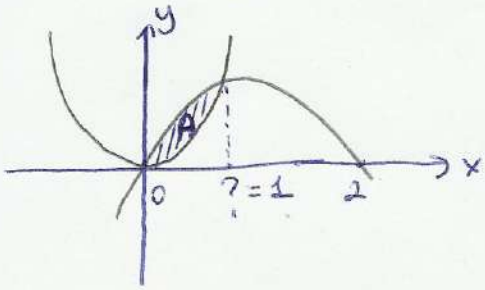
Example: Find the area of the region bounded above by $y=e^x$, bounded below by $y=x$, and bounded on the sides by $x=0$ and $x=1$.



$$\int_0^1 (e^x - x)\,dx = \left(e^x - \frac{x^2}{2}\right)\Big|_0^1$$

$$= e - \frac{1}{2} - 1 = e - \frac{3}{2}$$

Example: Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$.



We first find the points of intersection of the parabolas.

$$x^2 = 2x - x^2$$

$$\Rightarrow 2x^2 = 2x \quad \Rightarrow 2(x)(x-1) = 0$$

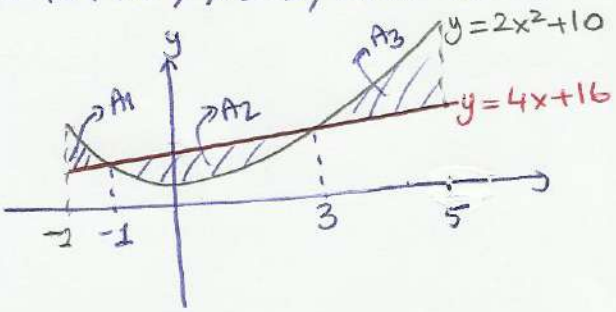
$$x = 0, \ \boxed{x = 1}$$

Hence the total area is $A = \int_0^1 \left[(2x - x^2) - x^2\right] dx = \int_0^1 (2x - 2x^2)\,dx$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Big|_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

Example: Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$, and $x = 5$.



Firstly, find the intersection points.

$$4x + 16 = 2x^2 + 10$$

$$\Rightarrow 2x^2 - 4x - 6 = 2(x^2 - 2x - 3)$$

$$= 2(x-3)(x+1) = 0$$

$$x = -1, \ x = 3$$

Total area $= A_1 + A_2 + A_3$

$$= \int_{-2}^{-1} \left[(2x^2 + 10) - (4x + 16)\right] dx + \int_{-1}^{3} \left[(4x + 16) - (2x^2 + 10)\right] dx + \int_{3}^{5} \left[(2x^2 + 10) - (4x + 16)\right]$$

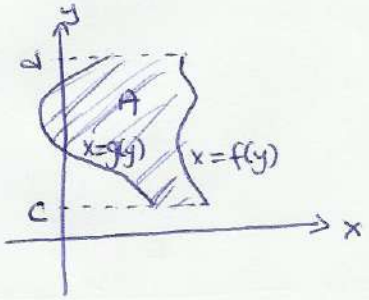
$$= \left(\frac{2x^3}{3} - \frac{4x^2}{2} - 6x\right)\Big|_{-2}^{-1} + \left(\frac{4x^2}{2} + 6x - \frac{2x^3}{3}\right)\Big|_{-1}^{3} + \left(\frac{2x^3}{3} - 6x - \frac{4x^2}{2}\right)\Big|_{3}^{5}$$

$$= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \frac{142}{3}$$

Example: Determine the area of the region enclosed by

$x=\frac{1}{2}y^2-3$ and $y=x-1$.

Note: If we are going to determine the area between $x=f(y)$ and $x=g(y)$ on the interval $[c,d]$ with $f(y)\geq g(y)$,



we have $A=\int_c^d\left[\begin{pmatrix}\text{right}\\ \text{function}\end{pmatrix}-\begin{pmatrix}\text{left}\\ \text{function}\end{pmatrix}\right]dy$

In this example we have $x=\frac{1}{2}y^2-3$ and $x=y+1$. Find the intersection points.

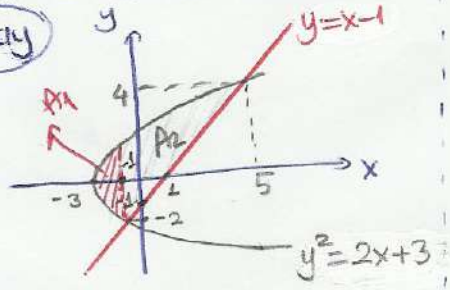
1. way

$y+1=\frac{1}{2}y^2-3 \Rightarrow 2y+2=y^2-6$

$y^2-2y-8=(y-4)(y+2)=0$

$y=4\ ,\ y=-2.$

2. way



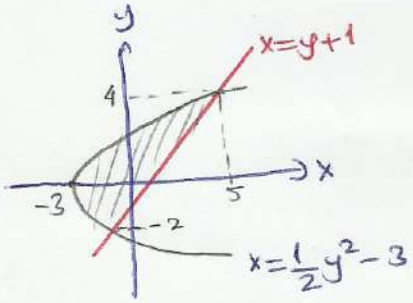
$y^2=2x+3 \Rightarrow y=\pm\sqrt{2x+3}$

"+" gives the upper portion of the parabola and "−" gives the lower portion

Total area $=A_1+A_2$

$=\int_{-3}^{-1}\left(\sqrt{2x+6}-(-\sqrt{2x+6})\right)dx$

$+\int_{-1}^{5}\left(\sqrt{2x+6}-(x-1)\right)dx$



$A=\int_{-2}^{4}\left[(y+1)-\left(\frac{1}{2}y^2-3\right)\right]dy$

$=\left(\frac{y^2}{2}+4y-\frac{y^3}{6}\right)\Big|_{-2}^{4}=18.$

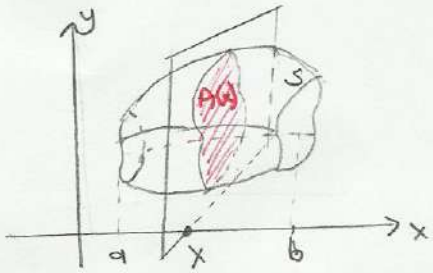
Exercises: 1) Determine the area of the region bounded by $x=-y^2+10$ and $x=(y-2)^2$.

2) Determine the area of the region bounded by $y=xe^{-x^2}$, $y=x+1$, $x=2$, and the y-axis.

3) Determine the area of the region enclosed by $y=x^2$ and $y=\sqrt{x}$

Volume

- To find the volume of a solid S, we first intersect S with a plane and obtain a plane region that is called a cross-section of S. Let $A(x)$ be the area of the cross-section of S in a plane perpendicular to the x-axis and passing through the point x, $x \in [a,b]$.



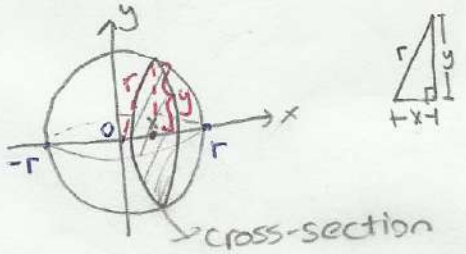
$A(x)$ is a continuous function.

Then the volume of S is

$$V = \int_a^b A(x)\,dx.$$

$A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Example: Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.



The cross-section is a disk with radius y.

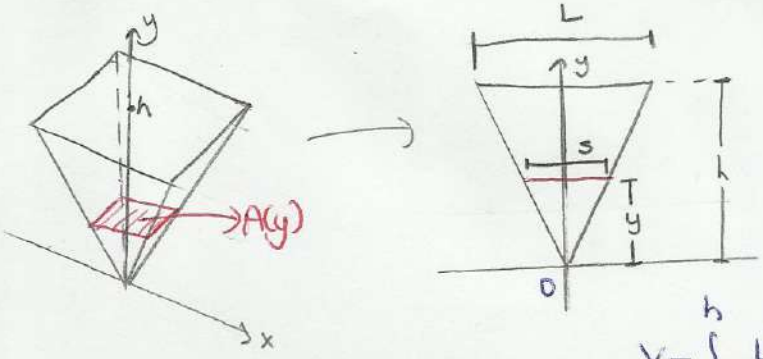
$y^2 + x^2 = r^2 \Rightarrow y = \sqrt{r^2 - x^2}$. So the cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2).$$

Hence $V = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = 2\pi \int_0^r (r^2 - x^2)\,dx = 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$

$$= 2\pi \left[r^3 - \frac{r^3}{3} \right] = \frac{4\pi r^3}{3}.$$

Example: Find the volume of a pyramid whose base is a square with sides of length L and whose height is h.



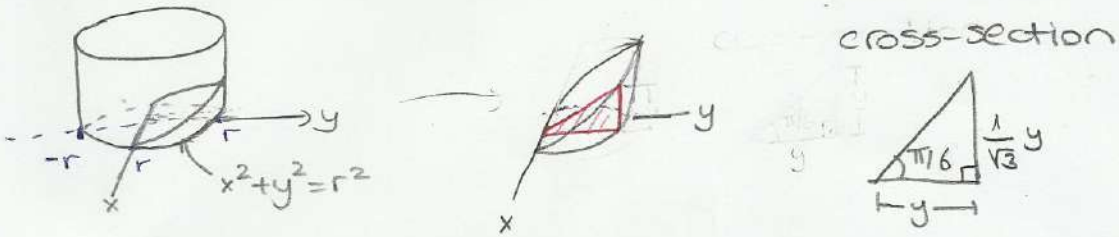
$A(y) = s^2$

$\frac{s}{L} = \frac{y}{h} \Rightarrow s = \frac{L}{h} y$

$A(y) = s^2 = \frac{L^2}{h^2} y^2, \quad 0 \le y \le h$

$$V = \int_0^h \frac{L^2}{h^2} y^2\,dy = \frac{1}{3} L^2 h.$$

Example: Find the volume of a wedge cut out of a cylinder of radius r if the angle between the top and bottom is $\frac{\pi}{6}$.



The cross-sectional area $A(x) = \frac{1}{2}y\left(\frac{1}{\sqrt{3}}y\right) = \frac{1}{2}\sqrt{r^2-x^2}\left(\frac{1}{\sqrt{3}}\sqrt{r^2-x^2}\right)$

$$= \frac{1}{2\sqrt{3}}(r^2-x^2), \quad x \in [-r, r]$$

So $V = \int_{-r}^{r} \frac{1}{2\sqrt{3}}(r^2-x^2)dx = \frac{2}{2\sqrt{3}}\left(r^2x - \frac{1}{3}x^3\right)\Big|_0^r = \frac{2r^3}{3\sqrt{3}}$

Volume of solids of Revolution

- Method of Rings (or method of disks) (~~other~~ Washer method)

We calculate the volume of a solid of revolution by using the formula

$$V = \int_a^b A(x)dx \quad \text{or} \quad V = \int_c^d A(y)dy$$

and we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

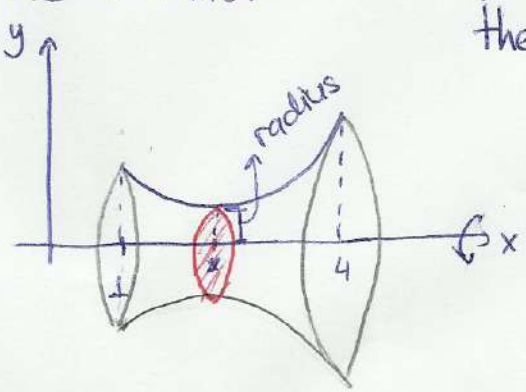
- If the cross-section is a disk, we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2,$$

- If the cross-section is a ring (or washer), we find the inner radius r_{in} and outer radius r_{out} and the are of the ring is

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

Example: Determine the volume of the solid obtained by the region bounded by $y=x^2-4x+5$, $x=1$, $x=4$, and the x-axis about the x-axis.



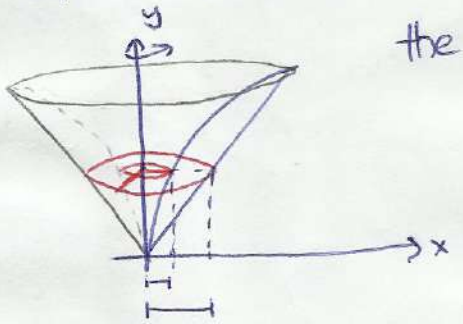
the cross-section is a disk.

radius of the disk is $y=x^2-4x+5$, $1\le x\le 4$

Hence $V=\int_1^4 \pi (x^2-4x+5)^2 dx$

$=\frac{78\pi}{5}$

Example: Determine the volume of the solid obtained by rotating the portion of the region bounded by $y=\sqrt[3]{x}$ and $y=\frac{x}{4}$ that lies in the first quadrant about the y-axis.



the cross-section is a ring (washer)

$y=\sqrt[3]{x} \Rightarrow x=y^3$

$y=\frac{x}{4} \Rightarrow x=4y$

outher radius $r_{out}=4y$

inner radius $r_{in}=y^3$

intersection point $y^3=4y \Rightarrow y(y^2-4)=0$

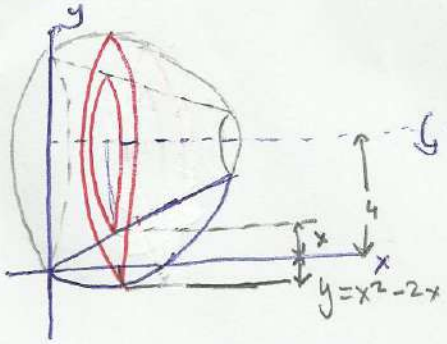
$y=0$, $y=2$, $y=-2$

first quadrant

$V=\int_0^2 \pi\left[(4y)^2-(y^3)^2\right]dy$

$=\frac{512\pi}{21}$

Example: Determine the volume of the solid obtained by rotating the region bounded by $y=x^2-2x$ and $y=x$ about th line $y=4$.



the cross-section is a ring (washer)

inner radius $r_{in}=4-x$

outer radius $r_{out}=4-(x^2-2x)=-x^2+2x+4$

intersection point $x^2-2x=x \Rightarrow x(x-3)=0$

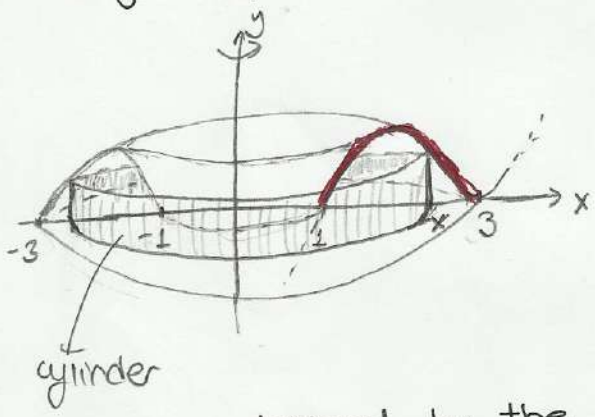
$x=0$, $x=3$

$V=\int_0^3 \pi\left[(-x^2+2x+4)^2-(4-x)^2\right]dx=\frac{153\pi}{5}$

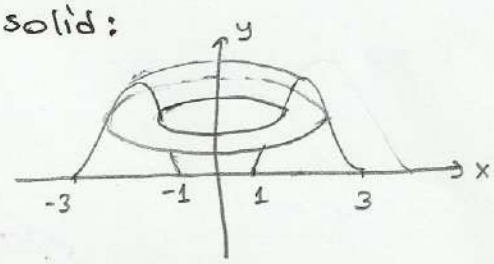
Method of Cylinders (Method of shells)

In the previous method, we only used cross-sections that were in the shape of a disk or a ring. But we can use any shape for the cross-sections as long as it can be expanded or contracted to completely cover the solid we are looking at. And also, we cannot always use rings/disks.

Example: Determine the volume of the solid obtained by rotating the region bounded by $y=(x-1)(x-3)^2$ and the x-axis about the y-axis.



• If we were to use rings on this solid:



Here, both the inner and outer radius are defined by the same function. This can be dealt but it usually means more work than other methods. And also, in order to use rings, we would need to put this function into the form $x=f(y)$. That is not easy in general and in this example.

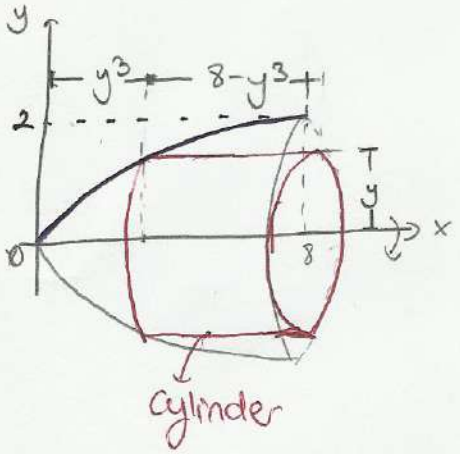
The surface area of the cylinder is

$$A(x) = 2\pi \text{ (radius) (height)}$$
$$= 2\pi \cdot x \cdot (x-1)(x-3)^2$$

The first cylinder will cut into the solid at $x=1$ and as we increase x to $x=3$ we will completely cover both sides of the solid. Hence here $1 \le x \le 3$.

$$V = \int_1^3 A(x)\,dx = \int_1^3 2\pi x(x-1)(x-3)^2\,dx = 2\pi\int_1^3 (x^4-7x^3+15x^2-9x)\,dx$$
$$= \frac{24\pi}{5}$$

Example: Determine the volume of the solid obtained by rotating the region bounded by $y=\sqrt[3]{x}$, $x=8$, and the x-axis about the x-axis.



Here we may use the disk method.

But let us use method of cylinders.

To use this method the equation should be written in $x=f(y)$ form.

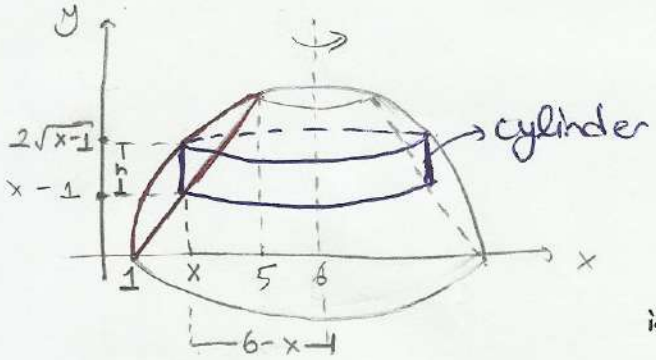
$y=\sqrt[3]{x} \Rightarrow x=y^3$

radius $=y$
height $=8-y^3$

$A(y)=2\pi(\text{radius})(\text{height})=2\pi y(8-y^3)$, $0\le y\le 2$

$$V=\int_0^2 2\pi y(8-y^3)\,dy=\frac{96\pi}{5}.$$

Example: Determine the volume of the solid obtained by rotating the region bounded by $y=2\sqrt{x-1}$ and $y=x-1$ about the line $x=6$



height $=h=2\sqrt{x-1}-(x-1)$
radius $=6-x$

$A(x)=2\pi(6-x)\left[2\sqrt{x-1}-(x-1)\right]$

intersection points.

$2\sqrt{x-1}=x-1$

$\Rightarrow 4(x-1)=x^2-2x+1$

$\Rightarrow x^2-6x+5=(x-5)(x-1)=0$

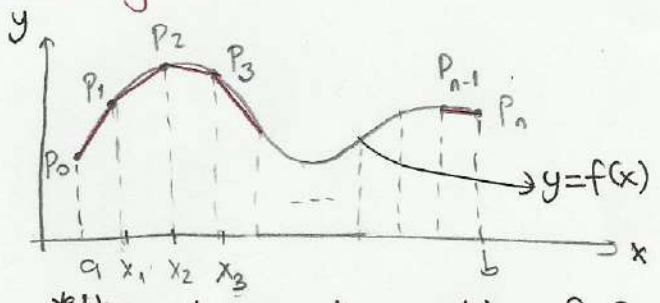
$x=1,\ x=5$

Hence here $1\le x\le 5$.

$$V=\int_1^5 2\pi\left[12\sqrt{x-1}-2x\sqrt{x-1}-6(x-1)+x(x-1)\right]dx$$

$$=\frac{272\pi}{15}$$

Arc length:



*the polygon with vertices $P_0, P_1, \dots, P_n$

C is a curve defined by $y=f(x)$, f is continuous, $a \le x \le b$.

Divide the interval $[a,b]$ into n subintervals with equal width Δx.

The length L of C is approximately the length of this polygon, and the approximation gets better as we let n increase.

Hence we define the length L by

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| \quad \text{(if the limit exists)}$$

$\Delta x = x_i - x_{i-1} \ \forall i$

Define $\Delta y = y_i - y_{i-1}$

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

* Assume that f has a continuous derivative (smooth function)

Apply the Mean Value Theorem to f on the interval $[x_{i-1}, x_i]$.

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}, \quad x_i^* \in [x_{i-1}, x_i]$$

$$= \frac{\Delta y_i}{\Delta x} \quad \Rightarrow \quad \Delta y_i = f'(x_i^*)\,\Delta x.$$

Thus we have $|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2}$

$$= \Delta x \sqrt{1 + [f'(x_i^*)]^2}$$

$$\Rightarrow L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

$$\Rightarrow \boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.}$$

Example: Determine the length of $y=\ln(\sec x)$ between $0 \le x \le \pi/4$.

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x. \Rightarrow \left(\frac{dy}{dx}\right)^2 = (f'(x))^2 = \tan^2 x.$$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx \; \frac{(\sec x + \tan x)}{(\sec x + \tan x)}$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln(\sqrt{2}+1)$$

Note: If a curve has the equation $x=g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y we have

$$L = \int_c^d \sqrt{1+[g'(y)]^2}\, dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Example: Find the length of the arc of the parabola $y^2=x$ from $(0,0)$ to $(1,1)$.

$$x=g(y)=y^2 \Rightarrow \frac{dx}{dy} = 2y$$

$$L = \int_0^1 \sqrt{1+(2y)^2}\, dy = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\, du = \frac{1}{2}\int_0^{\tan^{-1}(2)} \sqrt{\sec^2\theta} \cdot \sec^2\theta\, d\theta$$

$2y=u \Rightarrow 2dy=du$

$y=0 \Rightarrow u=0$

$y=1 \Rightarrow u=2$

$u=\tan\theta$, $du=\sec^2\theta\, d\theta$

$1+u^2 = 1+\tan^2\theta = \sec^2\theta$

$u=2 \Rightarrow \theta = \tan^{-1}(2)$

$u=0 \Rightarrow \theta=0$

$\tan^{-1}(2) = \alpha$

$$\frac{1}{2}\int_0^{\alpha} \underbrace{\sec^3\theta}_{\sec^2\theta\cdot\sec\theta}\, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta\Big|_0^{\alpha} - \int_0^{\alpha} \sec\theta \underbrace{\tan^2\theta}_{(\sec^2\theta-1)}\, d\theta\right]$$

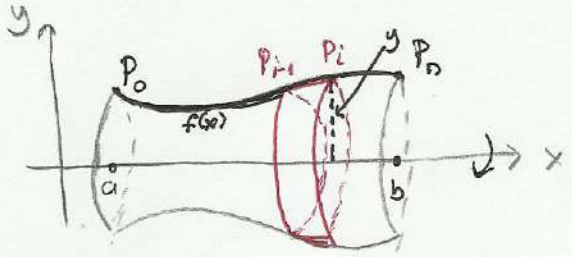
$u=\sec\theta$, $dv=\sec^2\theta\, d\theta$

$du=\sec\theta\tan\theta$, $v=\tan\theta$

$$\Rightarrow \int_0^{\alpha} \sec^3\theta = \frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln|\sec\alpha+\tan\alpha|$$

$$\Rightarrow L = \frac{1}{2}\left[\frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln|\sec\alpha+\tan\alpha|\right], \quad \alpha=\tan^{-1}(2).$$

Area of a surface of revolution



The solid is obtained by rotating the curve $y=f(x)$, $a \le x \le b$ about the x-axis.

$f(x)$: non-negative, has a continuous derivative.

In order to define its surface area, we divide the interval $[a,b]$ into n subintervals with endpoints $x_0, x_1, ---, x_n$ and equal width Δx.

$P_i(x_i, y_i)$: points lying on the curve, $i = 0, 1, ---, n$.

The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis. The result is a band as it is seen in the figure.

height of the band $= |P_{i-1}P_i|$

average radius of the band $= \dfrac{y_{i-1}+y_i}{2}$

So its surface area is $2\pi \cdot \left(\dfrac{y_{i-1}+y_i}{2}\right) |P_{i-1}P_i|$.

Remember that while we are considering the arclength we have found

$$|P_{i-1}P_i| = \sqrt{1+[f'(x_i^*)]^2}\ \Delta x \quad , \quad x_i^* \in [x_{i-1}, x_i].$$

when Δx is small, $y_i = f(x_i) \approx f(x_i^*)$, since f is continuous.

$$y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$$

Thus, $$2\pi \left(\frac{y_{i-1}+y_i}{2}\right) |P_{i-1}P_i| \approx 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\ \Delta x .$$

The area of the complete surface of revolution is

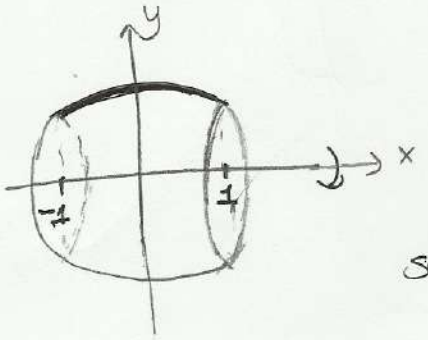
$\displaystyle\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\ \Delta x$. The approximation gets better as $n \to \infty$.

So we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\ \Delta x = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\ dx .$$

f nonnegative
has a continuous deriv.

Example: The curve $y=\sqrt{4-x^2}$, $-1\le x\le 1$ is an arc of the circle $x^2+y^2=4$. Find the area of the surface obtained by rotating this arc about the x-axis.



$$\frac{dy}{dx} = \frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

$$\text{surface area } S = \int_{-1}^{1} 2\pi\cdot\sqrt{4-x^2}\cdot\sqrt{1+\frac{x^2}{4-x^2}}\,dx$$

$$= 2\pi\int_{-1}^{1}\sqrt{4-x^2}\cdot\frac{2}{\sqrt{4-x^2}}\,dx$$

$$= 4\pi\, x\Big|_{-1}^{1} = 8\pi.$$

Note: If the function is given in $x=g(y)$ form, $c\le y\le d$, then the formula becomes

$$S = \int_c^d 2\pi\, g(y)\sqrt{1+[g'(y)]^2}\,dy = \int_c^d 2\pi\, x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Sequences

A sequence is a list of numbers written in a specific order.

Notation: $\{a_1, a_2, \dots, a_n, a_{n+1}, \dots\}$, $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$

Example: 1) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \dots\right\}$

2) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=1}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \dots\right\}$

Definition: • We say that $\lim_{n\to\infty} a_n = L$ if we can make a_n as close to L as we want for all sufficiently large n.

• We say that $\lim_{n\to\infty} a_n = \infty$ if we can make a_n as large as we want for all sufficiently large n.

• We say that $\lim_{n\to\infty} a_n = -\infty$ if we can make a_n as large and negative as $n\to\infty$.

- If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is convergent.
- If $\lim_{n\to\infty} a_n$ does not exist or infinite we say the sequence is divergent.

Theorem: Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n)=a_n$ and $\lim_{x\to\infty} f(x)=L$ then $\lim_{n\to\infty} a_n=L$

Properties If a_n, b_n are convergent series:

1) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

2) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$, c constant

3) $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq$

5) $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$, provided $a_n \geq 0$, $p>0$

The Sandwich (Squeeze) Theorem: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$ the $\lim_{n\to\infty} a_n = 0$.

Proof: Note that $-|a_n| \leq a_n \leq |a_n|$ and $\lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0$. So we have $\lim_{n\to\infty} (-|a_n|) = \lim_{n\to\infty} |a_n| = 0$ and hence by the squeeze theorem we must also have $\lim_{n\to\infty} a_n = 0$.

Theorem: The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$ and diverges for all other value of r.

Asistana
Soyle (Riemann Sum)

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \end{cases}$$

Example: Determine if the following sequences converge or diverge.

1) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \underbrace{\lim_{n\to\infty} \frac{1}{n}}_{\to 0}} = 1$ convergent

2) $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ Note that both numerator and denominator approach infinity as $n\to\infty$. We cannot apply L'Hopital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply L'Hopital's Rule to the related function $f(x) = \frac{\ln x}{x}$. $\quad \lim_{x\to\infty} \frac{\ln x}{x} \underset{L'R}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1/x}{1} = 0.$

$\Rightarrow$ By theorem on page 116, $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$ convergent

3) $\left\{\underbrace{\frac{(-1)^n}{n}}_{a_n}\right\}_{n=1}^{\infty}$ $\quad \underbrace{\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right|}_{\lim_{n\to\infty}|a_n|} = \lim_{n\to\infty} \frac{1}{n} = 0 \xRightarrow[\text{on page 116}]{\text{by Thm.}} \lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$ convergent

4) $\{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}_{n=}^{\infty}.$

It is clear that $\lim_{n\to\infty} (-1)^n$ does not exist. The terms just alternate between 1 and -1. So the sequence is divergent.

Theorem: If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Example: Find $\lim_{n\to\infty} \sin(\pi/n)$.

sine function is continuous everywhere hence also at 0.

$\lim_{n\to\infty} \sin(\pi/n) = \sin\left(\lim_{n\to\infty} (\pi/n)\right) = \sin 0 = 0.$

Example: Discuss the convergence of the sequence $a_n = \frac{n!}{n^n}$.

Both numerator and denominator approach ∞. But here we do not have corresponding function to apply L'Hopital's Rule.

$$a_n = \frac{n!}{n^n} = \frac{1.2.3.4.\cdots n}{n.n.n.n.\cdots n} = \frac{1}{n}\underbrace{\left(\frac{2.3.4.\cdots n}{n.n.n.\cdots n}\right)}_{\leq 1}$$

Hence $0 < a_n \leq \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$

by the squeeze thm $\lim_{n\to\infty} a_n = 0$.

Definition: A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$, it is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotonic if it is either increasing or decreasing.

Definition: A sequence $\{a_n\}$ is bounded above if there is a numbe M such that $a_n \leq M$ for all $n \geq 1$,

It is bounded below if there is a number m such that

$m \leq a_n$ for all $n \geq 1$,

If it is bounded above and below, then $\{a_n\}$ is a bounded sequen

Example: Determine if the following sequences are monotonic and/or bounded. Find the limits of the sequences.

1) $\{-n^2\}_{n=0}^{\infty}$ $\quad a_n = -n^2 \quad -n^2 > -(n+1)^2$, $a_n > a_{n+1}$ $\Rightarrow$ Hence the sequence is a decreasing and hence monotonic sequence.

We have $a_n = -n^2 \leq 0$ for every n. Hence it is bounded above. But the sequence is not bounded below. Hence the sequence is not bounded.

$\lim_{n\to\infty} (-n^2) = -\infty$. The sequence is divergent.

2) $\{(-1)^{n+1}\}_{n=1}^{\infty}$ The sequence terms alternate between 1 and -1 and so the sequence is neither an increasing or a decreasing sequence, hence not a monotonic sequence.

The sequence is bounded above by 1 and below by -1. Hence it is bounded.

$\lim_{n\to\infty} (-1)^{n+1}$ does not exist. The sequence is divergent.

Example: Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

1. way Show $a_{n+1} < a_n$.

2. way We can use the related function $f(x) = \frac{x}{x^2+1}$.

$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ whenever $x^2 > 1$. Thus f is decreasing on $(1, \infty)$ and so $f(n) > f(n+1)$. Therefore $\{a_n\}$ is decreasing.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

Example: Show that the sequence $\left\{\frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots(2n)}\right\}_{n=1}^{\infty}$ is convergent

- we first show that it is a decreasing (hence monotonic) sequence

$a_{n+1} < a_n \Rightarrow \frac{a_{n+1}}{a_n} < 1$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{1.3.5.\ldots.(2n-1)}(2n+1)}{\cancel{2.4.6.\ldots(2n)}(2n+2)} \cdot \frac{\cancel{2.4.6.\ldots(2n)}}{\cancel{1.3.5.\ldots(2n-1)}} = \frac{2n+1}{2n+2} < 1$$

$\Rightarrow a_{n+1} < a_n$ so the sequence is decreasing (hence monotonic)

- $0 < \frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)} = \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\ldots\frac{(2n-1)}{2n} < 1$

The sequence is bounded below and above. Hence it is bounded. So by the theorem it is convergent.

Example: Investigate the convergence of the sequence $\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty}$.

$$0 \le \frac{2^n}{n!} = \frac{\overbrace{2.2.2.\cdots.2}^{n}}{1.2.3.\cdots n} = \frac{2}{1}.\frac{2}{2}.\frac{2}{3}.\frac{2}{4}.\cdots.\frac{2}{n}$$

$$\le 2.1.\overbrace{\frac{2}{3}.\frac{2}{3}.\cdots.\frac{2}{3}}^{n-2} = 2\left(\frac{2}{3}\right)^{n-2}$$

$$= \frac{9}{2}\left(\frac{2}{3}\right)^{n}$$

$$\Rightarrow \underbrace{0}_{\text{as } n\to\infty \;\to\; 0} \le \frac{2^n}{n!} \le \underbrace{\frac{9}{2}\left(\frac{2}{3}\right)^{n}}_{\text{as } n\to\infty \;\to\; 0}$$

By the squeeze theorem $\lim_{n\to\infty} \frac{2^n}{n!} = 0$ convergent.

Exercises. Determine the convergence and divergence of the following sequences. Explain your answer.

1) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

2) $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

Series:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We define the following:

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$

The S_n are called partial sums and indeed they form a sequence $\{S_n\}_{n=1}^{\infty}$.

Now we consider the limit of the sequence of partial sums, $\{S_n\}_{n=1}^{\infty}$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \underbrace{\sum_{i=1}^{\infty} a_i}_{\text{infinite series}}.$$

By the relation $\lim_{n\to\infty} S_n = \sum_{i=1}^{\infty} a_i$, we can give the following remark.

If the sequence of partial sums, $\{S_n\}_{n=1}^{\infty}$, is convergent, then the infinite series $\sum_{i=1}^{\infty} a_i$ is also called convergent.

If $\lim_{n\to\infty} S_n$ is divergent then $\sum_{i=1}^{\infty} a_i$ is divergent.

Example: "Geometric Series"

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \quad , a \neq 0.$$

Each term is obtained from the preceding one by multiplying it by the common ratio r.

If $r=1$, then $S_n = a + a + \cdots + a = na$ and $\lim_{n\to\infty} S_n = \pm\infty$.

Hence geometric series diverges for $r=1$.

If $r \neq 1$, we have
$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$rS_n = ra + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

$$\Rightarrow S_n - rS_n = a - ar^n \Rightarrow S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

If $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$ so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) = \frac{a}{1-r}$$

Thus when $-1 < r < 1$, the geometric series is convergent and $\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad , -1<r<1}$ * $\frac{a}{1-r} = \frac{\text{first term}}{1 - \text{common ratio}}$

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent so $\{S_n\}$ is divergent so $\sum_{i=1}^{\infty} a_i$ is divergent.

Example: Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

we can write the series as $\sum_{k=1}^{\infty} 5\left(-\frac{2}{3}\right)^{k-1} = \frac{\text{first term}}{1-\text{common ratio}} = \frac{5}{1-(-2/3)}$

$= 3.$

first term $=5$

common ratio $= -2/3 = r$ $\quad -1 < -2/3 < 1$

Example: Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Let us try to write the nth term of the series in the form ar^{n-1}.

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} 4^n 3^{-(n-1)} = \sum_{n=1}^{\infty} 4 \cdot \frac{4^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \left(\frac{4}{3}\right)^{n-1}$$

$a=4$, $r = \frac{4}{3} > 1$. Hence the series diverges.

Example: Determine if the following series is convergent or divergent. If it converges find its value.

$$\sum_{n=1}^{\infty} n.$$

The sequence of partial sums, $S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$. Therefore, the sequence of partial sums diverges to ∞ and so the series also diverges.

Example: Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} (-1)^n.$$

Let us write the first few partial sums, $S_0 = 1$

$S_1 = 1 - 1 = 0$

$S_2 = 1 - 1 + 1 = 1$

$S_3 = 1 - 1 + 1 - 1 = 0$

i.e. $\{S_n\}_{n=0}^{\infty} = \{1, 0, 1, 0, 1, 0, 1, \ldots\}$. $\lim_{n\to\infty} S_n$ does not exist hence $\{S_n\}_{n=0}^{\infty}$ is divergent, so $\sum_{n=0}^{\infty} (-1)^n$ is divergent.

Example: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)}$$

or

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1} \Rightarrow S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

*Notice that the terms cancel in pairs. This is an example of a telescoping sum.

Example: Show that the "harmonic series" $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

For this example we will consider, $S_2, S_4, S_8, \ldots, S_{2^n}, \ldots$

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 1$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}$$

In general $S_{2^n} > 1 + \frac{n}{2}$. This shows that $S_{2^n} \to \infty$ as $n \to \infty$ and so $\{S_n\}$ is divergent. Therefore the harmonic series diverges.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

* The converse of this theorem is not correct in general. For instance, for the harmonic series $\sum_{n=1}^{\infty} 1/n$, we have $a_n = 1/n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} 1/n$ is divergent.

The test for divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Show that the series $\sum_{n=1}^{\infty} \frac{n^3}{3n^3+6}$ diverges.

$$a_n = \frac{n^3}{3n^3+6}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n^3}}{\cancel{n^3}\left(3+\underbrace{\frac{6}{n^3}}_{\to 0}\right)} = \frac{1}{3} \neq 0$$

so the series diverges by the test of divergence.

Properties: If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum c a_n$ (c is a constant), $\sum (a_n \pm b_n)$ and

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n.$$

Example: Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{1}{5^n} \right)$.

The series $\sum_{n=1}^{\infty} \frac{1}{5^n}$ is a geometric series with $a = 1/5$ (first term), $r = 1/5$ (common factor)

so, $\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1/5}{1 - 1/5} = \frac{1}{4}$

on page 123, we've proved that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, so $\sum_{n=1}^{\infty} \frac{4}{n(n+1)} = 4$

Hence $\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{1}{5^n} \right) = \frac{1}{4} + 4 = \frac{17}{4}$.

Integral test:

Suppose that $f(x)$ is a continuous, positive, and decreasing function on the interval $[k,\infty)$ and that $f(n)=a_n$, then

1) If $\int_k^\infty f(x)dx$ is convergent so is $\sum_{n=k}^\infty a_n$.

2) If $\int_k^\infty f(x)dx$ is divergent so is $\sum_{n=k}^\infty a_n$.

Example: For what values of p is the series $\sum_{n=1}^\infty \frac{1}{n^p}$ convergent?

- If $p<0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$
- If $p=0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$

In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$. Hence by the Test for Divergence, the series diverges.

If $p>0$, $f(x)=\frac{1}{x^p}$ is continuous, positive, and decreasing on $[1,\infty)$

Hence we can apply the integral test.

Remember that we proved

$$\int_1^\infty \frac{1}{x^p}dx \text{ converges if } p>1 \text{ and diverges if } p\leq 1,$$

previously.

Hence $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent for $p>1$ and diverges for $p\leq 1$. $\boxed{\text{p-test.}}$

Example: Determine if $\sum_{n=2}^\infty \frac{1}{n\ln n}$ is convergent or divergent?

Consider the function $f(x)=\frac{1}{x\ln x}$. It is continuous, positive, and decreasing on $[2,\infty)$. Hence we can use the integral test.

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u} = \lim_{b\to\infty} \ln|u|\Big|_{\ln 2}^{\ln b}$$

$u=\ln x$

$du=\frac{1}{x}dx$

$x=2 \Rightarrow u=\ln 2$

$x=b \Rightarrow u=\ln b$

$$=\lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2|$$

$$=\infty.$$

Hence by the integral test, the series is divergent.

Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also conv.

(ii) If $\sum a_n$ is divergent and $a_n \le b_n$ for all n, then $\sum b_n$ is also div.

Example: Determine if $\sum_{n=1}^{\infty} \underbrace{\frac{n^2+2}{n^4+5}}_{a_n}$ is convergent or divergent.

$$\underbrace{\frac{n^2+2}{n^4+5}}_{a_n} < \underbrace{\frac{n^2+2}{n^4}}_{b_n} \qquad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2+2}{n^4} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{conv. by} \\ \text{p-test} \\ p=2>1}} + 2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^4}}_{\substack{\text{conv. by} \\ \text{p-test} \\ p=4>1}}$$

Hence $\sum_{n=1}^{\infty} b_n$ is convergent.

By the comparison test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$ is also convergent.

Example: $\sum_{n=0}^{\infty} \underbrace{\frac{1}{3^n+n}}_{a_n}$. conv. / div.?

$$a_n = \frac{1}{3^n+n} < \underbrace{\frac{1}{3^n}}_{b_n}, \quad \sum_{n=0}^{\infty} \frac{1}{3^n} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}}_{\substack{\text{geometric} \\ \text{series} \\ a=1,\ r=\frac{1}{3}<1}} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \text{ convergent.}$$

Hence by the comparison test, $\sum_{n=0}^{\infty} \frac{1}{3^n+n}$ is also convergent.

Example: $\sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2-\cos^2(n)}}_{a_n}$ conv./div.?

$$a_n = \frac{n}{n^2-\cos^2(n)} > \frac{n}{n^2} = \underbrace{\frac{1}{n}}_{b_n}, \quad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ harmonic series}$$

divergent by p-test. $p=1$

Hence $\sum_{n=1}^{\infty} a_n$ is also divergent by the comparison test.

Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with <u>positive</u> terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c>0$, then either both series converge or both diverge.

Examples: Determine if the following series converges or diverges.

1) $\displaystyle\sum_{n=2}^{\infty} \underbrace{\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}}_{a_n}$. Fractions involving only polynomials or polynomials under roots will behave in the same way as the largest power. So, the terms in this series will behave as $\dfrac{n^2}{\sqrt[3]{n^7}} = \dfrac{n^2}{n^{7/3}} = \underbrace{\dfrac{1}{n^{1/3}}}_{b_n}$. This is a nice choice for our second series in the limit comparison test.

$$\lim_{n\to\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \cdot n^{1/3} = \lim_{n\to\infty} \frac{4n^{7/3}+n^{4/3}}{n^{7/3}\sqrt[3]{1+\frac{1}{n^4}}} = \lim_{n\to\infty} \frac{n^{7/3}\left(4+1/n\right)}{n^{7/3}\sqrt{1+\frac{1}{n^4}}} = 4 \neq 0,\infty$$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/3}}$ is divergent by the p-test, $p = 1/3 < 1$.

Hence by limit comparison test, $\displaystyle\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ is also divergent.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. $a_n = \dfrac{1}{2^n-1}$. Choose $b_n = \dfrac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/2^n-1}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 \neq 0,\infty.$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}}_{\substack{\text{geometric series} \\ a=1,\ r=\frac{1}{2}<1}} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1 \text{ convergent.}$$

Hence by limit comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is also convergent.

Alternating Series

The convergence tests that we considered so far apply ONLY to series with positive terms.

An alternating series is a series whose terms are alternately positive and negative.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots$

nth term of an alternating series $\sum_{n=1}^{\infty} a_n$ is of the form

$a_n = (-1)^{n-1} b_n$ or $a_n = (-1)^n b_n$ where $b_n = |a_n|$.

The Alternating Series Test: If the alternating series

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ $b_n > 0$ satisfies

(i) $b_{n+1} \leq b_n$ for all n (i.e. $\{b_n\}$ is decreasing)

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Examples: Determine if the following series is convergent or divergent.

1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \underbrace{\frac{1}{n}}_{b_n}$ $b_n = \frac{1}{n}$ (i) $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ ✓

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

Hence the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent.

2) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5} = \sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{n^2}{n^2+5}}_{b_n}$ $b_n = \frac{n^2}{n^2+5}$ (ii) $\lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 0$.

Since this condition isn't met we will need to use another test to check convergence.

$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5}$ This limit does not exist, so the series diverges by the Test for Divergence.

3) $\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4} = \sum_{n=0}^{\infty} (-1)^{n-3} \cdot \underbrace{\frac{\sqrt{n}}{n+4}}_{b_n}$ $\quad b_n = \frac{\sqrt{n}}{n+4}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{n} + \underbrace{\frac{4}{\sqrt{n}}}_{\to 0}\right)} = 0.$

(i) For showing $b_{n+1} \leq b_n$ for all n, we can use another way.

consider the function $f(x) = \frac{\sqrt{x}}{x+4} \Rightarrow f'(x) = \frac{4-x}{2\sqrt{x}\,(x+4)^2} \leq 0$ for $x \geq 4$.

Hence the function is increasing on $0 \leq x \leq 4$ and decreasing on $x \geq 4$. Therefore, since $f(n) = b_n$ we know as well that the b_n are also increasing on $0 \leq n \leq 4$ and decreasing on $n \geq 4$

The b_n are then eventually decreasing and so the other condition is met.

Hence by the Alternating Series Test, the series is convergent.

Absolute Convergence and the Ratio and Root Tests.

Definition: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent since

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-test $(p=2>1)$.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent but it is not absolutely convergent since $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-test. $(p=1)$ "harmonic series".

Definition: A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Theorem: If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

This series has both positive and negative terms, but it is not an alternating series. To apply the tests, we consider $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$.

$$\left| \frac{\cos n}{n^2} \right| \leq \underbrace{\frac{1}{n^2}}_{b_n}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent by p-test } (p=2>1)$$

Hence by Comparison test, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ is also convergent.

And by the Theorem, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

The Ratio Test:

a) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

b) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

c) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive.

Example: Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence

$$a_n = (-1)^n \frac{n^3}{3^n}$$

$$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3} \left(\frac{n+1}{n} \right) = \frac{1}{3} \left(1 + \frac{1}{n} \right)^3 = \frac{1}{3} < 1$$

Hence by the Ratio test, the series is convergent.

Example: Determine if the following series is convergent or divergent. $\sum_{n=0}^{\infty} \underbrace{\frac{n+2}{2n+7}}_{a_n}$

$a_n = \frac{n+2}{2n+7}$ $\quad L = \lim_{n\to\infty} \left| \frac{n+3}{2(n+1)+7} \cdot \frac{2n+7}{n+2} \right| = \lim_{n\to\infty} \frac{(n+3)(2n+7)}{(2n+9)(n+2)}$

$$= \lim_{n\to\infty} \frac{2n^2+13n+21}{2n^2+13n+18} = \lim_{n\to\infty} 1 + \frac{3}{2n^2+13n+18} = 1$$

So, the Ratio Test tells us nothing here.

But, note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{2n+7} = \frac{1}{2} \neq 0$. Thus by the Divergence test, this serie is divergent.

Example: $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$, conv. / div.?

$a_n = \frac{n^2}{(2n-1)!}$ $\quad L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2(n+1)-1)!} \cdot \frac{(2n-1)!}{n^2} \right|$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{\cancel{(2n-1)!}}{\cancel{(2n-1)!}(2n)(2n+1)} = 0 < 1.$$

Hence by the Ratio test, $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$ is absolutely convergent and so convergent.

The Root Test: Suppose that we have the series $\sum a_n$. Define

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}.$$

Then,

a) if $L<1$, the series is absolutely convergent and hence convergent

b) if $L>1$, the series is divergent

c) if $L=1$, the Root Test is inconclusive.

Example: Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$.

$$a_n=\left(\frac{2n+3}{3n+2}\right)^n \quad ; L=\lim_{n\to\infty}\left(|a_n|\right)^{1/n}=\lim_{n\to\infty}\frac{2n+3}{3n+2}=\lim_{n\to\infty}\frac{\cancel{n}\left(2+\overset{\to 0}{\frac{3}{n}}\right)}{\cancel{n}\left(3+\underset{\to 0}{\frac{2}{n}}\right)}=\frac{2}{3}<1.$$

Thus the given series is absolutely convergent, so convergent by the Root Test.

Example: Test the convergence of the series $\sum_{n=1}^{\infty}\frac{n^n}{3^{1+2n}}$.

$$a_n=\frac{n^n}{3^{1+2n}} \quad . \quad L=\lim_{n\to\infty}\left(|a_n|\right)^{1/n}=\lim_{n\to\infty}\left(\frac{n^n}{3^{1+2n}}\right)^{1/n}=\lim_{n\to\infty}\frac{n}{3^{\frac{1}{n}+2}}$$

$$=\lim_{n\to\infty}\frac{n}{\underset{\to 1}{3^{1/n}}\cdot 3^2}=\infty.$$

Hence by the Root test, the given series is divergent.

Power Series:

A power series about a point x_0, is any series that can be written in the form

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n.$$

c_n: constant coefficients.

Theorem: For a given series $\sum_{n=0}^{\infty} c_n (x-x_0)^n$, there are only three possibilities:

(i) The series converges only when $x=x_0$.

(ii) The series converges for all x.

(iii) There is a positive number R (called radius of convergence) such that the series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

The interval of all x's, including the endpoints if needed to be, for which the power series converges is called the Interval of convergence of the series.

Examples: Determine the radius of convergence and interval of convergence for the following power series.

1) $\sum_{k=1}^{\infty} \frac{x^k}{k^k}$.

By ratio test,

$$L = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)^{k+1}} \cdot \frac{k^k}{x^k} \right| = \lim_{k\to\infty} \frac{|x|}{k+1} \left(\frac{k}{k+1} \right)^k = \lim_{k\to\infty} \frac{|x|}{k+1} \cdot \frac{1}{\underbrace{\left(1+\frac{1}{k}\right)^k}_{e}}$$

$= 0 < 1$ for any $x \in \mathbb{R}$.

Hence the given series is convergent for any x. So radius of convergence $R = \infty$

interval of convergence $(-\infty, \infty)$.

2) $\sum_{k=1}^{\infty} k^k x^k$

By ratio test,

$$L = \lim_{k\to\infty} \left| \frac{(k+1)^{k+1} x^{k+1}}{k^k x^k} \right| = \lim_{k\to\infty} |x| (k+1) \underbrace{\left(1+\frac{1}{k}\right)^k}_{e} = \infty > 1$$

for any $x \in \mathbb{R}$ except $x=0$

Hence the given series is divergent for any x except $x=0$.

So radius of convergence $R=0$.

interval of convergence $\{0\}$.

3) $\sum_{k=0}^{\infty} x^k$.

By ratio test

$L = \lim_{k\to\infty} \left| \frac{x^{k+1}}{x^k} \right| = |x|$. For convergence, we must have $L = |x| < 1$. check the endpoints. For $x=1$, $\sum_{k=0}^{\infty} 1^k = \sum_{k=0}^{\infty} 1$ divergent

For $x=-1$, $\sum_{k=0}^{\infty} (-1)^k$ divergent.

Hence, the given series is divergent for $|x| \geq 1$ and convergent for $|x|<1$. So radius of convergence $R=1$

interval of convergence $|x|<1 \Rightarrow (-1,1)$.

4) $\sum_{k=0}^{\infty} \frac{(x+1)^k}{2^k}$.

By the root test,

$$L = \lim_{k\to\infty} \left(\left| \frac{(x+1)^k}{2^k} \right| \right)^{1/k} = \lim_{k\to\infty} \frac{|x+1|}{2} = \frac{|x+1|}{2}$$

For convergence, we must have $L = \frac{|x+1|}{2} < 1 \Rightarrow |x+1| < 2$ radius of convergence

$|x+1| < 2 \Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1$.

Check the endpoints.

For $x = -3$; $\sum_{k=0}^{\infty} \frac{(-2)^k}{2^k} = \sum_{k=0}^{\infty} (-1)^k$ divergent

For $x = 1$; $\sum_{k=0}^{\infty} \frac{2^k}{2^k} = \sum_{k=0}^{\infty} 1$ divergent.

So; radius of convergence $= 2$

interval of convergence $= (-3, 1)$

5) $\sum_{k=1}^{\infty} \frac{(x-1)^k}{\sqrt{k}}$

By the ratio test,

$$L = \lim_{k\to\infty} \left| \frac{(x-1)^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{(x-1)^k} \right| = \lim_{k\to\infty} |x-1| \cdot \sqrt{\frac{k}{k+1}} = |x-1|.$$

For convergence, we must have $L = |x-1| < 1$ radius of convergence.

$|x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$

Check the endpoints.

For $x = 0 \Rightarrow \sum_{k=1}^{\infty} (-1)^k \cdot \underbrace{\frac{1}{\sqrt{k}}}_{b_k}$ alternating series

$b_{k+1} \le b_k$

$\frac{1}{\sqrt{k+1}} \le \frac{1}{\sqrt{k}}$ ✓

$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$ ✓

By the Alternating Series test, this series is convergent.

For $x = 2 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, $p = \frac{1}{2} < 1$ By p-test this series is divergent

So; radius of convergence $R = 1$

interval of convergence $= [0, 2)$.

6) $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$

By the ratio test;

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n\ln n}{x^n} \right| = \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)}$$

$$= |x| \underbrace{\left(\lim_{n\to\infty} \frac{n}{n+1}\right)}_{1} \underbrace{\left(\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right)}_{\text{Apply L'R. } \frac{\infty}{\infty}} = |x| \cdot 1 \cdot \underbrace{\left(\lim_{n\to\infty} \frac{1/n}{1/(n+1)}\right)}_{1} = |x|.$$

For convergence, we must have $|x| < 1$. $\to$ Radius of convergence.

$|x| < 1 \Rightarrow -1 < x < 1$.

check the end points.

- For $x = -1 \Rightarrow \sum_{n=2}^{\infty} (-1)^n \cdot \underbrace{\frac{1}{n\ln n}}_{b_n}$ alternating series.

$b_n > 0$ for $n \geq 2$ ✓

consider the function $f(x) = \frac{1}{x\ln x}$, $f'(x) = \frac{-(1+\ln x)}{(x\ln x)^2} < 0$ for $x \geq 2$.

So b_n is decreasing. ✓

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n\ln n} = 0$ ✓

Hence by the Alternating series test, this series is convergent.

- For $x = 1 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\ln n}$.

Use the integral test. Consider the function $f(x) = \frac{1}{x\ln x}$ continuous for $x \geq 2$, positive for $x \geq 2$, decreasing.

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \lim_{b\to\infty} \ln|u| \Big|_{\ln 2}^{\ln b}$$

$u = \ln x$
$du = \frac{dx}{x}$

$$= \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2| = \infty$$

divergent

Hence by the integral test, this series is divergent.

So; radius of convergence $R = 1$

interval of convergence $= [-1, 1)$.

Exercises Find the radius of convergence and interval of convergence of the following series.

1) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$; Answer $R = 1/3$, $\left(-\frac{1}{3}, \frac{1}{3}\right]$

2) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$; Answer $R = 3$, $(-5, 1)$

3) $\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$; Answer $R = \infty$, $(-\infty, \infty)$

4) $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n}$; Answer $R = \sqrt{3}$, $(-\sqrt{3}, \sqrt{3})$

Representations of functions as Power Series

1) $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k$, $|x| < 1$

geometric series $a=1$, $r=x$

2) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, $|x| < 1$

geometric series $a=1$, $r=-x^2$

3) $\frac{x^3}{x+2} = x^3 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)} = x^3 \cdot \underbrace{\frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]}}_{\substack{\text{geometric} \\ \text{series } a=1 \\ r=-x/2}} = \frac{x^3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \frac{x^3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{2^{n+1}}$, $|x| < 2$

4) $\frac{1}{(1-x)^2} = ?$ From 1) we know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Differentiate each side of the equation. $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \underset{\text{or}}{=} \sum_{n=0}^{\infty} (n+1) x^n$, $|x| < 1$

5) $\ln(1-x) = ?$

$-\ln(1-x) = \int \frac{1}{1-x}\, dx = \int \left(\sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \underset{\text{or}}{=} \sum_{n=1}^{\infty} \frac{x^n}{n} + C$, $|x| <$

To find C, put $x=0$. $-\ln(1-0) = C \Rightarrow C = 0$. So

$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, $|x| < 1$.

Taylor and Maclaurin Series

Question: which functions have power series representations? How can we find such representations?

Suppose that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^2 + c_4(x-a)^4 + \cdots \quad |x-a|<R.$$

Try to determine the coefficients c_n.

Put $x=a \Rightarrow \boxed{f(a) = c_0}$

Differentiate $f(x)$.

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \quad |x-a|<R$$

Put $x=a \Rightarrow \boxed{f'(a) = c_1}$

Differentiate $f'(x)$.

$$f''(x) = 2c_2 + 2.3c_3(x-a) + 3.4\,c_4(x-a)^2 + \cdots \quad |x-a|<R$$

Put $x=a \Rightarrow f''(a) = 2c_2 \Rightarrow \boxed{c_2 = \frac{f''(a)}{2}}$

Differentiate $f''(x)$.

$$f'''(x) = 2.3c_3 + 2.3.4.\,c_4(x-a) + \cdots \quad |x-a|<R$$

Put $x=a \Rightarrow f'''(a) = 2.3\,c_3 \Rightarrow \boxed{c_3 = \frac{f'''(a)}{2.3} = \frac{f'''(a)}{3!}}$

If we continue in that way, we get $c_n = \frac{f^{(n)}(a)}{n!}$.

So,

Theorem: <u>If f has a power series</u> expansion at a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a|<R.$$

This series is called the Taylor series of f at a (or about a or centered at a). For the special case $a=0$, the Taylor series is called a Maclaurin series.

Example: Find the Maclaurin series of the function $f(x)=e^x$ and its radius of convergence.

$f(x)=e^x$ $\quad f^{(n)}(0)=e^0=1$, for all n.

$f'(x)=e^x$

$\vdots$

$f^{(n)}(x)=e^x$

Therefore, the Maclaurin series of $f(x)=e^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!}}_{a_n}.$$

By the ratio test,

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}|x|\cdot\frac{1}{n+1} = 0<1 \text{ for any } x\in\mathbb{R}.$$

So, the radius of convergence $R=\infty$.

Question: Under what circumstances is a function equal to the sum of its Taylor series?

This question will not be answered here. Please check the textbooks (Stewart Calculus - Early Transcendentals page 737). But, for this example we have $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.

Example: Find the Maclaurin series for $\sin x$.

$f(x)=\sin x \quad f(0)=0$

$f'(x)=\cos x \quad f'(0)=1$

$f''(x)=-\sin x \quad f''(0)=0$

$f'''(x)=-\cos x \quad f'''(0)=-1$

$f^{(4)}(x)=\sin x \quad f^{(4)}(0)=0$

Here we have $f^{(2k)}(0)=0$, $k=0,1,\ldots$

$f^{(2k+1)}(0)=(-1)^k$, $k=0,1,\ldots$

Hence the Maclaurin series for $\sin x$ is $\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$.

In this example, we also have $\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for all x.

Similarly, $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ for all x.