Name:

BBM 205, Spring 2015 11.3.2015

Midterm I

SOLUTIONS

(2 pts)
1. The notation $\exists! x\, P(x)$ denotes

"There exists a unique x such that P(x) is true."

What are the truth values of these statements?

a) $\exists! x\, P(x) \rightarrow \exists x\, P(x)$ $\boxed{\text{True}}$

b) $\forall x\, P(x) \rightarrow \exists! x\, P(x)$ $\boxed{\text{False}}$

(2 pts)
2. Prove that if n is a positive integer, then n is even if and only if $7n+4$ is even.

n, even $\rightarrow$ $7n+4$, even:

Since $n=2k$ for some k, integer

$7n+4 = 7 \cdot 2k + 4 = 2(7k+2)$ also even.

$7n+4$ even $\rightarrow$ n, even

$7n+4 = 2m$ where m, integer

$7n = 2m - 4 = 2(m-2)$, even

Since $2 \nmid 7$, $2 \mid n$. n is even.

(2 pts)
3. Let p, q and r be the propositions

p: You get an A on the final exam

q: You do every exercise in this book

r: You get an A in this class

Write these propositions using p, q and r and logical connectives.

a) You get an A in this class, but you do not do every exercise in this book.

b) You get an A on the final exam, you do every exercise in this book, and you get an A in this class.

c) To get an A in this class, it is necessary for you to get an A on the final exam.

d) Getting an A on the final exam and doing every exercise in this book is sufficient for getting an A in this class.

a) $r \wedge \bar{q}$

c) $r \rightarrow p$

b) $p \wedge q \wedge r$

d) $(p \wedge q) \rightarrow r$

(2 pts)
4. Show that this conditional statement is a tautology by using truth table: $(p \wedge q) \rightarrow (p \rightarrow q)$

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
0	0	0	1	1
1	0	0	0	1
0	1	0	1	1
1	1	1	1	1

(1 pt)

5. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

If $x = \frac{a}{b}$ for two integers a, b with $\gcd(a,b) = 1$,

then $\frac{1}{x} = \frac{b}{a}$ also rational.

(3 pts)

6. Determine the truth value of the statement $\forall x \exists y (xy = 1)$ if the domain for the variables consists of

a) the nonzero real numbers

b) the nonzero integers

c) the positive real numbers.

a) $\forall x$, $y = \frac{1}{x}$ so that $xy = 1$. True

b) If $x = 2$, $y = \frac{1}{2}$ so that $xy = 1$. But $y \notin \mathbb{Z}^+$

False

c) $\forall x$, $y = \frac{1}{x}$ so that $xy = 1$. True

(3pts)

7. State the converse, contrapositive, and inverse of each of these conditional statements.

a) When $\underbrace{\text{I stay up late}}_{p}$, it is necessary that $\underbrace{\text{I sleep}}_{q}$ until noon. $\}\ p \to q$

b) A $\overbrace{\text{positive integer is a prime}}^{p}$ only if it has $\underbrace{\text{no divisors other than 1 and itself}}_{q}$. $\}\ p \to q$

$\boxed{\text{converse}}$ $(q \to p)$ a) If I sleep... then I stay up late.

b) If no divisors other than 1... then prime.

$\boxed{\text{contrapositive}}$ $(\bar{q} \to \bar{p})$ a) If I do not sleep... then I do not stay up late.

b) If there is <u>any</u> divisor other than 1 and itself, then a positive integer is <u>not</u> a prime.

$\boxed{\text{inverse}}$ $(\bar{p} \to \bar{q})$ a) If I <u>do not</u> stay up late, then I <u>do not</u> sleep....

b) If a positive integer is <u>not</u> a prime, then it has <u>some</u> divisor other than 1 and itself.

(3pts)

8. Let $Q(x)$ be the statement "$x+1 > 2x$". If the domain consists of <u>all integers</u>, what are these truth values?

a) $Q(-1)$ b) $Q(1)$ c) $\forall x\ \neg Q(x)$

d) $\exists x\ Q(x)$ e) $\forall x\ Q(x)$ f) $\exists x\ \neg Q(x)$

a) $-1+1=0>-2$ <u>True</u>

b) $1+1 \not> 2\cdot 1$ <u>False</u>

c) As in (a), $\exists x\ Q(x)$ where $x=-1$, So, <u>false</u>

d) $Q(-1)$ is true as in (a). <u>True</u>

e) $Q(1)$ is false as in (b). <u>False</u>

f) $Q(1)$ is false as in (b). <u>True</u>

(2 pts)

9. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x\ (x^2 \neq x)$ b) $\forall x\ (|x| > 0)$ c) $\forall x\ (x^2 \neq 2)$

a) False.

If $x = 0$, $x^2 = x$

b) False.

If $x = 0$,

$|x| \ngtr 0$

c) False.

If $x = \sqrt{2}$,

$x^2 = 2$.

BBM 205

Spring 2015 Butunleme Exam

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Name: SOLUTIONS

1. (3 points) Solve the recurrence relation with the given initial condition below. $a_n = 2a_{n-1} + 8a_{n-2}$; $a_0 = 4$, $a_1 = 10$.

Let $a_n = t^n$ for $n \geq 0$

$t^n - 2t^{n-1} - 8t^{n-2} = 0$

$t^2 - 2t - 8 = 0$

$(t-4)(t+2) = 0$

$t_1 = 4$, $t_2 = -2$

Solution: $S_n = A t_1^n + B t_2^n$

where $S_0 = 4 = A \cdot 4^0 + B \cdot (-2)^0 = A + B$

$S_1 = 10 = A \cdot 4 - 2B$

$\times 2$

$+\ 8 = 2A + 2B$

$18 = 6A$

$\boxed{3 = A}$

$B = \frac{10 - 4A}{-2} = \frac{10-12}{-2} = \boxed{1 = B}$

Solution: $S_n = 3 \cdot 4^n + (-2)^n$

2. (3 points) (a) (1 point) How many bit strings of length seven either begin with two 0's or end with three 1's?

$A = \{\text{strings begin with } 00\}$, $|A| = 2^5$
$B = \{\text{strings end with } 111\}$, $|B| = 2^4$
$A \cap B = \{\text{strings as } 00\ldots111\}$, $|A \cap B| = 2^2$

$|A \cup B| = |A| + |B| - |A \cap B| = 2^5 + 2^4 - 2^2$

(b) (1 point) How many subsets with more than two elements does a set with 100 elements have?

$$2^{100} - \underset{\binom{100}{0}}{\underset{\shortparallel}{1}} - \binom{100}{1} - \binom{100}{2}$$

(c) (1 point) How many ways are there to select three **unordered** elements from a set with five (different) elements when **repetition is allowed**?

Same as finding the number of solutions

$x_1, x_2, x_3, x_4, x_5 \geq 0$ with $x_1 + x_2 + x_3 + x_4 + x_5 = 3$

$$\binom{3+5-1}{5-1} = \binom{7}{4}$$

3. (3 points) Suppose that there are nine students in a discrete mathematics class at a small college.

(a) (1.5 points) Show that the class must have at least five male students or at least five female students.

By pigeonhole principle, assuming the boxes are box-1 $\cong$ male and box 2 $\cong$ female, one box must contain at least $\left\lceil \frac{9}{2} \right\rceil = 5$ students.

(b) (1.5 points) Show that the class must have at least three male students or at least seven female students.

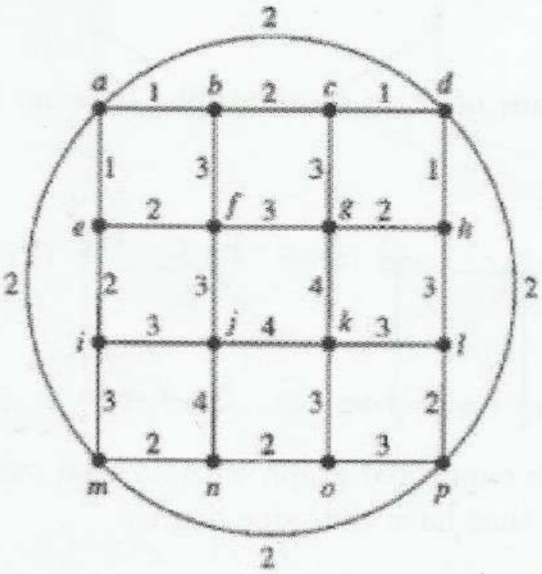
Proof by contradiction: Assume not. Then, the class has at most 2 male and at most 6 female students, adding up to at most 8 students, contradiction.

4. (4 points) Use

(a) (2 points) Kruskal's algorithm

(b) (2 points) Prim's algorithm

to find a minimum spanning tree for the weighted graph below.

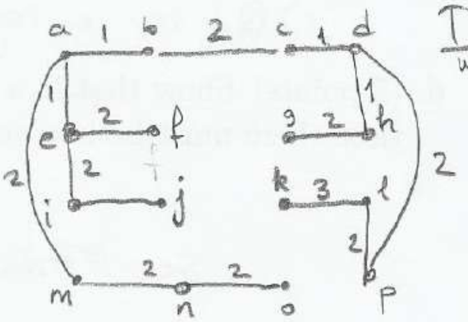


<u>Note:</u> The tree in (a), (b) is NOT unique.

a) Kruskal's algorithm

<u>edges picked in order:</u>

ab (1), ae (1), cd (1), dh (1)
bc (2), ef (2), gh (2), ei (2),
am (2), dp (2), lp (2), mn (2),
no (2), ij (3), kl (3)

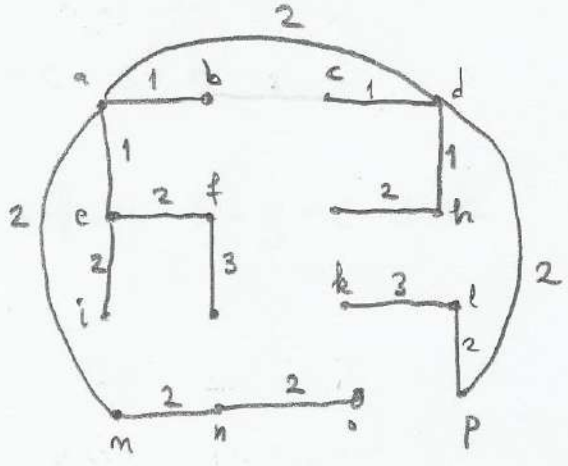


$\frac{\text{Total:}}{\text{weight}}$ $\boxed{28}$ obtained using Kruskal's Algo.

b) Prim's algorithm:

<u>edges picked in order:</u> (if we start at the edge ab)

ab(1), ae (1), ad (2), cd (1),
dh (1), am (2), ei (2), ef (2),
hg (2), mn (2), dp (2), lp (2),
no (2), fj (3), kl (3)



$\frac{\text{Total:}}{\text{weight}}$ $\boxed{28}$

5. (3 points) (a) (1.5 points) Write the chromatic number of the graphs below depending on the values of m and n.

a) K_n b) C_n c) $K_{m,n}$

a) n b) 2 if n, even
3 if n, odd
c) Always 2
independent of $m \geq 1$
and $n \geq 1$.

(b) (1.5 points) For which values of n do these graphs have an Euler circuit?

a) K_n b) C_n c) Q_n

a) K_n is $(n-1)$-regular and has E.C. only if n, odd.
b) All n
c) Q_n is n-regular and has E.C. only if n, even.

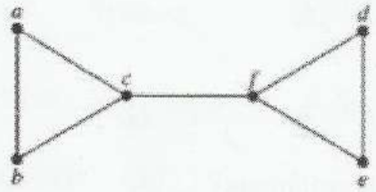
6. (2 points) Show that in a simple connected graph with at least two vertices there must be two vertices that have the same degree.

See Final Exam, question 1.

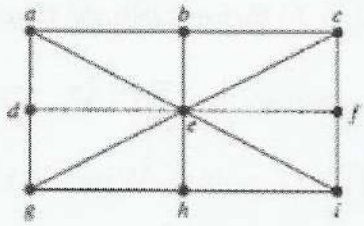
7. (4 points) (a) (2 points) Determine whether the given graph has a Hamilton cycle. Construct such a cycle when one exists.

(b) (2 points) If no Hamilton cycle exists, determine whether the graph has an Hamilton path and construct such a path if one exists.

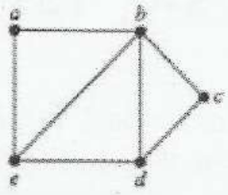
a)



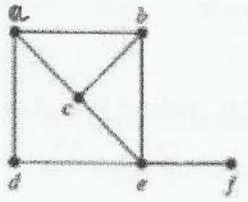
d)



b)



c)



a) No H. C.

H. path: a-b-c-f-d-e

b) H. cycle: a-b-c-d-e-a

c) No H. C.

H. path: d-a-b-c-e-f

d) H. cycle: a-b-c-f-i-h-g-e-d-a

8. (3 points) Let $P(n)$ be the statement that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

where n is an integer greater than 1. Show that $P(n)$ is true for all $n \geq 2$ using induction by following the steps below.

(a) (1 point) Show that $P(2)$ is true.

$\frac{5}{4} = 1 + \frac{1}{4} < 2 - \frac{1}{2} = \frac{3}{2}$, True.

(b) (1 point) What is the inductive hypothesis?

Assume that for all $i \leq n$, $P(i)$ is true.

(c) (1 point) Complete the inductive step.

Show that $P(n+1)$ is true using (b);

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \underset{\text{by I.H. (b)}}{<} 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \overset{?}{<} 2 - \frac{1}{n+1}$$

YES, because $\frac{1}{(n+1)^2} < \frac{1}{n} - \frac{1}{n+1}$ (arithmetic skipped)

9. (3 points) (a) (1 point) Show that $x^2 + 4x + 17$ is $O(x^3)$.

See final exam, question 13 (a), (b).

(b) (2 points) Show that x^3 is **not** $O(x^2 + 4x + 17)$.

10. (3 points) Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers.

Assume not. Then, if $\dfrac{a_1+a_2+\cdots+a_n}{n} = M$ (*),

we have $a_i < M$ for all $i=1,2,\ldots,n$.

This gives $a_1+a_2+\cdots+a_n < M\cdot n$, contradiction with (*).

11. (3 points) Show that if G is a bipartite simple graph with n vertices and e edges, then $e \leq n^2/4$.

$\boxed{\text{Solution 1}}$: Use induction on n.

Base step: True for $n=2$ or $e \leq \frac{2^2}{4} = 1$

Inductive Hypothesis (I. H.): Assume true for all bipartite simple graphs with $i \leq n$ vertices.

Inductive Step: Show for all graphs with $n+1$ vertices.

- Let G be any bipartite, simple graph with $n+1$ vertices and e edges.
- Remove from G a vertex x with minimum degree $d(x)$.
- Clearly, $d(x) \leq \frac{n+1}{2}$ (if n, odd) or $d(x) \leq \frac{n}{2}$ (if n, even)
- By I. H., $e(G-x) \leq \frac{n^2}{4}$.

$$\begin{array}{rl} & e(G-x) \leq \frac{n^2}{4} \\ + & d(x) \leq \frac{n+1}{2} \quad \text{if } n, \text{odd} \\ \hline & e(G) \leq \left\lfloor \frac{n^2}{4} + \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n^2+2n+2}{4} \right\rfloor \leq \frac{(n+1)^2}{4} \quad (n, \text{odd}) \end{array}$$

if n, even:

$$\begin{array}{rl} & e(G-x) \leq n^2/4 \\ + & d(x) \leq n/2 \\ \hline & e(G) \leq \left\lfloor \frac{n^2}{4} + \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2+2n}{4} \right\rfloor \leq \frac{(n+1)^2}{4} \end{array}$$

$\boxed{\text{Solution 2}}$: (shorter)

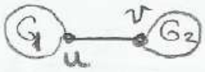
If $G=(X,Y)$ with $|X|=x$, $|Y|=y$ and $x+y=n$, then

$e(G) \leq x\cdot y = \underbrace{x\cdot(n-x)}$

maximize for x when $0 \leq x \leq n$ (by MAT123)

we observe $x(n-x) \leq \frac{n^2}{4}$

12. (3 points) Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.



$\boxed{\Rightarrow}$ If v is a cut vertex, then $G_2 - \{v\}$ is not empty. Therefore, v is not pendant.

$\boxed{\Leftarrow}$

If v is not pendant, then G_2 is not empty in the figure and removing v disconnects u from $G_2 - \{v\}$. Therefore, v is a cut vertex.

13. (3 points) Let $S(n,k)$ denote the number of functions from $\{1,\ldots,n\}$ onto $\{1,\ldots,k\}$. Show that $S(n,k)$ satisfies the recurrence relation

$$S(n,k) = k^n - \sum_{i=1}^{k-1} C(k,i)S(n,i).$$

Let $A_i = \{$functions that have exactly i numbers from $\{1,2,\ldots,k\}$ as its value$\}$

Therefore, $|A_i| = \binom{k}{i} \cdot S(n,i)$ and

$$S(n,k) = k^n - \sum_{i=1}^{k-1} |A_i| = k^n - \sum_{i=1}^{k-1} \binom{k}{i} S(n,i).$$

The number of all functions from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,k\}$.

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Spring 2015 Final Exam

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1. (3 points) Use **pigeonhole principle** to show that in any simple connected graph, there are two vertices that have the same degree.

Since the graph is connected, the degrees may vary from 1 to n-1.

Because the graph is simple, the degree is at most n-1 for each vertex.

n pigeons
$n-1$ pigeonholes (degrees) $\Big\} \left\lceil \frac{n}{n-1} \right\rceil = 2$ means there are two vertices of the same degree.

2. (2 points) Use a **proof by contraposition** to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

$$\underbrace{x+y \geq 2}_{p} \rightarrow \underbrace{x \geq 1 \text{ or } y \geq 1}_{q \vee r}$$

Contrapositive: $\overline{q \vee r} \rightarrow \overline{p}$ that is $\bar{q} \wedge \bar{r} \rightarrow \bar{p}$

It is same to prove the contrapositive as proving $p \rightarrow q \vee r$.

If $\bar{q}$: $x < 1$

AND

$\bar{r}$: $+ \; y < 1$

then $x + y < 2$ which is $\bar{p}$. Done.

3. (7 points) (a) (1 point) How many license plates can be made using either three letters followed by three digits or four letters followed by two digits? 10 digits, 29 letters : $29^3 \cdot 10^3 + 29^4 \cdot 10^2$

(b) (.5 points) How many different functions are there from a set with 10 elements to a set with 5 elements? 5^{10}

(c) (1.5 points) How many permutations of the letters ABCDEFG contain

a) the string BCD? A, BCD, E, F, G $\boxed{5!}$

b) the strings ABC and CDE? ABCDE, F, G $\boxed{3!}$

c) the strings CBA and BED? CBA CBED not possible $\boxed{0}$

(d) (1 point) Show that if n and k are integers with $1 \le k \le n$, then $\binom{n}{k} \le n^k/2^{k-1}$.

k of them

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1) \le n^k}{k\cdot(k-1)\cdots 3\cdot 2\cdot 1 \ge 2^{k-1}}$$ Therefore, $\binom{n}{k} \le \frac{n^k}{2^{k-1}}$

(e) (1 point) How many different ways are there to choose 6 donuts from the 21 varieties at a donut shop? $\binom{21+6-1}{6-1} = \binom{26}{5}$

(f) (1 point) How many different strings can be made from the letters in ABRACADABRA, using all letters? 5A's, 2B's, 2R's, 1 C, 1D } 11 letters $\boxed{\frac{11!}{5!\,2!\,2!\,1!\,1!}}$

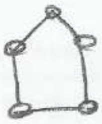
(g) (1 point) A bowl contains 10 red balls and 10 blue balls. A person selects balls at random without looking at them. How many balls must be selected to be sure of having at least three balls of the same color?

By pigeonhole principle, to have at least 3 pigeons in the same pigeonhole (color), the smallest n that satisfies $\left\lceil \frac{n}{2} \right\rceil = 3$ is $\boxed{n=5.}$

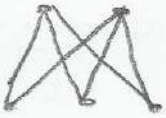
4. (8 points) (a) (2 points) Draw these graphs: K_4, C_5, $K_{2,3}$, Q_3.

K_4:

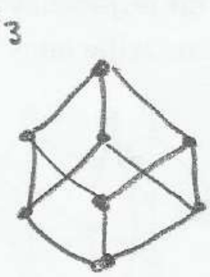
C_5:



$K_{2,3}$:



Q_3



(b) (1.5 points) For which values of n are these graphs bipartite?
a) K_n b) C_n c) Q_n

K_n is bipartite if $n = 1, 2$
C_n is bip. if n, even
Q_n is bip. for all $n \geq 1$.

(c) (2 points) How many vertices and how many edges do these graphs have?
a) K_n b) C_n c) $K_{m,n}$ d) Q_n

a) n vxs, $\binom{n}{2}$ edges
b) n vxs, n edges
c) $m+n$ vxs, $m \cdot n$ edges
d) 2^n vxs, $n \cdot 2^{n-1}$ edges

(d) (1.5 points) Find the degree sequence of each of the following graphs:
a) K_4 b) C_5 c) $K_{2,3}$

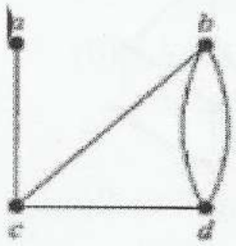
a) 3, 3, 3, 3
b) 2, 2, 2, 2, 2
c) 2, 2, 2, 3, 3

(e) (1 point) Determine whether each of these sequences is the degree sequence of a graph. For those that are, draw a graph having the given degree sequence.
a) 5,4,3,2,1,0
b) 1,1,1,1,1,1

a) $5+4+3+2+1 = 15$, not even. Since degree sum is even for all graphs, this deg. seq. is <u>NOT</u> graphical.

b)  is an example.

5. (2 points) Represent the graph below using
   a) an adjacency list,
   b) an adjacency matrix.

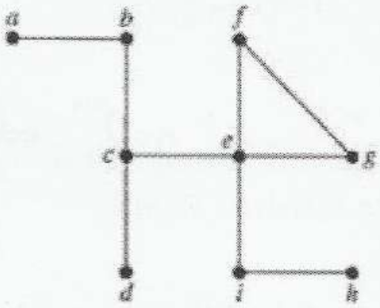


a)

vertex, v	its neighbors, $N(v)$
a	c
b	c, d
c	a, b, d
d	b, c

b)
$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 1 & 2 \\ c & 1 & 1 & 0 & 1 \\ d & 0 & 2 & 1 & 0 \end{array}$$

6. (1 point) Find all cut-vertices of the graph below.



cut vertices are b, c, e, i.

7. (1 point) A simple graph is called k-regular if every vertex has degree k. Show that if a bipartite graph $G = (V, E)$ is k-regular for some positive integer k and (V_1, V_2) is a bipartition of V, then $|V_1| = |V_2|$.

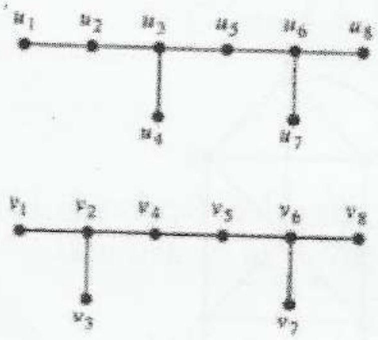
(1). The number of edges is the degree sum of the vertices in V_1, therefore $k \cdot |V_1|$.

(2). The number of edges is also the degree sum of the vertices in V_2, therefore $k \cdot |V_2|$.

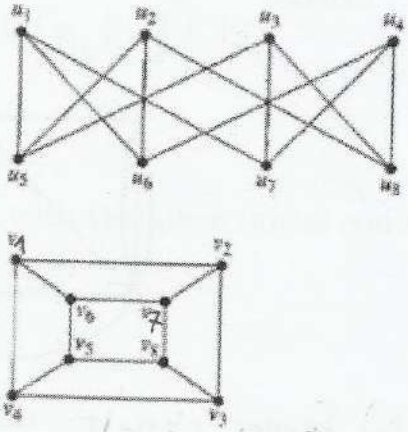
By (1) and (2) number of edges $= k \cdot |V_1| = k \cdot |V_2|$.
Therefore, $|V_1| = |V_2|$.

8. (4 points) Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

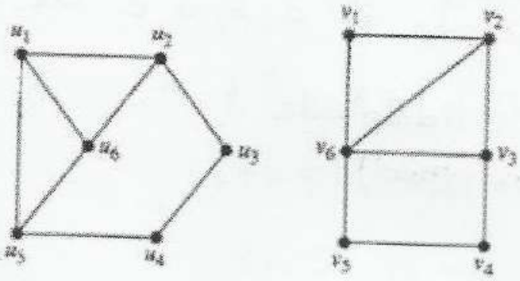
a)



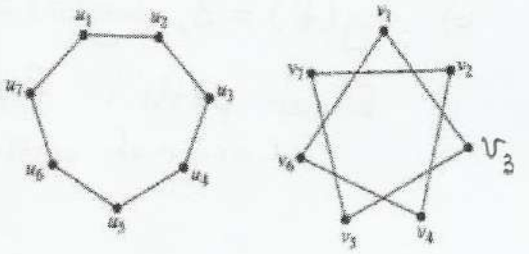
b)



c)



d)



a) NOT isomorphic.
One reason is, on graph - I, there are two paths of length 5. On graph - II, there are four paths of length 5.

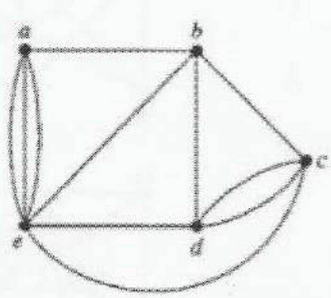
b) isomorphic. Isomorphic relation defined by the function f as: $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_7$, $f(u_5) = v_2$, $f(u_6) = v_4$, $f(u_7) = v_6$, $f(u_8) = v_8$

c) NOT isomorphic: One reason is, on graph - I there is no vertex of degree 4. On graph - II, there is a vertex of degree 4.

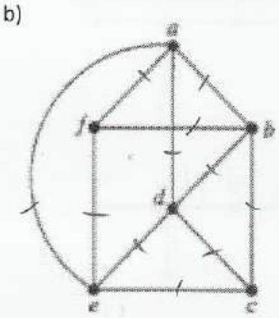
d) isomorphic. Isomorphic relation defined by the function f as: $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_7$, $f(u_5) = v_2$, $f(u_6) = v_4$, $f(u_7) = v_6$.

9. (2 points) Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

a)



b)



a) Euler circuit: a b c e d c d b e a e a
(order of vertices to travel)

b) deg(f) = 3, deg(c) = 3, degrees of a, b, d, e are even.

Euler path : f b a e c b d a f e d c
(order of vertices to travel)

10. (2 points) (a) (1 point) For which values of m and n does the complete bipartite graph $K_{m,n}$ have a Hamilton circuit?

for all $m = n$, $m, n \geq 1$

(b) (1 point) Can you find a simple graph with n vertices (and $n \geq 3$) that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n-1)/2$?

As observed in (a), $K_{m,r}$ does not have a Ham. circ. if $m = r+1$. This graph has $n = m + r = 2r+1$ vertices and each vertex has degree either $r = \frac{n-1}{2}$ or $m = \frac{n-1}{2} + 1$.

11. (3 points) (a) (1 point) Derive a recurrence relation for $C(n,k) = \binom{n}{k}$, the number of k-element subsets of an n-element subset. Specifically, write $C(n+1,k)$ in terms of $C(n,i)$ for appropriate i.

$$C(n+1,k) = C(n,k) + C(n,k-1)$$

(b) (2 points) Solve the recurrence relation with the given initial condition below. $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 5$, $a_1 = 16$.

Let $a_n = t^n$ for $n \geq 0$

$$t^n - 7t^{n-1} + 10t^{n-2} = 0$$

$$t^2 - 7t + 10 = 0$$

$$(t-5)(t-2) = 0$$

$$t_1 = 5, \quad t_2 = 2$$

Solution: $S_n = A t_1^n + B t_2^n$

where $S_0 = 5 = A \cdot 5^0 + B \cdot 2^0 = A + B$ (I)

$S_1 = 16 = A \cdot 5^1 + B \cdot 2^1 = 5A + 2B$ (II)

By (I) $\cdot$ 2 : $10 = 2A + 2B$

Subtract from (II): $16 = 5A + 2B$

$$\begin{array}{r} 16 = 5A + 2B \\ -\ 10 = 2A + 2B \\ \hline 6 = 3A \end{array}$$

$$\boxed{\begin{array}{l} 2 = A \\ 3 = B \end{array}}$$

$$\boxed{a_n = 2 \cdot 5^n + 3 \cdot 2^n}$$

12. (2 points) Let f_i be the ith Fibonacci number. **Use induction** to prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

Base step: $n=1$ $\quad \underset{1}{f_1^2} = \underset{1}{f_1} \cdot \underset{1}{f_2}$ True.

Ind. Hypo. : For $i \leq n$, assume that $f_1^2 + f_2^2 + \cdots + f_i^2 = f_i f_{i+1}$

Ind. Step: For $n+1$, $f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 \overset{\text{By Ind. Hypo.}}{=} f_n \cdot f_{n+1} + f_{n+1}^2 =$

$= f_{n+1}(f_n + f_{n+1}) = f_{n+1} \cdot f_{n+2}$ True.

13. (3 points) (a) (1 point) Show that $x^2 + 4x + 17$ is $O(x^3)$.

$x^2 + 4x + 17 \leq 22x^3$ for all $x \geq 1$.

(b) (2 points) Show that x^3 is **not** $O(x^2 + 4x + 17)$.

Proof by contradiction:

Assume that $x^3 = O(x^2 + 4x + 17)$. By definition of O-notation, there are constants $C^{>0}$ and $k^{>1}$ such that $x^3 \leq C \cdot (x^2 + 4x + 17)$ for all $x \geq k$.

Divide both sides by x^3, we get

$$1 \leq C \cdot \left(\frac{1}{x} + \frac{4}{x^2} + \frac{17}{x^3}\right) \leq C \cdot \frac{22}{x}$$

But if $x = 23 \cdot C$, then $1 \leq C \cdot \frac{22}{23 \cdot C} = \frac{22}{23}$ is NOT true

Contradiction.

BBM 205 - Discrete Structures Midterm
Date: 19.11.2015, Time: 10:00 - 11:45

Ad Soyad / Name:
Ögrenci No /Student ID:
Şube /Section:

Question	1	2	3	4	5	6	Total
Points	15	15	10	20	20	20	100
Grade							

1. (15 points) There are 13 squares of side 1 positioned inside a circle of radius 2. Prove that at least 2 of the squares have a common point. (Let $\pi = 3.14$. The area inside a circle with radius r is πr^2.)

Solution: If no 2 of the squares have a common point, then the total area they cover is equal to the sum of their individual areas, which is 13. On the other hand, a circle of radius 2 has area

$$\pi r^2 = \pi 2^2 = 4\pi \approx 12.56 < 13,$$

so it's impossible for the squares to cover an area of 13. Thus in fact some 2 of them must share a common point.

2. (15 points) Let $f(n) = n + 1$ and $g(n) = n^2$. Prove that $g(n) \neq O(f(n))$.

Solution: Suppose that

$$n^2 \leq c(n+1)$$

eventually, that is, for all n greater than or equal to some number n_0. Dividing by n, we have

$$n \leq c(1 + \frac{1}{n}) \leq 2c$$

for all $n \geq n_0$. This is impossible because $2c$ is a constant.

3. (10 points) Find the minimum number of ordered pairs of integers (a, b) that guarantees that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \equiv a_2 \pmod 5$ and $b_1 \equiv b_2 \pmod 5$. Explain your answer.

Solution: There are $k = 5 \cdot 5 = 25$ different (a, b) pairs. If we pick N pairs such that

$$\left\lceil \frac{N}{k} \right\rceil \geq 2,$$

then there will be at least two identical pairs. The smallest N satisfying this condition is 26.

4. (20 points) Find the solution of the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = 7$.

Solution: The characteristic polynomial is

$$r^2 - 8r + 16$$

Factoring gives us

$$r^2 - 8r + 16 = (r-4)(r-4),$$

so $r_0 = 4$.
Using the initial conditions, we get a system of equations

$$\begin{aligned} a_0 &= 1 &&= \alpha_1 \\ a_1 &= 7 &&= 4\alpha_1 + 4\alpha_2. \end{aligned}$$

Solving the second, we get $\alpha_2 = 3/4$. And the solution is

$$a_n = 4^n + \frac{3}{4} n 4^n.$$

5. (20 points) Let x be any real number greater than -1. Prove that

$$(1+x)^n \geq 1+nx$$

for every integer $n \geq 0$.

Solution: Let x be any real number greater than -1. Let $P(n) : (1+x)^n \geq 1+nx$.
Basis step: To verify that $P(0)$ is true: Notice that

$$(1+x)^0 = 1 \geq 1 + 0 \cdot x,$$

so $P(0)$ is true.
Induction step: Assume that $P(k)$ is true. We show that $P(k+1)$ is true. By the inductive hypothesis, we have $(1+x)^k \geq 1+kx$. Then,

$$(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx) = 1+(k+1)x+kx^2 \geq 1+(k+1)x.$$

Therefore, $P(k+1)$ is also true. Thus by Mathematical Induction $(1+x)^n \geq 1+nx$ for every integer $n \geq 0$.

6. (20 points) A bagel shop (simit dükkanı) has onion (Soğanlı) bagels, poppy seed (Haşhaşlı) bagels, egg (Yumurtalı) bagels, salty (Tuzlu) bagels, pumpernickel (Çavdarlı) bagels, sesame seed (Susamlı) bagels, raisin (Üzümlü) bagels, and plain (Sade) bagels. How many ways are there to choose

(a) (4 points) six bagels?

Solution: $\binom{6+7}{7} = \binom{13}{7}$

(b) (4 points) a dozen bagels? (a dozen = 12)

Solution: $\binom{12+7}{7} = \binom{19}{7}$

(c) (4 points) two dozen bagels?

Solution: $\binom{24+7}{7} = \binom{31}{7}$

(d) (4 points) a dozen bagels with at least one of each kind?

Solution: $\binom{4+7}{7} = \binom{11}{7}$

(e) (4 points) a dozen bagels with at least three egg bagels and no more than two salty bagels?

Solution: $\binom{6+9}{6} + \binom{6+8}{6} + \binom{6+7}{6} = \binom{15}{6} + \binom{14}{6} + \binom{13}{6}$

BBM 205

Spring 2015 Exam 3

SHOW YOUR WORK TO RECEIVE FULL CREDIT.
KEEP YOUR CELLPHONE TURNED OFF.

Name: SOLUTIONS

1. (4 points) For each $f(n)$ and $g(n)$, fill in the blank $f(n) = ...(g(n))$ and $g(n) = ...(f(n))$ with one of the symbols O, Ω or Θ. There can be more than one correct symbol.

   (a) (1 point) $f(n) = 5\log n$, $g(n) = n$; $f(n) = O(g(n))$, $g(n) = \Omega(f(n))$

   (b) (1 point) $f(n) = 2^n + n^2$, $g(n) = n^3 + 3^n$: $f(n) = O(g(n))$, $g(n) = \Omega(f(n))$

   (c) (1 point) $f(n) = (x^3+1)/(x+1)$, $g(n) = x^2$: $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$

   So, $f(n) = \Theta(g(n))$
   $g(n) = \Theta(f(n))$

   (d) (1 point) $f(n) = n!$, $g(n) = 2^n$

   $f(n) = \Omega(g(n))$, $g(n) = O(f(n))$

2. (3 points)
Show that, for any positive integer k, $1^k + 2^k + 3^k + \cdots + n^k = \Theta(n^{k+1})$ by using the definition of Θ-notation.

Step 1: Show $\sum_{i=1}^{n} i^k = O(n^{k+1})$

Since $1^k + 2^k + \dots + n^k \leq 1 \cdot n \cdot n^k = n^{k+1}$, true.

Step 2: Show $\sum_{i=1}^{n} i^k = \Omega(n^{k+1})$

$$1^k + 2^k + \dots + n^k \geqslant \underbrace{\left\lceil \frac{n}{2} \right\rceil^k + \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)^k + \dots + (n-1)^k + n^k}_{\left\lfloor \frac{n}{2} \right\rfloor \text{ of them}} \geqslant \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil^k \geqslant$$

$$\geqslant \frac{n^{k+1}}{2^{k+1}} = \underbrace{\frac{1}{2^{k+1}}}_{\text{a constant, since } k \text{ is fixed}} \cdot n^{k+1}$$

$\boxed{\text{By Steps 1 and 2: } \sum_{i=1}^{n} i^k = \Theta(n^{k+1}).}$

3. (3 points) Find the smallest integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

(a) (1 point) $f(x) = 2x^3 + x^2 \log x$, $f(x) = O(x^3)$.

(b) (1 point) $f(x) = 3x^3 + (\log x)^4$. $f(x) = O(x^3)$

(c) (1 point) $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$ $f(x) = O(x)$.

4. (2 points) (a) (1 point) Let d be a positive integer. Use pigeonhole principle to show that among any group of $d+1$ integers there are two with exactly the same remainder when they are divided by d.

There are $d+1$ integers (pigeons) and d remainders (pigeonholes). Since $\left\lceil \frac{d+1}{d} \right\rceil = 2$, there will always be two integers with the same remainder.

(b) (1 point) Let $n_1, n_2, \ldots, n_t$ be positive integers. Use pigeonhole principle to show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some i $(1 \le i \le t)$, the ith box contains at least n_i objects.

If this is not true, then

box 1 contains $\le n_1 - 1$ objects

box 2 contains $\le n_2 - 1$ objects

$\vdots$ $\vdots$

box t contains $\le n_t - 1$ objects,

$+$

then all boxes contain $\le (n_1 + n_2 + \ldots + n_t) - t$ objects, contradiction.

5. (2 points)

(a) (1 point) Find the coefficient of x^5 in $(4-3x)^{21}$.

$(4-3x)^{21} = \sum_{i=0}^{21} 4^i (-3x)^{21-i} \cdot \binom{21}{i}$ } when $i=16$, coefficient of x^5 is $\boxed{\binom{21}{16} \cdot 4^{16} \cdot (-3)^5.}$

(b) (1 point) Find the coefficient of $x^3y^2z^5$ in $(x+y+z)^{10}$.

$(x+y+z)^{10} = \sum_{i,j,k} x^i y^j z^k \cdot \binom{10}{i,j,k}$. When $i=3, j=2, k=5$ the coeff. of $x^3y^2z^5$ is $\boxed{\binom{10}{3,2,5} = \frac{10!}{3!\,2!\,5!}}$

6. (2 points) Use Pascal's identity to prove the following whenever n and r are positive integers.

$$\sum_{k=0}^{k=r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

Claim:

$$\sum_{k=0}^{k=r} \binom{n+k}{k} = \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}$$

Show using Induction:

For $r=0$ (Base step): $\sum_{k=0}^{k=0} \binom{n+k}{k} = \binom{n}{0} = 1 \quad \binom{n+0+1}{0} = 1$ True.

Ind. Hypo. Assume $\sum_{k=0}^{k=i} \binom{n+k}{k} = \binom{n+i+1}{i}$ for all $i < r$.

Ind. Step: $\sum_{k=0}^{k=r} \binom{n+k}{k} = \left[\sum_{k=0}^{k=r-1} \binom{n+k}{k}\right] + \binom{n+r}{r} \underset{\text{By Ind. Hypo.}}{=} \binom{n+r-1+1}{r-1} + \binom{n+r}{r} \overset{\text{By Pascal's Identity}}{=} \binom{n+r+1}{r}$

7. (1 point) How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are nonnegative integers?

The solutions for $x_1+x_2+x_3 \leq 11$, will also be solutions for $x_1+x_2+x_3+x_4 = 11$ (*), where x_1, x_2, x_3, x_4 are nonnegative integers and $x_4 := 11-(x_1+x_2+x_3)$.

The number of solutions for (*) is $\binom{11+3}{3} = \binom{14}{3}$.

8. (1 point) Seven women and nine men are on the faculty in the computer science department at a school.

   (a) (.5 points) How many ways are there to select a committee of <u>five</u> members of the department if at least one woman must be on the committee?

   $\binom{16}{5} - \binom{9}{5}$ or $\sum_{i=1}^{5} \binom{7}{i}\binom{9}{5-i}$

   (b) (.5 points) How many ways are there to select a committee of <u>five</u> members of the department if at least one woman and at least one man must be on the committee?

   $\binom{7}{1} \cdot \binom{9}{1} \cdot \binom{14}{3}$

9. (2 points) How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

   (a) (.5 points) both the balls and boxes are labelled?

   $\binom{7}{5} \cdot 5! = P(7,5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$ ways

   (b) (.5 points) the balls are labelled, but the boxes are unlabelled?

   1 way

   (c) (.5 points) the balls are unlabelled, but the boxes are labelled?

   $\binom{7}{5}$

   (d) (.5 points) both the balls and boxes are unlabelled?

   1 way

Name:

BBM 205, Spring 2015 11.3.2015

Midterm I

(2 pts)

1. The notation $\exists! x\, P(x)$ denotes

"There exists a unique x such that P(x) is true."

What are the truth values of these statements?

a) $\exists! x\, P(x) \rightarrow \exists x\, P(x)$

b) $\forall x\, P(x) \rightarrow \exists! x\, P(x)$

(2 pts)

2. Prove that if n is a positive integer, then n is even if and only if 7n+4 is even.

(2 pts)

3. Let p, q and r be the propositions

p: You get an A on the final exam

q: You do every exercise in this book

r: You get an A in this class

Write these propositions using p, q and r and logical connectives.

a) You get an A in this class, but you do not do every exercise in this book.

b) You get an A on the final exam, you do every exercise in this book, and you get an A in this class.

c) To get an A in this class, it is necessary for you to get an A on the final exam.

d) Getting an A on the final exam and doing every exercise in this book is sufficient for getting an A in this class.

a) c)

b) d)

(2 pts)

4. Show that this conditional statement is a tautology by using truth table: $(p \wedge q) \rightarrow (p \rightarrow q)$

(1 pt)

5. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

(3 pts)

6. Determine the truth value of the statement $\forall x \exists y (xy = 1)$ if the domain for the variables consists of

a) the nonzero real numbers

b) the nonzero integers

c) the positive real numbers.

(3pts)

7. State the converse, contrapositive, and inverse of each of these conditional statements.

   a) When I stay up late, it is necessary that I sleep until noon.

   b) A positive integer is a prime only if it has no divisors other than 1 and itself.

(3pts)

8. Let $Q(x)$ be the statement "$x+1 > 2x$". If the domain consists of all integers, what are these truth values?

   a) $Q(-1)$ b) $Q(1)$ c) $\forall x \neg Q(x)$

   d) $\exists x\, Q(x)$ e) $\forall x\, Q(x)$ f) $\exists x \neg Q(x)$

(2 pts)

9. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x\ (x^2 \neq x)$ b) $\forall x\ (|x| > 0)$ c) $\forall x\ (x^2 \neq 2)$

BBM 205 - Discrete Structures: Midterm 1 - ANSWERS
Date: 3.11.2015, Time: 16:00 - 17:30

Ad Soyad / Name:
Ögrenci No /Student ID:
Şube /Section:

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	12	4	8	24	6	12	9	15	10	100
Score:										

1. Determine the truth value of each of these statements if the domain consists of real numbers. **Give a short explanation for each answer to receive full credit.**

   (a) (3 points) $\exists x(x^3 = -1)$

   Solution: True. Example, $x = -1$.

   (b) (3 points) $\exists x(x^4 < x^2)$

   Solution: True. Example, any $x \in (-1, 1)$.

   (c) (3 points) $\forall x((-x^2) = x^2)$

   Solution: False. Example, any real number x other than 0.

   (d) (3 points) $\forall x(2x > x)$

   Solution: False. Example, any real number x that is not positive.

2. (4 points) What is the cardinality of each of these sets?
   a) $\{a\}$
   b) $\{\{a\}\}$
   c) $\{a, \{a\}\}$
   d) $\{a, \{a\}, \{a, \{a\}\}\}$

> **Solution:** 1, 1, 2, 3

3. Determine whether each of these arguments is valid. If it is valid, show the steps of your conclusion. If it is not valid, give a logical error.
   (a) (4 points)

If n is a real number such that $n > 2$, then $n^2 > 4$.
$n \leq 2$

$n^2 \leq 4$

> **Solution:** This conclusion is not valid. One example is when $n = -2.01$. Although $n \leq 2$, it is not true that $n^2 \leq 4$. This particular question shows that the inverse of a true statement is not always true. Here, the inverse of the first statement is false as shown by this example.

   (b) (4 points)

If it snows today, the university will close.
The university is not closed today.

Therefore, it did not snow today.

> **Solution:** There are various ways to show that this conclusion is valid.
> Let p be that *it snows today* and q be that *the university is close*. By modus tollens, we know that
>
> $$\begin{array}{l} p \implies q \\ \underline{\neg q \qquad} \\ \\ \neg p \end{array}$$

4. Answer the following questions. Write your answer clearly and do not calculate the any number (for example, leave 2^{10} as it is instead of calculating $2^{10} = 1024$).

(a) (4 points) How many ways are there to travel in xyz space from the origin $(0, 0, 0)$ to the point $(12, 8, 4)$ by taking steps **two units** in the positive x, positive y, or positive z direction?

Solution: Since each step has two units, there are 6 steps in x-dir., 4 steps in y-dir. and 2 steps in z-dir. to take. Thus, the answer is the following.

$$\frac{12!}{6!4!2!}$$

(b) (4 points) How many ways are there to pack twelve **identical** DVD's into four **distinguishable** boxes so that each box contains at least one DVD?

Solution: Initially by placing one DVD in each box, we can distribute the remaining eight DVD's into the boxes in the following number of ways.

$$\binom{8+4-1}{4-1} = \binom{11}{3}$$

(c) (4 points) Find the coefficient of x^4 in $(2 + 5x)^{13}$.

Solution: By the binomial theorem, we have

$$(2 + 5x)^{13} = \sum_{i=0}^{i=13} \binom{13}{i} 2^i (5x)^{13-i}.$$

Here, the power of x is 4 only when $i = 9$. So, the answer is $\binom{13}{9} 2^9 5^4$.

(d) (4 points) How many functions are there $f : A \to B$, where $|A| = 3$, $|B| = 7$?

Solution: Let $A = \{a_1, a_2, a_3\}$. Since we have 7 choices for each element in the string $(f(a_1), f(a_2), f(a_3))$, by product rule, the answer is 7^3.

(e) (4 points) How many one-to-one functions are there $f : A \to B$, where $|A| = 3$, $|B| = 7$?

Solution: Let $A = \{a_1, a_2, a_3\}$. Since the values in the string $(f(a_1), f(a_2), f(a_3))$ are pairwise different in a one-to-one function, the answer is $P(7, 3) = 7 \times 6 \times 5$.

(f) (4 points) How many subsets with an odd number of elements does a set with 10 elements have?

Solution:

$$\sum_{i=0}^{i=4} \binom{10}{2i+1}$$

5. Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ and if the domain for x and y consists of the following sets. Give a short explanation for your answer.

   (a) (2 points) the positive real numbers

   Solution: False. Let $P(x, y)$ be the statement $x \leq y^2$. The explanation requires a proof by contradiction. Assume that there is an $x = \epsilon$ that satisfies $P(x, y)$ for all y in the domain. Since $P(\epsilon, 1)$ is also true, clearly $0 < \epsilon \leq 1^2 = 1$. However, this implies that $\epsilon \not\leq (\epsilon/2)^2$ and $P(\epsilon, \epsilon/2)$ is not true, a contradiction.

   (b) (2 points) the integers

   Solution: True. Let x be an integer at most 0 and $x \leq y^2$ is satisfied for all integer y.

   (c) (2 points) the nonzero real numbers

   Solution: True. Let x be a negative integer and $x \leq y^2$ is satisfied for all nonzero real number y.

6. (12 points) Determine whether each of these statements is true or false.

a)$0 \in \emptyset$	b)$\emptyset \in \{0\}$
c)$\{0\} \subset \emptyset$	d)$\emptyset \subset \{0\}$
e)$\{0\} \in \{0\}$	f)$\{\emptyset\} \subset \{\emptyset\}$

Solution: F, F, F, T, F, T

7. (9 points) Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$.

Solution: This can be shown by using pigeonhole principle.
Let pigeons be the elements of S and pigeonholes be the values of the function f. Since there are $|S|$ pigeons and at most $|T|$ pigeonholes, the fact that $|S| > |T|$ implies that the statement is true.

8. How many bit strings of length 10 contain

(a) (4 points) an equal number of 0's and 1's?

Solution:

$$\binom{10}{5}$$

(b) (5 points) at most four 1's?

Solution:

$$\sum_{i=0}^{i=4} \binom{10}{i}$$

(c) (6 points) at least four 1's, where the number of 1's is even?

Solution:

$$\sum_{i=2}^{i=5} \binom{10}{2i}$$

9. How many permutations of the letters ABCDEFGH contain

(a) (2 points) the string ED?

Solution:

$$7!$$

(b) (2 points) the strings BA and FGH?

Solution:

$$5!$$

(c) (2 points) the strings CAB and BED?

Solution:

$$4!$$

(d) (2 points) the strings BCA and ABF?

Solution:

$$0$$

(e) (2 points) the strings AB, DE and GH?

Solution:

$$5!$$

1)

In Orange County, 51% of the adults are males. (It doesn't take too much advanced mathematics to deduce that the other 49% are females.) One adult is randomly selected for a survey involving credit card usage.

a) Find the prior probability that the selected person is a male.
b) It is later learned that the selected survey subject was smoking a cigar. Also, 9.5% of males smoke cigars, whereas 1.7% of females smoke cigars (based on data from the Substance Abuse and Mental Health Services Administration).

Use this additional information to find the probability that the selected subject is a male.

Solution

Let's use the following notation:

M = male $\quad\quad$ $\overline{M}$ = female (or not male)

C = cigar smoker $\quad\quad$ $\overline{C}$ = not a cigar smoker.

a. Before using the information given in part b, we know only that 51% of the adults in Orange County are males, so the probability of randomly selecting an adult and getting a male is given by $P(M) = 0.51$.

b. Based on the additional given information, we have the following:

$P(M) = 0.51$	because 51% of the adults are males
$P(\overline{M}) = 0.49$	because 49% of the adults are females (not males)
$P(C\|M) = 0.095$	because 9.5% of the males smoke cigars (That is, the probability of getting someone who smokes cigars, given that the person is a male, is 0.095.)
$P(C\|\overline{M}) = 0.017.$	because 1.7% of the females smoke cigars (That is, the probability of getting someone who smokes cigars, given that the person is a female, is 0.017.)

Let's now apply Bayes' theorem by using the preceding formula with M in place of A, and C in place of B. We get the following result:

$$P(M \mid C) = \frac{P(M)\cdot P(C|M)}{[P(M)\cdot P(C|M)]+[P(\overline{M})\cdot P(C|\overline{M})]}$$

$$= \frac{0.51\cdot 0.095}{[0.51\cdot 0.095]+[0.49\cdot 0.017]}$$

$$= 0.85329341$$

$$= 0.853 \text{ (rounded)}$$

2)

Proof that, if a binary tree of height h has t terminal vertices, then

$$\lg t \leq h$$

Solution

Here, log function is base 2. We want to show that $t \leq 2^h$.

Using induction, base case with h=0 is clear.

Assuming that the statement is true when height is h-1, we consider a tree with height h, call it T. Remove all terminal vertices and call this new tree T'. The height of T' is h-1 and by inductive hypothesis, there are at most 2^{h-1} terminal vertices. Since, the removed terminal vertices in T are the children of terminal vertices in T', there can be at most 2^h of them.

3)

Assume that the function f, g and h take on only positive values. Prove that, if $f(n) = \Theta(g(n))$ and $g(n) = \Theta(\mathrm{h(n)})$, then $f(n) = \Theta(h(n))$.

Solution

There are constants c_1 and c_2 such that

$$c_1 h(n) \leq f(n) \leq c_2 h(n)$$

for large enough n. The constants in both inequalities are obtained by the given assumptions. (The details are omitted here.)

4) Show that a simple graph is a tree if and only if it is connected, but the deletion of any of its edges makes the graph disconnected.

Solution

→ Direction is clear, since in any tree there is a unique path between any pair of vertices and if we remove the edge uv, u and v are disconnected.

<-- Show by contrapositive: "If not a tree, then either the graph is not connected or there is a cycle by definition." Since the graph is connected, assume there is a cycle C. Then the deletion of any edge uv on that cycle would leave the graph still connected. Because if any two vertices x and y were connected using a path P that contains the edge uv, then after deleting the edge uv, x and y are still connected by using a path in the union of P and the remaining u,v-path on C. So, the deletion of any edge does not make the graph disconnected.

5) Use mathematical induction to show that given n+1 non-negative integers none exceeding 2n, there is at least one integer that divides another integer in this set.

Solution

Clearly true for n=1.

Assume true for all i less than or equal to n. We will prove for n+2 numbers not exceeding

2(n+1).
Case 1: If all numbers are at most 2n, then we are done by Ind. Hypo.
Case 2: If all numbers but one number are at most 2n, again done by Ind. Hypo.
Case 3: If at least two numbers are greater than 2n, then there are exactly two numbers greater than 2n. This means 2n+1 and 2n+2 in this set.
If n+1 is also in this set, then we are done, because n+1 divides 2n+2.
Otherwise, we remove 2n+2 from the set and add n+1 to the set. Call this new set A. By ind. Hypo, there are two integers in A that divide each other. If one of these integers is n+1, then the other number divides n+1 and this number divides also 2n+2, done.

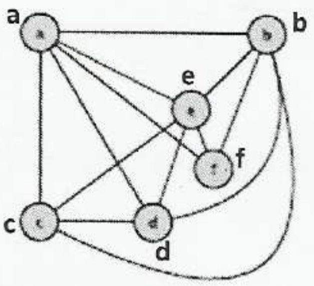
BBM 205 - Discrete Structures Final Exam
Date: 7.1.2016, Time: 9:30 - 11:00 (90 minutes)

Please solve 5 out of 6 questions. Circle the questions you answered.

Ad Soyad / Name: SOLUTIONS
Ögrenci No /Student ID:
Şube /Section: 1, 2, Hasmet Gürçay (Morning, Afternoon)
3, Lale Özkahya

Question	1	2	3	4	5	6	Total
Points	20	20	20	20	20	20	100
Grade							

1. (20 points) Consider the following graph G.



(a) Find a Hamilton path in G.

(b) Find a subgraph of G that is isomorphic to K_4.

(c) Find a subgraph of G that is homeomorphic to K_5. Is G planar?

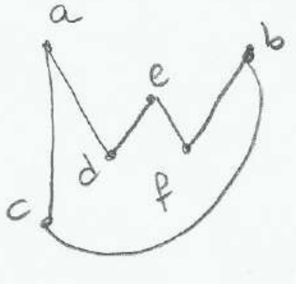
(d) Is G Eulerian? Justify your answer.

(e) Find two non-isomorphic spanning trees of G.

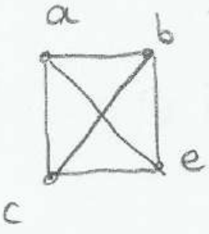
(f) Give a list of 5 non-negative integers that cannot be the degree list of a graph on 5 vertices.

You can use the back of this page to answer Question 1.

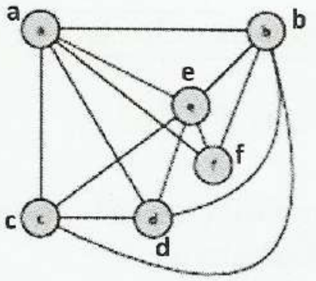
a)



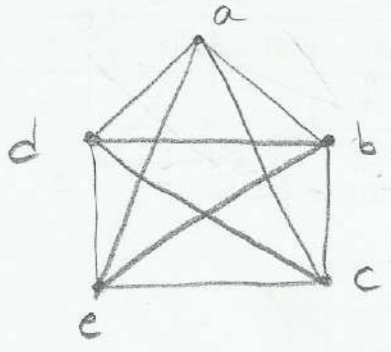
b)



(The figure copied from Page 1)



c)

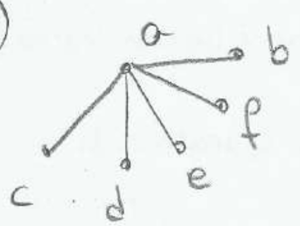


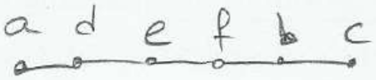
$\cong K_5$

By Kuratowski's Thm., G is not planar.

d) There are vertices a, b, e, f that have an odd degree. Therefore, G is not Eulerian.

e)





f) $\{1, 1, 1, 1, 1\}$

2. (20 points) Find the solution of the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = 7$.

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

$$t^2 - 8t + 16 = 0$$

$$(t-4)^2 = 0$$

$$S_n = A \cdot 4^n + B \cdot n\, 4^n$$

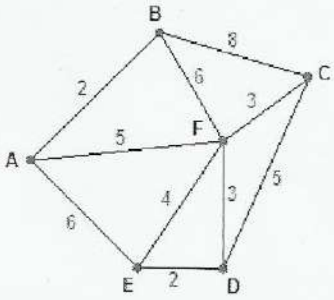
$$S_0 = 1 = A$$

$$S_1 = 7 = 4A + 4B$$

$$\therefore\ B = \frac{3}{4}$$

$$a_n = S_n = 4^n + \frac{3}{4} n\, 4^n$$

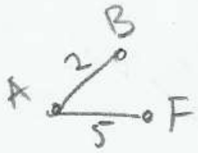
3. (20 points) Find a minimum spanning tree by using Prim's Algorithm for the following graph. SHOW EACH STEP OF THE ALGORITHM IN DETAIL.



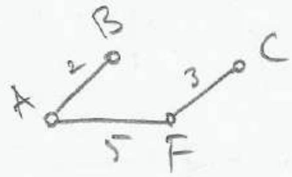
(There can be different order of steps.)

Start by picking the edge $\underline{AB}$, since it has the minimum weight 2.

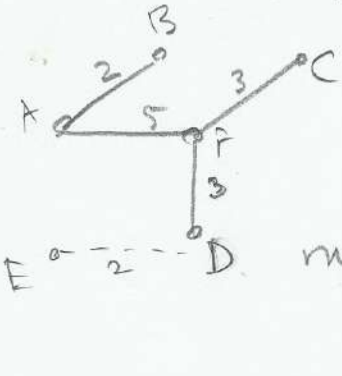
$\min\{6,5,6,8\} = 5$. Add the edge AF.



$\min\{6,4,8,3\} = 3$. Add the edge $\underline{FC}$



$\min\{6,4,3,5\} = 3$ Add the edge $\underline{FD}$.



$\min\{6,4,2\} = 2$ Add the edge $\underline{ED}$.

Done.

4. (20 points) Show that $2^n \geq n^2$ for every integer $n \geq 4$.

Use induction:

Base step: $n=4$

$2^4 \geq 4^2$ since $16=16$.

Ind. Hypo. Assume that $2^{n-1} \geq (n-1)^2$. (*)

Ind. Step: To show $2^n \geq n^2$, multiply each side of (*) with 2:

$$\underbrace{2 \cdot 2^{n-1}}_{=2^n} \geq \underbrace{2(n-1)^2}_{2n^2-4n+2}$$

$$= 2^n \geq \underbrace{2n^2-4n+2}_{=n^2+n^2-4n+2}$$

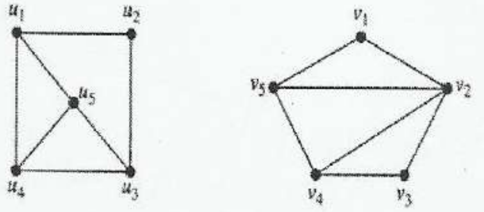
$$= n^2 + \underbrace{n^2-4n+2}_{\geq 0 \text{ when } n \geq 4}$$

Therefore,

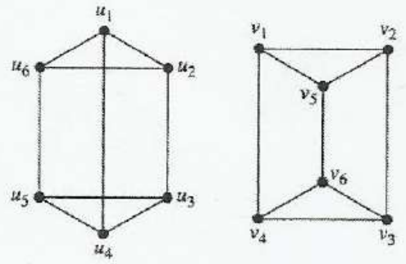
$2^n \geq n^2$. ∎

5. (20 points) Determine whether the given pair of graphs is isomorphic or not. Justify your answer.

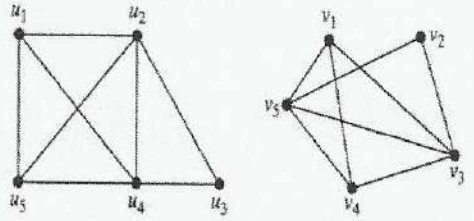
a)



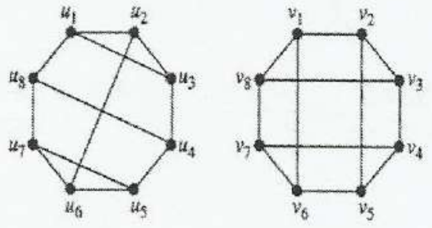
b)



c)



d)

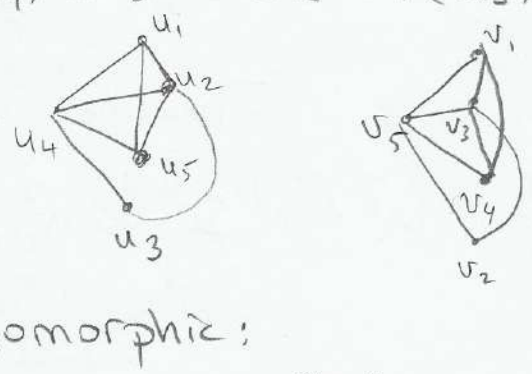


G $\qquad$ H

a) not isomorphic : v_2 has degree 4. No such vertex in the first graph.

b) isomorphic ;
$f(u_1) = v_5$, $f(u_6) = v_1$, $f(u_2) = v_2$,
$f(u_4) = v_6$, $f(u_5) = v_4$, $f(u_3) = v_3$.

c) isomorphic ;
$f(u_4) = v_5$ and $f(u_3) = v_2$ (Only possibility)



$f(u_1) = v_1$
$f(u_5) = v_4$
$f(u_2) = v_3$

d) not isomorphic :

Many reasons are possible.
One reason is there is a K_3 (triangle) in the first graph, but no triangle in the second graph.

6. (20 points) Encode (Find a code for) the sentence HAVE A HAPPY NEW YEAR using Huffman Code.

HAVE␣A␣HAPPY␣NEW␣YEAR

Frequencies:

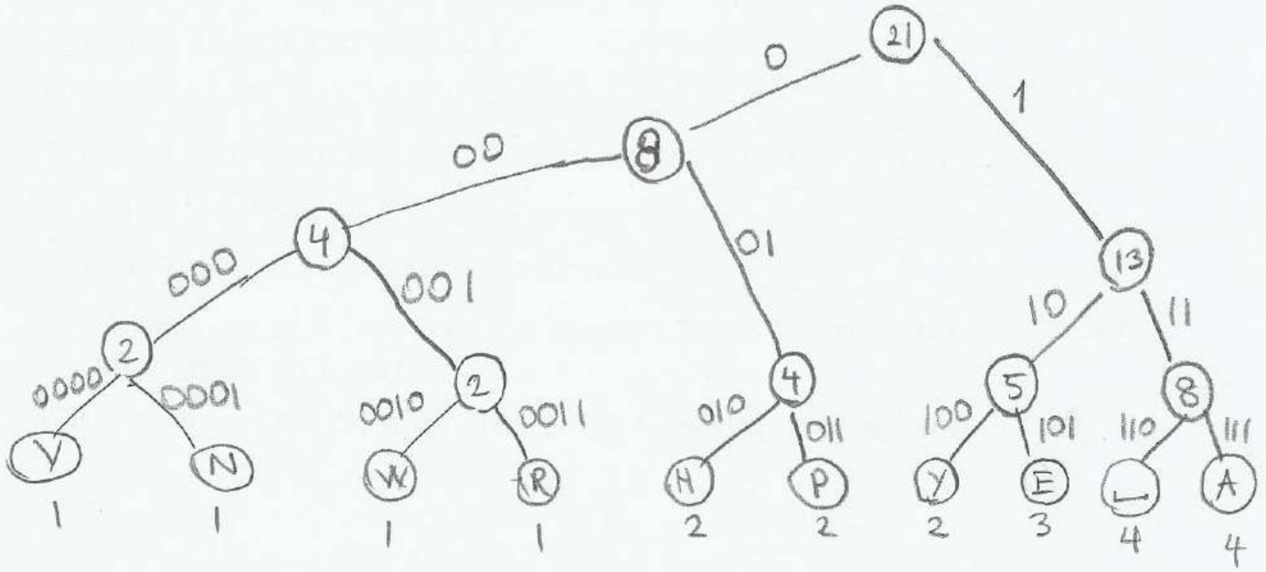
H - 2 P - 2

A - 4 Y - 2

V - 1 N - 1

E - 3 W - 1

␣ - 4 R - 1



V = 0000
N = 0001
W = 0010

R = 0011
H = 010
P = 011

Y = 100
E = 101
␣ = 110

A = 111

Use these codes to encode the given sentence

BBM 205 - Discrete Structures: Final Exam - ANSWERS
Date: 12.1.2017, Time: 15:00 - 17:00

Ad Soyad / Name:
Ögrenci No /Student ID:

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	6	16	8	8	10	9	6	8	14	5	10	100
Score:												

1. (6 points) The complementary graph $\bar{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. A simple graph G is called self-complementary if G and $\bar{G}$ are isomorphic. Show that if G is a self-complementary simple graph with n vertices, then $n \equiv 0$ or $1 \mod 4$.

   (Bir G çizgesinin tersi $\bar{G}$ aynı köşe kümesine sahip olup, $\bar{G}$'de kenar olan çiftler sadece ve sadece G'de kenar olmayan köşe çiftleridir. Bir G çizgesinin kendini-tamamlayan olması, G ve $\bar{G}$'nin birbirine izomorf olmasıdır. Eğer G kendini-tamamlayan n köşeli bir çizgeyse, $n \equiv 0$ veya $1 \mod 4$ olduğunu gösterin.)

   Solution: Since G and $\bar{G}$ are isomorphic, $|E(G)| = |E(\bar{G}|$. Also, $|E(G)| + |E(\bar{G}| = 2|E(G)| = \binom{n}{2}$, we get $4|E(G)| = n(n-1)$. This implies, either $4|n$ or $4|(n-1)$.

2. (a) (2 points) Write a necessary and sufficient condition for a graph G to have an Eulerian circuit. (Bir G çizgesinin Euler döngüsüne sahip olabilmesi için gerekli ve yeterli bir koşul yazın.)

   Solution: The degree of every vertex in the graph is even.

(b) (2 points) Write a necessary and sufficient condition for a graph G to have an Eulerian path. (Bir G çizgesinin Euler yoluna sahip olabilmesi için gerekli ve yeterli bir koşul yazın.)

   Solution: The degree of every vertex except two vertices is even.

(c) (2 points) Write a necessary and sufficient condition for a bipartite graph G with parts V_1 and V_2 to have a complete matching with respect to V_1. (Iki-parçalı olup parçaları V_1 ve V_2 olan bir G çizgesinin, V_1'i tamamlayan bir eşleştirmeye sahip olabilmesi için gerekli ve yeterli bir koşul yazın.)

   Solution: For every subset S of V_1, $|N(S)| \geq |S|$.

(d) (6 points) Use Euclidean algorithm to find gcd(210, 648). (Euclid algoritmasını kullanarak gcd(210, 648)'i bulun.)

   Solution: Iter.1: $648 \equiv 18 \mod 210$
   Iter. 2: $210 \equiv 12 \mod 18$
   Iter. 3: $18 \equiv 6 \mod 12$.
   Iter. 4: $12 \equiv 0 \mod 6$. Thus, $\gcd(210, 648) = 6$.

(e) (4 points) Draw the graph represented by the adjacency matrix below. (Asağıdaki komşuluk matrisi ile temsil edilen çizgeyi çizin.)

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

   Solution: Let $V(G) = \{a, b, c, d, e\}$.
   The edge set will be $E(G) = \{ab, ac(3), ae(4), bb(2), bc, bd(3), cc, ce, de(2), ee(3)\}$, where the multiplicity of each edge is indicated in the paranthesis next to it. (If you have drawn with different multiplicities, because of degree-sum condition, that also works.)

3. (8 points) Show that for every simple graph G, if $|E(G)| > (n-1)(n-2)/2$, then G is connected. (Her basit G çizgesi için, eğer $|E(G)| > (n-1)(n-2)/2$ ise, G'nin bağlı olacağını gösterin.)

Solution: Solution 1: (Proof by contradiction) Since the degree-sum of the graph is $2|E(G)|$, the average degree is $d_{avg} = 2|E(G)|/n$. Because, $|E(G)| > (n-1)(n-2)/2$, $d_{avg} > (n-1)(n-2)/n = n-3+2/n$. Since the maximum degree of G, call it M, is at least d_{avg}, $M \geq \lceil n-3+2/n \rceil = n-2$.

Let v have degree M, thus $\deg(v) \geq n-2$, meaning v has at least $n-2$ neighbors. If G is not connected, then this is only possible, if $\deg(v) = n-2$ and one component contains $v \cup N(v)$ and another component is an isolated vertex. But if there is an isolated vertex, $|E(G)| = (n-1)(n-2)/2$, contradiction. Thus, G is connected.

Solution 2: (Proof by contrapositive: If G is not connected, show that $|E(G)| \leq (n-1)(n-2)/2$.)
If G is not connected, let G_1 and G_2 be two subgraphs of G that are disconnected from each other and their union is G. If G_1 has k vertices, then G_2 has $n-k$ vertices. Also, $|E(G_1)| \leq \binom{k}{2}$ and $|E(G_2)| \leq \binom{n-k}{2}$, thus

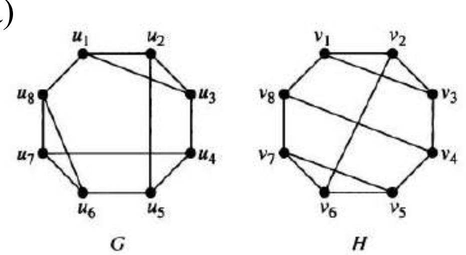
$$|E(G)| = |E(G_1)| + |E(G_2)| \leq \binom{k}{2} + \binom{n-k}{2}.$$

This is maximized, when $k = 1$. Thus, $|E(G)| \leq \binom{n-1}{2}$.

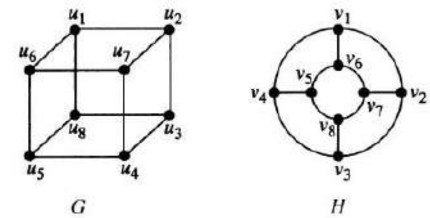
4. (8 points) Draw these graphs: a) K_6, b) $K_{2,3}$, c) C_6, d)Q_3. (Verilen çizgeleri çizin.)

5. (10 points) Either show that the pairs of graphs in the figure are isomorphic by finding an isomorphism function or explain why they cannot be isomorphic. (Asağıda verilen çizge çiftlerinin izomorfizma fonksiyonunu bularak izomorf olduğunu gösterin ya da neden izomorf olamayacaklarını açıklayın.)

a)



b)



Solution: In part (a), the graphs are isomorphic. Let the isomorphism function f be defined as, $f(u_1) = v_2$, $f(u_2) = v_3$, $f(u_3) = v_1$, $f(u_4) = v_8$, $f(u_5) = v_4$, $f(u_6) = v_5$, $f(u_7) = v_7$, $f(u_8) = v_6$.

In part (b), the graphs are isomorphic. Let the isomorphism function f be defined as, $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_7) = v_3$, $f(u_6) = v_4$, $f(u_8) = v_6$, $f(u_3) = v_7$, $f(u_4) = v_8$, $f(u_5) = v_5$.

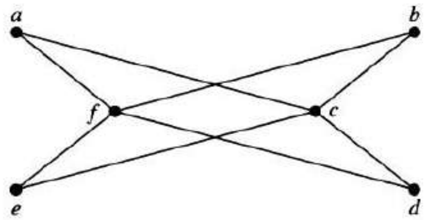
6. (9 points) Show **by using induction** that every m-ary tree with height h has at most m^h leaves. (**Tümevarım yöntemi kullanarak**, her m-lik ve yüksekliği h olan ağacın en fazla m^h yaprağı olacağını gösterin.)

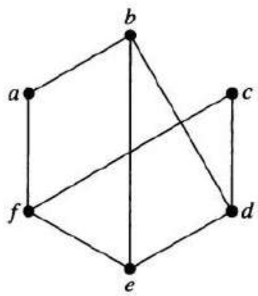
> **Solution:** Let $P(h)$ be the statement that every m-ary tree with height h has at most m^h leaves. We use induction on h.
> **Base step:** $P(1)$ is true, since a tree with height 1 has at most m leaves.
> **Inductive Hypothesis (I.H.)** Assume that $P(h-1)$ is true.
> **Inductive step:** Let T be a tree with height h. Remove all leaves in T. Call this new tree T'. The height of T' is exactly $h-1$. By I.H., T' has at most m^{h-1} leaves. These are all parents of the leaves in T. Therefore, the number of leaves in T is at most the total number of children of the leaves in T', which is at most $m \cdot m^{h-1} = m^h$.

7. (6 points) Let T be a full m-ary tree with n vertices, i internal vertices and ℓ leaves. Show that $\ell = (m-1)i + 1$. (T, dolu m-lik, n köşeli, i internal köşeli ve ℓ yapraklı bir ağaç ise, $\ell = (m-1)i + 1$ olduğunu gösterin.)

> **Solution:** Since T is full
> $$n = m \cdot i + 1. \tag{1}$$
> We also know that $n = \ell + i$, thus
> $$\ell = n - i. \tag{2}$$
> By substituting the value of n in (1) into (2), we obtain $\ell = (m \cdot i + 1) - i = (m-1) \cdot i + 1$.

8. (8 points) Determine whether the following two graphs are bipartite or not. Justify your answer. (Asağıda verilen iki çizgenin iki-parçalı olup olmadığına karar verin. Cevabınızı açıklayın.)





Solution: The first graph is bipartite, since it has a proper 2-coloring by using color red on $\{a, b, e, d\}$ and color blue on $\{c, f\}$. The second graph is not bipartite, since there is a triangle on the vertices $\{b, d, e\}$ and we cannot 2-color a triangle properly.

9. A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 1209805608 is not valid. Let a_n be the number of valid n-digit codewords. (Bir bilgisayar sisteminde, geçerli olan şifre tanımı her basamağı 0'dan 9'a kadar olan rakamlardan oluşan ve çift sayıda 0 içeren sayı dizisidir. Örnek olarak 1230407869 geçerli olup, 1209805608 geçerli olmayan bir şifredir. a_n, n uzunluğundaki geçerli tüm şifrelerin sayısı olsun.)

(a) (8 points) Find a recurrence relation and initial conditions for a_n. (a_n dizisi için rekürsif bir ilişki ve başlangıç koşulları bulun.)

Solution: We group the set of valid passwords of length n into two groups A and B, where A is the set containing passwords ending with 0, B is the set containing passwords not ending with 0. All passwords in A should contain odd number of 0's in the first n-1 digits, therefore, $|A| = 10^{n-1} - a_{n-1}$. All passwords in B should have last digit from $\{1, 2, \ldots, 9\}$ and should contain even number of 0's in the first n-1 digits, therefore, $|B| = 9a_{n-1}$. Alltogether, $a_n = (10^{n-1} - a_{n-1}) + 9a_{n-1} = 8a_{n-1} + 10^{n-1}$. The initial condition is $a_1 = 9$.

(b) (6 points) Solve this recurrence relation. (Bu rekürsif ilişkiyi çözün.)

Solution: By substituting $a_{n-1} = 8a_{n-2} + 10^{n-2}$ into a_n, we obtain

$$a_n = 8(8a_{n-2} + 10^{n-2}) + 10^{n-1} = 8^2 a_{n-2} + 8 \cdot 10^{n-2} + 10^{n-1}.$$

Then, we substitute $a_{n-2} = 8a_{n-3} + 10^{n-3}$ into the above equality, which gives

$$a_n = 8(8(8a_{n-3} + 10^{n-3}) + 10^{n-2}) + 10^{n-1} = 8^3 a_{n-3} + 8^2 \cdot 10^{n-3} + 8 \cdot 10^{n-2} + 10^{n-1}.$$

After that, we substitute the recursive formula for a_{n-3} and continue until there is only a_1 in the expression for a_n. At that final step, we have

$$a_n = 8^{n-1} \cdot 9 + \sum_{i=0}^{i=n-2} 8^i 10^{n-1-i}.$$

10. (5 points) Determine the number of multiplications used to find x^{2^i} starting with x and successively squaring (to find x^2, x^4, and so on). (Verilen bir x için x^{2^i} değerini hesaplarken x'den başlayıp arka arkaya kare alınarak x^{2^i}'in bulunmasında kaç defa kare alındığını bulun.)

Solution: The number of multiplications will be the value of i in the power.

11. Give as good a big-Omega estimate as possible for each of these functions. A good estimate means, for example, if you know a function is both $\Omega(n^3)$ and $\Omega(n^{3.5})$, then $\Omega(n^{3.5})$ is a better estimate than $\Omega(n^3)$. (Asağıdaki fonksiyonlar için olabileceğinin en iyisi büyük-Omega tahmininde bulunun. En iyi tahmin ile anlatılmak istenen, örneğin bir fonksiyon hem $\Omega(n^3)$ hem $\Omega(n^{3.5})$ ise, o zaman $\Omega(n^{3.5})$'nin daha iyi bir tahmin olmasıdır.)

(a) (2 points) $(n! + 5^n)(n^3 + 2)$

Solution: $\Omega(n^3 n!)$

(b) (2 points) $(n \log n + n^2)(n + 1)$

Solution: $\Omega(n^3)$

(c) (2 points) $(n! + 2^n)(17 \log n + 19)$

Solution: $\Omega(n! \log n)$

(d) (2 points) $(n^3 + n^2 \log n)(n^3 \log n + 1) + (n^3 + \log(n^2 + 1))(n^3 + 2)$

Solution: $\Omega(n^6 \log n)$

(e) (2 points) $(n^n + n2^n + 5^n)(n^2 + 8)$

Solution: $\Omega(n^{n+2})$

BBM 205 - Discrete Structures: Midterm 1
Date: 24.10.2017, Time: 16:00 - 17:30

Ad Soyad / Name:
Ögrenci No /Student ID:

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	12	10	7	12	12	12	10	10	100
Score:										

1. (15 points) Determine by using a truth table whether the following statements are a tautology (true for all combinations of x and y), a contradiction (false for all combinations of x and y) or neither. (Asagidaki ifadelerin totoloji (her x ve y degeri icin dogru), celiski (her x ve y degeri icin yanlis) ya da bunlardan hicbiri oldugunu dogruluk tablosu kullanarak gosterin.

   (a) $x \wedge (x \implies y) \wedge (\neg y)$

   (b) $x \implies (x \vee y)$

   (c) $x \vee y) \wedge (\neg(x \wedge y))$

   Solution: See pages 1-2 in the webpage:
   www.eecs70.org/static/homeworks/hw01-sol.pdf

2. (12 points) Let p, q and r be the propositions (p, q ve r ifadeleri asagidaki gibidir.)
p: You get an A in the final exam (Finalden A aliyorsun)
q: You do every exercise in the book (Kitaptaki her alistirmayi cozuyorsun)
r: You get an A in this class (Dersten A aliyorsun)
Write the following propositions using p, q, r and logical connectives. (Asagidaki onermeleri p, q, r ve mantiksal baglaclar kullanarak yazin.)

(a) You get an A in this class, but you do not do every exercise in the book. (Dersten A aliyorsun, ama kitaptaki her alistirmayi cozmuyorsun.)

Solution: $p \wedge \neg q$

(b) You get an A on the final exam, you do every exercise in the book, and you get an A in this class. (Finalden A aliyorsun, kitaptaki her alistirmayi cozuyorsun, ve dersten A aliyorsun.)

Solution: $p \wedge q \wedge r$

(c) To get an A in this class, it is necessary for you to get an A on the final exam. (Dersten A alman icin finalden A alman gerekli.)

Solution: $r \implies p$

(d) Getting an A on the final exam and doing every exercise in the book are sufficient for getting an A in this class. (Finalden A almak ve kitaptaki her alistirmayi cozmek, dersten A almak icin yeterli.)

Solution: $(p \wedge q) \implies r$

3. (10 points) Show using proof by contrapositive for any real number t that if t is irrational, then $5t$ is irrational. (Kontrapozitif ile ispat yontemini kullanarak her t reel sayisi icin eger t irrasyonel ise $5t$'nin de irrasyonel olacagini gosterin.)

Solution: Assume that for some number t, $5t$ is rational. Then we can write $5t = a/b$ for some integers a and b. This implies that $t = a/(5b)$ and thus t is rational. So, the contrapositive of this statement is true, hence the statement is true.

4. (7 points) Prove or disprove that for any integer n, if $n^2 \equiv 0 \pmod 4$, then $n \equiv 0 \pmod 4$. (Her n tamsayisi icin eger $n^2 \equiv 0 \pmod 4$ ise $n \equiv 0 \pmod 4$ oldugunun dogru ya da yanlis oldugunu gosterin.)

Solution: The statement is false, for example when $n = 6$. Although $6^2 = 36 \equiv 0 \pmod 4$, $6 \equiv 2 \not\equiv 0 \pmod 4$

5. (12 points) Prove using induction that, for $n \geq 1$, $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$. (Verilen ifadeyi her $n \geq 1$ icin tumevarim kullanarak gosterin.)

Solution: Let $P(n)$ be the statement given in the question. We show that $P(n)$ is true for all $n \geq 1$.
Base step:
For $n = 1$, it is true that $1 \cdot 1! = 2! - 1$.
Inductive Hypothesis (I.H.):
Assume that $P(n-1)$ is true, so $1 \cdot 1! + 2 \cdot 2! + \cdots + (n-1) \cdot (n-1)! = n! - 1$.
Inductive step:
We can substitute what we have from I.H. as below:
$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n! - 1) + n \cdot n!$.
Since the righthand side can be simplified as $(n!-1)+n \cdot n! = (n+1)n!-1 = (n+1)!-1$ we obtain $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n! - 1) + n \cdot n! = (n+1)n! - 1 = (n+1)! - 1$, hence $P(n+1)$ is true.

6. (12 points) Show using proof by contradiction the following statement: Let x and y be two positive integers. If $xy < 36$, then either $x < 6$ or $y < 6$. (Celiski ile ispat yontemini kullanarak verilen ifadenin dogrulugunu gosterin: Her x ve y pozitif tamsayisi icin, eger $xy < 36$ ise ya $x < 6$ ya da $y < 6$ dogrudur.)

Solution: Our premises are that $xy < 36$ and the negation of (either $x < 6$ or $y < 6$), thus $x \geq 6$ and $y \geq 6$. If we multiply the two inequalities $x \geq 6$ and $y \geq 6$, we obtain $x \cdot y \geq 6 \cdot 6 = 36$, which contradicts with the first premise. Done.

7. (12 points) Use induction to prove that for all non-negative integers n,

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Yukaridaki ifadenin her en az sifir olan n tamsayisi icin dogrulugunu tumevarim kullanarak gosterin.)

Solution: Let $P(n)$ be the statement given in the question. We show that $P(n)$ is true for all $n \geq 0$.

Base step:

$P(0)$ is true, since $0^2 = 0(0+1)(0+1)/6$.

Inductive Hypothesis:

Assume that $P(n-1)$ is true, so

$$\sum_{k=0}^{(} n-1)k^2 = \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} = \frac{(n-1)n(2n-1)}{6}.$$

Inductive step:

We can substitute what we have from I.H. as below:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \left(\sum_{k=0}^{(n-1)} k^2\right) + n^2 = \frac{(n-1)n(2n-1)}{6} + n^2.$$

By simplifying, we observe that the lefthand-side above equals

$$\frac{(n-1)n(2n-1)}{6} + n^2 = \frac{(n-1)n(2n-1) + 6n^2}{6} = \frac{n(2n^2+3n+1)}{6} =$$
$$= \frac{n(n+1)(2n+1)}{6}.$$

Done.

8. (10 points) The Fibonacci number F_n is described as: $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show by using induction that for all non-negative integers n, F_n is even if and only if n is divisible by 3. (Fibonacci sayisi yukaridaki soruda tanimlanmistir. Tumevarim kullanarak her sifirdan buyuk n tamsayisi icin, F_n'in cift olmasinin ve n'in 3'e tam bolunebilmesinin denk kosullar oldugunu gosterin.)

Solution: See Reading for week-3 by Jeff Erickson, page 16.

9. (10 points) Show by using induction that every non-negative integer can be written as the sum of distinct powers of 2. (Tumevarim kullanarak her sifirdan buyuk tamsayinin 2'nin farkli kuvvetlerinin toplami olarak yazilabilecegini gosterin.)

Solution: See Reading for week-3 by Jeff Erickson, page 13.

BBM 205 - Discrete Structures: Final Exam
Date: 15.1.2018, Time: 9:30 - 11:30
SOLUTION SHEET

Name:
Student ID:

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	22	6	8	21	10	9	14	16	8	124
Score:											

Unless specified differently, you are expected to show all your work in your answers to receive full credit from the questions.

1. (10 points) Show by using induction on the number of vertices that every tree can be colored properly by 2 colors. (Information you can use: Every tree has at least two vertices with degree 1.)

Solution: Base step: True for trees with one and two vertices.
Inductive hypothesis (I.H.): Assume that every tree on $n-1$ vertices can be 2-colored.
Inductive step: Assume that we removed a vertex v with degree 1 from our tree T on n vertices, call this new tree T', which can be 2-colored by I.H..We can add v back by connecting it to its neighbor in T and obtain a 2-coloring of T by using the 2-coloring of T'. Since v has only one neighbor in T, it uses only one color. We can color v by the unused color on $N(v)$ and obtain a 2-coloring of T.

2. **In the questions below, you are not expected to explain your answer.**

(a) (3 points) How many edges are there in a graph with 8 vertices each of degree 5?

Solution: $8 \cdot 5/2 = 20$.

(b) (4 points) How many vertices and how many edges do these graphs have: a) K_n, b) C_n, c) $K_{m,n}$, d) Q_n

Solution: a) $(n, n(n-1)/2))$, b) (n, n), c) $(m+n, mn)$, d) $(2^n, n2^{n-1})$.

(c) (3 points) For which values of n are these graphs bipartite: a) K_n, b) C_n, c) Q_n

Solution: a) only when $n = 1, 2$, b) n, even, c) for every n

(d) (3 points) For which values of m and n does the complete graph $K_{m,n}$ have a Hamilton circuit?

Solution: When $m = n$.

(e) (3 points) Give a list of six nonnegative integers that cannot be the degree list of a graph on six vertices.

Solution: $\{3, 3, 4, 2, 2, 3\}$

(f) (3 points) If G is a simple graph with 15 edges and $\bar{G}$ has 13 edges, how many vertices does G have?

Solution: We have one equation with one unknown to solve, let n be the number of vertices in such a graph G. Then, $|E(G)| + |E(\bar{G})| = 15 + 13 = 28$. Therefore, $n(n-1)/2 = 28$, which gives that $n = 8$.

(g) (3 points) Draw all nonisomorphic simple graphs with three vertices.

Solution: They are C_3, $K_{1,2}$, K_2 and the empty graph with no edges.

3. (6 points) Let G be a graph with n vertices and e edges. Let M be the maximum degree in G. Show that $2e/n \leq M$.

Solution: We can use proof by contradiction: assume that this is not true for some graph G with n vertices and e edges with degree list $\{d_1, d_2, \ldots, d_n\}$. This means $2e/n > M$ and thus

$$(d_1 + d_2 + \cdots + d_n) > nM \tag{1}$$

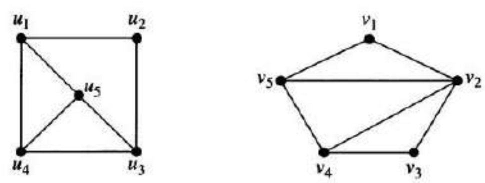
But we also know that for every degree $d_i \leq M$, since M is the maximum degree of G. Thus,

$$(d_1 + d_2 + \cdots + d_n) \leq nM. \tag{2}$$

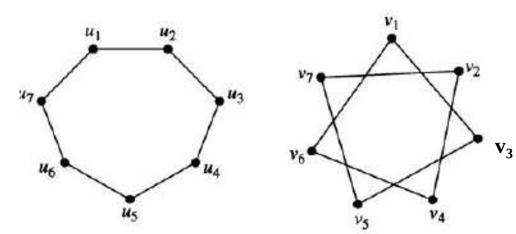
As seen above, the equations (1) and (2) cannot be both true, contradiction.

4. (8 points) Determine whether the following pairs of graphs are isomorphic or not. (If isomorphic, provide the isomorphism function, if not isomorphic, give an explanation.)

a)



b)



Solution: The pair in part (a) cannot be isomorphic, because the graph on the right has a vertex with degree 4, but the graph on the left doesn't.
The pair in part (b) are isomorphic. The isomorphism function follows as:

$f(u_1) = v_1,\ f(u_2) = v_3,\ f(u_3) = v_5,\ f(u_4) = v_7,\ f(u_5) = v_2,\ f(u_6) = v_4,\ f(u_7) = v_6.$

5. In the questions below, all dice and coins are fair, unless specified differently.

(a) (3 points) What is the probability that a die comes up six when it is rolled?

Solution:
$$\frac{1}{6}$$

(b) (3 points) What is the probability that a randomly selected day of the year is in April?

Solution:
$$\frac{30}{365}.$$

(c) (3 points) What is the probability that the sum of numbers that the sum of the numbers on two dice is even when they are rolled?

Solution:
$$\frac{1+3+5+5+3+1}{36} = \frac{18}{36}$$

(d) (4 points) What is the probability that when a coin is flipped six times in a row, the number of heads is at least 3?

Solution:
$$\frac{\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}}{2^6} = \frac{20+15+6+1}{2^6} = \frac{42}{64}.$$

(e) (4 points) What is the probability that n precedes 1 and $n-1$ precedes 2, when we randomly select a permutation of $\{1, 2, \ldots, n\}$?

Solution: There are $4! = 16$ ways to permute these four numbers and out of these permutations, only six of them satisfy the given conditions: $(n, 1, n-1, 2)$, $(n-1, 2, n, 1)$, $(n, n-1, 1, 2)$, $(n, n-1, 2, 1)$, $(n-1, n, 1, 2)$, $(n-1, n, 2, 1)$. By symmetry, for each permutation, there are equal number of ways to arrange the remaining elements and choose the positions of the four numbers. Hence, the probability of this event is $6/16$.

(f) (4 points) Suppose that E and F are events such that $Pr(E) = 0.7$ and $Pr(F) = 0.5$. Show that $Pr(E \cup F) \geq 0.7$ and $Pr(E \cap F) \geq 0.2$.

Solution: Since $Pr(E \cup F) \geq Pr(E) = 0.7$, done. On the other hand, $Pr(E \cap F) = Pr(E) + Pr(F) - Pr(E \cup F) \geq 0.7 + 0.5 - 1 = 0.2$.

6. (a) (3 points) Write Euler's formula for planar graphs.

Solution: If a graph with n vertices and e edges has a planar drawing with f faces, then $n - e + f = 2$.

(b) (7 points) Use Euler's formula to show that $K_{3,3}$ is not planar.

Solution: Observation1: Since $K_{3,3}$ does not contain a C_3 as a subgraph, every face of its planar drawing would have at least four edges.
Observation2: Since the sum of the face lengths in a planar drawing is 2e, by Obs. 1, $4f \leq 2e$.
By Euler's formula, we have $6 - 9 + f = 2$, hence $f = 5$. By using that in Obs.2, we get $4 \cdot 5 \leq 2 \cdot 9$, which gives $20 \leq 18$, contradiction.

7. Suppose you have balls numbered $1, \ldots, n$, where n is a positive integer at least 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) (2 points) What is the probability that the first ball is 1 and the second ball is 2?

Solution:

$$\frac{1}{n}\frac{1}{n} = \frac{1}{n^2}$$

(b) (3 points) What is the probability that the second ball's number is exactly one greater than the first ball's number?

Solution: This event contains the pairs $(i, i+1)$ for $1 \leq i \leq n-1$. Hence, the probability is $\frac{n-1}{n^2}$.

(c) (4 points) What is the probability that the second ball is strictly less than the first ball's number?

Solution: This event contains the pairs (i, j), where for each j, the value of i can be a number from $\{1, \ldots, j-1\}$. Hence, the total number of pairs is $\sum_{j=1}^{j=n}(j-1) = (n-1)n/2$. Therefore, its probability is $\frac{(n-1)n}{2n^2} = \frac{n-1}{2n}$.
One can find the cardinality of this event also by making use of the symmetry, since after excluding the pairs (i, i), there remain equal number of pairs (i, j) for which either $i < j$ or $i > j$. Hence, this event has $(n^2 - n)/2 = (n-1)n/2$ pairs as also calculated with a different method above.

8. In the questions below, all dice and coins are fair, unless specified differently.

(a) (5 points) Suppose that E, F_1, F_2, and F_3 are events from a sample space and that F_1, F_2, and F_3 are mutually disjoint and their union is S. Find $Pr(F_2|E)$ if $Pr(E|F_1) = 2/7$, $Pr(E|F_2) = 3/8$, $Pr(E|F_3) = 1/2$, $Pr(F_1) = 1/6$, $Pr(F_2) = 1/2$, and $Pr(F_3) = 1/3$.

Solution: By Bayes' theorem, we have

$$Pr(F_2|E) = \frac{Pr(F_2)Pr(E|F_2)}{Pr(F_1)Pr(E|F_1) + Pr(F_2)Pr(E|F_2) + Pr(F_3)Pr(E|F_3)} = \frac{(1/2)(3/8)}{(1/6)(2/7) + (1/2)(3/8) + (1/3)(1/2)} = \frac{7}{15}.$$

(b) (5 points) What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled. Show all your work.

Solution: Let X be the sum of the numbers on the dice. Since $E(X) = \sum_{i=2}^{i=12} i \cdot Pr(X = i)$, we have

$$E(X) = 2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+5\cdot\frac{4}{36}+6\cdot\frac{5}{36}+7\cdot\frac{6}{36}+8\cdot\frac{5}{36}+9\cdot\frac{4}{36}+10\cdot\frac{3}{36}+11\cdot\frac{2}{36}+12\cdot\frac{1}{36} = 7$$

(c) (4 points) When one red die and one black die are rolled, show that the probability of the events A = sum is 7 and B = red die is 1 are independent.

Solution: To show this, we need to show that $Pr(A \cap B) = Pr(A)Pr(B)$. Since $|A| = 6$, $|B| = 6$ and $|A \cap B| = 1$, we have $Pr(A \cap B) = 1/36$, $Pr(A) = 6/36$, and $Pr(B) = 6/36$. These give $Pr(A \cap B) = Pr(A)Pr(B)$, so A and B are independent events.

The questions in the following are to receive extra credit and you are free not to solve them.

9. Zeynep and Umut are flipping coins for fun. Zeynep flips a fair coin k times and Umut flips $n-k$ times. In total there are n coin flips.

   (a) (6 points) Use a combinatorial proof to show that

   $$\sum_{i=0}^{i=k} \binom{k}{k-i}\binom{n-k}{i} = \binom{n}{k}.$$

   You may assume $n-k \geq k$.

   (b) (6 points) Prove that Zeynep and Umut flip the same number of heads is equal to the probability that there are a total of k heads.

10. (10 points) For a graph with n vertices, if the degree of each vertex is at least $n/2$, then G is connected.