Definition: A linear transformation $L: V \to W$ is said to be injective (one-to-one) if it is a one-to-one function, namely $L(u) = L(v)$ implies $u = v$.
It is called surjective (or onto) if $L(V) = W$.

Injective linear transformations are also called monomorphisms. Surjective linear transformations are also called epimorphisms

Theorem: For a linear transformation $L: V \to W$ the following statements are equivalent;
(a) L is injective
(b) $\ker L = \{0_V\}$

Definition: A linear map $L: V \to W$ which is one-to-one and onto is called an isomorphism.
If there is an isomorphism L from V to W then we say that V and W are isomorphic.
When $W = V$, then an isomorphism $L: V \to W$ is called an automorphism.

Corollary: Let V and W be finite dimensional vector spaces. Then V is isomorphic to W $\Leftrightarrow \dim V = \dim W$.

Example: Let L be the linear operator on $\mathbb{R}^3$ given by

$$L(x_1,x_2,x_3) = (x_1 - x_2 + x_3,\ x_1 + 2x_2 - x_3,\ -x_1 + 4x_2 - 3x_3)$$

What must the value of a if $(a,-2a,1)$ is in

(a) $\ker L$? (b) $\operatorname{Im} L$?

Solution: $(a,-2a,1) \in \ker L \iff L(a,-2a,1) = (0,0,0)$

$$\iff \left.\begin{array}{l} a + 2a + 1 = 0 \\ a - 4a - 1 = 0 \\ -a - 8a - 3 = 0 \end{array}\right\} \quad \begin{array}{l} 3a = -1 \\ a = -1/3 \end{array}$$

(b) $(a,-2a,1) \in \operatorname{Im} L \iff \exists v \in V$ st $L(v) = (a,-2a,1)$

$\iff$ The system

$$\begin{aligned} x_1 - x_2 + x_3 &= a \\ x_1 + 2x_2 - x_3 &= -2a \\ -x_1 + 4x_2 - 3x_3 &= 1 \end{aligned}$$

has a solution.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 2 & -1 & -2a \\ -1 & 4 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 3 & -2 & -3a \\ 0 & 0 & 0 & 4a+1 \end{array}\right]$$

$$4a+1 = 0 \iff a = -1/4$$

Theorem and Definition Let $L: V \to W$ be a linear transformation. Then:

(i) The subset

$$\ker L = \{ v \in V \mid L(v) = 0 \}$$

is a subspace of V, called the kernel of L.
(null space)

(ii) The subset

$$\operatorname{Im} L = L(V) = \{ L(v) : v \in V \}$$

is a subspace of W called the image (range) of L.

proof: $L(0_V) = 0_W \Rightarrow 0_V \in \ker L$.

$v_1, v_2 \in \ker L \Rightarrow v_1 + v_2 \in \ker L$ (?)

$$L(v_1+v_2) = L(v_1) + L(v_2) \underset{\substack{v_1 \in \ker L \\ v_2}}{=} 0 + 0 = 0 \Rightarrow v_1 + v_2 \in \ker L.$$

$v_1 \in \ker L,\ c \in F \Rightarrow c v_1 \in \ker L$ (?)

$$L(cv_1) \underset{L \text{ lin. tr.}}{=} c L(v_1) \underset{v_1 \in \ker L}{=} c \cdot 0_V = 0_V. \Rightarrow c v_1 \in \ker L.$$

Selected solutions for SSEA 51, Homework 1

LA 3.8. Let $\{u, v, w\}$ be a linearly independent set. Is

$$\{u - v, v - w, u - w\}$$

a linearly independent set? Show that it is or show why it is not.

Solution. Since $\{u, v, w\}$ is linearly independent, we know that if we have an equation of the form

$$c_1 u + c_2 v + c_3 w = \vec{0},$$

then necessarily $0 = c_1 = c_2 = c_3$. This is our given information.

Let's find out whether $\{u - v, v - w, u - w\}$ is linearly independent or not. To do so, we consider the equation

$$d_1(u - v) + d_2(v - w) + d_3(u - w) = \vec{0}.$$

Must all the coefficients be zero? Let's regroup the terms according to the vectors u, v, and w. We get

$$(d_1 + d_3)u + (-d_1 + d_2)v + (-d_2 - d_3)w = \vec{0}.$$

We did this regrouping because now we can now apply the given information. Since $\{u, v, w\}$ is linearly independent, the coefficients above must all be zero. That is, we have

$$\begin{aligned} d_1 + d_3 &= 0 \\ -d_1 + d_2 &= 0 \\ -d_2 - d_3 &= 0. \end{aligned}$$

The first equation gives $d_1 = -d_3$. We plug into the second equation to get $d_3 + d_2 = 0$, or $d_2 = -d_3$. This is also the same as the third equation. Note we can pick d_3 to be any real number and still solve this system for d_1 and d_2. Let's pick d_3 to be nonzero, say $d_3 = 1$. Then we get $d_1 = -1$ and $d_2 = -1$.

We plug this back into the equation

$$d_1(u - v) + d_2(v - w) + d_3(u - w) = \vec{0}$$

to see that

$$-1(u - v) - 1(v - w) + 1(u - w) = \vec{0}.$$

Hence the set $\{u - v, v - w, u - w\}$ is linearly dependent.

Remark: if you were able to see the linear dependency

$$-1(u - v) - 1(v - w) + 1(u - w) = \vec{0}$$

just by staring at the vectors, then that's fine too. But recognize that this approach won't work when the vectors happen to be linearly independent.

LA 3.10. If $S = \{v_1, \ldots, v_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$, then any subset of S must be linearly independent.

Solution. This is true. Let's prove it.

Suppose $S = \{v_1, \ldots, v_k\}$ is linearly independent. This means that if we have an equation of the form

$$c_1 v_1 + \ldots + c_k v_k = \vec{0},$$

then necessarily $0 = c_1 = \ldots = c_k$. This is our given information.

Now, suppose we have a subset of S of size $m < k$. Without loss of generality, let's relabel the vectors in this subset to be $\{v_1, \ldots, v_m\}$. We need to show that $\{v_1, \ldots, v_m\}$ is linearly independent, and so we consider the equation

$$d_1 v_1 + \ldots + d_m v_m = \vec{0}.$$

Note that we can add $0v_{m+1} + \ldots + 0v_k = \vec{0}$ without changing anything. So we also have

$$d_1 v_1 + \ldots + d_m v_m + 0v_{m+1} + \ldots + 0v_k = \vec{0}.$$

But now this is in the same form as our given information. Since $\{v_1, \ldots, v_k\}$ is linearly independent, all the coefficients above must be zero (including the last few coefficients that we already knew were zero). That is, necessarily $0 = d_1 = \ldots d_m = 0$. So we have shown that $\{v_1, \ldots, v_m\}$ is linearly independent. Hence any subset of S is linearly independent.

LA 3.12. If $\operatorname{span}(v_1, v_2, v_3) = \mathbb{R}^3$, then $\{v_1, v_2, v_3\}$ must be a linearly independent set.

Solution. Let's prove the contrapositive, which is the same as proving this statement. That is, we'll prove that if $\{v_1, v_2, v_3\}$ is linearly dependent then $\operatorname{span}(v_1, v_2, v_3)$ is not all of $\mathbb{R}^3$.

Suppose $\{v_1, v_2, v_3\}$ is linearly dependent. Then by definition, at least one of the vectors v_1, v_2, or v_3 is a linear combination of the other two. Without loss of generality, let's relabel the vectors so that v_3 is a linear combination of v_1 and v_2. That is, $v_3 = c_1 v_1 + c_2 v_2$ for some $c_1, c_2 \in \mathbb{R}$.

You can use the equation $v_3 = c_1 v_1 + c_2 v_2$ to show that $\operatorname{span}(v_1, v_2, v_3) = \operatorname{span}(v_1, v_2)$. I haven't written out all the details of why this is true, but I will if you ask me. The span of two vectors is either a point, a line, or a plane, and hence not all of $\mathbb{R}^3$. Hence $\operatorname{span}(v_1, v_2, v_3) = \operatorname{span}(v_1, v_2)$ is not all of $\mathbb{R}^3$. This proves the contrapositive, which is the same as proving the original statement.

0.1 Matrices

Exercise 0.1.1 Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}.$$
Find A^{-1} (the inverse of A) if it exists.

Solution:
$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -4 & -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow[-4R_2+R_3]{R_2+R_1}$$
$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 5 & -4 & 1 \end{array}\right] \xrightarrow[-R_2]{\frac{1}{2}R_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 5/2 & -2 & 1/2 \end{array}\right] \xrightarrow{-R_3+R_2}$$
$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 5/2 & -2 & 1/2 \end{array}\right].$$

So $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1/2 & 1 & -1/2 \\ 5/2 & -2 & 1/2 \end{bmatrix}$. □

Exercise 0.1.2 Let B, C, and D be $n \times n$ matrices such that BC is right invertible and D is a right inverse of BC. Show that B is right invertible and find a right inverse of B.

Solution: $BCD = I$ and hence CD is a right inverse of B. By the theorem, if a square matrix has a right inverse then it is invertible. Thus B is invertible and $B^{-1} = CD$. □

Exercise 0.1.3 Let
$$A = \begin{bmatrix} 3 & -3 & 7 & 2 \\ 1 & -1 & 3 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & k & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}.$$
a) Find a row reduced echelon matrix R that is row equivalent to A.

b) Find the value(s) of k (if exist) for which A is row equivalent to B.

Solution: a)

$$\begin{bmatrix} 3 & -3 & 7 & 2 \\ 1 & -1 & 3 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 3 & -3 & 7 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 3 & -3 & 7 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow$$

$$\xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{-2R_3+R_2} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_2]{3R_2+R_1}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

b) $B \sim A \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & k & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow k = 1.$ □

Exercise 0.1.4 Let C, D, L, M, and K be 2×4 matrices such that $C \xrightarrow{R_1 \leftrightarrow R_2} L \xrightarrow{2R_2} K$ and $D \xrightarrow{2R_2+R_1} M \xrightarrow{3R_1} K$.

Find an invertible matrix P such that $PC = D$ and write P as a product of four elementary matrices (accordingly to the diagrams above).

Solution:

$$C \xrightarrow[\mathcal{E}_1]{R_1 \leftrightarrow R_2} L \xrightarrow[\mathcal{E}_2]{2R_2} K \xrightarrow[\mathcal{E}_3]{\frac{1}{3}R_1} M \xrightarrow[\mathcal{E}_4]{-2R_2+R_1} D.$$

$$P = \mathcal{E}_4(I) \cdot \mathcal{E}_3(I) \cdot \mathcal{E}_2(I) \cdot \mathcal{E}_1(I) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 1/3 \\ 2 & 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} -4 & 1/3 \\ 2 & 0 \end{bmatrix}. \quad \square$$

Exercise 0.1.5 Let $A = \begin{bmatrix} -2 & 6 & 2 & -2 \\ 1 & -1 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1/4 & 3/2 \\ 1/4 & 1/2 \end{bmatrix}$.

a) Find P^{-1}.

b) Find a row reduced echelon matrix R and the invertible matrix Q such that $A = QR$.

Solution: a) $\left[\begin{array}{cc|cc} 1/4 & 3/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|cc} 1/4 & 3/2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow[-R_2]{4R_1}$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 6 & 4 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{-6R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 6 \\ 0 & 1 & 1 & -1 \end{array}\right].$

$$P^{-1} = \begin{bmatrix} -2 & 6 \\ 1 & -1 \end{bmatrix}.$$

b) $\left[\begin{array}{cccc|cc} -2 & 6 & 2 & -2 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|cc} 1 & -1 & 0 & 1 & 0 & 1 \\ -2 & 6 & 2 & -2 & 1 & 0 \end{array}\right] \xrightarrow{2R_1+R_2}$

$\rightarrow \left[\begin{array}{cccc|cc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 4 & 2 & 1 & 1 & 2 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{cccc|cc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1/2 & 0 & 1/4 & 1/2 \end{array}\right] \xrightarrow{R_2+R_1}$

$\rightarrow \left[\begin{array}{cccc|cc} 1 & 0 & 1/2 & 1 & 1/4 & 3/2 \\ 0 & 1 & 1/2 & 0 & 1/4 & 1/2 \end{array}\right] = [R|P].$

So we have $PA = R$ consequently $A = P^{-1}R$ and finally

$$Q = P^{-1} = \begin{bmatrix} -2 & 6 \\ 1 & -1 \end{bmatrix}; \qquad R = \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}. \quad \square$$

Exercise 0.1.6 Let $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & -1 & 3 & 4 \\ 2 & 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 3 & 3 \\ r & r & 3 & 1 \\ 1 & 1 & 6 & 5 \end{bmatrix}$.

a) Find the row reduced echelon form of A.

b) Find $r \in \mathbb{R}$ for which the matrices A and B are row equivalent.

Solution: a) $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & -1 & 3 & 4 \\ 2 & 2 & 3 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow[-R_2+R_3]{\frac{1}{3}R_2}$

$$\to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

b) $B = \begin{bmatrix} 0 & 0 & 3 & 3 \\ r & r & 3 & 1 \\ 1 & 1 & 6 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 6 & 5 \\ r & r & 3 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{-rR_1+R_2}$

$$\to \begin{bmatrix} 1 & 1 & 6 & 5 \\ 0 & 0 & 3-6r & 1-5r \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 3-6r & 1-5r \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_2]{-\frac{1}{3}R_3}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3-6r & 1-5r \end{bmatrix} \sim R \text{ if and only if } 3-6r = 1-5r.$$

Thus $3 - r = 1$ and $r = 2$. □

Exercise 0.1.7 Given $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, solve the following matrix equations.

a) Find a matrix X such that $AX = B$.

b) Find a matrix Y such that $YA = B$.

c) Find a matrix Z such that $AZ^TB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: $\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2}$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right] = [I|A^{-1}].$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1/3 \end{array}\right] \xrightarrow{-2R_2+R_1}$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2/3 \\ 0 & 1 & 0 & 1/3 \end{array}\right] = [I|B^{-1}].$$

a) $AX = B \Rightarrow X = A^{-1}B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -9 \\ -1 & 4 \end{bmatrix}.$

b) $YA = B \Rightarrow Y = BA^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}$.

c) $AZ^TB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow Z^T = A^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}B^{-1} = A^{-1}B^{-1}$.

So $Z^T = A^{-1}B^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & -11/3 \\ -1 & 4/3 \end{bmatrix}$.

Finally $Z = (Z^T)^T = \begin{bmatrix} 3 & -1 \\ -11/3 & 4/3 \end{bmatrix}$. $\square$

Exercise 0.1.8 Let A be a square matrix. Show that $A^T - A$ is a skew-symmetric matrix.

Solution: $(A^T - A)^T = A^{TT} - A^T = A - A^T = -(A^T - A)$. $\square$

Exercise 0.1.9 Let B be a square matrix. Show that B^TB is a symmetric matrix.

Solution: $(B^TB)^T = B^TB^{TT} = B^TB$. $\square$

Exercise 0.1.10 Find x, y, z, and t if $-2\begin{bmatrix} x & -1 \\ 3 & 1 \end{bmatrix} + 3\begin{bmatrix} 2 & y \\ z & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ z & t \end{bmatrix}$.

Solution:
$$\left\{\begin{array}{rcl} -2x+6 & = & 2 \\ -6+3z & = & z \\ 2+3y & = & 0 \\ -2+12 & = & t \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} -2x & = & -4 \\ 2z & = & 6 \\ 3y & = & -2 \\ t & = & 10 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} x & = & 2 \\ y & = & -2/3 \\ z & = & 3 \\ t & = & 10 \end{array}\right. . \ \square$$

Exercise 0.1.11 Find all matrices of the form $X = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ satisfying $X^2 - I = 0$.

Solution: $X^2 = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Hence $a^2 = 1$. Thus $a = \pm 1$ and $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $X = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. $\square$

Exercise 0.1.12 Show that

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^n = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^n) \\ 0 & (-1)^n \end{bmatrix}$$

for any positive integer n.

[Hint: 1) Show that it is true for $n = 1$. 2) Show that when it is true for $n = m$ then it is true also for $n = m + 1$.]

1-st solution: If $n = 1$ then

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^1 = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^1) \\ 0 & (-1)^1 \end{bmatrix}.$$

If $n = m + 1$ then

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^{m+1} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}^m \cdot \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^m) \\ 0 & (-1)^m \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 3-\frac{3}{2}(1-(-1)^m) \\ 0 & (-1)^{m+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2}(1+(-1)^m) \\ 0 & (-1)^{m+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^{m+1}) \\ 0 & (-1)^{m+1} \end{bmatrix}.$$

2-nd solution: $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus $A^3 = A$, $A^4 = I$, ... Hence

$$A^n = \begin{cases} A \text{ if } n \text{ is odd} \\ I \text{ if } n \text{ is even} \end{cases}.$$

Remark that the same is true for $B_n = \begin{bmatrix} 1 & \frac{3}{2}(1-(-1)^n) \\ 0 & (-1)^n \end{bmatrix}$, namely

$$B^n = \begin{cases} A \text{ if } n \text{ is odd} \\ I \text{ if } n \text{ is even} \end{cases}.$$

Hence $A^n = B_n$ for all $n \geq 0$. $\square$

Exercise 0.1.13 Let A, B, C, and D be 3×3 matrices such that $A \xrightarrow{2R_1+R_2} B$ and $D \xrightarrow{R_1\leftrightarrow R_3} C \xrightarrow{-R_2+R_3} B$.

a) Find an invertible matrix P such that $PA = D$.

b) Write P as a product of three elementary matrices (accordingly to the three row operations in the diagrams above).

Solution: a) $A \xrightarrow[\mathcal{E}_1]{2R_1+R_2} B \xrightarrow[\mathcal{E}_2]{R_2+R_3} C \xrightarrow[\mathcal{E}_3]{R_1\leftrightarrow R_3} D$.

$$P = \mathcal{E}_3\mathcal{E}_2\mathcal{E}_1(I) = \mathcal{E}_3\mathcal{E}_2\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \mathcal{E}_3\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

b) $P = \mathcal{E}_3\mathcal{E}_2\mathcal{E}_1(I) = \mathcal{E}_3(I)\cdot\mathcal{E}_2(I)\cdot\mathcal{E}_1(I) =$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

Exercise 0.1.14 Find the values of x, y, and z for which the following matrix is skew-symmetric $\begin{bmatrix} x+y-3 & -1 & 3 \\ x & 0 & -2 \\ -3 & x+z & z-x \end{bmatrix}$.

(A matrix A is called skew-symmetric if $A^T = -A$ and A is called symmetric if $A^T = A$).

Solution: $A^T = -A \Rightarrow$

$$\begin{bmatrix} x+y-3 & x & -3 \\ -1 & 0 & x+z \\ 3 & -2 & z-x \end{bmatrix} = \begin{bmatrix} -x-y+3 & 1 & -3 \\ -x & 0 & 2 \\ 3 & -x-z & -z+x \end{bmatrix}.$$

So $x = 1$ and $x+z = 2$ since $z = 2-1 = 1$.

$x+y-3 = -x-y+3 \Rightarrow x+y = 3 \Rightarrow y = 3-1 = 2$.

Thus $x = 1$, $y = 2$, $z = 1$. $\square$

Exercise 0.1.15 Given a real or complex square matrix A. Find a symmetric matrix S and a skew-symmetric matrix K such that $A = S + K$.

(Hint: First show that for any matrix B the matrix $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric).

Solution: $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$ hence $B + B^T$ is symmetric.

$(B - B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T)$ hence $B - B^T$ is skew-symmetric.

$A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T + \frac{1}{2}A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = S + K$, where $S = \frac{1}{2}(A + A^T)$ is symmetric and $K = \frac{1}{2}(A - A^T)$ is skew-symmetric. □

Exercise 0.1.16 Apply 0.1.15 to the matrix $A = \begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix}$.

Solution: $A = S + K$.

$$S = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix} + \begin{bmatrix} 2i & -i \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 2i & -i/2 \\ -i/2 & -1 \end{bmatrix}.$$

$$K = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix} - \begin{bmatrix} 2i & -i \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 0 & i/2 \\ -i/2 & 0 \end{bmatrix}.$$

Thus $\begin{bmatrix} 2i & 0 \\ -i & -1 \end{bmatrix} = \begin{bmatrix} 2i & -i/2 \\ -i/2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & i/2 \\ -i/2 & 0 \end{bmatrix}$. □

Exercise 0.1.17 Let $A = \begin{bmatrix} 1 & 2 & -2 & 7 \\ -1 & 1 & 2 & -1 \\ 1 & 5 & -2 & 13 \end{bmatrix}$.

a) Find a row-reduced echelon matrix R which is row equivalent to A.

b) Find an invertible matrix P such that $R = PA$.

c) Find an invertible matrix Q such that $A = QR$.

Solution: a) and b). $[A|I] =$

$$= \left[\begin{array}{rrrr|rrr} 1 & 2 & -2 & 7 & 1 & 0 & 0 \\ -1 & 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & 5 & -2 & 13 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{R_1+R_2} \left[\begin{array}{rrrr|rrr} 1 & 2 & -2 & 7 & 1 & 0 & 0 \\ 0 & 3 & 0 & 6 & 1 & 1 & 0 \\ 0 & 3 & 0 & 6 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{\frac{1}{3}R_2}$$

$$\rightarrow \left[\begin{array}{rrrr|rrr} 1 & 2 & -2 & 7 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{rrrr|rrr} 1 & 0 & -2 & 3 & 1/3 & -2/3 & 0 \\ 0 & 1 & 0 & 2 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right] =$$

$[R|P]$, where

$$R = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad P = \begin{bmatrix} 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

c) $R = PA \Rightarrow P^{-1}R = P^{-1}PA = A$. Hence $Q = P^{-1}$. Calculate P^{-1}:

$$[P|I] = \left[\begin{array}{rrr|rrr} 1/3 & -2/3 & 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3R_1]{3R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2]{2R_1+R_3}$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & -3 & 3 & 0 \\ 0 & -5 & 1 & 6 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{3}R_2]{5R_2+R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 5 & 1 \end{array}\right] \xrightarrow{2R_2+R_1}$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 5 & 1 \end{array}\right] = [I|Q = P^{-1}].$$

So $Q = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$. $\square$

Exercise 0.1.18 Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ -3 & 3 & -6 & -3 & 3 \\ 2 & -2 & 5 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 & -2 & x \\ 0 & 0 & 0 & 0 & 1 \\ 1 & y & 2 & 1 & z \end{bmatrix}$.

a) Find the row reduced echelon matrices R and S which are row equivalent to A and B respectively. At each step write the elementary row operation that you use.

b) Find the values of x, y, z, for which the matrices A and B are row equivalent.

c) By using the row operation in a) properly, write $B = \mathcal{E}_k \dots \mathcal{E}_2\mathcal{E}_1 A$ with $k \le 10$, where $\mathcal{E}_1$, $\mathcal{E}_2$, $\dots$, $\mathcal{E}_k$ are elementary matrices.

d) Show that the system $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ is consistent for each p, q, r.

e) Solve $AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: a) $A = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ -3 & 3 & -6 & -3 & 3 \\ 2 & -2 & 5 & 0 & 0 \end{bmatrix} \xrightarrow[-2R_1+R_3]{3R_1+R_2} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix} \to$

$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[-2R_2+R_1]{\frac{1}{3}R_3} \begin{bmatrix} 1 & -1 & 0 & 5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R.$

$B = \begin{bmatrix} 0 & 0 & 1 & -2 & x \\ 0 & 0 & 0 & 0 & 1 \\ 1 & y & 2 & 1 & z \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & y & 2 & 1 & z \\ 0 & 0 & 1 & -2 & x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to$

$\xrightarrow[-xR_3+R_2]{-zR_3+R_1} \begin{bmatrix} 1 & y & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & y & 0 & 5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = S.$

b) For two matrices to be row equivalent, they should have the same row reduced echelon matrix. Thus $R = S$ and $y = -1$; x and z can be any numbers.

c) $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot$

$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot$

$\cdot \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ -3 & 3 & -6 & -3 & 3 \\ 2 & -2 & 5 & 0 & 0 \end{bmatrix}.$

d) $\left[\begin{array}{rrrrr|c} 1 & -1 & 2 & 1 & 0 & p \\ -3 & 3 & -6 & -3 & 3 & q \\ 2 & -2 & 5 & 0 & 0 & r \end{array}\right] \xrightarrow[3R_1+R_2]{-2R_1+R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & p \\ 0 & 0 & 0 & 0 & 3 & q+3p \\ 0 & 0 & 1 & -2 & 0 & r-2p \end{array}\right] \rightarrow$

$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & p \\ 0 & 0 & 1 & -2 & 0 & r-2p \\ 0 & 0 & 0 & 0 & 3 & q+3p \end{array}\right].$

So the system is consistent for each p, q, r.

e) $\left[\begin{array}{rrrrr|c} 1 & -1 & 2 & 1 & 0 & 1 \\ -3 & 3 & -6 & -3 & 3 & 1 \\ 2 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3R_1+R_2]{-2R_1+R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & -2 & 0 & -1 \end{array}\right] \rightarrow$

$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 3 & 4 \end{array}\right].$

$$\left\{\begin{array}{rcr} x_1 - x_2 + 2x_3 + x_4 & = & 1 \\ 3x_5 & = & 4 \\ x_3 - 2x_4 & = & -1 \end{array}\right. \Leftrightarrow$$

$x_1 = 1 + x_2 - 2x_3 - x_4 = 1 + x_2 - 4x_4 + 2 - x_4 = 3 + x_2 - 5x_4$; $x_5 = 4/3$; $x_3 = 2x_4 - 1$.

$$X = \begin{bmatrix} 3 + x_2 - 5x_4 \\ x_2 \\ -1 + 2x_4 \\ x_4 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 4/3 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}. \square$$

Exercise 0.1.19 Consider the following list of statements. In each case, either prove the statement is true or give an example showing that it is false.

a) For a square matrix A, $A + A^T$ is symmetric.

b) For a square matrix A, $A - A^T$ is skew-symmetric.

c) For square matrices A and B, $(A + B)(A - B) = A^2 - B^2$.

d) If $A^2 = I$ then $A = I$ or $A = -I$.

e) For square matrices A and B, if $AB = 0$ then $BA = 0$.

f) A square matrix P is called idempotent if $P^2 = P$. If P is idempotent so is $Q = P + AP - PAP$ for any square matrix A.

g) If A^2 is invertible then A is invertible.

Solution: a) True. $(A + A^T)^T = A^T + (A^T)^T = A^T + A$.

b) True. $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.

c) False. Consider as example $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$. We have

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \neq$$

$$\neq \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^2 - \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.$$

d) False. An example A or B from c).

e) False. As example take $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. We have

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

f) True. We have $P \cdot Q = P^2 + PAP - P^2AP = P + PAP - PAP = P$ and $Q \cdot P = P^2 + AP^2 - PAP^2 = P + AP = PAP = Q$. Thus $Q^2 = (QP)Q = Q(PQ) = QP = Q$, so Q is idempotent.

g) True. If A^2 is invertible then there is B such that $A^2B = I$ (and $BA^2 = I$) then $A(AB) = I$ then A has a left inverse. By Theorem 2.2.1 in the book, A is invertible. □

Exercise 0.1.20 Let $C = [A|B]$ be the augmented matrix of a system $AX = B$ of linear equations with a square coefficient matrix A. Assume that C is row equivalent to a matrix D with a zero row. Show that the matrix A is not invertible.

Solution: Let $C \sim D$ then $PC = D$ for some invertible matrix P. That is $P[A|B] = [PA|PB] = C$. Hence the square matrix PA has a zero row. Since $A \sim PA$, then A is not invertible. □

Exercise 0.1.21 Let A, B be $n \times n$ matrices such that AB is invertible. Show that A is invertible.

Solution: Let C be the inverse of AB, that is $CAB = ABC = I$. Hence $A(BC) = I$. Thus a square matrix A has a right inverse, namely BC. Then BC is also a left inverse and hence $A^{-1} = BC$ □

Exercise 0.1.22 Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 6 & 4 & 2 \\ -1 & 2 & -2 & 0 \end{bmatrix}$. Find a row-reduced echelon matrix B which is row equivalent to A. Find an invertible matrix P such that $B = PA$.

Solution: Remark that matrix A is not square, thus there are infinitely many such matrix P. We find one of them.

$$[A|I] = \left[\begin{array}{rrrr|rrr} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 6 & 4 & 2 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_3+R_2} \left[\begin{array}{rrrr|rrr} 0 & 4 & 1 & 1 & 1 & 0 & 1 \\ 0 & 8 & 2 & 2 & 0 & 1 & 1 \\ -1 & 2 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_3]{-2R_1+R_2} \left[\begin{array}{rrrr|rrr} 0 & 4 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 \\ 1 & -2 & 2 & 0 & 0 & 0 & -1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{\frac{1}{4}R_1;\ R_3 \leftrightarrow R_2}$$

$$\left[\begin{array}{rrrr|rrr} 1 & -2 & 2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 \end{array}\right] \xrightarrow{2R_2+R_1}$$

$$\left[\begin{array}{rrrr|rrr} 1 & 0 & 5/2 & 1/2 & 1/2 & 0 & -1/2 \\ 0 & 1 & 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 \end{array}\right] = [B|P].$$

Hence $P = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 1/4 & 0 & 1/4 \\ -2 & 1 & -1 \end{bmatrix}$. □

Exercise 0.1.23 Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 6 & 4 \\ -1 & 2 & -2 \end{bmatrix}$. Is C invertible? If no, explain why C is not invertible.

Solution: Since $C \xrightarrow[R_3+R_2]{R_3+R_1 \;\; -2R_1+R_2} \begin{bmatrix} 0 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix}$, then C is row equivalent to a matrix with zero row. Hence C is not invertible. □

Exercise 0.1.24 Let $D = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$. Is D invertible? If yes, find D^{-1}. If no, explain why C is not invertible.

Solution: $[D|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 3 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow[2R_3+R_1]{R_2+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/3 & -5/3 & 2 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/3 & -5/3 & 2 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] = [I|D^{-1}].$$

Hence $D^{-1} = \begin{bmatrix} 7/3 & -5/3 & 2 \\ -2/3 & 1/3 & 0 \\ -1 & 1 & -1 \end{bmatrix}$. □

0.2 Systems of linear equations

Exercise 0.2.1 Find the general solution $[x_1\ x_2\ x_3\ x_4\ x_5]^T$ of the following system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & = & 2 \\ & & & & x_3 & + & 4x_4 & - & 2x_5 & = & -1 \\ x_1 & + & 2x_2 & + & x_3 & + & x_4 & - & x_5 & = & 1 \end{array}$$

Solution:

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 1 & 2 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 0 & 0 & 1 & 4 & -2 & -1 \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\rightarrow \left[\begin{array}{rrrrr|r} \underline{1} & 2 & 0 & -3 & 1 & 2 \\ 0 & 0 & \underline{1} & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent; the variables x_2, x_4, x_5 are free. Find the fundamental solutions of the system.

1) $x_2 = 1$, $x_4 = 0$, $x_5 = 0$. Then $\left\{\begin{array}{lcl} x_1 + 2 & = & 0 \\ x_3 & = & 0 \end{array}\right.$ and $X_1 = \left[\begin{array}{r} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right].$

2) $x_2 = 0$, $x_4 = 1$, $x_5 = 0$. Then $\left\{\begin{array}{lcl} x_1 - 3 & = & 0 \\ x_3 + 4 & = & 0 \end{array}\right.$ and $X_2 = \left[\begin{array}{r} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{array}\right].$

3) $x_2 = x_4 = 0$, $x_5 = 1$. Then $\left\{\begin{array}{lcl} x_1 + 1 & = & 0 \\ x_3 - 2 & = & 0 \end{array}\right.$ and $X_3 = \left[\begin{array}{r} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{array}\right].$

Find a partial solution of the system.

$x_2 = x_4 = x_5 = 0$. Then $\begin{cases} x_1 &= 2 \\ x_3 &= -1 \end{cases}$ and $V = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

Thus the general solution is $X = V + x_2 \cdot X_1 + x_4 \cdot X_2 + x_5 \cdot X_3$ □

Exercise 0.2.2 Find the value(s) of t for which $[t\ 0\ -1\ 0\ 0]^T$ is a solution of the system in 0.2.1.

Solution:

$$\begin{bmatrix} t \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} -2x_2 + 3x_4 - x_5 = t-2 \\ x_2 = 0 \\ -4x_4 + 2x_5 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{cases} \quad \Rightarrow \quad t-2=0 \quad \Rightarrow \quad t=2. \quad \square$$

Exercise 0.2.3 Find x, y, and z (if exist) for which

$$x \cdot \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + y \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix}.$$

Solution: The correspondent system of linear equations is the following:

$$\begin{cases} x + y + z = 2 \\ 2x + 2y + 3z = 5 \\ x + z = -3 \end{cases}$$

Solve it:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 1 & 0 & 1 & -3 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

The system is consistent. It is equivalent to $\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ & & -y & & & = & -5 \\ & & & & z & = & 1 \end{array}\right.$.

So $x = -4$, $y = 5$, $z = 1$ □

Exercise 0.2.4 Find the value(s) of t for which the following matrix equation has no solution.

$$x \cdot \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} + y \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + z \cdot \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & t \end{bmatrix}.$$

Solution: The correspondent system of linear equations is the following:

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ 2x & + & 2y & + & 3z & = & 5 \\ x & & & + & z & = & -3 \\ 4x & + & 3y & + & 5z & = & t \end{array}\right.$$

Solve it:

$$\left[\begin{array}{ccc|r} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 1 & 0 & 1 & -3 \\ 4 & 3 & 5 & t \end{array}\right] \xrightarrow{\text{see 0.2.3}} \left[\begin{array}{ccc|r} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 4 & 3 & 5 & t \end{array}\right] \xrightarrow[-R_2]{-4R_1+R_4} \left[\begin{array}{ccc|r} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & t-8 \end{array}\right]$$

$$\xrightarrow{R_2+R_4} \left[\begin{array}{ccc|r} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & t-3 \end{array}\right] \xrightarrow{-R_3+R_4} \left[\begin{array}{ccc|r} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & t-4 \end{array}\right].$$

Consequently, the system is inconsistent if $t \neq 4$, thus our equation has no solution if $t \neq 4$. □

Exercise 0.2.5 a) Find the value(s) of r such that the following system of linear equations

$$\left\{\begin{array}{rcrcrcrcr} 2x & + & 3y & + & 7z & + & 11t & = & 1 \\ x & + & 2y & + & 4z & + & 7t & = & 2r \\ 5x & & & + & 10z & + & 5t & = & r-1 \end{array}\right.$$

is consistent.

b) Find fundamental solutions of the following homogeneous system and write down the general solution in terms of them.

$$\begin{cases} 2x + 3y + 7z + 11t = 0 \\ x + 2y + 4z + 7t = 0 \\ 5x + 10z + 5t = 0 \end{cases}.$$

Solution: a) $\left[\begin{array}{cccc|c} 2 & 3 & 7 & 11 & 1 \\ 1 & 2 & 4 & 7 & 2r \\ 5 & 0 & 10 & 5 & r-1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{-5R_2+R_3;-2R_2+R_1}$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 7 & 2r \\ 0 & -1 & -1 & -3 & 1-4r \\ 0 & -10 & -10 & -30 & -1-9r \end{array}\right] \xrightarrow[-R_2]{-10R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 7 & 2r \\ 0 & 1 & 1 & 3 & 4r-1 \\ 0 & 0 & 0 & 0 & 31r-11 \end{array}\right].$$

Hence it is consistent iff $31r - 11 = 0$ or $r = \frac{11}{31}$.

b) $\begin{bmatrix} 2 & 3 & 7 & 11 \\ 1 & 2 & 4 & 7 \\ 5 & 0 & 10 & 5 \end{bmatrix} \rightarrow \ldots \rightarrow \begin{bmatrix} \underline{1} & 2 & 4 & 7 \\ 0 & \underline{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

So, the variables z and t are free. Find the fundamental solutions of the system.

$$\begin{cases} x + 2y + 4z + 7t = 0 \\ y + z + 3t = 0 \end{cases}.$$

1) $z = 1$, $t = 0$. Then $X_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$

2) $z = 0$, $t = 1$. Then $X_2 = \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$

The general solution is $X = z \cdot X_1 + t \cdot X_2 = \begin{bmatrix} -2z - t \\ -z - 3t \\ z \\ t \end{bmatrix}$. $\square$

Exercise 0.2.6 Let $A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Consider the homogeneous system $AX = 0$. Find for the system:

a) Free variable(s) and basic variable(s).

b) Fundamental solution(s).

c) The general solution.

d) Is the system $AX = B$ consistent for $B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$?

Solution: a) $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} \underline{1} & 0 & -2 & 3 \\ 0 & \underline{1} & -1 & 1 \\ 0 & 0 & \underline{1} & 0 \end{bmatrix}$.

So, the variables x_1, x_2, and x_3 are basic. The variable x_4 is free.

b) Put $x_4 = 1$. We have $\begin{cases} x_1 - 2x_3 + 3x_4 & = 0 \\ x_2 - x_3 + x_4 & = 0 \\ x_3 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 + 3 & = 0 \\ x_2 + 1 & = 0 \\ x_3 & = 0 \end{cases}$.

The fundamental solution is $X_1 = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

c) $X = x_4 \cdot X_1 = \begin{bmatrix} -3x_4 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix}$.

d) $\left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} \underline{1} & 0 & -2 & 3 & 1 \\ 0 & \underline{1} & -1 & 1 & 2 \\ 0 & 0 & \underline{1} & 0 & -2 \end{array}\right]$.

So the system is consistent. □

Exercise 0.2.7 Find fundamental solutions of the following homogeneous system

$$\begin{cases} x_1 & - & x_2 & - & x_3 & - & x_4 & + & x_5 & = & 0 \\ 4x_1 & - & 4x_2 & - & x_3 & - & 9x_4 & + & 6x_5 & = & 0 \\ & & & & 3x_3 & - & 5x_4 & + & 2x_5 & = & 0 \\ & & x_2 & + & x_3 & & & + & x_5 & = & 0 \end{cases}.$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ 4 & -4 & -1 & -9 & 6 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_1+R_2} \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4}$$

$$\to \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & -5 & 2 \\ 0 & 0 & 3 & -5 & 2 \end{bmatrix} \xrightarrow{-R_3+R_4} \begin{bmatrix} \underline{1} & -1 & -1 & -1 & 1 \\ 0 & \underline{1} & 1 & 0 & 1 \\ 0 & 0 & \underline{3} & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{A}.$$

So, the variables x_4 and x_5 are free. Fundamental solutions are:

1) Solutions of $\tilde{A}X = 0$ corresponding to $x_4 = 1$, $x_5 = 0$.

$$\begin{cases} x_1 - x_2 - x_3 & = & 1 \\ x_2 + x_3 & = & 0 \\ 3x_3 & = & 5 \end{cases} \text{ and } F_1 = \begin{bmatrix} 1 \\ -5/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix}.$$

2) Solutions of $\tilde{A}X = 0$ corresponding to $x_4 = 0$, $x_5 = 1$.

$$\begin{cases} x_1 - x_2 - x_3 & = & -1 \\ x_2 + x_3 & = & -1 \\ 3x_3 & = & -2 \end{cases} \text{ and } F_2 = \begin{bmatrix} -2 \\ -1/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix}. \square$$

Exercise 0.2.8 Find the relations satisfied by a and b if the system $AX = B$ is consistent, where A is the coefficient matrix from 0.2.7 and $B = \begin{bmatrix} a \\ b \\ 2 \\ 3 \end{bmatrix}$.

Solution: Augmented matrix is

$$[A|B] = \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -1 & 1 & a \\ 4 & -4 & -1 & -9 & 6 & b \\ 0 & 0 & 3 & -5 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{see sol. 0.2.7}}$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -1 & 1 & a \\ 0 & 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & 3 & -5 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & b-4a-2 \end{array}\right] = [\tilde{A}|\tilde{B}].$$

So $AX = B$ is equivalent to $\tilde{A}X = \tilde{B}$. Hence $AX = B$ is consistent iff $b-4a-2 = 0$. $\square$

Exercise 0.2.9 Write the general solution of the system in 0.2.8 for $a = 1$ (in terms of fundamental solutions).

Solution: If $a = 1$ then $b - 4 \cdot 1 - 2 = 0$ and hence $b = 6$. The general solution of $AX = B$ is the same as the general solution of $\tilde{A}X = \tilde{B}$ that is $X = x_4 \cdot F_1 + x_5 \cdot F_2 + X'$, where X' is any partial solution of $\tilde{A}X = \tilde{B}$. To find X', one may take $x_4 = 0$ and $x_5 = 0$ in $\tilde{A}X = \tilde{B}$:

$$\begin{cases} x_1 - x_2 - x_3 & = & 1 \\ x_2 + x_3 & = & 3 \\ 3x_3 & = & 2 \end{cases}. \text{ Hence } X' = \begin{bmatrix} 4 \\ 7/3 \\ 2/3 \\ 0 \\ 0 \end{bmatrix} \text{ and}$$

$$X = x_4 \cdot \begin{bmatrix} 1 \\ -5/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -2 \\ -1/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 7/3 \\ 2/3 \\ 0 \\ 0 \end{bmatrix}. \quad \square$$

Exercise 0.2.10 Given a system

$$\begin{cases} x & + & 3y & - & 2z & = & 1 \\ -x & - & 5y & + & 3z & = & -1 \\ 2x & - & 8y & + & 3z & = & \alpha \end{cases}.$$

a) Determine the value(s) (if exist) of α which makes the system consistent.

b) Find fundamental solutions of the corresponding homogeneous system.

c) Write down a general solution for those α when the system is consistent to the given non-homogeneous system in terms of fundamental solutions that you found in b).

Solution: a) $\left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ -1 & -5 & 3 & -1 \\ 2 & -8 & 3 & \alpha \end{array}\right] \xrightarrow[-2R_1+R_3]{R_1+R_2} \left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -14 & 7 & \alpha-2 \end{array}\right] \rightarrow$

$\xrightarrow{-7R_2+R_3} \left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & \alpha-2 \end{array}\right].$

So only $\alpha = 2$ makes the system consistent.

b) The free variable is z. Put $z = 1$. We have $\left\{\begin{array}{rcl} x+3y-2z & = & 0 \\ -2y+z & = & 0 \end{array}\right.$.

$-2y + 1 = 0 \Rightarrow y = 1/2$; $x + 3 \cdot 1/2 - 2 = 0 \Rightarrow x = 1/2$.

Hence there is only one fundamental solution for $z = 1$: $F = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$.

c) A general solution is $X = z \cdot F + X_p$, where X_p is a partial solution of the system ($\alpha = 2$).

$\left\{\begin{array}{rcl} x+3y-2z & = & 1 \\ -2y+z & = & 0 \end{array}\right.$.

For X_p take $z = 0$ then $X_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus a general solution is $X = z \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. □

Exercise 0.2.11 Find the conditions on a, b, c, and d for which the matrix system

$$x_1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 7 & 7 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has

a) no solution;

b) infinitely many solutions.

c) Find the general solution of the equation for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution: The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 1 & 1 & 3 & 7 & b \\ 1 & -1 & 1 & -3 & c \\ 1 & -1 & 1 & -3 & d \end{array}\right] \xrightarrow[-R_3+R_4]{-R_1+R_2} \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 0 & 0 & 0 & 0 & b-a \\ 1 & -1 & 1 & -3 & c \\ 0 & 0 & 0 & 0 & d-c \end{array}\right] \xrightarrow{-R_1+R_3}$$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 0 & 0 & 0 & 0 & b-a \\ 0 & -2 & -2 & -10 & c-a \\ 0 & 0 & 0 & 0 & d-c \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{-\frac{1}{2}R_3} \left[\begin{array}{rrrr|r} 1 & 1 & 3 & 7 & a \\ 0 & 1 & 1 & 5 & a/2-c/2 \\ 0 & 0 & 0 & 0 & b-a \\ 0 & 0 & 0 & 0 & d-c \end{array}\right] \xrightarrow{-R_2+R_1}$$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 2 & a/2+c/2 \\ 0 & 1 & 1 & 5 & a/2-c/2 \\ 0 & 0 & 0 & 0 & b-a \\ 0 & 0 & 0 & 0 & d-c \end{array}\right].$$

a) If $b-a \neq 0$ or $d-c \neq 0$ the system has no solution.

b) If $b-a=0$ and $d-c=0$ the system has infinitely many solutions.

c) We have $b=a=c=d=1$. Then the augmented matrix of the system is equivalent to $\left[\begin{array}{rrrr|r} \underline{1} & 0 & 2 & 2 & a/2+c/2=1 \\ 0 & \underline{1} & 1 & 5 & a/2-c/2=0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The fundamental solutions are solutions of the correspondent homogeneous system. $\begin{cases} x_1+2x_3+2x_4 &= 0 \\ x_2+x_3+5x_4 &= 0 \end{cases}$. So $x_1=-2x_3-2x_4$ and $x_2=-x_3-5x_4$.

For $x_3=0$ and $x_4=1$: $F_1 = \begin{bmatrix} -2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$.

For $x_3 = 1$ and $x_4 = 0$: $F_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

A partial solution find from $\begin{cases} x_1 + 2x_3 + 2x_4 & = & 1 \\ x_2 + x_3 + 5x_4 & = & 0 \end{cases}$ with $x_3 = x_4 = 0$: $V = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

And finally the general solution is $X = x_3 \cdot F_1 + x_4 \cdot F_2 + V$. $\square$

Exercise 0.2.12 Find the values of a and b for which the system

$$\begin{cases} x & + & 2y & - & z & + & at & = & 1 \\ & & ay & + & (b+1)z & & & = & b \\ & & & & z & + & at & = & b \\ & & & & & & (a-1)t & = & b \end{cases}$$

has

i) No solution.

ii) A unique solution.

iii) Infinitely many solutions.

Solution: i) If $a = 0$, $b \neq 0$ then the system has **no solution** since

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & b+1 & 0 & b \\ 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & -1 & b \end{array}\right] \xrightarrow{(b+1)R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & b+1 & 0 & b \\ 0 & 0 & b+1 & 0 & b(b+1) \\ 0 & 0 & 0 & -1 & b \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & b+1 & 0 & b \\ 0 & 0 & 0 & 0 & b^2 \neq 0 \\ 0 & 0 & 0 & -1 & b \end{array}\right].$$

If $a = 1$, $b \neq 0$ then the system has **no solution** since

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & b+1 & 0 & b \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 & b \neq 0 \end{array}\right].$$

ii) If $a \neq 0$, $a \neq 1$, and b is arbitrary then the coefficient matrix $\begin{bmatrix} 1 & 2 & -1 & a \\ 0 & a & b+1 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & a-1 \end{bmatrix}$ is invertible. In this case our system has a **unique solution**.

iii) If $a = 0$, $b = 0$ then the system has **infinitely many solutions** since y is a free variable.

If $a = 1$, $b = 0$ then the system has **infinitely many solutions** since t is a free variable. □

Exercise 0.2.13 Find two different pairs (a, b) of values of a and b for which the homogeneous system

$$\left\{\begin{array}{ccccccccccc} x & + & 2y & - & z & + & (a+1)t & + & bu & = & 0 \\ & & ay & + & (b+1)z & & & + & 2au & = & 0 \\ & & & & z & + & bt & + & au & = & 0 \end{array}\right.$$

has three fundamental solutions. For one of such a pair (a, b) of numbers a and b find all three correspondent fundamental solutions.

Solution: To get more than 2 free variables, we must have $a = 0$. Moreover, then our system becomes:

$$\left\{\begin{array}{ccccccccccc} x & + & 2y & - & z & + & t & + & bu & = & 0 \\ & & & & (b+1)z & & & & & = & 0 \\ & & & & z & + & bt & & & = & 0 \end{array}\right.$$

Thus, in order to have 3 free variables, one must also take $b = 0$ or $b = -1$. So the required pairs are $(a, b) = (0, 0)$ and $(a, b) = (0, -1)$.

1) $a = 0$, $b = 0$.

$[x\ y\ z\ t\ u] : \begin{bmatrix} \underline{1} & 2 & -1 & 1 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 \end{bmatrix}$

$$\left\{\begin{array}{rcrcrcrcl} x & + & 2y & - & z & + & t & = & 0 \\ & & & & z & & & = & 0 \end{array}\right. ;$$

y, t, and u are free.

$$y = 1,\ t = 0,\ u = 0,\ F_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$y = 0,\ t = 1,\ u = 0,\ F_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix};$$

$$y = 0,\ t = 0,\ u = 1,\ F_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

2) $a = 0$, $b = -1$.

$$[x\ y\ z\ t\ u] : \begin{bmatrix} \underline{1} & 2 & -1 & 1 & -1 \\ 0 & 0 & \underline{1} & -1 & 0 \end{bmatrix}$$

$$\left\{\begin{array}{rcrcrcrcrcl} x & + & 2y & - & z & + & t & - & u & = & 0 \\ & & & & z & - & t & & & = & 0 \end{array}\right. ;$$

y, t, and u are free.

$$y = 1,\ t = 0,\ u = 0,\ G_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$y = 0,\ t = 1,\ u = 0,\ G_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix};$$

$$y=0,\ t=0,\ u=1,\ G_3=\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}.\ \square$$

Exercise 0.2.14 a) Determine whether the system $Ax = B$ is consistent,

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + 5x_5 = 0 \\ -5x_1 - 10x_2 + 3x_3 + 3x_4 + 55x_5 = -8 \\ x_1 + 2x_2 + 2x_3 - 3x_4 - 5x_5 = 14 \\ -x_1 - 2x_2 + x_3 + x_4 + 15x_5 = -2 \end{cases}$$

b) If it is consistent, find the general solution of the form $x_h + x_p$, where x_h is the solution of $Ax = 0$ and x_p is the solution of $Ax = B$. What is the dimension of the solution space of the system $Ax = 0$, please, explain.

Solution: a) Take the coefficient matrix.

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & 5 & 0 \\ -5 & -10 & 3 & 3 & 55 & -8 \\ 1 & 2 & 2 & -3 & -5 & 14 \\ -1 & -2 & 1 & 1 & 15 & -2 \end{array}\right] \xrightarrow[5R_1+R_2]{-R_1+R_3;\ R_1+R_4} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & 5 & 0 \\ 0 & 0 & 8 & 8 & 80 & -8 \\ 0 & 0 & 1 & -4 & -10 & 14 \\ 0 & 0 & 2 & 2 & 20 & -2 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{4}R_2+R_4]{-\frac{1}{8}R_2+R_3;\ \frac{1}{8}R_2} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & 10 & -1 \\ 0 & 0 & 0 & -5 & -20 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-\frac{1}{5}R_3]{R_3+R_2;\ -R_2+R_1}$$

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 6 & 2 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{rrrrr|r} \underline{1} & 2 & 0 & 0 & -5 & 1 \\ 0 & 0 & \underline{1} & 0 & 6 & 2 \\ 0 & 0 & 0 & \underline{1} & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So the system is consistent and, since x_2 and x_5 are free variables, there are infinitely many solutions.

b) From (a) we have the system $Ax = 0$ is equivalent to that with the coefficient matrix $\left[\begin{array}{ccccc|c} \underline{1} & 2 & 0 & 0 & -5 & 0 \\ 0 & 0 & \underline{1} & 0 & 6 & 0 \\ 0 & 0 & 0 & \underline{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. So we have the solution

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5x_5 - 2x_2 \\ x_2 \\ -6x_5 \\ -4x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ -6 \\ -4 \\ 1 \end{bmatrix}.$$

The dimension of the solution space is 2.

Find x_p from $\left[\begin{array}{ccccc|c} \underline{1} & 2 & 0 & 0 & -5 & 1 \\ 0 & 0 & \underline{1} & 0 & 6 & 2 \\ 0 & 0 & 0 & \underline{1} & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 + 5x_5 - 2x_2 \\ x_2 \\ 2 - 6x_5 \\ -3 - 4x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ -6 \\ -4 \\ 1 \end{bmatrix}. \quad \square$$

0.3 Determinants

Exercise 0.3.1 Evaluate the following determinants.

$$A = \begin{vmatrix} 1 & 2 & 0 & 4 & 2 \\ 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 4 & 1 \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 0 & 4 & 0 \\ 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix}.$$

Solution: $A = (-1)^{2+3} \cdot 4 \cdot \begin{vmatrix} 1 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 4 & 1 \end{vmatrix} = (-4) \cdot 3 \cdot \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} =$

$$= (-12) \cdot \left(2 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\right) = (-12) \cdot (2 \cdot (-1) - 1 \cdot 0) = 24.$$

$$B = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 5 & 4 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 3 \\ 5 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2 \cdot (-12) \cdot (-2) = 48. \ \square$$

Exercise 0.3.2 Compute the adjoint (adjugate) of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 4 & 3 & 0 \end{bmatrix}$.

1-st solution: $a_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -9$; $a_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 3 \\ 4 & 0 \end{vmatrix} = 12$; $a_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & 2 \\ 4 & 3 \end{vmatrix} = -8$; $a_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = 3$; $a_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = -4$; $a_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 1$; $a_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$; $a_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3$; $a_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$.

$$A_0 = \begin{bmatrix} -9 & 12 & -8 \\ 3 & -4 & 1 \\ 1 & -3 & 2 \end{bmatrix}; \qquad \operatorname{adj}(A) = A_0^T = \begin{bmatrix} -9 & 3 & 1 \\ 12 & -4 & -3 \\ -8 & 1 & 2 \end{bmatrix}.$$

2-nd solution: $adj(A) = |A| \cdot A^{-1}$, $|A| = 2 \cdot \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -8 + 3 = -5.$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-4R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & -4 & -4 & 0 & 1 \end{array}\right] \xrightarrow[-R_2]{R_2 \leftrightarrow R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 4 & 0 & -1 \\ 0 & 2 & 3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 4 & 0 & -1 \\ 0 & 0 & -5 & -8 & 1 & 2 \end{array}\right] \xrightarrow{\frac{4}{5}R_3+R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -12/5 & 4/5 & 3/5 \\ 0 & 0 & -5 & -8 & 1 & 2 \end{array}\right] \xrightarrow[-R_2+R_1]{\frac{1}{5}R_3+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/5 & -3/5 & -1/5 \\ 0 & 1 & 0 & -12/5 & 4/5 & 3/5 \\ 0 & 0 & -5 & -8 & 1 & 2 \end{array}\right] \xrightarrow{-\frac{1}{5}R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/5 & -3/5 & -1/5 \\ 0 & 1 & 0 & -12/5 & 4/5 & 3/5 \\ 0 & 0 & 1 & 8/5 & -1/5 & -2/5 \end{array}\right] \Rightarrow A^{-1} = \frac{1}{5} \begin{bmatrix} 9 & -3 & -1 \\ -12 & 4 & 3 \\ 8 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow adj(A) = |A| \cdot A^{-1} = (-5) \cdot \frac{1}{5} \begin{bmatrix} -9 & 3 & 1 \\ 12 & -4 & -3 \\ -8 & 1 & 2 \end{bmatrix}. \ \square$$

Exercise 0.3.3 Given that $\begin{vmatrix} 1 & a & 1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 9$, compute the following determinants

$$A = \begin{vmatrix} a+1 & a+2 & b+1 \\ b & 1 & b \\ 2a & 4 & 2b \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} x^2 & ax^3 & x^4 \\ bx & x^2 & bx^3 \\ ax & 2x^2 & bx^3 \end{vmatrix}.$$

Solution: $A = 2 \cdot \begin{vmatrix} a+1 & a+2 & b+1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & a & 1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 2 \cdot 9 = 18.$

$B = x \cdot x^2 \cdot x^3 \cdot \begin{vmatrix} x & ax & x \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = x^6 \cdot x \cdot \begin{vmatrix} 1 & a & 1 \\ b & 1 & b \\ a & 2 & b \end{vmatrix} = 9x^7. \ \square$

Exercise 0.3.4 Compute the following determinant $\begin{vmatrix} 1 & 2 & -3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 7 & -8 & 9 \\ 3 & -2 & 1 & 10 & 9 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 & 5 & 5 \end{vmatrix}$.

Solution:
$$\left|\begin{array}{rrr|rrr} 1 & 2 & -3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 7 & -8 & 9 \\ 3 & -2 & 1 & 10 & 9 & 8 \\ \hline 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 & 5 & 5 \end{array}\right| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 0 \\ 3 & -2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{vmatrix} \cdot (-1) =$$
$= (-42) \cdot \left((-3) \cdot \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \right) = (-42) \cdot ((-3) \cdot (-13) + (-1)) =$
$-42 \cdot 38 = -1596.$ □

Exercise 0.3.5 Use Cramer's rule to solve for z: $\begin{cases} 2x & - & y & - & z & = & 0 \\ 2x & - & y & + & 4z & = & -1 \\ -x & + & 2y & + & z & = & 2 \end{cases}$.

Solution: The augmented matrix is $\left[\begin{array}{rrr|r} 2 & -1 & -1 & 0 \\ 2 & -1 & 4 & -1 \\ -1 & 2 & 1 & 2 \end{array}\right]$.

$$z = \frac{\begin{vmatrix} 2 & -1 & 0 \\ 2 & -1 & -1 \\ -1 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ 2 & -1 & 4 \\ -1 & 2 & 1 \end{vmatrix}} = \frac{2 \cdot \begin{vmatrix} -1 & -1 \\ 2 & 2 \end{vmatrix} + (-1)(-1) \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}{2 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} + (-1) \cdot (-1) \cdot \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}} =$$
$$= \frac{3}{3 \cdot (-9) + 6 - 3} = \frac{3}{-15} = -\frac{1}{5}. \quad \square$$

Exercise 0.3.6 Given that $\begin{vmatrix} a & b & c & d \\ b & c & 0 & b \\ c & 0 & b & c \\ d & b & a & d \end{vmatrix} = 6$, compute the determinant

$$D = \begin{vmatrix} a+c & 3bx & c & d \\ d+a & 3bx & a & d \\ c+b & 0 & b & c \\ bx^2 & 3cx^3 & 0 & bx^2 \end{vmatrix}.$$

Solution: $D = \begin{vmatrix} a & 3bx & c & d \\ d & 3bx & a & d \\ c & 0 & b & c \\ bx^2 & 3cx^3 & 0 & bx^2 \end{vmatrix} = 3x \cdot \begin{vmatrix} a & b & c & d \\ d & b & a & d \\ c & 0 & b & c \\ bx^2 & 3cx^3 & 0 & bx^2 \end{vmatrix} = 3x^3 \cdot \begin{vmatrix} a & b & c & d \\ d & b & a & d \\ c & 0 & b & c \\ b & c & 0 & b \end{vmatrix} = [R_2 \leftrightarrow R_4] = -3x^3 \begin{vmatrix} a & b & c & d \\ b & c & 0 & b \\ c & 0 & b & c \\ d & b & a & d \end{vmatrix} = -3x^3 \cdot 6 = -18x^3.$ $\square$

Exercise 0.3.7 Compute the determinants of the following matrices

$$A = \begin{bmatrix} 2 & 1 & 2 & -1 & 1 & 2 \\ 1 & 2 & 1 & 1 & -1 & 3 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 7 & 5 & 2 & -1 \\ 0 & 0 & 3 & 2 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 & 2 & 4 \\ 2 & -1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & -5 & 6 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}.$$

Solution:

$$|A| = \left|\begin{array}{cc|cc|cc} 2 & 1 & 2 & -1 & 1 & 2 \\ 1 & 2 & 1 & 1 & -1 & 3 \\ \hline 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ \hline 0 & 0 & 7 & 5 & 2 & -1 \\ 0 & 0 & 3 & 2 & 1 & 2 \end{array}\right| = |A| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 3 \cdot 5 \cdot 5 = 75.$$

$$B \xrightarrow[R_1 \leftrightarrow R_4]{R_2 \leftrightarrow R_5; R_3 \leftrightarrow R_6} \begin{bmatrix} 2 & -1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & -5 & 6 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 & 2 & 4 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_6} \begin{bmatrix} 2 & -1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & -5 & 6 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} =$$

C.

$|B| = (-1)^4 \cdot |C| = |C| = 2 \cdot 1 \cdot 3 \cdot 3 \cdot (-2) \cdot 2 = -72.$ □

Exercise 0.3.8 Let C be a 11×11 skew-symmetric matrix. Find $\det(C)$.

Solution: Since C is a skew-symmetric, $C = -C^T$ and $\det(C) = (-1)^{11} \cdot \det(C^T) = -\det(C)$. Hence $2\det(C) = 0$, so $\det(C) = 0$. □

Exercise 0.3.9 Let A be a 3×3 matrix and let B be obtained from A by applying the following elementary row operations: $2R_1 + R_2$, $2R_2$, $-R_2 + R_3$, and $R_1 \leftrightarrow R_3$. Evaluate the followings if $B = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$:

a) $\det(A)$.

b) $\det(3A^{-1}B^T)$.

c) $\operatorname{adj}(B)$.

d) Express the matrix $\operatorname{adj}(2A)$ in terms of A^{-1}.

Solution: a) $B = \mathcal{E}_4 \cdot \mathcal{E}_3 \cdot \mathcal{E}_2 \cdot \mathcal{E}_1 \cdot A$, where $\mathcal{E}_1 = 2R_1 + R_2$, $\mathcal{E}_2 = 2R_2$, $\mathcal{E}_3 = -R_2 + R_3$, and $\mathcal{E}_4 = R_1 \leftrightarrow R_3$. Thus $\det(B) = -2\det(A)$.

$\det(B) = \begin{vmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (-1) \cdot (-2) \cdot 2 = 4 = -2\det(A)$ hence $\det(A) = -2$.

b) $\det(3A^{-1}B^T) = 3^3 \det(A^{-1}) \det(B^T) = 27 \cdot (\det(A))^{-1} \cdot \det(B) = 27 \cdot \frac{1}{-2} \cdot 4 = -54$.

c) $b_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = -4$; $b_{12} = (-1)^{1+2} \cdot \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = 2$; $b_{13} = (-1)^{1+3} \cdot \begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = -1$; $b_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = 0$; $b_{22} = (-1)^{2+2} \cdot \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2$; $b_{23} = (-1)^{2+3} \cdot \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $b_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 0 & 0 \\ -2 & 0 \end{vmatrix} = 0$; $b_{32} = (-1)^{3+2} \cdot \begin{vmatrix} -1 & 0 \\ -1 & 0 \end{vmatrix} = 0$; $b_{33} = (-1)^{3+3} \cdot \begin{vmatrix} -1 & 0 \\ -1 & -2 \end{vmatrix} = 2$.

So the cofactor matrix of B is $\begin{bmatrix} -4 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Thus

$$\operatorname{adj}(B) = \begin{bmatrix} -4 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} -4 & 0 & 0 \\ 2 & -2 & 0 \\ -1 & 1 & 2 \end{bmatrix}.$$

d) By using the fact $A \cdot \operatorname{adj}(A) = \det(A) \cdot I$ for every square matrix, we have $2A \cdot \operatorname{adj}(2A) = \det(2A) \cdot I \Rightarrow \operatorname{adj}(2A) = 1/2 \cdot 2^3 \cdot \det(A) \cdot A^{-1} \cdot I$. Thus

$$\operatorname{adj}(2A) = -8 \cdot A^{-1}. \quad \square$$

Exercise 0.3.10 Consider the following list of statements. In each case either prove the statement if it is true or give an example showing that it is false.

i) If $\det(A) = 0$ then A has two equal rows.

ii) If R is the row reduced echelon form of A then $\det(R) = \det(A)$.

iii) $\det(A^T) = -\det(A)$.

iv) If $\det(A) = \det(B)$ and matrices A and B have the same size, then $A = B$.

v) If $\det(A) \neq 0$ and $AB = AC$ then $B = C$.

vi) $\det(I + A) = 1 + \det(A)$.

vii) If $\det(A) = 1$ then $\operatorname{adj}(A) = A$.

viii) There is no invertible 17×17 skew-symmetric matrix.

Solution: i) False. For example, $A = \begin{bmatrix} 1 & 1/2 \\ 4 & 2 \end{bmatrix}$ has no equal rows, but $\det(A) = 0$.

ii) False. An example: $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3/2 \\ 0 & 5/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$. We see that $\det(A) = 5$ but $\det(R) = 1$.

iii) False. An example: $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, but $\det(A^T) = \det(A) = 2$.

iv) False. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have $\det(A) = \det(B) = 1$. but $A \neq B$.

v) True. If $\det(A) \neq 0$ then A is invertible, i.e. there exists A^{-1}. So if $AB = AC$, we can write $A^{-1}(AB) = A^{-1}(AC)$ that is equal to $IB = IC$ or $B = C$.

vi) False. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\right) = 6 \neq 3 = 1 + \det(A).$$

vii) False. An example: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq \operatorname{adj}(A) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

viii) True. Let A is 17×17 skew-symmetric matrix. Then $A^T = -A$ and $\det(A) = \det(A^T) = \det(-A) = (-1)^{17} \cdot \det(A) = -\det(A)$. So $\det(A) = -\det(A)$ hence $\det(A) = 0$, that means A is not invertible. $\square$

Exercise 0.3.11 Calculate the determinant of A where

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & 13 & 6 & 8 \\ 0 & 2 & -1 & 21 & 0 & 7 \\ 1 & 1 & 2 & 17 & -5 & 3 \end{bmatrix}.$$

Solution: $A \xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_6; R_2 \leftrightarrow R_5} \begin{bmatrix} 1 & 1 & 2 & 17 & -5 & 3 \\ 0 & 2 & -1 & 21 & 0 & 7 \\ 3 & 1 & 0 & 13 & 6 & 8 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{bmatrix}$. Hence

$$|A| = (-1)^3 \cdot \left|\begin{array}{ccc|ccc} 1 & 1 & 2 & 17 & -5 & 3 \\ 0 & 2 & -1 & 21 & 0 & 7 \\ 3 & 1 & 0 & 13 & 6 & 8 \\ \hline 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{array}\right| = (-1) \cdot \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 3 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 2 & 3 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} =$$

$$= (-1) \cdot \left(1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right) \cdot \left((-2) \cdot \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}\right) =$$

$$= (-1) \cdot (1 + 3 \cdot (-5)) \cdot (6 + 2 \cdot 5) = (-1) \cdot (-14) \cdot 16 = 224. \ \square$$

Exercise 0.3.12 Use Crammer's rule to solve for u: $\begin{cases} x+y &= 0 \\ y-u &= -1 \\ x+z &= 0 \\ x-y &= 1 \end{cases}$.

Solution: We have $A \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} - (-1) \cdot$$
$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -2 \neq 0.$$

$$|A_u| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} -$$
$$\left((-1) \cdot \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\right) = 0 - (0 + 1) = -1.$$

$$u = \frac{|A_u|}{|A|} = \frac{-1}{-2} = 1/2. \quad \square$$

0.1 Vector Spaces

Exercise 0.1.1 Show that the following functions x, $1+x$, $x+\sin^2 x$, x^3-x, and $x+\cos^2 x$ defined on $\mathbb{R}$ are linearly dependent.

Solution:

$x+(1+x)-(x+\sin^2 x)-(x+\cos^2 x) = 1-(\sin^2 x+\cos^2 x) = 1-1 = 0.$ $\square$

Exercise 0.1.2 Compute the dimension of the vector subspace

$$V = \operatorname{span}\{(-1,2,3,0),(5,4,3,0),(3,1,0,0)\}$$

of $\mathbb{R}^4$.

Solution: $$\begin{bmatrix} -1 & 2 & 3 & 0 \\ 5 & 4 & 3 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[3R_2+R_3]{5R_1+R_2} \begin{bmatrix} -1 & 2 & 3 & 0 \\ 0 & 14 & 18 & 0 \\ 0 & 7 & 9 & 0 \end{bmatrix} \xrightarrow[-2R_3+R_2]{R_3\leftrightarrow R_2} \begin{bmatrix} \underline{-1} & 2 & 3 & 0 \\ 0 & \underline{7} & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $\dim(V) = 2$. $\square$

Exercise 0.1.3 Find a basis for the row space of A and find the dimension of the row space of A, where $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

Solution: $$A \xrightarrow[R_1+R_4]{-R_1+R_2;-R_1+R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[-R_3+R_5]{R_3\leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basis is $\{(1,0,1,0,0),(0,1,-1,1,0),(0,0,2,0,1),(0,0,0,1,1)\}$. The dimension is 4. $\square$

Exercise 0.1.4 Extend $\{1+x^2, x-x^3\}$ to a basis for the space of polynomials of degree ≤ 3.

Solution: Let $v_1 = 1 + x^2$, $v_2 = x - x^3$, $v_3 = 1$, $v_4 = x$, $v_5 = x^2$, and $v_6 = x^3$. Consider the matrix $[[v_1][v_2][v_3][v_4][v_5][v_6]]$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{R_2+R_4} \begin{bmatrix} \underline{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \underline{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & \underline{1} & 0 & 1 \end{bmatrix}.$$

We see that the basis is $\{1 + x^2, x - x^3, 1, x\}$. $\square$

Exercise 0.1.5 Find coordinates (the coordinate matrix $[u]_C$) of $u = x - x^2 + x^3$ with respect to the basis $C = \{w_1, w_2, w_3\}$ of the vector space $W = \text{span}\{w_1, w_2, w_3\}$, where $w_1 = x + x^2$, $w_2 = x - x^2$, and $w_3 = x + x^2 + 2x^3$.

Solution:

$$u = x - x^2 + x^3 = w_2 + x^3 = w_2 + \frac{1}{2}(w_3 - w_1) = -\frac{1}{2} \cdot w_1 + 1 \cdot w_2 + \frac{1}{2} \cdot w_3$$

and

$$[u]_C = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix}. \qquad \square$$

Exercise 0.1.6 Let $w \in W$ be such that $[w]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, where C is the basis for W defined in 0.1.5. Find the polynomial w.

Solution:

$$w = 1 \cdot w_1 + 2 \cdot w_2 + 0 \cdot w_3 = (x + x^2) + 2(x - x^2) = 3x - x^2. \qquad \square$$

Exercise 0.1.7 Compute the transition matrix $P = P_{B \to C}$ from the basis $B = \{x, x^2, x^3\}$ for W to the basis $C = \{w_1, w_2, w_3\}$ for W defined in 0.1.5.

1-st Solution:

$$2x = w_1 + w_2 \Rightarrow [x]_C = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix};$$

$$2x^2 = w_1 - w_2 \Rightarrow [x^2]_C = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix};$$

$$2x^3 = w_3 - w_1 \Rightarrow [x^3]_C = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

Hence

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

2-nd Solution: $[w_1\ w_2\ w_3 \mid x\ x^2\ x^3] =$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2]{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-R_2-R_3+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] = [I|P]. \quad \square$$

Exercise 0.1.8 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$.

a) Find a basis for the solution space of $AX = 0$.

b) Find a basis for $\mathbb{R}^3$ that contains the basis constructed in part (a).

Solution: a) $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^{3\times 1}$, $AX = 0$.

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} \underline{1} & 1 & 1 \\ 0 & \underline{-2} & -2 \end{bmatrix}$, so x_3 is free. Find the fundamental solution.

Set $x_3 = 1$ then $x_2 = -1$, $x_1 = 0$. So $X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

b) Consider the matrix $[X_1\ e_1\ e_2\ e_3]$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \underline{-1} & 0 & 1 & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & 1 \end{bmatrix}$$

Hence the basis $\mathbb{R}^{3\times 1}$ is $\{X_1, e_1, e_2\} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. □

Exercise 0.1.9 Let $\{t, u, v, w\}$ be a basis for a vector space V. Find $\dim(U)$, where $U = \operatorname{span}\{t + 2u + v + w, t + 3u + v + 2w, 3t + 4u + 2v, 3t + 5u + 2v + w\}$.

Solution: Let $v_1 = t + 2u + v + w$, $v_2 = t + 3u + v + 2w$, $v_3 = 3t + 4u + 2v$, $v_4 = 3t + 5u + 2v + w$. Consider the coordinate matrix $[v_1\ v_2\ v_3\ v_4]$:

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} \underline{1} & 1 & 3 & 3 \\ 0 & \underline{1} & -2 & -1 \\ 0 & 0 & \underline{-1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $\{v_1, v_2, v_3\} = \{t + 2u + v + w, t + 3u + v + 2w, 3t + 4u + 2v\}$ is a basis for U. It has 3 vectors. Hence $\dim(U) = 3$. □

Exercise 0.1.10

a) Show that $\mathcal{C} = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$ is a basis for $\mathbb{R}^3$.

b) Let $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Find the change of coordinate matrices (that is transition matrices) from $\mathcal{C}$ to $\mathcal{B}$, and from $\mathcal{B}$ to $\mathcal{C}$.

Solution: a) The determinant $\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2 \neq 0$. It means that the rows are linearly independent, so $\mathcal{C}$ is linearly independent in $\mathbb{R}^3$, consequently $\mathcal{C}$ is a basis for $\mathbb{R}^3$ since $\dim(\mathbb{R}^3) = 3$ and $\mathbb{R}^3 \supseteq \operatorname{span}(\mathcal{C})$, where $\mathcal{C}$ has 3 vectors and finally $\dim(\langle \mathcal{C} \rangle) = 3$.

b) Set $u_1 = (1, 1, 0)$, $u_2 = (1, -1, 0)$, $u_3 = (0, 0, 1)$. Then $\mathcal{C} = \{u_1, u_2, u_3\}$.

$$P_{\mathcal{C}\to\mathcal{B}} = [[u_1]_{\mathcal{B}}\ [u_2]_{\mathcal{B}}\ [u_3]_{\mathcal{B}}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$P_{\mathcal{B}\to\mathcal{C}} = P_{\mathcal{C}\to\mathcal{B}}^{-1}.$$

$$[P_{\mathcal{C}\to\mathcal{B}}|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I|P_{\mathcal{B}\to\mathcal{C}}]. \qquad \square$$

Exercise 0.1.11 Let the set $\{u, v, w\}$ be linearly independent. Show that the set $\{u + 2v, v - 3w, u - v + w\}$ is linearly independent.

Solution: Denote $B = \{u, v, w\}$ is a basis for $\operatorname{span}(B)$. Then we can write

$$[u + 2v]_B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad [v - 3w]_B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \quad [u - v + w]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Since

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & -3 & 1 \end{bmatrix} = 1 \cdot \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} = -2 - 6 \neq 0.$$

So the set $\{u + 2v, v - 3w, u - v + w\}$ is linearly independent. $\square$

Exercise 0.1.12 Find the value(s) of α if $\begin{bmatrix} \alpha & 2 \\ 0 & 6-\alpha \end{bmatrix}$ is contained in the space

$$\operatorname{span}\left\{ \begin{bmatrix} -1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha & -1 \\ 0 & \alpha^2-\alpha-1 \end{bmatrix}, \begin{bmatrix} \alpha+1 & -3 \\ 0 & \alpha^2-4 \end{bmatrix} \right\}.$$

Solution:

$$\begin{bmatrix} \alpha & 2 \\ 0 & 6-\alpha \end{bmatrix} = x \cdot \begin{bmatrix} -1 & \alpha \\ 0 & 1 \end{bmatrix} + y \cdot \begin{bmatrix} \alpha & -1 \\ 0 & \alpha^2-\alpha-1 \end{bmatrix} + z \cdot \begin{bmatrix} \alpha+1 & -3 \\ 0 & \alpha^2-4 \end{bmatrix}.$$

Thus the following system must be consistent:

$$\left\{ \begin{array}{rcrcrcr} -x & + & \alpha y & + & (\alpha+1)z & = & \alpha \\ \alpha x & - & y & - & 3z & = & 2 \\ x & + & (\alpha^2-\alpha-1)y & + & (\alpha^2-4)z & = & 6-\alpha \end{array} \right.$$

$$\left[\begin{array}{rrr|r} -1 & \alpha & \alpha+1 & \alpha \\ \alpha & -1 & -3 & 2 \\ 1 & \alpha^2-\alpha-1 & \alpha^2-4 & 6-\alpha \end{array} \right] \xrightarrow[R_1+R_3]{\alpha R_1+R_2}$$

$$\to \left[\begin{array}{rrr|r} -1 & \alpha & \alpha+1 & \alpha \\ 0 & \alpha^2-1 & \alpha^2+\alpha-3 & 2+\alpha^2 \\ 0 & \alpha^2-1 & \alpha^2+\alpha-3 & 6 \end{array} \right] \xrightarrow{-R_2+R_3}$$

$$\to \left[\begin{array}{rrr|r} -1 & \alpha & \alpha+1 & \alpha \\ 0 & \alpha^2-1 & \alpha^2+\alpha-3 & 2+\alpha^2 \\ 0 & 0 & 0 & 4-\alpha^2 \end{array} \right].$$

Hence $4-\alpha^2=0$ or $\alpha=\pm 2$. $\square$

Exercise 0.1.13 Given the matrix $\begin{bmatrix} 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$. Show that the dimension of the column space of this matrix is equal to 3. Justify your answer.

Solution:

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow[R_2\leftrightarrow R_3]{R_1\leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_2}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} \underline{1} & 1 & 1 & -1 & 0 \\ 0 & \underline{-1} & -3 & 3 & 0 \\ 0 & 0 & \underline{-3} & 3 & 1 \end{bmatrix}.$$

Hence $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the column space. Therefore its dimension is equal to 3. □

Exercise 0.1.14 Find the value(s) of $\alpha \in \mathbb{R}$ such that $\dim(\text{span}(A)) = 2$, where $A = \{1+2x^2+x^4, 2+x+4x^2+x^3+5x^4, 1+x+2x^2+x^3+\alpha x^4\}$. Justify your answer.

Solution: Put the coefficients in 3×5 matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 2 & 1 & 4 & 1 & 5 \\ 1 & 1 & 2 & 1 & \alpha \end{bmatrix} \xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & \alpha-1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} \underline{1} & 0 & 2 & 0 & 1 \\ 0 & \underline{1} & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & \underline{\alpha-4} \end{bmatrix}.$$

The dimension $\dim(\text{span}(A))$ is the same with the dimension of column or row space of the matrix. To make it equal to 2 one must have $\alpha - 4 = 0$ or $\alpha = 4$. □

Exercise 0.1.15 Given two bases $B = \{u+v, u-v, w\}$ and $C = \{u+w, v, v-w\}$ for the vector space spanned by $\{u, v, w\}$.

a) Find the transition matrix $P_{B\to C}$ from B to C.

b) Find the transition matrix $P_{C\to B}$ from C to B.

Solution: a)

$$[u+v]_C = [(u+w)+(v-w)]_C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$[u-v]_C = [(u+w)+(v-w)-2v]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

$$[w]_C = [v-(v-w)]_C = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Hence $P_{B\to C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

b) $P_{C\to B} = (P_{B\to C})^{-1}$. Write $[P_{B\to C}|I] =$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2]{-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \to$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] = \left[I|\,(P_{B\to C})^{-1}\right].$$

Hence

$$P_{C\to B} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & -2 \end{bmatrix}. \qquad \square$$

Exercise 0.1.16 a) Determine whether the following subsets are subspace (giving reasons for your answers).

(i) $U = \{A \in \mathbb{R}^{2\times 2} | A^T = A\}$ in $\mathbb{R}^{2\times 2}$. ($\mathbb{R}^{2\times 2}$ is the vector space of all real 2×2 matrices under usual matrix addition and scalar-matrix multiplication.)

(ii) $W = \{(x, y, z) \in \mathbb{R}^3 | x \geq y \geq z\}$ in $\mathbb{R}^3$.

b) Find a basis for U. What is the dimension of U? (Show all your work by explanations.)

c) What is the dimension of $\mathbb{R}^{2\times 2}$? Extend the basis of U to a basis for $\mathbb{R}^{2\times 2}$.

Solution: a-i)

1) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U$ since $A = A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $U \neq \emptyset$.

2) Let $A, B \in U$. Then $A = A^T$ and $B = B^T$. Then $A + B \in U$, since $(A + B)^T = A^T + B^T = A + B$. So U is closed under addition.

3) Let $c \in \mathbb{R}$ and $A \in U$. Then $A = A^T$. Then $c \cdot A \in U$ since $(c \cdot A)^T = c \cdot A^T = c \cdot A$. Thus U is closed under scalar multiplication.

So we proved that U is a subspace in $\mathbb{R}^{2\times 2}$.

a-ii) W is not a subspace in $\mathbb{R}^3$. Since $(2,1,1) \in W$, however $(-1)\cdot(2,1,1) = (-2,-1,-1) \notin W$, that is W is not closed under scalar multiplication..

b) Let $A \in U$. Then $A = A^T$ i.e. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ for all $a,b,c,d \in \mathbb{R}$. Thus a and d are arbitrary real numbers and $c = b$. So any matrix $A \in U$ can be written as

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent, and any matrix in U can be written as a linear combination of these matrices, these matrices form a basis for U, namely $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for U. Thus $\dim(U) = 3$.

c) The space $\mathbb{R}^{2\times 2}$ has the standard basis

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

therefore $\dim(\mathbb{R}^{2\times 2}) = 4$. We can extend the basis B for U to a basis for $\mathbb{R}^{2\times 2}$ by

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4}$$

$$\to \left[\begin{array}{ccc|cccc} \underline{1} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \underline{-1} & 0 & 1 \end{array}\right].$$

Thus a basis for $\mathbb{R}^{2\times 2}$ containing vectors of B is

$$D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}. \quad \square$$

Exercise 0.1.17 Let $p \in \mathcal{P}_2$. The coordinate matrix of p relative to the standard ordered basis $B = \{1, x, x^2\}$ is $[p]_B = [2, -1, 5]^T$. Find the change of coordinate matrix from the ordered basis $B = \{1, x, x^2\}$ to the ordered basis $C = \{1, 1-x, 1+x+x^2\}$ and the coordinate matrix of p relative to C, $[p]_C$.

Solution: $[p]_B = [2, -1, 5]^T$ then $p = 2 \cdot 1 + (-1) \cdot x + 5 \cdot x^2$.

$$[P_{C\to B}|I] = \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_3+R_1]{-R_3+R_2} = \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\xrightarrow[-R_2]{R_2+R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I|P_{B\to C}],$$

where

$$P_{B\to C} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is the change of coordinate matrix from the basis B to the basis C.

$[p]_C = P_{B\to C} \cdot [p]_B$. Thus

$$[p]_C = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \\ 5 \end{bmatrix}. \quad \square$$

Exercise 0.1.18 Let $B = \{u, v\}$ be a basis of $\mathbb{R}^2$ and let $A = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. Show that A is invertible iff $C = \{\alpha u + \beta v, \gamma u + \delta v\}$ is a basis of $\mathbb{R}^2$.

Solution: First, assume A is invertible. Let $c_1 \cdot (\alpha u + \beta v) + c_2 \cdot (\gamma u + \delta v) = 0$. Then $(c_1\alpha + c_2\gamma)u + (c_1\beta + c_2\delta)v = 0$ since u and v are linearly independent.

So we have the system

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}$$

By our assumption, there exists A^{-1}. Hence this homogeneous system has only trivial solution, namely $c_1 = c_2 = 0$. So C consists of two linearly independent vectors and consequently it is a basis for $\mathbb{R}^2$ since $\dim(\mathbb{R}^2) = 2$.

Conversely, assume C is a basis for $\mathbb{R}^2$, then the system (1) has only the trivial solution, so $AX = 0$ consequently $RX = 0$ for some R which is the row echelon reduced matrix. If $RX = 0$ has only trivial solution then $R = I$ which proves that A is invertible. $\square$

Exercise 0.1.19 Consider the following list of statements. In each case either prove the statement if it is true or give an example showing that it is false.

i) If V is a subspace of $\mathbb{R}^3$ containing two linearly independent vectors, then V is equal to all of $\mathbb{R}^3$.

ii) If vectors v_1 and v_2 are linearly dependent and $u \notin \operatorname{span}(v_1, v_2)$ then the vectors $u + v_1$ and $u + v_2$ are linearly dependent.

iii) If vectors v_1 and v_2 are linearly independent and $u \notin \operatorname{span}(v_1, v_2)$ then the vectors $u + v_1$ and $u + v_2$ are linearly independent.

Solution: i) False.

$\dim(\mathbb{R}^3) = 3$, so we need at least 3 vectors to span $\mathbb{R}^3$. Consider $V = \{v_1, v_2\} = \{(1, 1, 0), (0, 1, 0)\}$. The vectors v_1 and v_2 are linearly independent since $c_1(1, 1, 0) + c_2(0, 1, 0) = (c_1, c_1 + c_2, 0) = (0, 0, 0)$ iff $c_1 = c_2 = 0$.

But $V \neq \mathbb{R}^3$ since $(0, 0, 1) \in \mathbb{R}^3$ but $(0, 0, 1) \neq k_1(1, 1, 0) + k_2(0, 1, 0) = (k_1, k_1 + k_2, 0) = (0, 0, 0)$. Thus $(0, 0, 1) \notin V$.

ii) False.

v_1 and v_2 are linearly dependent means that $v_1 = k \cdot v_2$. So $u + v_1 = u + k \cdot v_2$.

$$c_1(u + k \cdot v_2) + c_2(u + v_2) = 0 \Rightarrow (c_1 + c_2)u + (kc_1 + c_2)v_2 = 0.$$

But $u \notin \operatorname{span}(v_1, v_2)$ hence $c_1 + c_2 = 0 = kc_1 + c_2$ i.e. $(k - 1)c_1 = 0$ that is $k = 1$ or $c_1 = 0$.

So when $k \neq 1$ we have $c_1 = c_2 = 0$ i.e. $u+v_1$ and $u+v_2$ are linearly independent.

For example, $u = (1, 1)$, $v_1 = (2, 0)$, $v_2 = (1, 0)$. Then $u + v_1 = (3, 1)$ and $u + v_2 = (2, 1)$ are not linearly dependent.

iii) True.

If $c_1(u+v_1)+c_2(u+v_2) = 0$ then $(c_1+c_2)u+c_1v_1+c_2v_2 = 0$. But v_1 and v_2 are linearly independent and $u \notin \operatorname{span}(v_1, v_2)$, hence $c_1 = c_2 = 0$ and $c_1 + c_2 = 0$, so $u + v_1$ and $u + v_2$ are linearly independent.. $\square$

Exercise 0.1.20 Given three ordered bases $B = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$, $C = \{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\}$, and $D = \{\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3\}$ with the transition matrix $P_{C\to D} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, satisfying $\mathrm{v}_1 = \mathrm{u}_1 + \mathrm{u}_2 + \mathrm{u}_3$, $\mathrm{v}_2 = \mathrm{u}_2 + \mathrm{u}_3$, and $\mathrm{v}_3 = \mathrm{u}_1 - \mathrm{u}_2$.

a) Write down the vector $2\mathrm{u}_1 - 3\mathrm{u}_2 + 4\mathrm{u}_3$ as a linear combination of w_1, w_2, and w_3.

b) Find the transition matrix $P_{D\to C}$.

c) Let $\bar{C} = \{\mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_1\}$ and $\bar{D} = \{\mathrm{w}_3, \mathrm{w}_2, \mathrm{w}_1\}$. Find the transition matrix $P_{\bar{C}\to\bar{D}}$.

d) Find the transition matrix $P_{B\to D}$.

Solution: a) $\mathrm{v} = 2\mathrm{u}_1 - 3\mathrm{u}_2 + 4\mathrm{u}_3$ then $[\mathrm{v}]_C = [2, -3, 4]^T$. So

$$[\mathrm{v}]_D = P_{C\to D} \cdot [\mathrm{v}]_C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 26 \end{bmatrix}.$$

And thus $\mathrm{v} = 3\mathrm{w}_1 + 8\mathrm{w}_2 + 26\mathrm{w}_3$.

b) $P_{D\to C} = P^{-1}_{C\to D}$.

$$[P_{C\to D}|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2]{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-3R_2+R_3]{-R_2+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right] \xrightarrow[-2R_3+R_2]{R_3+R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5/2 & 1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right] = [I|P_{D\to C}].$$

c) $P_{\bar{C}\to\bar{D}} = [[\mathrm{u}_2]_{\bar{D}}[\mathrm{u}_3]_{\bar{D}}[\mathrm{u}_1]_{\bar{D}}] = \begin{bmatrix} 4 & 9 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$\mathrm{u}_2 = \mathrm{w}_1 + 2\mathrm{w}_2 + 4\mathrm{w}_3 = 4\mathrm{w}_3 + 2\mathrm{w}_2 + \mathrm{w}_1 \Rightarrow [\mathrm{u}_2]_{\bar{D}} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

$$\mathrm{u}_3 = \mathrm{w}_1 + 3\mathrm{w}_2 + 9\mathrm{w}_3 = 9\mathrm{w}_3 + 3\mathrm{w}_2 + \mathrm{w}_1 \Rightarrow [\mathrm{u}_3]_{\bar{D}} = \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}.$$

$$\mathrm{u}_1 = \mathrm{w}_1 + \mathrm{w}_2 + \mathrm{w}_3 = \mathrm{w}_3 + \mathrm{w}_2 + \mathrm{w}_1 \Rightarrow [\mathrm{u}_1]_{\bar{D}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

d) $P_{B\to D} = P_{C\to D} \cdot P_{B\to C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 6 & 5 & -1 \\ 14 & 13 & -3 \end{bmatrix}.$

Here we used $\mathrm{v}_1 = \mathrm{u}_1 + \mathrm{u}_2 + \mathrm{u}_3$, $\mathrm{v}_2 = \mathrm{u}_2 + \mathrm{u}_3$, and $\mathrm{v}_3 = \mathrm{u}_1 - \mathrm{u}_2$, hence $P_{B\to C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$. $\square$

0.2 Inner Product Spaces

Exercise 0.2.1 Find a non-zero polynomial of degree ≤ 2 orthogonal to the set $\{1, x\}$ with respect to the integral inner product $(p|q) = \int_0^1 p(x)q(x)\,dx$.

Solution: Let $p(x) = ax^2 + bx + c$. Then we want $0 = (1|p)$ and $0 = (x|p)$. That is $0 = (1|p) = \int_0^1 (ax^2 + bx + c)\,dx$ which gives $2a + 3b + 6c = 0$ and $0 = (x|p) = \int_0^1 (ax^3 + bx^2 + cx)\,dx$ yields $3a + 4b + 6c = 0$. Consider the matrix

$$\begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 6 \\ 0 & -1/2 & -3 \end{bmatrix}.$$

If we take $c = 1$ then $-\frac{1}{2}b = 3$. i.e. $b = 6$ and $a = 6$. So one of the required polynomial is $6x^2 - 6x + 1$. $\square$

Exercise 0.2.2 Orthogonalize by the Gram – Schmidt process the basis $\{v_1, v_2, v_3\} = \{(1,0,1), (0,1,0), (0,-1,2)\}$ for $\mathbb{R}^3$ with respect to the standard inner product $((x_1, y_1, z_1)|(x_2, y_2, z_2)) = x_1x_2 + y_1y_2 + z_1z_2$.

Solution: Choose $w_1 = v_1 = (1,0,1)$.

$$w_2 = v_2 - \frac{(v_2|w_1)}{(w_1|w_1)} \cdot w_1 = (0,1,0) - 0 \cdot w_1 = (0,1,0).$$

$$w_3 = v_3 - \frac{(v_3|w_1)}{(w_1|w_1)} \cdot w_1 - \frac{(v_3|w_2)}{(w_2|w_2)} \cdot w_2 = (0,-1,2) - \frac{2}{2} \cdot (1,0,1) - \frac{-1}{1} \cdot (0,1,0) =$$

$$= (0,-1,2) - (1,0,1) + (0,1,0) = (-1,0,1).$$

The answer is $\{(1,0,1), (0,1,0), (-1,0,1)\}$. $\square$

Exercise 0.2.3 Let $\mathbb{R}^{2\times 2}$ be the vector space of all real 2×2 matrices with inner product given by

$$(A|B) = tr(B^T \cdot A),$$

where tr is the trace of a matrix (i.e. sum of the diagonal entries of a matrix). Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

a) Find $(A|B)$ and $\|B\|$, where $\|\cdot\|$ denotes the norm (length) induced by the above inner product.

b) Are A and B orthogonal?

c) Determine the scalar c such that $A - cB$ is orthogonal to A.

Solution: a)

$$(A|B) = tr(B^T{\cdot}A) = tr\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) = tr\left(\begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}\right) = 0{+}2 = 2.$$

$$\|B\| = \sqrt{(B|B)} = [tr(B^T \cdot B)]^{1/2} = \left[tr\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right)\right]^{1/2} =$$

$$= \left[tr\left(\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}\right)\right]^{1/2} = \sqrt{4+2} = \sqrt{6} = \|B\|.$$

b) A and B are not orthogonal since, by part a), $(A|B) = 2 \neq 0$.

c) $A - cB$ and A are orthogonal iff $(A - cB|A) = 0$.

$$(A - cB|A) = (A|A) - c(B|A) = tr(A^T \cdot A) - c(A|B) =$$

$$= tr\left(\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) - c \cdot 2 = tr\left(\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\right) - 2c = 3 - 2c.$$

We have $A - cB$ and A are orthogonal iff $3 - 2c = 0$ so $c = 3/2$. □

Exercise 0.2.4 Let u_1 and u_2 be two vectors in an inner product space V such that $\|u_1\| = \|u_2\| = 1$, $(u_1|u_2) = 0$.

a) Find the cosine of the angle between the vectors $2u_1 + 3u_2$ and $4u_1 - 2u_2$.

b) Find a vector $v \in \text{span}(u_1, u_2)$ such that $v \perp (2u_1 + 3u_2)$ and $\|v\| = 1$.

Solution: a)

$$\cos\theta = \frac{(2u_1 + 3u_2|4u_1 - 2u_2)}{\sqrt{(2u_1 + 3u_2|2u_1 + 3u_2)\cdot(4u_1 - 2u_2|4u_1 - 2u_2)}}.$$

$(2u_1 + 3u_2|4u_1 - 2u_2) = 8(u_1|u_1) - 6(u_2|u_2) = 8 - 6 = 2.$

$(2u_1 + 3u_2|2u_1 + 3u_2)\cdot(4u_1 - 2u_2|4u_1 - 2u_2) = (4+9)\cdot(16+4) = 260.$

We used facts that $(u_1|u_2) = 0$ and $\|u_1\| = \sqrt{(u_1|u_1)} = \|u_2\| = \sqrt{(u_2|u_2)} = 1$. Thus

$$\cos\theta = \frac{2}{\sqrt{260}}.$$

b) Let $v = x\cdot u_1 + y\cdot u_2$. Find x and y.

$$0 = (v|2u_1 + 3u_2) = (xu_1 + yu_2|2u_1 + 3u_2) = 2x + 3y \Rightarrow y = -\frac{2}{3}x.$$

$$1 = (v|v) = \|v\|^2 = (xu_1 + yu_2|xu_1 + yu_2) = x^2 + y^2 = x^2 + \left(-\frac{2}{3}\right)^2 x^2 = \frac{13}{9}x^2.$$

So we have

$$x^2 = \frac{9}{13} \Rightarrow x = \pm\sqrt{\frac{9}{13}} = \pm\frac{3}{\sqrt{13}}.$$

Finally

$$v = \frac{3}{\sqrt{13}}u_1 - \frac{2}{\sqrt{13}}u_2 \qquad \text{and} \qquad v = -\frac{3}{\sqrt{13}}u_1 + \frac{2}{\sqrt{13}}u_2. \quad \square$$

Exercise 0.2.5 Let $v_1 = (1,1,1,1)$, $v_2 = (1,1,2,0)$, and $v_3 = (2,3,0,0)$ be vectors in $\mathbb{R}^4$ equipped with the standard inner product.

a) Find the orthogonal complement for span$\{v_1, v_2\}$ in $\mathbb{R}^4$.

b) Find the orthogonal basis to span$\{v_1, v_2, v_3\}$.

c) Find the orthogonal projection of $(1,1,-1,-1)$ to span$\{v_1, v_2\}$.

Solution: a) $(v_1|(x,y,z,u)) = 0$ and $(v_2|(x,y,z,u)) = 0$. So we have the system
$\left\{\begin{array}{lcl} x + y + z + u & = & 0 \\ x + y + 2z & = & 0 \end{array}\right.$.

Or in matrix notation $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Find the fundamental solutions of this system. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} \underline{1} & 1 & 1 & 1 \\ 0 & 0 & \underline{1} & -1 \end{bmatrix}$.
The variables y and u are free.

So we have $F_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $F_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Finally

$$\mathrm{span}\{v_1, v_2\}^{\perp} = \langle(-1,1,0,0),(-2,0,1,1)\rangle.$$

b) By the Gram –Schmidt, $w_1 = v_1 = (1,1,1,1)$.

$$w_2 = v_2 - \frac{(v_2|w_1)}{(w_1|w_1)} \cdot w_1 = (1,1,2,0) - \frac{4}{4} \cdot (1,1,1,1) = (0,0,1,-1).$$

$$w_3 = v_3 - \frac{(v_3|w_1)}{(w_1|w_1)} \cdot w_1 - \frac{(v_3|w_2)}{(w_2|w_2)} \cdot w_2 =$$

$$= (2,3,0,0) - \frac{5}{4} \cdot (1,1,1,1) - \frac{0}{2} \cdot (0,0,1,-1) = \left(\frac{3}{4}, \frac{7}{4}, -\frac{5}{4}, -\frac{5}{4}\right).$$

The orthogonal basis is $\left\{(1,1,1,1),(0,0,1,-1),\left(\frac{3}{4}, \frac{7}{4}, -\frac{5}{4}, -\frac{5}{4}\right)\right\}$.
c) Since $((1,1,-1,-1)|v_1) = 0$ and $((1,1,-1,-1)|v_2) = 0$ then

$$(1,1,-1,-1) \in \mathrm{span}\{v_1,v_2\}^{\perp}$$

and hence

$$\mathrm{pr}_{\mathrm{span}\{v_1,v_2\}}((1,1,-1,-1)) = (0,0,0,0). \qquad \square$$

Exercise 0.2.6 Let $\mathbb{R}^4$ be the inner product space relative to the standard inner product. Let $B = \{(1,1,0,0),(0,1,1,0),(1,-1,1,1)\}$ be a basis for $L = \mathrm{span}(B)$.

a) Orthogonalize the basis B by means of the Gram –Schmidt orthogonalization process.

b) Find the closest vector to $g = (1,1,1,0)$ in L.

Solution: a) $w_1 = v_1 = (1,1,0,0)$.

$$w_2 = v_2 - \frac{(v_2|w_1)}{(w_1|w_1)} \cdot w_1 = (0,1,1,0) - \frac{1}{2} \cdot (1,1,0,0) = \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right).$$

$$w_3 = v_3 - \frac{(v_3|w_1)}{(w_1|w_1)} \cdot w_1 - \frac{(v_3|w_2)}{(w_2|w_2)} \cdot w_2 = (1,-1,1,1) - 0 \cdot w_1 - 0 \cdot w_2 = (1,-1,1,1).$$

The obtained orthogonal basis for L is

$$\{w_1, w_2, w_3\} = \left\{(1,1,0,0), \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right), (1,-1,1,1)\right\}.$$

b) The vector closest to g is the orthogonal projection of g in L, that is

$$\mathrm{pr}_L(g) = \frac{(g|w_1)}{(w_1|w_1)} \cdot w_1 + \frac{(g|w_2)}{(w_2|w_2)} \cdot w_2 + \frac{(g|w_3)}{(w_3|w_3)} \cdot w_3 = \frac{2}{2} \cdot w_1 + \frac{2}{3} \cdot w_2 + \frac{1}{4} \cdot w_3 =$$

$$= (1,1,0,0) + \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0\right) + \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(1 - \frac{1}{12}, 1 + \frac{1}{12}, \frac{11}{12}, \frac{1}{4}\right) =$$

$$= \frac{1}{12} \cdot (11, 13, 11, 3). \quad \square$$

Exercise 0.2.7 Consider the vector space $\mathbb{R}^3$ with the standard inner product and let $S = \{(2,-1,1), (1,2,3), (3,1,4)\}$.

a) Find a basis for the orthogonal complement $S^\perp$ of S.

b) Find the orthogonal projection of $(1,1,1)$ on the subspace spanned by S.

Solution: a) All vectors $v \in S^\perp$ satisfy $(v|u) = 0$, where $u \in S$. So to find a basis of $S^\perp$ we need to solve the system $\begin{cases} 2x - y + z &= 0 \\ x + 2y + 3z &= 0 \\ 3x + y + 4z &= 0 \end{cases}$.

Or in matrix notation $A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Find the fundamental solutions of this system.

$$A \xrightarrow[-R_2+R_3]{-R_1+R_3} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{-2R_2+R_1} \begin{bmatrix} \underline{1} & 2 & 3 \\ 0 & \underline{-5} & -5 \\ 0 & 0 & 0 \end{bmatrix}.$$

The variable z is free.

So we have $P_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Hence $\{u = (-1, -1, 1)\}$ is a basis for $S^{\perp}$.

b) Note $v = (1, 1, 1)$. Since

$$v = \mathrm{pr}_{\langle S \rangle}(v) + \mathrm{pr}_{\langle S^{\perp} \rangle}(v)$$

then

$$\mathrm{pr}_{\langle S \rangle}(v) = v - \mathrm{pr}_{\langle S^{\perp} \rangle}(v).$$

$$\mathrm{pr}_{\langle S^{\perp} \rangle}(v) = \frac{(v|u)}{\|u\|^2} \cdot u = \frac{((1,1,1)|(-1,-1,1))}{((-1,-1,1)|(-1,-1,1))} \cdot (-1,-1,1) =$$

$$= \left(-\frac{1}{3}\right) \cdot (-1,-1,1) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right).$$

Hence

$$\mathrm{pr}_{\langle S \rangle}(v) = (1,1,1) - \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right). \qquad \square$$

Exercise 0.2.8 If v and w are two vectors of an inner product space, prove that

$$\|\mathrm{v} + \mathrm{w}\|^2 + \|\mathrm{v} - \mathrm{w}\|^2 = 2(\|\mathrm{v}\|^2 + \|\mathrm{w}\|^2).$$

Solution:

$$\|\mathrm{v} + \mathrm{w}\|^2 + \|\mathrm{v} - \mathrm{w}\|^2 = (\mathrm{v} + \mathrm{w}|\mathrm{v} + \mathrm{w}) + (\mathrm{v} - \mathrm{w}|\mathrm{v} - \mathrm{w}) =$$

$$= [(\mathrm{v}|\mathrm{v}) + 2(\mathrm{v}|\mathrm{w}) + (\mathrm{w}|\mathrm{w})] + [(\mathrm{v}|\mathrm{v}) - 2(\mathrm{v}|\mathrm{w}) + (\mathrm{w}|\mathrm{w})] =$$

$$= 2[(\mathrm{v}|\mathrm{v}) + (\mathrm{w}|\mathrm{w})] = 2(\|\mathrm{v}\|^2 + \|\mathrm{w}\|^2). \quad \square$$

Exercise 0.2.9 Let $\mathbb{R}^4$ be the inner product space with the standard inner product $(\cdot|\cdot)$. Let $S = \operatorname{span}\{(1,1,0,1),(1,0,1,0),(0,1,-1,1)\} \subseteq \mathbb{R}^4$.

a) Find a basis B for the orthogonal complement to S in $\mathbb{R}^4$.

b) Applying the Gram –Schmidt orthogonalization to the basis B constructed in a), find an orthonormal basis for the orthogonal complement $S^{\perp}$ of S.

c) Find the orthogonal projection of v $= (0,0,0,1)$ on S.

Solution: a)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{-R_1+R_3} \begin{bmatrix} \underline{1} & 0 & 1 & 0 \\ 0 & \underline{1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The third and forth variables are free. We have $x + z = 0$ and $y - z + t = 0$. Then $y = z - t$ and $x = -z$.

Find the fundamental vectors of the system.

$$z = 0, t = 1. \qquad P_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$z = 1, t = 0. \qquad P_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

So $B = \{P_1, P_2\}$ is a basis for $S^{\perp}$.

b) $w_1 = P_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$

$$w_2 = P_2 - \frac{(P_2|w_1)}{(w_1|w_1)} \cdot w_1 = P_2 + \frac{1}{2} \cdot w_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ 1 \\ 1/2 \end{bmatrix}.$$

$$\overline{w}_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\overline{w}_2 = \frac{w_2}{\|w_2\|} = w_2 \cdot \left(\sqrt{1+1/4+1+1/4}\right)^{-1} = \begin{bmatrix} -\sqrt{2}/\sqrt{5} \\ 1/\sqrt{10} \\ \sqrt{2}/\sqrt{5} \\ 1/\sqrt{10} \end{bmatrix}$$

So $B_{ort} = \{\overline{w}_1, \overline{w}_2\}$.

c) $\mathrm{pr}_{\langle S\rangle}(\mathrm{v}) = \mathrm{v} - \mathrm{pr}_{\langle S^{\perp}\rangle}(\mathrm{v})$.

$$\mathrm{pr}_{\langle S^{\perp}\rangle}(\mathrm{v}) = (\mathrm{v}|\overline{w}_1)\cdot\overline{w}_1 + (\mathrm{v}|\overline{w}_2)\cdot\overline{w}_2 = \frac{1}{\sqrt{2}}\overline{w}_1 + \frac{1}{\sqrt{10}}\overline{w}_2 =$$

$$= \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1/5 \\ 1/10 \\ 1/5 \\ 1/10 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \\ 1/5 \\ 3/5 \end{bmatrix}.$$

$$\mathrm{pr}_{\langle S\rangle}(\mathrm{v}) = \mathrm{v} - \begin{bmatrix} -1/5 \\ -2/5 \\ 1/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ -1/5 \\ 2/5 \end{bmatrix}. \quad \square$$

Exercise 0.2.10 Given a basis $B = \{1, t+t^2, t-t^2\}$ for $V = \mathcal{P}_2(\mathbb{R})$. The inner product $(.|.)$ in the vector space V is defined by $(u|v) = [u]_B^T[v]_B$, where $[u]_B^T$ is the transpose of the coordinate matrix $[u]_B$ of a vector u with respect to the basis B.

a) Show that B is an orthonormal basis for V with respect to the inner product $(.|.)$.

b) Find the norm of $v = 1+t+t^2$ with respect to the given inner product $(.|.)$.

c) Find the cosine of the angle between $v = 1+t+t^2$ and $u = 2t$ with respect to the given inner product $(.|.)$.

d) Find the orthogonal projection of $w = 1-t+2t^2$ onto $S = \mathrm{Span}\{1, 1+t^2\}$ with respect to the given inner product $(.|.)$.

Solution: a) Denote $v_1 = 1$, $v_2 = t + t^2$, $v_3 = t - t^2$.

Then $[v_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[v_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $[v_3]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$(v_1|v_1) = [1\ 0\ 0] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$, $(v_2|v_2) = 1$, $(v_3|v_3) = 1$.

Consequently $\|v_1\| = \sqrt{(v_1|v_1)} = 1$, $\|v_2\| = 1$, $\|v_3\| = 1$.

$(v_1|v_2) = [1\ 0\ 0] \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$, $(v_2|v_3) = 0$, $(v_1|v_3) = 0$.

Hence $B = \{v_1, v_2, v_3\}$ is orthonormal basis.

b) $[v]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ since $v = 1 + t + t^2 = 1 \cdot 1 + 1 \cdot (t + t^2) + 0 \cdot (t - t^2)$.

$(v|v) = [v]_B^T [v]_B = [1\ 1\ 0] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2$, $\|v\| = \sqrt{(v|v)} = \sqrt{2}$.

c) $[u]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ since $u = 2t = 0 \cdot 1 + 1 \cdot (t + t^2) + 1 \cdot (t - t^2)$, $[v]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$\|u\| = \sqrt{(u|u)} = \sqrt{2}$, $\|v\| = \sqrt{2}$. $(u|v) = [u]_B^T [v]_B = [0\ 1\ 1] \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1$.

$$\cos \hat{vu} = \cos \hat{uv} = \frac{(u|v)}{\|u\| \cdot \|v\|} = \frac{2}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}.$$

d) $w = 1 - t + 2t^2 = \alpha \cdot 1 + \beta \cdot (t + t^2) + \gamma \cdot (t - t^2)$ then

$$\begin{cases} \alpha = 1 \\ \beta + \gamma = -1 \\ \beta - \gamma = 2 \end{cases} \sim \begin{cases} \alpha = 1 \\ \beta + \gamma = -1 \\ 2\beta = 1 \end{cases} \sim \begin{cases} \alpha = 1 \\ \beta = 1/2 \\ \gamma = -1 + 1/2 = -3/2 \end{cases}.$$

Hence $[w]_B = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$.

$\mathrm{pr}_S(w) = (w|w_1)w_1 + (w|w_2)w_2$, where $\{w_1.w_2\}$ is an orthonormal basis for $S = \mathrm{Span}\{1, 1 + t^2\}$. $\square$

Exercise 0.2.11 a) Find a basis for the orthogonal complement of $S = \{(1,2,-1,3),(2,2,1,-3),(1,0,2,-6)\}$ in $\mathbb{R}^4$ with respect to the standard inner product in $\mathbb{R}^4$.

b) Let $w_1 = (1,1,-1,-1)$, $w_2 = (1,2,1,2)$, and $w_3 = (1,1,2,1)$. Find an orthonormal basis for $W = \mathrm{Span}\{w_1,w_2,w_3\}$ with respect to the standard inner product in $\mathbb{R}^4$.

Solution: a) Denote $v_1 = (1,2,-1,-1)$, $v_2 = (2,2,1,-3)$, and $v_3 = (1,0,2,-6)$. Consider the system $\begin{cases} (v|v_1) = x + 2y - z + 3t = 0 \\ (v|v_2) = 2x + 2y + z - 3t = 0 \\ (v|v_3) = x + 0 \cdot y + 2z - 6t = 0 \end{cases}$. Find the fundamental solution of this system. It is a basis for $S^\perp$.

$$\begin{bmatrix} 1 & 0 & 2 & -6 \\ 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & -3 \end{bmatrix} \xrightarrow[-R_1+R_2]{-2R_1+R_3} \begin{bmatrix} 1 & 0 & 2 & -6 \\ 0 & 2 & -3 & 9 \\ 0 & 2 & -3 & 9 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} \underline{1} & 0 & 2 & -6 \\ 0 & \underline{2} & -3 & 9 \\ 0 & 0 & 0 & 01 \end{bmatrix}.$$

$x = 6t - 2z$, $y = \frac{3}{2}z - \frac{9}{2}t$. The variables z and t are free. The fundamental solution is

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2z + 6t \\ \frac{3}{2}z - \frac{9}{2}t \\ z \\ t \end{bmatrix} = z \begin{bmatrix} -2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ -9/2 \\ 0 \\ 1 \end{bmatrix}.$$

The basis of $S^\perp$ is $\{(-2,3/2,1,0),(6,-9/2,0,1)\}$.

b) $x_1 = w_1 = (1,1,-1,-1)$.

$x_2 = w_2 - \frac{(w_2|x_1)}{(x_1|x_1)} \cdot x_1 = (1,2,1,2) - 0 \cdot x_1 = (1,2,1,2)$.

$x_3 = w_3 - \frac{(w_3|x_1)}{(x_1|x_1)} \cdot x_1 - - \frac{(w_3|x_2)}{(x_2|x_2)} \cdot x_2 = (1,1,2,1) - \frac{-1}{4}(1,1,-1,-1) - \frac{7}{10}(1,2,1,2) = \left(\frac{5}{4},\frac{5}{4},\frac{7}{4},\frac{3}{4}\right) - \left(\frac{7}{10},\frac{14}{10},\frac{7}{10},\frac{14}{10}\right) = \left(\frac{11}{20},-\frac{3}{20},\frac{21}{20},-\frac{13}{20}\right)$.

$\|x_1\| = \sqrt{(x_1|x_1)} = \sqrt{4} = 2$, $\|x_2\| = \sqrt{(x_2|x_2)} = \sqrt{10}$, $\|x_3\| = \sqrt{(x_3|x_3)} = \sqrt{\frac{11^2+3^2+21^2+13^2}{20^2}} = \frac{\sqrt{640}}{20} = \frac{8\sqrt{10}}{20}$.

The required orthonormal basis is

$$\left\{ \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|} \right\} = \left\{ \frac{1}{2}(1,1,-1,-1), \frac{1}{\sqrt{10}}(1,2,1,2), \frac{1}{8\sqrt{10}}(11,-3,21,-13) \right\}. \quad \square$$

Exercise 0.2.12 Let $u_1 = t - t^2$, $u_2 = t + t^2$, $u_3 = 2$, $w_1 = 1$, $w_2 = t$, and $w_3 = t^2$.

a) Show that $B = \{u_1, u_2, u_3\}$ is a basis for the vector space $\mathcal{P}_2(\mathbb{R})$ of polynomials of degree ≤ 2.

b) Find the transition matrix $P_{C\to B}$, where $B = \{u_1, u_2, u_3\}$ and $C = \{w_1, w_2, w_3\}$.

c) Calculate the coordinate matrix $[3 - 2t + t^2]_B$, where $B = \{u_1, u_2, u_3\}$.

d) Given $[v]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find the polynomial $v \in \mathcal{P}_2(\mathbb{R})$.

Solution: a) Consider the coordinate matrix $[u_1\ u_2\ u_3]$ in the standard basis of $\mathcal{P}_2(\mathbb{R})$:

$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$. So the vectors u_1, u_2 and u_3 are linearly independent hence they form a basis for $\mathcal{P}_2(\mathbb{R})$.

b) $[v]_B = P_{C\to B}[v]_C = [[w_1]_B\ [w_2]_B\ [w_3]_B][v]_C$

$$\begin{cases} w_1 = 1 = 0 \cdot u_1 + 0 \cdot u_2 + 1/2 \cdot u_3 \\ w_2 = t = 1/2 \cdot u_1 + 1/2 \cdot u_2 + 0 \cdot u_3 \\ w_3 = t^2 = -1/2 \cdot u_1 + 1/2 \cdot u_2 + 0 \cdot u_3 \end{cases}.$$

$$P = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \end{bmatrix}.$$

c) $3 - 2t + t^2 = x_1u_1 + x_2u_2 + x_3u_3 = x_1(t - t^2) + x_2(t + t^2) + x_3 \cdot 2$ hence $(x_2 - x_1)t^2 + (x_2 + x_1)t + 2x_3 = t^2 - 2t + 3$ then $x_2 - x_1 = 1$, $x_2 + x_1 = -2$, and $2x_3 = 3$. Finally $x_1 = -3/2$, $x_2 = -1/2$, $x_3 = 3/2$ and

$$[3 - 2t + t^2]_B = \begin{bmatrix} -3/2 \\ -1/2 \\ 3/2 \end{bmatrix}.$$

d)

$$[t - t^2\ \ t + t^2\ \ 2] \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = t - t^2 + 2(t + t^2) + 6 = t^2 + 3t + 6.$$

The required polynomial is $t^2 + 3t + 6$. □

0.1 Diagonalization and its Applications

Exercise 0.1.1 Let A be a 3×3-matrix whose characteristic polynomial is $\Delta_A(t) = t^3 - 2t^2 + 3t + 2$.

a) Express A^{-1} in term of powers A^0, A^1, and A^2 of A.

b) Express A^5 in term of powers A^0, A^1, and A^2 of A.

Solution: a) By the Cayley — Hamilton theorem, $A^3 - 2A^2 + 3A + 2I = 0$. Hence

$$I = -\frac{3}{2}A + A^2 - \frac{1}{2}A^3 = A \cdot \left(-\frac{3}{2}I + A - \frac{1}{2}A^2\right).$$

Thus

$$A^{-1} = -\frac{3}{2}I + A - \frac{1}{2}A^2.$$

b) Since $A^3 = 2A^2 - 3A - 2I$, we have

$$A^5 = A^2 \cdot A^3 = A^2 \cdot (2A^2 - 3A - 2I) = 2A^1 \cdot A^3 - 3A^3 - 2A^2 =$$

$$= 2A(2A^2-3A-2I)-3(2A^2-3A-2I)-2A^2 = 4A^3-6A^2-4A-6A^2+9A+6I-2A^2 =$$

$$= 4A^3 - 14A^2 + 5A + 6I = 4(2A^2 - 3A - 2I) - 14A^2 + 5A + 6I =$$

$$= 8A^2 - 12A - 8I - 14A^2 + 5A + 6I = -2I - 7A - 6A^2. \quad \square$$

Exercise 0.1.2 Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

a) Find an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

b) Find the matrix $P^{-1}A^{-1}P$ for the matrix P which is found in a).

c) Find an invertible matrix Q such that $Q^{-1}A^3Q = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix}$.

Solution: a) We have $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 2$, where λ is from $(A - \lambda I)X = 0$.

$$\lambda_1 = -1 \Rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}; \ y \text{ is free, hence } P_1 = \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}.$$

$$\lambda_2 = 3 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}; \ z \text{ is free, hence } P_2 = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

$$\lambda_3 = 2 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}; \ x \text{ is free, hence } P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} -\frac{1}{3} & \frac{3}{2} & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

b) $P^{-1}A^{-1}P = (P^{-1}AP)^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$

c) $Q^{-1}A^3Q = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (-1)^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix} = D^3$ for $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Hence $Q^{-1}AQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$$\lambda_1 = 2 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}; \ x \text{ is free, hence } Q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\lambda_2 = -1 \Rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}; \ y \text{ is free, hence } Q_2 = \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}.$$

$$\lambda_3 = 3 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}; \ z \text{ is free, hence } Q_3 = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

$$Q = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

Exercise 0.1.3 Let $B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & x & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

What must be value of x so that B is diagonalizable?

Solution: $\Delta_B(t) = \det(B - \lambda I) = (\lambda - 2)(\lambda - 3)^2(\lambda - 1)$; $\lambda_2 = \lambda_3 = 3$.

B is diagonalizable iff there are two linearly independent eigenvectors corresponding to $\lambda = 3$.

$$\lambda = 3,\ (B - \lambda I)X = 0, \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In order to have two fundamental solutions, x must be zero. $\square$

Exercise 0.1.4 Find all eigenvalues of the matrix $A = \begin{bmatrix} 5 & 5 \\ 6 & 4 \end{bmatrix}$.

Solution:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 5 & -5 \\ -6 & \lambda - 4 \end{vmatrix} = (\lambda-5)(\lambda-4)-30 = \lambda^2-9\lambda+20-30 = \lambda^2-9\lambda-10.$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{121}}{2} = 10, 1. \quad \square$$

Exercise 0.1.5 The eigenvalues of the matrix $B = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = 5$. Show that B is not diagonalizable.

Solution: 1) $\lambda = \lambda_1 = \lambda_2 = -2$, $(\lambda I - B)X = 0$, $\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$.

$$\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ -8 & -3 & 8 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$ the variable z is free.

So there is only one fundamental solution $F_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

2) $\lambda = \lambda_3 = 5$, $(\lambda I - B)X = 0$, $\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$

$\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} -1 & -3 & 8 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -3 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free.

So there is only one fundamental solution $F_2 = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$.

Since we have only two linearly independent eigenvectors correspondent to λ_1, λ_2, and λ_3, the matrix B is not diagonalizable. $\square$

Exercise 0.1.6 The eigenvalues of the real symmetric matrix $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ are $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 2$. Diagonalize C by means of an orthogonal matrix.

Solution: $\lambda = \lambda_1 = \lambda_2 = -1$, $\lambda I - C = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables y and z are free. $P_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda = \lambda_3 = 2$, $\lambda I - C = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1+R_2]{\frac{1}{2}R_1+R_3} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \to$
$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $P_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Use the Gram — Schmidt orthogonalization.

$$Q_1 = P_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \quad Q_2 = P_2 - \frac{(P_2|Q_1)}{(Q_1|Q_1)} \cdot Q_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

$$Q_3 = P_3 - \frac{(P_3|Q_1)}{(Q_1|Q_1)} \cdot Q_1 - \frac{(P_3|Q_2)}{(Q_2|Q_2)} \cdot Q_2 = P_3 - 0 \cdot Q_1 - 0 \cdot Q_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Normalizing $\{Q_1, Q_2, Q_3\}$ one gets

$$\tilde{Q_1} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \tilde{Q_2} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad \tilde{Q_3} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Hence the orthogonal matrix which diagonalizes C is

$$\tilde{Q} = [\tilde{Q_1}, \tilde{Q_2}, \tilde{Q_3}] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}. \qquad \square$$

Exercise 0.1.7 Determine whether or not the following matrix is diagonalizable, and if it is, find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

a) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix}$. b) $B = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Solution: a) $|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & -3 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda^2 - 4) = (\lambda - 2)^2(\lambda + 2) = 0$. The eigenvalues are $\lambda_{1,2} = 2$ and $\lambda_3 = -2$.

$\lambda = 2$, $(2I - A)X = 0$ with $X = [x, y, z]^T$. $2I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \underline{1} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables x and z are free. $P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda = -2$, $(-2I - A)X = 0$. $-2I - A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{3} & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $P_3 = \begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix}$.

Hence $P = [P_1\ P_2\ P_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

b) $|\lambda I - B| = \begin{vmatrix} \lambda - 4 & 0 & 1 \\ 0 & \lambda - 3 & 0 \\ -1 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 3)((\lambda - 4)(\lambda - 2) + 1) = (\lambda - 3)(\lambda^2 - 6\lambda + 9) = (\lambda - 3)^3 = 0$. So, the only eigenvalue is $\lambda = 3$.

$\lambda = 3$, $(3I - B)X = 0$. $3I - B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \underline{-1} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables y and z are free.

Thus we have only two fundamental solutions which are not enough for diagonalizing a matrix. Hence B is not diagonalizable.□

Exercise 0.1.8 Given a diagonal matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and an orthogonal matrix $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find a real symmetric matrix A such that $P^{-1}AP = D$.

Solution: Since P is orthogonal, $P^{-1} = P^T$. Then $A = PDP^{-1} = PDP^T$.

$$PD = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 2 & 0 & 1 \end{bmatrix}, \qquad P^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

$$A = PD \cdot P^T = \left(\frac{1}{\sqrt{2}}\right)^2 \cdot \begin{bmatrix} 2 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}. \quad \square$$

Exercise 0.1.9 The characteristic polynomial of an invertible 3×3-matrix A is given by $x^3 - 3x^2 - 6x + 8$.

a) Write A^{-1} as a polynomial matrix in A.

b) Write A^5 as a linear combination of I, A, A^2, A^3.

Solution: a) By the Cayley-Hamilton theorem, $A^3 - 3A^2 - 6A + 8I = 0$. Hence $A^2 - 3A - 6I + 8A^{-1} = 0$ and

$$A^{-1} = -\frac{1}{8}A^2 + \frac{3}{8}A^1 + \frac{3}{4}A^0 = \frac{3}{4}I + \frac{3}{8}A^1 - \frac{1}{8}A^2.$$

b) From (a), $A^2 - 3A - 6I + 8A^{-1} = 0$ then $A^5 - 3A^4 - 6A^3 + 8A^2 = 0$ and thus $A^5 = 3A^4 + 6A^3 - 8A^2$.

Analogously $A^4 - 3A^3 - 6A^2 + 8A = 0$ and thus $A^4 = 3A^3 + 6A^2 - 8A$. So we have

$$A^5 = 3(3A^3 + 6A^2 - 8A) + 6A^3 - 8A^2 = 9A^3 + 18A^2 - 24A + 6A^3 - 8A^2 =$$

$$= -24A + 10A^2 + 15A^3. \quad \square$$

Exercise 0.1.10 Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

a) Find eigenvalues of A.

b) Determine whether A is invertible or not.

c) Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

Solution: a) $0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = (-\lambda) \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 2 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 2 & -\lambda \\ 2 & 2 \end{vmatrix} = (-\lambda)(\lambda^2 - 4) - 2(-2\lambda - 4) + 2(4 + 2\lambda) = (\lambda + 2)(-\lambda^2 + 2\lambda + 8) = -(\lambda + 2)^2(\lambda - 4)$ then $\lambda_{1,2} = -2$ and $\lambda_3 = 4$.

b) The determinant $|A| = \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} = (-2) \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} = (-2)(-4) + 2 \cdot 4 = 16$. Since $16 \neq 0$, the matrix A is invertible.

c) $\lambda = 4$, $\lambda I - A = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 6 & -6 \\ -2 & 4 & -2 \\ 0 & -6 & 6 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 2 \\ 0 & 6 & -6 \end{bmatrix}$,

the variable z is free. $y = z$; $x = 2y - z = z$. $P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda = -2$, $\lambda I - A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the variables y and z are free. $x = -y - z$. The fundamental solutions are $P_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Use the Gram — Schmidt orthogonalization.

$Q_1 = P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $Q_2 = P_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ since $P_2 \perp P_1$.

$$Q_3 = P_3 - \frac{(P_3|Q_1)}{(Q_1|Q_1)} \cdot Q_1 - \frac{(P_3|Q_2)}{(Q_2|Q_2)} \cdot Q_2 = P_3 - \frac{1}{2}P_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

Normalizing $\{Q_1, Q_2, Q_3\}$ one gets

$$\tilde{Q_1} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \tilde{Q_2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \tilde{Q_3} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

$$P = [\tilde{Q_1}, \tilde{Q_2}, \tilde{Q_3}] = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}. \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \qquad \square$$

Exercise 0.1.11 Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

a) Determine the characteristic polynomial and all eigenvalues of the matrix A.

b) Find the eigenvectors of the matrix A.

c) Diagonalize the matrix A by means of an orthogonal matrix Q such that $Q^T AD$ is diagonal.

Solution: a) $|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda-1)^2 - (\lambda+1) =$ $(\lambda+1)\cdot\lambda\cdot(\lambda-2) = \lambda^3 - \lambda^2 - 2\lambda$ is a characteristic polynomial of A and $\lambda_1 = -1$, $\lambda_2 = 0$, and $\lambda_3 = 2$ are the eigenvalues of A.

b) $\lambda = -1$, $\lambda I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, the variable x is free. $y = z = 0$. $P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\lambda = 0$, $\lambda I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $x = 0$, $y = -z$. $P_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

$\lambda = 2$, $\lambda I - A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the variable z is free. $x = 0$, $y = z$. $P_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

c) The vectors P_1,P_2, and P_3 are orthogonal so we need only to norming them.

$$\tilde{P_1} = \frac{P_1}{\|P_1\|} = P_1 \cdot 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\tilde{P_2} = \frac{P_2}{\|P_2\|} = P_2 \cdot 1/\sqrt{2} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\tilde{P_3} = \frac{P_3}{\|P_3\|} = P_3 \cdot 1/\sqrt{2} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

So the required orthogonal matrix is

$$Q = [\tilde{P}_1 \ \tilde{P}_2 \ \tilde{P}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad Q^{-1} = Q^T = Q.$$

$$D = Q^T A Q = QAQ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \qquad \square$$

0.2 Linear Transformations

Exercise 0.2.1 Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$.

a) Is T one-to-one? Explain.

b) Find a basis for the kernel (= null space) $N = T^{-1}(0)$ of T.

c) Extend the basis which is found in b) to a basis for $\mathbb{R}^3$.

d) Find the dimension of $N = T^{-1}(0)$ and the dimension of $T(\mathbb{R}^3)$.

e) Find the matrix representation of T with respect to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^2$.

Solution: a) T is not one-to-one, since, for example, $T(1, -1, 1) = (0, 0) = T(0, 0, 0)$.

b) $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3) = (0, 0)$. Thus $\left\{ \begin{array}{rcl} x_1 + x_2 & = & 0 \\ x_2 + x_3 & = & 0 \end{array} \right.$.

$\begin{bmatrix} \underline{1} & 1 & 0 \\ 0 & \underline{1} & 1 \end{bmatrix}$; x_3 is free. $P_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is a single fundamental solution. Hence $B = \{P_1^T\} = \{(1, -1, 1)\}$ is a basis for $T^{-1}(0)$.

c) $[P_1\ e_1\ e_2\ e_3] =$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} \underline{1} & 1 & 0 & 0 \\ 0 & \underline{1} & 1 & 0 \\ 0 & 0 & \underline{1} & 1 \end{bmatrix}.$$

Hence $\{P_1, e_1, e_2\}$ is a basis for $\mathbb{R}^{3\times 1}$.

d) $\dim(N) = 1$ and $\dim(T(\mathbb{R}^3)) = 3 - 1 = 2$.

e) $T(1, 0, 0) = (1, 0) = 1 \cdot (1, 0) + 0 \cdot (0, 1)$;

$T(0, 1, 0) = (1, 1) = 1 \cdot (1, 0) + 1 \cdot (0, 1)$;

$T(0, 0, 1) = (0, 1) = 0 \cdot (1, 0) + 1 \cdot (0, 1)$.

$$A_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \qquad \square$$

Exercise 0.2.2 The linear transformation T of $\mathbb{R}^3$ id given by $T(x,y,z) = (x+y, 2y+2z, -x+z)$.

a) Find the matrix representation of T relative to the standard basis for $\mathbb{R}^3$.

b) Find a vector of norm one which is orthogonal to the vector space $T(\mathbb{R}^3)$ relative to the standard inner product in $\mathbb{R}^3$.

Solution: a) $T(e_1) = T(1,0,0) = (1,0,-1)$; $T(e_2) = T(0,1,0) = (1,2,0)$; $T(e_3) = T(0,0,1) = (0,2,1)$.

The required matrix is

$$A_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix}.$$

b) $(T(e_1)|(x,y,z)) = ((1,0,-1))|(x,y,z)) = 0 \quad \Rightarrow \quad x - z = 0;$

$(T(e_2)|(x,y,z)) = ((1,2,0))|(x,y,z)) = 0 \quad \Rightarrow \quad x + 2y = 0;$

$(T(e_3)|(x,y,z)) = ((0,2,1))|(x,y,z)) = 0 \quad \Rightarrow \quad 2y + z = 0.$

We have $\begin{cases} x - z &= 0 \\ x + 2y &= 0 \\ 2y + z &= 0 \end{cases}.$

$$[[x][y][z]] = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[-R_2+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since z is free, we find the fundamental solution that is $F = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$, $\|F\| = \sqrt{(F|F)} = 3/2$. The required vector is

$$P = \pm \frac{F}{\|F\|} = \pm \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}. \qquad \square$$

Exercise 0.2.3 Let $L : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ be the linear transformation given by $L(A) = \frac{1}{2}(A - A^T)$.

a) Write a basis and find the dimension of the $\mathrm{Ker}(L) = \{A \in \mathbb{R}^{2\times 2} |\ L(A) = 0_{\mathbb{R}^{2\times 2}}\}$.

b) Write a basis and find the dimension of the $\mathrm{Im}(L) = \{L(A) |\ A \in \mathbb{R}^{2\times 2}\}$.

c) Find the matrix representation of L with respect to the standard ordered basis for $\mathbb{R}^{2\times 2}$ which is $\{E_{11}, E_{12}, E_{21}, E_{22}\}$.

Solution: a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We have $L(A) = \frac{1}{2}\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} = 0$, hence $b = c$. Thus

$$\mathrm{Ker}(L) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} |\ b = c \right\} = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} |\ a, b, d \in \mathbb{R} \right\}.$$

And the standard basis for $\mathrm{Ker}(L)$ is

$a = 1$, $b = d = 0$: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $a = 0$, $b = 1$, $d = 0$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $a = b = 0$, $d = 1$: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Hence $\dim(\mathrm{Ker}(L)) = 3$.

b) $L(A) = L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

$\begin{cases} 2x = 0 \\ 2y = b - c \\ 2z = c - b \\ 2t = 0 \end{cases}$, so $y = -z$, and z is free. We have $P = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ (is the fundamental solution).

Thus the basis for $\mathrm{Im}(L)$ is $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$, and hence $\dim(\mathrm{Im}(L)) = 1$.

c) $B = \{E_{11}, E_{12}, E_{21}, E_{22}\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$$A_L = [[L(E_{11})]_B \ [L(E_{12})]_B \ [L(E_{21})]_B \ [L(E_{22})]_B] =$$
$$= \left[\left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right]_B \left[\begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} \right]_B \left[\begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \right]_B \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right]_B \right] =$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \square$$

Exercise 0.2.4 Let $L : V \longrightarrow V$ be a linear transformation such that $L(\mathbf{v}_1) = 2\mathbf{v}_1$ and $L(\mathbf{v}_2) = -\mathbf{v}_2$, where $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2 \neq 0$.

a) Show that $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly independent.

b) Show that there is no $\lambda \in \mathbb{R}$ such that $L(\mathbf{v}_1 + \mathbf{v}_2) = \lambda(\mathbf{v}_1 + \mathbf{v}_2)$.

c) Let $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ be a basis for V such that $\mathcal{B}_1 = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\mathrm{Ker}(L) = \{\mathbf{v} \in V | \ L(\mathbf{v}) = 0\}$. Show that the vectors $L(\mathbf{w}_3)$, $L(\mathbf{w}_4)$ are linearly independent.

Solution: a) Find $\alpha, \beta \in \mathbb{R}$ such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = 0$. It means that $0 = L(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha L(\mathbf{v}_1) + \beta L(\mathbf{v}_2) = 2\alpha\mathbf{v}_1 - \beta\mathbf{v}_2 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = 0$. Then $3\alpha\mathbf{v}_1 = 0$ that is $\alpha = 0$ (since $\mathbf{v}_1 \neq 0$ by the condition) and so $\beta = \alpha = 0$.

Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

b) If such a λ exists then $L(\mathbf{v}_1 + \mathbf{v}_2) = 2\mathbf{v}_1 - \mathbf{v}_2 = \lambda(\mathbf{v}_1 + \mathbf{v}_2)$ that is equivalent to $(2 - \lambda)\mathbf{v}_1 = \mathbf{v}_2 + \lambda\mathbf{v}_2 = (\lambda + 1)\mathbf{v}_2$.

If $\lambda \neq 2$ then $\mathbf{v}_1 = \frac{1+\lambda}{2-\lambda}\mathbf{v}_2$ that contradicts to a) ($\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent).

If $\lambda = 2$ then $(1 + \lambda)\mathbf{v}_2 = 0$ and $\mathbf{v}_2 = 0$ that contradicts to the condition that $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2 \neq 0$.

So there is no such a λ.

c) $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\mathrm{Ker}(L)$ then $L(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2) = 0 = \alpha L(\mathbf{w}_1) + \beta L(\mathbf{w}_2)$ for all $\alpha, \beta \in \mathbb{R}$.

If $L(\mathbf{w}_3)$ and $L(\mathbf{w}_4)$ are linearly dependent then there are γ and δ ($\gamma \neq 0$, $\delta \neq 0$) such that $\gamma L(\mathbf{w}_3) + \delta L(\mathbf{w}_4) = 0$ that is equivalent to $L(\gamma\mathbf{w}_3 + \delta\mathbf{w}_4) = 0$ i.e. $\gamma\mathbf{w}_3 + \delta\mathbf{w}_4 \in \mathrm{Ker}(L)$.

This means that there are $a, b \in \mathbb{R}$ ($a \neq 0$, $b \neq 0$) such that $\gamma\mathbf{w}_3 + \delta\mathbf{w}_4 = a\mathbf{w}_1 + b\mathbf{w}_2$ that contradicts to the condition that $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ are linearly independent as basis vectors in $\mathcal{B}$.$\square$