SECTION 1: PERMUTATION AND COMBINATION

1.1. Permutation and Combination

As known that probability theory started with finding the solution of gamble in literature. In these games, the mathematicians were interested in determining how many different possibilities there were, so then the basic principle of counting and the bases of probability have been come forward.

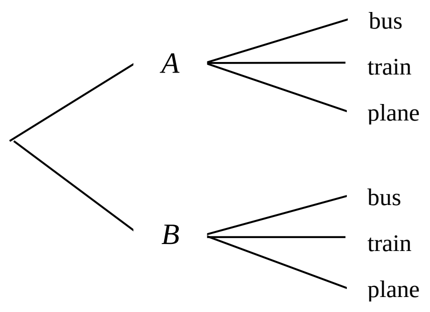
Let us start with the basic principle of counting.

Theorem 1.1: If an operation consists of two steps, of which the first can be done in n_1 ways and for each of these the second can be done in n_2 ways, then the whole operation can be done in $n_1.n_2$ ways. The theorem is called as "*a multiplication rule*".

Proof: We define the ordered pair (x_i, y_j) for each possibility of the first step x_i, the possibilities of second step y_j are arisen. Then the set of all possible of outcomes is composed of the following $n_1.n_2$ pairs.

$$\begin{array}{c}(x_1, y_1), (x_1, y_2), ..., (x_1, y_{n_2})\\ (x_2, y_1), (x_2, y_2), ..., (x_2, y_{n_2})\\ \cdots \\ (x_{n_1}, y_1), (x_{n_1}, y_2), ..., (x_{n_1}, y_{n_2})\end{array}$$

Example 1. 1: Suppose that someone wants to go to one of the two cities *A*, *B* using by bus, by train, or by plane. Find the number of different ways in which this can be done.
Using tree diagram,



This can be done by $2.3 = 6$ different ways.

Example 1.2: How many possible outcomes are there when we roll a coin and a dice?

$(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)$
$(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)$
In the expression in parenthesis, first one shows *H* for "*Head*" or *T* for "*Tail*" on the coin; second one shows the number on the dice. So 2.6=12 possible outcomes are occurred.

Theorem 1.2: If an operation consists of k steps, of which the first can be done in n_1 ways, for each of these the second step can be done in n_2 ways, for each of these the third step can be done in n_3 ways, and so forth, then the whole operation can be done in $n_1 \bullet n_2 \bullet ... \bullet n_k$ ways.

Example 1. 3: In how many different ways can one answer all the questions of true-false test consisting of 20 questions?

Altogether there are

$$2 \bullet 2 \bullet ... \bullet 2 = 2^{20} = 1,048,576$$

different ways in which one can answer all the questions; only one of these corresponds to the case where all questions are answered right.

Theorem 1.3: The numbers of permutations of n distinct objects is n!

Example 1. 4: How many permutations are there of the letters *a, b, c, d, e, f, g, i*?

There are 8.7.6.5.4.3.2.1=8!=40,320 ways in which they are arranged.

How many of them are meaningful in English?

Example 1. 5: How many permutations of two random numbers are there of 1, 2, 3, 4? (each number cannot be repeated)

There are two positions to fill, with four choices for the first position and then three choices for the second position. 4.3=12

Example 1. 6: How many permutations of three random numbers are there of 1, 2, 3, 4? (each number can be repeated)

Under replacement condition, the solution is 4.4.4=64.

Theorem 1.4: The numbers of permutations of n distinct objects taken r at a time is:

$$_{n}P_{r} = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!} \tag{1}$$

for $r = 0,1,...,n$

Proof: The formula in Eq.(1),

$$_{n}P_{r} = n(n-1)(n-2)...(n-r+1)\frac{(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Example 1. 7: Art Gallery owner exhibits the 4 ones of 6 paintings by Spanish painter “Frida” on its wall. How many exhibitions can be done using Frida’s paintings?

From Eq.(1), the solution is $_{6}P_{4} = 6 \bullet 5 \bullet 4 \bullet 3 = 360$.

Theorem 1.5: The numbers of permutations of n distinct objects arranged in a circle is $(n-1)!$

Example 1. 8: How many circular permutations are there of four persons (gamblers) playing bridge?

From Theorem 1.5, (4-1)!=3!=6.

Theorem 1.6: The number of permutations of n objects of which n_1 are of one kind, n_2 are of second kind,…, n_k are of a k^{th} kind, and $n_1+n_2+...+n_k=n$ is:

$$\frac{n!}{n_1!n_2!...n_k!}. \tag{2}$$

Example 1. 9: In how many ways can two paintings by Monet, three paintings by Renoir, and two paintings by Degas be hung side by side on a museum wall if we do not distinguish between the paintings by the same artists?

7 paintings are ordered by 7!=5040 ways. But we do not see two or more paintings successively. So we need to "subtract or other words divide" the successive paintings of any painter.

$\frac{7!}{2!3!2!}=210$ arrangements can be done.

Combination means the same as "subset" the number of combinations of r objects selected from a set of n distinct objects. In the combination, putting objects in order is not asked. In general there are r! permutations of objects in a subset of r objects, so that the ${}_nP_r$ permutations of r objects selected from a set of n distinct objects contain each subset r! times. Dividing ${}_nP_r$ by r! and then combination formula is shown as $\binom{n}{r}$.

Theorem 1.7: The number of combinations of n distinct objects taken r at a time is:

$${}_nC_r=\binom{n}{r}=\frac{{}_nP_r}{r!}=\frac{n!}{r!(n-r)!}$$

for $r=0,1,...,n$. In general, combination formula is written as:

$${}_nC_r=\binom{n}{r}=\frac{n(n-1)...(n-r+1)}{r!}.$$

Example 1. 10: In how many different groups with 3 persons from 10 people can be constituted?

$\binom{10}{3}=\frac{10!}{3!7!}=\frac{8\times9\times10}{3!}=120$ different groups can be constituted.

Example 1. 11: In how many different ways can six tosses of a coin yield two heads and four tails?

HHTTTT, HTHTTT, THHTTT,… $\dfrac{6!}{2!4!} = \dfrac{6.5}{2!} = 15$

Theorem 1.8: The number of ways in which a set of *n* distinct objects can be partitioned in to *k* subsets with n_1 objects in the first subset, n_2 objects in the second subset,…, and n_k objects in the k^{th} subset is:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \bullet n_2! \bullet \ldots \bullet n_k!}. \tag{3}$$

Proof:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_k} &= \binom{n}{n_1}\binom{n-n_1}{n_2} \bullet \ldots \bullet \binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k} \\ &= \frac{n!}{n_1!(n-n_1)!} \bullet \frac{(n-n_1)!}{\mathrm{n}_2!(n-n_1-n_2)!} \\ &\qquad \bullet \ldots \bullet \frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k!0!} \\ &= \frac{n!}{n_1! \bullet n_2! \bullet \ldots \bullet n_k!}\end{aligned}$$

1.2. Binomial Coefficients

When we multiply out $(x+y)^n$ term by term for *n* is positive integer, each term will be the product of *x*'s and *y*'s. For example,

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

The terms x^3, x^2y, xy^2, y^3 has the components 1, 3, 3 and 1 respectively. These components can be written as ***binomial coefficients*** $\binom{3}{0} = 1, \binom{3}{1} = 3, \binom{3}{2} = 3, \binom{3}{3} = 1$.

Theorem 1.9: For any positive integer n,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r. \tag{4}$$

Theorem 1.10: For any positive integer n and for $r = 0, 1, \ldots, n$,

$$\binom{n}{r} = \binom{n}{n-r} \tag{5}$$

Proof: $\binom{n}{n-r} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$

Theorem 1.11: For any positive integer n and for $r = 0,1,...,n$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

Proof: Substituting x=1 into $(x+y)^n$, let us write

$(1+y)^n = (1+y)(1+y)^{n-1} = (1+y)^{n-1} + y(1+y)^{n-1}$ and the coefficient of the y^r in $(1+y)^n$ is $\binom{n}{r}$ which is equal to sum up the coefficient $\binom{n-1}{r}$ of the y^r in $(1+y)^{n-1}$ and the coefficient $\binom{n-1}{r-1}$ of the y^{r-1} in $(1+y)^{n-1}$. So the proof is completed.

Alternatively, the proof of the theorem can be seen easily from Pascal triangle:

$$\begin{array}{lccccccccc}
n=0 \Rightarrow & & & & & 1 & & & & \\
n=1 \Rightarrow & & & & 1 & & 1 & & & \\
n=2 \Rightarrow & & & \boxed{1} & & \boxed{2} & & 1 & & \\
n=3 \Rightarrow & & 1 & & 3 & & 3 & & 1 & \\
n=4 \Rightarrow & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
\qquad \binom{2}{0} + \binom{2}{1} = \binom{3}{1}$$

Theorem 1.12: For any positive integer n and for $r = 0,1,...,n$,

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}.$$

Proof: Using the same technique as Theorem 1.11,

$(1+y)^{m+n} = (1+y)^m(1+y)^n$, and the coefficient of y^k in $(1+y)^{m+n}$ is $\binom{m+n}{k}$ which is equal to the multiplying of the coefficients of term $(1+y)^m(1+y)^n$.

Note: For many other properties, see Miller and Miller, 1999.

Example 1. 11: Let we show that,

a) $\sum_{r=0}^{n} \binom{n}{r} = 2^n$, when x=1 and y=1 are substituted in the term of $(x+y)^n$, we get this result.

b) $\sum_{r=0}^{n}(-1)^r\binom{n}{r}=0$, when y=-x and x=1 are substituted in the term of $(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$, we get this result.

c) $\sum_{r=0}^{n}\binom{n}{r}(a-1)^r=a^n$, when x=1, y=a-1 are substituted in the term of $(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$, we get this result.

Example 1. 12: There are four routes, A, B, C, and D, between a person's home and the place where he works, but route B is one way, so he cannot take it on the way to work, and route C is one way, so he cannot take it on the way home.

a) Draw a tree diagram showing the various ways the person can go to and from work.
AA, AD, DA, DD, AB, DB, CA, CD, CB that is 3 $\times$3=9
First letter says us the route is used to go work from home,
Second letter says us the route is used to go back home from work.

b) Draw a tree diagram showing the various ways he can go to and from work without taking the same route both ways.
AD, DA, AB, DB, CA, CD, CB so that is 7.

Example 1. 13: A person with $2 in her pocket bets $1, even money, on the flip of coin, and she continuous to bet $1 as long as she has any money. Draw a tree diagram to show the various things that can happen during the first four flips of the coin. After the fourth flip of the coin, in how many of the cases will she be

a) Exactly no money,

b) Exactly $2

Example 1. 14: Seven people are in an elevator which stops at ten floors. No two people get off at the same floor. Before the elevator begins to travel, each of them pushes a button for his or her floor. In how many ways can the elevator buttons be lighted?

It doesn't matter what order the buttons were pressed, just which 7 of the 10 floors were selected. $\binom{10}{7}$

Example 1. 15: How many positive integers less than 1000 using the digits 2, 3, and 4 in the order given?

$3+3^2+3^3$=3+3.3+3.3.3=39 (repeated)
$3+{}_3P_2+3!$ =3+3.2+3.2.1=15 (unrepeated)

Example 1. 16: On Friday morning, the pro shop of the tennis club has 14 identical cans of tennis balls. If they are all sold by Sunday night and we are interested only in how many were sold on each day?

Friday, Saturday, Sunday

$$\sum_{x=0}^{14}\sum_{y=0}^{14-x}\binom{14}{x}\binom{14-x}{y}\binom{14-x-y}{14-x-y}=\sum_{x=0}^{14}\sum_{y=0}^{14-x}\binom{14}{x,y,14-x-y}$$

Example 1. 17: Rework Example 1. 16 given that at least two of the cans of tennis balls were sold on each of three days

$$\sum_{x=0}^{8}\sum_{y=0}^{8-x}\binom{8}{x}\binom{8-x}{y}\binom{8-x-y}{8-x-y}=\binom{8}{x,y,8-x-y}$$

Example 1. 18: If eight persons are having dinner together, in how many different ways can three order chickens, four order steaks, one order lobster? (Assume that they are sitting circle table)

$$\frac{(8-1)!}{3!4!}=\frac{4!5.6.7}{4!.6}=35$$

Example 1.19: How many 5 card hands can be made if there must be 3 of one face value and 2 other cards with different face values?

$$\binom{3}{1}.\binom{4}{3}.\binom{4}{1}\binom{4}{1}$$

Example 1. 20: In selecting an ace, king, queen, and jack from an ordinary deck of 52 cards, how many ways may we choose if: (a) they must be of different suits? (b) they may or may not be of different suits? (c) they must be of the same suit?

There are 13 cards of each symbol.

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Diller	♥	♦	♣	♠
Türkçe Fransızca İngilizce	Kupa Cœur Hearts	Karo Carreau Diamonds	Sinek Trèfle Clubs	Maça Pique Spades

SECTION 2: PROBABILITY

2.1 Introduction

The oldest way of defining probabilities are based on called the classical probability definition which is that if there are N equally possibilities, of which one must occur and m are regarded as favorable, or as “success”, then the probability of a success is given by the ratio $\frac{m}{N}$.

Probability is based on observations of certain events. **Probability of an event** is the ratio of the number of observations of the event to the total numbers of the observations. **An experiment** is a situation involving chance or probability that leads to results called outcomes. **An outcome** is the result of a single trial of an experiment. **The probability of an event** is the measure of the chance that the event will occur as a result of an experiment.

The probability of an event tells us that how likely the event will happen. Situations in which each outcome is equally likely, then we can find the probability using probability formula. Probability is a chance of prediction. If the probability that an event will occur is "x", then the probability that the event will not occur is "1 - x". If the probability that one event will occur is "a" and the independent probability that another event will occur is "b", then the probability that both events will occur is "ab". Probability of an event A can be written as:

P(A) =Number of favorable outcomes /Total number of possible outcomes.
(http://www.probabilityformula.org/)

Even the classic definition of probability is somehow useful for finding the events or occurrences probabilities but it has limited applicability. In many situations, outcomes are not equally likely to occur. It is widely used if we are concerned with the outcome of an election, if we are concerned with a person’s recovery from a disease or if we are concerned with sales of a product in a day.

For example, let assume that we are interested in the probability of a disease occurrences in a population (the people living in a region). In this population, there are N people and we know how many people having that disease in the population (assuming there are m people). So we can calculate the probability of the disease occurrences by the proportion of m to N (m/N). Many other examples can be given like that.

In the examples such as these the probability of an event (outcome or happening) is based on the ratio of same kind outcomes in total occurrences.

Beside the classic probability concept, Russian Mathematician Andrey Nikolaevich Kolmogorov (1950) introduced first a set of postulates (axioms) for probability and then the theory of probability was developed under the axiomatic probability concept.

Before giving these axioms, we need learn many definitions such as **sample space**, **random event.**

In the probability concept, our interest is based on random events or experiments. The word “random” or “randomness” is really important. Here we do not know which outcome can be occurred before an event happens but we know all possible outcomes with that event. In this explanation, we need a set of possibilities with related the event.

Definition 2.1. The set of all possible outcomes of an experiment is called the ***Sample Space*** for the experiment. It is usually denoted by the letter S (capital S).

Definition 2.2. Each outcome (point) or a set of outcomes of the sample space is referred as a ***random event*** briefly an event and denoted by letter A or B.

Example 2.1. Suppose a die is rolled and the number of dots on the upturned face is recorded.

Sample space for the example would be S={1, 2, 3, 4, 5, 6}

Example 2.2. Suppose a coin tossing experiment, the sample space S={T, H}

Example 2.3. Suppose rolling two dices, the sample space S={(1,1), (1,2),..,(1,6), (2,1),…, (2,6),…, (6,1), (6,2),…,(6,6)} so there are 6×6=36 points (outcomes) of the experiment and these consist of the sample space.

Example 2.4. A bowl contains 3 blue, 4 red and 5 black identical balls. 3 balls are chosen randomly from the bowl. How many points are found in the sample space?

In case of replacement, 3×3×3=27
In case of without replacement, 3×3×3=27

Example 2.5. Suppose that measuring the weight of fetus. The sample space for the example is S ={ x | x>0 }, where the number of elements in the set is uncountable.

Probabilities are values of a set function, also called a probability measure, for, as we shall see, this function assigns real numbers to the various subsets of a sample space S. As we shall formulate them here, the postulates (axioms) of probability apply only when the sample space S is discrete.

Kolmogorov's Axioms

Axiom 1. The probability of an event is a nonnegative real number; that is, P(A) ≥ 0 for subset A of S.
Axiom 2. P(S)=1.
Axiom 3. If $A_1, A_2, ...$ is a finite or infinite sequence of mutually exclusive events of S, then
$P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$ ($\cup$:union sign)

Axioms per se (kendi başına, kendiliğinden) require no proof but the axioms and frequency (classic) probability concept are really related. Since proportions are always positive or zero, the first axiom is complete agreement with frequency interpretation. The second axiom states indirectly that certainty is identified with probability of 1; after all, it is always assume that one of the possibilities in S must occur, and it is to certain event that we assign a probability of 1. For example, a coin with two Heads on both sides, the sample space S={ "H" }, so when filliping the coin, the result Head always occurs it means that an event A=S={ "H" } is certain to occur and the probability of A is 1. As far as the frequency interpretation is concerned, a probability of 1 implies that the event in question will occur 100 percent of the time.

Theorem 2.1. If A is an event in a discrete sample space S, then P(A) equals the sum of the probabilities of the individual outcomes comprising A.

Proof: Let $B_1, B_2, \ldots,$ be the finite or infinite sequence of outcomes that comprise the event *A*. Thus,

$$A = B_1 \cup B_2 \cup$$

And since the individual outcomes, $B_1, B_2, \ldots,$ are mutually exclusive, the third axioms of probability yields

$$P(A) = P(B_1) + P(B_2) + \ldots$$

This completes the proof.

Theorem 2.2. If an experiment can result in any one of *N* different equally likely outcomes, and if *m* of these outcomes together constitute event *A*, then the probability of event *A* is

$$P(A) = \frac{m}{N}$$

Proof: Let $B_1, B_2, \ldots, B_N$, represent the individual outcomes in *S*, each with probability 1/N. If *A* is the union of *m* of these mutually exclusive outcomes, and it does matter which ones, then

$$\begin{aligned} P(A) &= P(B_1 \cup B_2 \cup \ldots \cup B_m) \\ &= P(B_1) + P(B_2) + \ldots + P(B_m) \\ &= \underbrace{\frac{1}{N} + \frac{1}{N} + \ldots + \frac{1}{N}}_{m} = \frac{m}{N} \end{aligned}$$

Theorem 2.3. If *A* and A' are complementary (tümleyen) events in a sample space *S*, then

$$P(A') = 1 - P(A)$$

Proof: By the second axiom, $P(S) = 1$ and $A \cup A' = S$ so *A* and A' are mutually exclusive then,

$$\begin{aligned} &P(S) = 1 \\ &P(A \cup A') = 1 \\ &P(A) + P(A') = 1 \\ &P(A') = 1 - P(A) \end{aligned}$$

the proof is completed.

Theorem 2.4. $P(\varnothing) = 0$ for any sample space S.

Proof: Since S and $\varnothing$ are mutually exclusive and $S \cup \varnothing = S$ in accordance with the definition of empty set $\varnothing$, it follows that

$$\begin{aligned} &P(S) = P(S \cup \varnothing) \\ &P(S) = P(S) + P(\varnothing) \\ &P(\varnothing) = 0. \end{aligned}$$

Theorem 2.5. If A and B are events in a sample space S and $A \subset B$, then $P(A) \leq P(B)$.

Proof: Since $A \subset B$, we can write

$$B = A \cup (A' \cap B)$$

as can easily be verified by means of Venn diagram. Then, since A and $A' \cap B$ are mutually exclusive, we get

$$\begin{aligned} P(B) &= P(A) + P(A' \cap B) && \text{by Axiom 3} \\ &\geq P(A) && \text{by Axiom 1.} \end{aligned}$$

Theorem 2.6. $0 \leq P(A) \leq 1$ for any event A.

Proof: Using the Theorem 2.5 and the fact that $\varnothing \subset A \subset S$ for any event A in S, we have

$$P(\varnothing) \leq P(A) \leq P(S)$$

Then, $P(\varnothing) = 0$ and $P(S) = 1$ leads to the result that

$$0 \leq P(A) \leq 1$$

Theorem 2.7. If A and B are two events in a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: Assigning a, b, and c to the mutually exclusive events $A \cap B$, $A \cap B'$, and $A' \cap B$ as in the Venn diagram. We find that

$$\begin{aligned} P(A \cup B) &= \mathrm{a} + \mathrm{b} + \mathrm{c} \\ &= (\mathrm{a} + \mathrm{b}) + (\mathrm{c} + \mathrm{a}) - \mathrm{a} \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

Theorem 2.8. If A, B, and C are three events in a sample space S, then

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - \\ - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

Proof: Writing $A \cup B \cup C$ as $A \cup (B \cup C)$ and using the formula of Theorem 2.7 twice, once for $P[A \cup (B \cup C)]$ and once for $P(B \cup C)$, we get

$$\begin{aligned} P(A \cup B \cup C) &= P[A \cup (B \cup C)] \\ &= P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P[A \cap (B \cup C)] \end{aligned}$$

Then, using the distributive law that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and getting probability of this event

$$P[A\cap(B\cup C)] = P[(A\cap B)\cup(A\cap C)]$$
$$= P[(A\cap B)] + P[(A\cap C)] - P[(A\cap B)\cap(A\cap C)]$$
$$= P[(A\cap B)] + P[(A\cap C)] - P(A\cap B\cap C)]$$

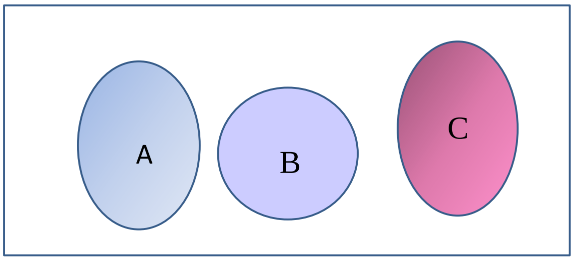
and hence that the result is substituted into the first Eq. The theorem is proofed now.

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) -$$
$$- P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

Example 2.6. Suppose $P(A) = 1/2$, $P(B) = 1/8$, $P(C) = 1/4$ where A, B, C are mutually exclusive. Determine the values of:

a) $P(A\cup B)$
b) $P(A\cup B\cup C)$
c) $P(A-B)$
d) $P(\bar{A}\cap\bar{B})$

Solution:



a) Since A and B are mutually exclusive $P(A\cup B) = P(A) + P(B) = 5/8$
b) Since A, B and C are mutually exclusive $P(A\cup B\cup C) = P(A) + P(B) + P(C) = 7/8$
c) $P(A-B) = P(A) = 1/2$
d) $P(\bar{A}\cap\bar{B}) = 1 - P(A\cup B) = 1 - P(A) - P(B) = 1 - 5/8 = 3/8$

Example 2.7. Suppose you play a game over and over again, each time with chance 1/N of winning the game, no matter what the results of previous games. What is the probability of at least one win in the n games?

Solution:

A = {at least one win in the n games}

$$P(A) = 1 - \left(1 - \frac{1}{N}\right)^n$$

Example 2.8. In a certain population, 10% of the people are rich, 5% are famous, and 3% are rich and famous. For a person picked at random from this population.

a) What is the chance that person is rich but not famous?
b) What is the chance that person is either rich or famous?

Solution:

a) P (rich but not famous) = P (rich) - P (rich and famous) = 10%-3%=7%
b) P (rich or famous) = P (rich) + P (famous)- P (rich and famous) = 10%+5%-3%=12%

Example 2.9. Event A, B, and C are such that
P(A)=0.7, P(B)=0.6 P(C)=0.5 P(A∩B)=0.4, P(A∩C)=0.3, P(B∩C)=0.2, P(A∩B∩C)=0.1
Find:

a) $P(\bar{A}\cap B)$
b) $P(\bar{A}\cap \bar{B}\cup C)$

Solution:

a) $P(\bar{A}\cap B)=P(B-A)=P(B)-P(A\cap B)=0.6-0.4=0.2$
b) $P(\bar{A}\cap \bar{B}\cup C)=1-P(A\cup B)+\left[P(A\cap C)+P(B\cap C)-P(A\cap B\cap C)\right]$
$=1-0.9+0.3+0.2-0.1=0.5$

Example 2.10. (Moore, 1960) the chieftain (kabile reisi) of a primitive tribe (ilkel kabile) of Indians (kızıldereliler) had feeling that some of his men were cheating him of tax payments. According to the laws of tribe, a tax had to be paid by any man who owned three pieces of taxable property: teepees (kızıldereli çadırı), horses and squaws (kızıldereli kadın). The tribe practiced monogamy (tek eşlilik), and no man owned more than one horse or one teepee. The chieftain knew that none would admit guilt, and so he decided on trick scheme. He inquired (soruşturmak) of the 2000 men in his tribe how many owned: a horse, a teepee; a squaw; a squaw and a horse; a squaw and teepee; a horse and a teepee. He knew that he could expect honest answer to these questions; the result of which are given in the table below. It was observed that every man admitted ownership of at least one piece of property. How many men should be paying taxes? (Barr and Zehna, 1971, page, 19)

property	Number of men owning property
Horse	700
Teepee	1600
Squaw	1000
Squaw and horse	700
Squaw and teepee	500
Horse and teepee	200

(For student)

SECTION 3: CONDITIONAL PROBABILITY AND INDEPENDENT EVENTS

3.1 Conditional Probability

Conditional probability provides probability of events in a restricted sample space. For example, let us think rolling a die, if we know that the face-up of the die is even, when we ask what the probability of the face-up of the die of 4 is?

Definition 3.1. If A and B are any two events in a sample space S and $P(A)\neq 0$, the conditional probability of B given A is

$$P\left(B|A\right)=\frac{P(A\cap B)}{P(A)}$$

Example 3.1. A die is loaded in such a way that each odd number is twice as likely to occur as each even number. The die is rolled, if it is given that the number of points is greater than 4 or equal 4, what is the probability of the number of points being 6?

The sample space S={1,2,3,4,5,6} hence we assign probability w to each even number and probability $2w$ to each odd number, we find that $2w+w+2w+w+2w+w=9w=1$, $w=1/9$.

A={the points are greater than 4 or equal 4}={4, 5, 6}, here the restricted sample space would be S′={4, 5, 6}

B={the point is 6}={6}

$A\cap B$={6}

The points 4 and 6 are even numbers; their probabilities are 1/9s.
The point 5 is odd number; its probability is 2/9.

$$P\left(B|A\right)=\frac{P(A\cap B)}{P(A)}=\frac{1/9}{4/9}=\frac{1}{4}$$

Example 3.2. A manufacturer of airplane parts knows from past experience that the probability is 0.80 that an order (sipariş) will be ready for shipment (sevkiyat) on time, and it is 0.72 that and order will be ready for shipment on time and will also be delivered on time. What is the probability that such an order will be delivered on time given that it was ready for shipment on time?

R shows the event that an order is ready for shipment on time.
D shows the event that an order is delivered on time.

$P(R)=0.80$, $P(R\cap D)=0.72$ then the reply in the question asked is:

$$P\left(D|R\right) = \frac{P(D \cap R)}{P(R)} = \frac{0.72}{0.80} = 0.90.$$

Theorem 3.1. If *A* and *B* are any two events in a sample space *S* and $P(A) \neq 0$, then

$$P(A \cap B) = P(A)P\left(B|A\right)$$

Example 3.3. Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards if we sample,

a) Without replacement,
b) With replacement.

a) If the first card is not replaced before the second card is drawn the probability getting two aces in succession is $\frac{4}{52}\frac{3}{51} = \frac{1}{221}$
b) If the first card is replaced before the second card is drawn the probability getting two aces in succession is $\frac{4}{52}\frac{4}{52} = \frac{1}{169}$

Theorem 3.2. If *A*, B and *C* are any three events in a sample space *S* and $P(A \cap B) \neq 0$, then

$$P(A \cap B \cap C) = P(A)P\left(B|A\right)P\left(C|A \cap B\right)$$

Proof: writing $A \cap B \cap C$ as $(A \cap B) \cap \mathrm{C}$ and using the formula of Theorem 3. 1 twice, we get

$$\begin{aligned} P(A \cap B \cap C) &= P[(A \cap B) \cap C] \\ &= P\left(A \cap B\right)P\left(C|A \cap B\right) \\ &= P(A)P\left(B|A\right)P\left(C|A \cap B\right) \end{aligned}$$

3.2 Independent Events

Informally speaking, two events are independent, if the occurrence or non-occurrence of either one does not affect the probability of the occurrence of the other.

Definition 3.2. Two events *A* and *B* are independent if only if

$$P(A \cap B) = P(A)P(B)$$

Example 3.4. A coin is tossed three times and A is the event that a head occurs on each of the first two tosses, B is the event that a tail occurs on the third toss. Find that A and B are independent.

S={ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

A={ HHH, HHT}

B={HHT, HTT, THT, TTT}

A∩B={HHT}

If they are independent, it must be P(A∩B)=P(A)P(B) and so $\frac{1}{8}=\frac{1}{4}\cdot\frac{1}{2}$ and we say that they are independent events.

Theorem 3.3. If *A* and B are independent, then A and B′ are also independent.

Proof: Since $A=(A\cap B)\cup(A\cap B')$ and A∩B and A∩ B′ are mutually exclusive events, and A and B are independent by assuming, we have

$$\begin{aligned} P(A) &= P[(A\cap B)\cup(A\cap B')] \\ &= P(A\cap B)+P(A\cap B') \\ &= P(A)P(B)+P(A\cap B') \end{aligned}$$

It follows that

$$\begin{aligned} P(A\cap B') &= P(A)[1-P(B)] \\ &= P(A)P(B') \end{aligned}$$

and hence that A and B′ are independent.

Example 3.5. A sharpshooter (keskin nişancı) hits a target with probability 0.75. Assuming independence, find the probabilities getting

a) a hit followed by two misses
b) two hits and a miss in any order

a) "HMM" hence $P(HMM)=0.75\times0.25^2$
b) "HHM", "HMH", "MHH" these are mutually exclusive and hitting or missing a target are independent events from **Theorem 3.3.**
$P(HHM)+P(HMH)+P(MHH)=3\times0.75^2\times0.25$

Definition 3.3. Events $A_1, A_2, ..., A_k$ are independent if and only if the probability of intersection of any 2,3,…, or k of these events equals the product of their respective probabilities.

For three events A, B, and C for example, independence requires that

$$\begin{aligned} &P(A\cap B)=P(A)P(B) \\ &P(A\cap C)=P(A)P(C) \\ &P(B\cap C)=P(B)P(C) \\ &\textit{and } P(A\cap B\cap C)=P(A)P(B)P(C) \end{aligned}$$

Note that three or more events can be **pairwise independent** without being independent.

Example 3.6. A coin is loaded so the probabilities of heads and tails are 0.52 and 0.48, respectively. If the coin is tossed three times, what are the probabilities of getting

a) All heads$\Rightarrow$ because of independency $P(HHH)=P(H)\underbrace{P\left(H|H\right)}_{P(H)}\underbrace{P\left(H|H\cap H\right)}_{P(H)}=0.52^3$

b) Two tails and a head in that order
$\Rightarrow$ because of independency $P(TTH)=P(T)\underbrace{P\left(T|T\right)}_{P(T)}\underbrace{P\left(H|T\cap T\right)}_{P(H)}=0.48^2\times 0.52$

Example 3.7. There are 90 applicants for a job with the news department of television station. Some of them are college graduates and some are not, some of them at least three years' experience and some have not, with the exact breakdown being

	College graduates	Not College graduates	Total
At least three years' experience	18	9	27
Less than three years' experience	36	27	63
Total	54	36	90

If the order in which the applicants are interviewed by the station manager is random, G is the event that the first applicant interviewed is a college graduate, and T is the event the first applicant interviewed has at least three years 'experience determine each of the probabilities given below:

a) P(G), P(T), P(T′)
b) $P\left(T|G\right)$
c) $P\left(G'|T'\right)$

a) $P(G)=\frac{54}{90}=\frac{3}{5}\quad P(G')=\frac{36}{90}=\frac{2}{5}\quad P(T)=\frac{27}{90}=\frac{3}{10}\quad P(T')=\frac{63}{90}=\frac{7}{10}$

b) $P\left(T|G\right)=\frac{P(G\cap T)}{P(G)}=\frac{18/90}{54/90}=\frac{18}{54}=\frac{1}{3}$

c) $P\left(G'|T'\right)=\frac{P(T'\cap G')}{P(T')}=\frac{27/90}{63/90}=\frac{27}{63}=\frac{3}{7}$

Example 3.8. Suppose that in Vancouver, B.C. the probability that a rainy day is followed by a rainy day is 0.80 and the probability that a sunny day is followed by a rainy day is 0.60. Find the probabilities that a rainy day is followed by

a) A rainy day, a sunny day, and another rainy day,
b) Two rainy days and then two sunny days

$P\left(R|R\right)=0.80\quad P\left(R|S\right)=0.60$

a) $P\left(RSR|R\right)=P\left(R|R\right)P\left(S|R\cap R\right)P\left(R|R\cap R\cap S\right)=0.80\times 0.20\times 0.60=0.096$

b) $P\left(RRSS|R\right)=P\left(R|R\right)P\left(R|R\cap R\right)P\left(S|R\cap R\cap R\right)P\left(S|R\cap R\cap R\cap S\right)$
$=0.80\times 0.80\times 0.20\times 0.40$
$=0.0512$

SECTION 4: BAYES' THEOREM

Example 4.1. The completion of construction job may be delayed because of a strike. The probabilities are 0.60 that there will be a strike, 0.85 that the construction job will be completed on time if there is no strike, and 0.35 that the construction job will be completed on time if there is a strike. What is the probability that the construction job will be completed on time? (Miller and Miller, 2004, page 48)

Example 4.2. The members of consulting firm rent cars from three rental agencies: 60% from agency I, 30% from agency II, and 10% from agency III. If 9% of the cars from agency I need a tune-up, 20% of the cars from agency II need a tune-up, and 6 % of the cars from agency III need a tune-up, what is the probability that a rental car delivered to the firm will need a tune-up?

Example 4.3. Suppose that a laboratory test on a blood sample yields one of two results, positive, negative. It is found that 95% of people with a particular disease produce a positive result. But 2% of people without disease will also produce a positive result (a false positive). Suppose that 1% of the population actually has the disease. What is the probability that person chosen at random from the population will have the disease, given that person's blood yields a positive result?

Example 4.4. A manufacturing process produces integrated circuit chips. Over the long run the fraction of bad chips produced by the process is round 20%. Thoroughly testing a chip to determine whether it is good or bad is rather expensive, so a cheap test is tried. All good chips will pass the chip test, but so will 10% of bad chips.

- **a)** Given a chip passes the cheap test, what is the probability that it is a good chip?
- **b)** If a company using this manufacturing process sell all chips which pass the cheap test, over the long run what percentage of chips sold will be bad?

EXERCISES:

1) The probability that a one-car-accident is due to faulty brakes is 0.04, the probability that a one-car accident is correctly attributed to faulty brakes is 0.82, the probability that a one-car accident is incorrectly attributed to faulty brakes is 0.03. what is the probability that

- **a)** a one car accident will be attributed to faulty brakes;
- **b)** a one car accident attributed to faulty brakes was actually due to faulty brakes.

2) In a certain community, 8 percent of all adults over 50 have diabetes. If a health service in this community correctly diagnoses 95 percent of all persons with diabetes as having the disease and incorrectly diagnoses 2 percent of all persons without diabetes as having the disease, find probabilities that

- **a)** the community health service will diagnose an adult over 50 as having diabetes;
- **b)** a person over 50 diagnosed by the health service as having diabetes actually has the disease.

3) An explosion at a construction site could have occurred as the result of **static electricity**, **malfunctioning of equipment**, **carelessness**, or **sabotage**. Interviews with construction engineers analyzing the risks involved led to the estimates that such an explosion would occur with probability 0.25 as a result of static electricity, 0.20 as a result of malfunctioning of equipment, 0.40 as result of carelessness, and 0.75 as a result of sabotage. It is also felt that the prior probabilities of four causes of the explosion are 0.20, 0.40, 0.25 and 0.15. Based on all this information, what is

- **a)** the most likely cause of the explosion;
- **b)** the least likely cause of the explosion?

4) A polygraph (lie detector) is said to be 90% reliable in the following sense: There is a 90% chance that a person who is telling the truth will pass the polygraph test; and there is a 90% chance that a person telling a lie will fail the polygraph test.

- **a)** Suppose a population consists of 5% liars. A random person takes a polygraph test, which concludes that he/she is lying. What is the probability that he/she is actually lying?
- **b)** Consider the probability that a person is actually lying given that the polygraph says that he/she is. Using the definition of reliability, how reliable must the polygraph test be in order that this probability is at least 85%?

5) Your friend has three dice. One die is fair. One die has fives on all six sides. One die has fives on three sides and four on three sides. A die is chosen at random. It comes up five. Find the probability that the chosen die is the fair one.

SECTION 4: BAYES' THEOREM

4. Bayes' Theorem (Bayes' Rule)

In probability theory and statistics, Bayes' theorem (alternatively Bayes' law or Bayes' rule) describes the probability of an event, based on conditions that might be related to the event. For example, if cancer is related to age, then, using Bayes' theorem, a person's age can be used to more accurately assess the probability that they have cancer.

One of the many applications of Bayes' theorem is Bayesian inference, a particular approach to statistical inference. When applied, the probabilities involved in Bayes' theorem may have different probability interpretations. With the Bayesian probability interpretation the theorem expresses how a subjective degree of belief should rationally change to account for evidence. Bayesian inference is fundamental to Bayesian statistics.

In many situations, the outcome of an experiment can be observed conditionally to other events. In other words, the outcome of an experiment could not be observed directly. For example, there are n_1 white balls and m_1 black balls are in the first urn, and there are n_2 white balls and m_2 black balls are in the second urn. What is the probability of a black ball drawn?

Theorem 4.1. If the events $B_1, B_2, ..., B_k$ constitute a partition of the sample space S and $P(B_i)\neq 0$, for i=1,2,…,k, then for any event of A in S

$$P(A) = \sum_{i=1}^{k} P(B_i)P\left(A|B_i\right)$$

This theorem is called the **rule of total probability** or the **rule of elimination**.

Example 4.1. The completion of construction job may be delayed because of a strike. The probabilities are 0.60 that there will be a strike, 0.85 that the construction job will be completed on time if there is no strike, and 0.35 that the construction job will be completed on time if there is a strike. What is the probability that the construction job will be completed on time? (Miller and Miller, 2004, page 48)

Solution:

A is the event that the construction job will be completed on time.
B is the event that there will be strike. $P(B)$=0.60
$P\left(A|B\right) = 0.35$ and $P\left(A|B'\right) = 0.85$

$$\begin{aligned}
P(A) &= P\left[(A\cap B)\cup(A\cap B')\right] \\
&= P(A\cap B) + P(A\cap B') \\
&= P(B)P\left(A|B\right) + P(B')P\left(A|B'\right) \\
&= (0.60)(0.35) + (0.40)(0.85) = 0.55
\end{aligned}$$

Example 4.2. The members of consulting firm rent cars from three rental agencies: 60% from agency I, 30% from agency II, and 10% from agency III. If 9% of the cars from agency I need a tune-up (ayar vermek), 20% of the cars from agency II need a tune-up, and 6 % of the cars from agency III need a tune-up, what is the probability that a rental car delivered to the firm will need a tune-up?

Solution:

$P(I)=0.60, \quad P\left(T \mid I\right)=0.09$

$P(II)=0.30, \quad P\left(T \mid II\right)=0.20$

$P(III)=0.10, \quad P\left(T \mid III\right)=0.06$

$$\begin{aligned} P(T) &= P(I)P\left(T \mid I\right)+P(II)P\left(T \mid II\right)+P(III)P\left(T \mid III\right) \\ &= (0.60)(0.09)+(0.30)(0.20)+(0.10)(0.06)=0.12 \end{aligned}$$

With 12% of cars delivered to the firm will need a tune-up.

Theorem 4.2. If the events $B_1, B_2, ..., B_k$ constitute a partition of the sample space S and $P(B_i)\neq 0$, for i=1,2,…,k, then for any event of A in S such that $P(A)\neq 0$

$$P\left(B_i \mid A\right)=\frac{P(B_i)P\left(A \mid B_i\right)}{\sum_{i=1}^{k} P(B_i)P\left(A \mid B_i\right)}, \quad for\ i=1,2,...,\mathrm{k}$$

The theorem is called the **Bayes' Theorem**. Where the unconditional probabilities $P(B_i)$ are called prior probabilities and the conditional probabilities $P\left(A \mid B_i\right)$ are called likelihoods, hence the probabilities $P\left(B_i \mid A\right)$ are called posterior probabilities.

Example 4.3. Suppose that a laboratory test on a blood sample yields one of two results, positive, negative. It is found that 95% of people with a particular disease produce a positive result. But 2% of people without disease will also produce a positive result (a false positive). Suppose that 1% of the population actually has the disease. What is the probability that person chosen at random from the population will have the disease, given that person's blood yields a positive result?

Solution:

$P\left(+ \mid D\right)=0.95, \quad P\left(+ \mid \bar{D}\right)=0.02, \quad P(D)=0.01, \quad P(\bar{D})=0.99$

$$\begin{aligned} P\left(D \mid +\right) &= \frac{P\left(+ \mid D\right)P(D)}{P\left(+ \mid D\right)P(D)P\left(+ \mid \bar{D}\right)P(\bar{D})} \\ &= \frac{(0.95)(0.01)}{(0.95)(0.01)+(0.02)(0.99)}=\frac{95}{293}\cong 32\% \end{aligned}$$

	Test result		total
	Positive(+)	**Negative(-)**	
Have a disease	100×95%=95	100×5%=5	100
Does not have disease	9900×2%=198	9900×98%=9702	9900
total	293	9707	10000

Example 4.4. A manufacturing process produces integrated circuit chips. Over the long run the fraction of bad chips produced by the process is round 20%. Thoroughly testing a chip to determine whether it is good or bad is rather expensive, so a cheap test is tried. All good chips will pass the cheap test, but so will 10% of bad chips.

a) Given a chip passes the cheap test, what is the probability that it is a good chip?
b) If a company using this manufacturing process sell all chips which pass the cheap test, over the long run what percentage of chips sold will be bad?

Solution:
Bad={a chip produced being a bad }

$P\left(+\middle|Good\right)=1,\quad P\left(+\middle|Bad\right)=0.10,\quad P(Bad)=0.20,\quad P(Good)=0.80$

$P(Bad)=1-P(Good)$

a)

$$P\left(Good\middle|+\right)=\frac{P\left(+\middle|Good\right)P(Good)}{P\left(+\middle|Good\right)P(Good)+P\left(+\middle|Bad\right)P(Bad)}$$

$$=\frac{(1)(0.80)}{(1)(0.80)+(0.10)(0.20)}=\frac{80}{82}\cong 98\%$$

b) $P\left(Bad\middle|+\right)=1-P\left(Good\middle|+\right)=1-0.98=2\%$

For example, suppose that if 1 000 000 chips were sold, %2 of them would be bad chips. That is, 20 000 bad chips were sold.

Exercises:

1) The probability that a one-car-accident is due to faulty brakes is 0.04, the probability that a one-car accident is correctly attributed to faulty brakes is 0.82, the probability that a one-car accident is incorrectly attributed to faulty brakes is 0.03. what is the probability that

   a) a one car accident will be attributed to faulty brakes;
   b) a one car accident attributed to faulty brakes was actually due to faulty brakes.

Solution:

a)

F={ faulty break } P(F)=0.04
A={ attributed to faulty brakes }
$P\left(A\middle|F\right)=0.82$ corresponds to the probability of the event about that a one-car accident is correctly attributed to faulty brakes.

$P\left(A\middle|\bar{F}\right)=0.03$ corresponds to the probability of the event about that a one-car accident is incorrectly attributed to faulty brakes.

$$
\begin{aligned}
P(A) &= P\left(A\middle|F\right)P(F)+P\left(A\middle|\bar{F}\right)P(\bar{F}) \\
&= (0.82)(0.04)+(0.03)(0.96) \\
&= 0.0616
\end{aligned}
$$

b)

$$
\begin{aligned}
P\left(F\middle|A\right) &= \frac{P\left(A\middle|F\right)P(F)}{P\left(A\middle|F\right)P(F)+P\left(A\middle|\bar{F}\right)P(\bar{F})} \\
&= \frac{(0.82)(0.04)}{0.0616}=\frac{0.0328}{0.0616}\cong 0.5325
\end{aligned}
$$

2) In a certain community, 8 percent of all adults over 50 have diabetes. If a health service in this community correctly diagnoses 95 percent of all persons with diabetes as having the disease and incorrectly diagnoses 2 percent of all persons without diabetes as having the disease, find probabilities that

   a) the community health service will diagnose an adult over 50 as having diabetes;
   b) a person over 50 diagnosed by the health service as having diabetes actually has the disease.

Solution:

I={having diabetes disease(illness)}, D={diagnosed diabetes }

$$P\left(D\middle|I\right)=0.95, \quad P\left(D\middle|\bar{I}\right)=0.02$$

a)

$$
\begin{aligned}
P(D) &= P(D\cap I)+P(D\cap\bar{I}) \\
&= P\left(D\middle|I\right)P(I)+P\left(D\middle|\bar{I}\right)P(\bar{I}) \\
&= (0.95)(0.08)+(0.02)(0.92)=0.0944
\end{aligned}
$$

b)

$$
\begin{aligned}
P\left(I\middle|D\right) &= \frac{P\left(D\middle|I\right)P(I)}{P(D)} \\
&= \frac{(0.95)(0.08)}{0.0944}\cong 0.8051
\end{aligned}
$$

3) An explosion at a construction site could have occurred as the result of **static electricity**, **malfunctioning of equipment**, **carelessness**, or **sabotage**. Interviews with construction engineers analyzing the risks involved led to the estimates that such an explosion would occur with probability 0.25 as a result of static electricity, 0.20 as a result of malfunctioning of equipment, 0.40 as result of carelessness, and 0.75 as a result of sabotage. It is also felt that the prior probabilities of four causes of the explosion are 0.20, 0.40, 0.25 and 0.15. Based on all this information, what is

   a) the most likely cause of the explosion;
   b) the least likely cause of the explosion?

Solution:

St: static electricity
Ma: malfunctioning of equipment
Ca: carelessness
Sa: sabotage

$$\left.\begin{array}{l} P\left(E\middle|St\right)=0.25,\quad P\left(E\middle|Ma\right)=0.20, \\ P\left(E\middle|Ca\right)=0.40,\quad P\left(E\middle|Sa\right)=0.75 \end{array}\right. \text{likelihoods}$$

$P(St)$=0.20, $P(Ma)$=0.40, $P(Ca)$=0.25, $P(Sa)$=0.15 priors

$$\begin{aligned} P(E) &= P\left(E\middle|St\right)P(St)+P\left(E\middle|Ma\right)P(Ma)+P\left(E\middle|Ca\right)P(Ca)+P\left(E\middle|Sa\right)P(Sa) \\ &= (0.25)(0.20)+(0.20)(0.40)+(0.40)(0.25)+(0.75)(0.15) \\ &= 0.3425 \end{aligned}$$

$$P\left(St\middle|E\right)=\frac{P\left(E\middle|St\right)P(St)}{P(\mathrm{E})} = \frac{(0.25)(0.20)}{0.3425}\cong 0.146 \qquad P\left(Ma\middle|E\right)=\frac{P\left(E\middle|Ma\right)P(Ma)}{P(\mathrm{E})} = \frac{(0.20)(0.40)}{0.3425}\cong 0.2336$$

$$P\left(Ca\middle|E\right)=\frac{P\left(E\middle|Ca\right)P(Ca)}{P(\mathrm{E})} = \frac{(0.25)(0.40)}{0.3425}\cong 0.292 \qquad P\left(Sa\middle|E\right)=\frac{P\left(E\middle|Sa\right)P(Sa)}{P(E)} = \frac{(0.75)(0.15)}{0.3425}= 0.3285$$

a) the most likely cause of the explosion at a construction site occurred as the result of **sabotage** with probability of 0.3285.
b) the least likely cause of the explosion at a construction site occurred as the result of **static electricity** with probability of 0.146.

4) A polygraph (lie detector) is said to be 90% reliable in the following sense: There is a 90% chance that a person who is telling the truth will pass the polygraph test; and there is a 90% chance that a person telling a lie will fail the polygraph test.

a) Suppose a population consists of 5% liars. A random person takes a polygraph test, which concludes that he/she is lying. What is the probability that he/she is actually lying?
b) Consider the probability that a person is actually lying given that the polygraph says that he/she is. Using the definition of reliability, how reliable must the polygraph test be in order that this probability is at least 85%?

Solution:

Po={ A polygraph says the person is liar} L={the person is actually liar}

a)

$$P\left(Po\middle|L\right)=0.90,\quad P\left(\overline{Po}\middle|\overline{L}\right)=0.90,\quad P(L)=0.05,\quad P(\overline{L})=0.95$$

$$P\left(L\middle|Po\right)=\frac{P\left(Po\middle|L\right)P(L)}{P\left(Po\middle|L\right)P(L)+P\left(Po\middle|\bar{L}\right)P(\bar{L})}$$

$$=\frac{(0.90)(0.05)}{(0.90)(0.05)+(0.10)(0.95)}=\frac{0.045}{0.14}=0.3214$$

b)

$$P\left(L\middle|Po\right)=\frac{P\left(Po\middle|L\right)P(L)}{P\left(Po\middle|L\right)P(L)+P\left(Po\middle|\bar{L}\right)P(\bar{L})}$$

$$0.85=\frac{(p)(0.05)}{(p)(0.05)+(1-p)(0.95)}$$

$$0.8075=0.815p$$

$$p=\frac{0.8075}{0.815}=0.9908$$

If the probability that a person is actually lying given that the polygraph says that he/she is increases, the reliability will increase. It means that there is a 99.08% chance that a person who is telling the truth will pass the polygraph test; and there is a 99.08% chance that a person telling a lie will fail the polygraph test.

5) Your friend has three dice. One die is fair. One die has fives on all six sides. One die has fives on three sides and four on three sides. A die is chosen at random. It comes up five. Find the probability that the chosen die is the fair one.

Solution:
One die is fair, F={fair die}
All sides are five, A={all sides of die are five}
Three sides are five, three sides are four T={ three sides of are five, three sides of die are four}
Die comes up five B={the face up of the die is five}

$$P\left(F\middle|B\right)=\frac{P\left(B\middle|F\right)P(F)}{P\left(B\middle|F\right)P(F)+P\left(B\middle|A\right)P(A)+P\left(B\middle|T\right)P(T)}$$

$$=\frac{(1/6)(1/3)}{(1/6)(1/3)+(1)(1/3)+(1/2)(1/3)}=\frac{1}{10}$$

SECTION 5: RANDOM VARIABLES

We often summarize the outcome from a random experiment by a simple number. In many of the examples of random experiments that we have considered, the ***sample space*** has been a description of possible outcomes. In some cases, descriptions of outcomes are sufficient, but in other cases, it is useful to associate a number with each outcome in the ***sample space.*** Because the particular outcome of the experiment is not known in advance, the resulting value of our variable is not known in advance. For this reason, the variable that associates a number with the outcome of a random experiment is referred to as a ***random variable***.

Definition: A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment. A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as x=70 milliamperes.

Sometimes a measurement (such as current in a copper wire (bakır tel) or length of a machined part) can assume any value in an interval of real numbers (at last theoretically). Then arbitrary (keyfi) precision in the measurement is possible. Of course, in practice, we might round off to the nearest tenth or hundredth of a unit. The random variable that represents this measurement is said to be a ***continuous random variable***. The range of the random variable includes all values in an interval of real numbers; that is, the range can be thought of as continuum.

In other experiments, we might record a count such as the number of transmitted bits that are received in error. Then the measurement is limited to integers. Or we might record that a proportion such as 0.0042 of the 10000 transmitted bits were received in error. Then the measurement is fractional (kesirli/oransal), but it is still limited to discrete points on the real line. Whenever the measurement is limited to discrete points on the real line, the random variable is said to be a ***discrete random variable.***

Definition: A **discrete random variable** is random variable with a finite (or countably infinite) range.
A **continuous random variable** is a random variable with an interval (either finite or infinite) of real numbers for its range.

In some cases, the random variable X is actually discrete but, because the range of possible values is so large, it might be more convenient to analyze X as a continuous random variable. For example, suppose that current measurements (akım ölçümleri) are read from a digital instrument that displays the current to the nearest one-hundredth of a milliampere. Because the possible measurements are limited, the random variable is discrete. However, it might be a more convenient, simple approximation to assume that the current measurements are values of a continuous random variable.

Examples of Random Variables:

Examples of continuous random variables: electrical current (elektrik akımı), length, pressure, temperature, time, voltage, weight
Examples of discrete random variables: number of scratches (çizikler) on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error.

The following are examples of discrete random variables:

1. The number of seizures (kriz, nöbet) an epileptic patient has in a given week:
   x=0, 1, 2, … (it is not known that the number of seizures for any epileptic patient in a given week, that's why it is defined as a random event.)

2. The number of voters in a sample of 500 who favor impeachment (görevini kötüye kullanma suçlaması/ithamı) of the president: x=0, 1, 2, …, 500.

3. The shoe size of a tennis player: $x = \cdots 5, 5\frac{1}{2}, 6, 6\frac{1}{2}, 7, 7\frac{1}{2} \cdots$ (assume that we think the customers of sport store, and shoe sizes are different from a customer to another)

4. The number of customers waiting to be served in a restaurant at a particular time: x=0,1,2,…. (for any particular time, the number of customers waiting to be served in a restaurant differs.)

Note that several of the examples of discrete random variables begin with the words *The number of…* This wording is very common, since the discrete random variables most frequently observed are counts. The following are examples of continuous random variables:

1. The length of time (in seconds) between arrivals at a hospital clinic: $0 \le x \le \infty$ (infinity)
2. The length of time (in minutes) it takes a student to complete a one-hour exam: $0 \le x \le 60$
3. The amount (in ounces) of carbonated beverage (içecek, meşrubat) loaded into a 12-ounce can (teneke kutu) in a can-filling operation: $0 \le x \le 12$
4. The depth (in feet) at which a successful oil-drilling (petrol sondaj) venture (girişimi) first strikes (çıkarmak) oil: $0 \le x \le c$, where c is the maximum depth obtainable
5. The weight (in pounds) of a food item bought in a supermarket: $0 \le x \le 500$ [Note: Theoretically, there is no upper limit on x, but it is unlikely that it would exceed 500 pounds.]

5.1. DISCRETE RANDOM VARIABLES

Many physical systems can be modeled by the same or similar random experiments and random variables. The distribution of the random variables involved in each of these common systems can be analyzed, and the results of that analysis can be used in different applications and examples.

Example 1: A voice communication system (ses haberleşme sistemi) for a business contains 48 external lines (dış hatlar). At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48. When the system is observed, if 10 lines are in use, x=10.

Example 2: In a semiconductor (yarı iletken) manufacturing (imalat) process, two wafers (silikon devre levhaları) from a lot (parti) are tested. Each wafer is classified as pass or fail. Assume that the probability that a wafer passes the test is 0.8 and that wafers are independent. The sample space for the experiment and associated probabilities are shown in Table 1. For example, because of the independence, the probability of the outcome that the first wafer tested passes and the second wafer tested fails, denoted as pf is $P(pf) = 0.8(0.2) = 0.16$. The random variable X is defined to be equal to the number of wafers that pass. The last column of the table shows the values of X that are assigned to each outcome in the experiment.

Table 1. Wafer Tests.

Outcome			
Wafer 1	Wafer 2	Probability	x
Pass	Pass	0.64	2
Fail	Pass	0.16	1
Pass	Fail	0.16	1
Fail	Fail	0.04	0

EXERCISES 1

Exercise 1.1: The random variable is the number of nonconforming solder (lehim) connections on a printed circuit board (anakart devresi) with 1000 connections. $x = \{0, 1, 2, \cdots, 1000\}$

Exercise 1.2: An electronic scale (elektronik tartı) that displays weights to the nearest pounds is used to weigh (tartmak) packages. The display shows only five digits. Any weight greater than the display can indicate is shown as 99999. The random variable is the displayed weight. $x = \{0, 1, 2, \cdots, 99999\}$

5.1.1. Probability Distributions and Probability Mass Functions for Discrete Random Variables

The **probability distribution** of a random variable X is a description of the probabilities associated with the possible values of X. For a discrete random variable, the distribution is often specified by just a list of the possible values along with the probability of each. In some cases, it is convenient to express the probability in terms of a formula.

Example 3: Consider the experiment of tossing two coins, and let X be the number of heads observed. Find the probability associated with each value of the random variable X, assuming that the two coins are fair.

Example 4: There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are $\{0,1,2,3,4\}$. Based on a model for the errors that is presented in the following section, probabilities for these values will be determined. Suppose that the probabilities are

$$P(X=0)=0.6561 \quad P(X=1)=0.2916 \quad P(X=2)=0.0486 \quad P(X=3)=0.0036 \quad P(X=4)=0.0001$$

The probability distribution of X is specified by the possible values along with the probability of each. A graphical description of the probability distribution of X is shown in Figure 1.

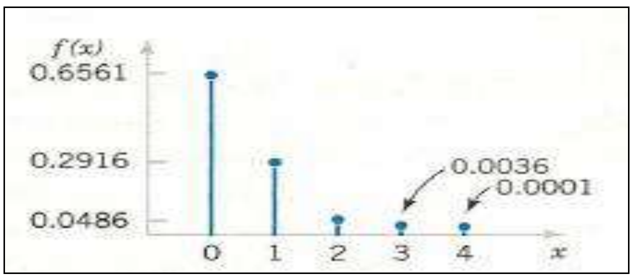


Figure 1. Probability distribution for bits in error.

<u>Definition</u>: For a discrete random variable X with possible values $x_1, x_2, \cdots, x_n$, a **probability mass function** is function such that

(1) $p(x_i) \geq 0$

(2) $\sum_{i=1}^{n} p(x_i) = 1$

(3) $p(x_i) = P(X = x_i)$

For example, in Example 4,
$p(0)=0.6561, p(1)=0.2916, p(2)=0.0486, p(3)=0.0036$ and $p(4)=0.0001$. Check that the sum of the probabilities in Example 4 is 1.

Example 5: Let the random variable X denote the number of semiconductor wafers that need to be analyzed in order to detect a large particle of contamination. Assume that the probability that a wafer contains a large particle is 0.01 and the wafers are independent. Determine the probability distribution of X. Let *p* denote a wafer in which a large particle is present, and let *a* denote a wafer in which it is absent. The sample space of the experiment is infinite, and it can be represented as all possible sequences that start with a string of *a*'s and end with *p*. That is,

$s = \{p, ap, aap, aaap, aaaap, aaaaap, and\ so\ forth\}$

EXERCISES 2

Exercise 2.1: The sample space of a random experiment is $\{a, bc, d, e, f\}$, and each outcome is equally likely. A random variable is defined as follows:

Outcome	a	b	c	d	e	f
x	0	0	1.5	1.5	2	3

Determine the probability mass function of X.

Exercise 2.2:

x	-2	-1	0	1	2
p(x)	1/8	2/8	2/8	2/8	1/8

a) $P(X \le 2) = ?$ **b)** $P(X > -2) = ?$ **c)** $P(-1 \le X \le 1) = ?$ **d)** $P(X \le -1\, or\ X = 2) = ?$

Exercise 2.3: $$p(x) = \frac{2x+1}{25}, \quad x = 0,1,2,3,4$$
$$= 0, \qquad otherwise$$

a) $P(X = 4) = ?$ **b)** $P(X \le 1) = ?$ **c)** $P(2 \le X < 4) = ?$ **d)** $P(X > -10) = ?$

Exercise 2.4: Marketing estimates that a new instrument for the analysis of soil samples will be very successful, moderately successful, or unsuccessful, with probabilities 0.3, 0.6, and 0.1, respectively. The yearly revenue associated with a very successful, moderately successful, or unsuccessful product is \$10 million, \$5 million, and \$1 million, respectively. Let the random variable X denote the yearly revenue of the product. Determine the probability mass function of X.

Exercise 2.5: An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent. Let the random variable X denote the number of parts that are correctly classified. Determine the probability mass function of X.

5.1.2. Cumulative Distribution Function of a Discrete Random Variable

<u>Definition:</u> The **cumulative distribution** function of a discrete random variable X, denoted as $F(x)$, is

$$F(x) = P(X \le x) = \sum_{x_i \le x} p(x_i)$$

For a discrete random variable X, $F(x)$ satisfies the following properties.

(1) $F(x) = P(X \le x) = \sum_{x_i \le x} p(x_i)$

(2) $0 \le F(x) \le 1$

(3) If $x \le y$, then $F(x) \le F(y)$

(4) Since $F(x)$ is a probability, the value of the distribution function is always between 0 and 1. Moreover,

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} P(X \le x) = 1$$
$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} P(X \le x) = 0$$

Example 6: Determine the probability mass function of X from the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.2 & -2 \le x < 0 \\ 0.7 & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

Figure 3 displays a plot of $F(x)$. From the plot, the only points that receive nonzero probability are -2, 0, and 2. The probability mass function at each point is the change in the cumulative distribution function at the point. Therefore,

$$p(-2) = 0.2 - 0 = 0.2 \qquad p(0) = 0.7 - 0.2 = 0.5 \qquad p(2) = 1.0 - 0.7 = 0.3$$

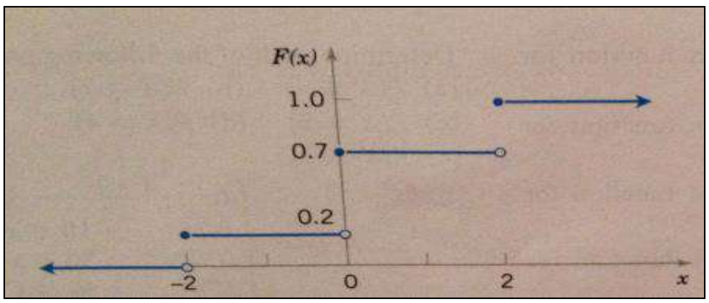


Figure 2. Cumulative distribution function for Example 6.

Example 7: Suppose that a day's production of 850 manufactured (üretilmiş) parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch (yığın). Let the random variable X equal the number of nonconforming parts in the sample. What is the cumulative distribution function of X?

EXERCISES 3

Exercise 3.1: Determine the cumulative distribution function for the random variable for

x	-2	-1	0	1	2
p(x)	1/8	2/8	2/8	2/8	1/8

also determine the following probabilities :

a) $P(X \le 1.25)$ **b)** $P(X \le 2.2)$ **c)** $P(-1.1 < X \le 1)$ **d)** $P(X > 0)$

Exercise 3.2: $F(x) = \begin{cases} 0 & x < 1 \\ 0.5 & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$

a) $P(X \le 3)$ **b)** $P(X \le 2)$ **c)** $P(1 \le X \le 2)$ **d)** $P(X > 2)$

Exercise 3.3: $F(x) = \begin{cases} 0 & x < -10 \\ 0.25 & -10 \le x < 30 \\ 0.75 & 30 \le x < 50 \\ 1 & x \ge 50 \end{cases}$

a) $P(X \le 50)$ **b)** $P(X \le 40)$ **c)** $P(40 \le X \le 60)$ **d)** $P(X < 0)$ **e)** $P(0 \le X \le 10)$
f) $P(-10 < X < 10) = 0.25 - 0.25 = 0$

5.1.3. Mean and Variance of a Discrete Random Variable

<u>Definition:</u> The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_x x\, p(x)$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 p(x) = \sum_x x^2 p(x) - \mu^2 = E(X^2) - \mu^2, \, where\ \mu = E(X)$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$

The ***variance of a random variable X*** is a measure of dispersion or scatter in the possible values for X. The variance of X uses weight $p(x)$ as the multiplier of each possible squared deviation $(x-\mu)^2$. Figure 5 illustrates probability distributions with equal means but different variances. Properties of summations and the definition of μ can be used to show the equality of the formulas for variance.

$$\begin{aligned} V(X) &= \sum_x (x-\mu)^2 p(x) = \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x) \\ &= \sum_x x^2 p(x) - 2\mu^2 + \mu^2 = \sum_x x^2 p(x) - \mu^2 = E(X^2) - \mu^2 = E(X^2) - \left[E(X)\right]^2 \end{aligned}$$

Example 8: In Example 4, there is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are $\{0,1,2,3,4\}$. Based on a model for the errors that is presented in the following section, probabilities for these values will be determined. Suppose that the probabilities are

$$\mathrm{P}(\mathrm{X}=0) = 0.6561 \quad \mathrm{P}(\mathrm{X}=1) = 0.2916 \quad \mathrm{P}(\mathrm{X}=2) = 0.0486 \quad \mathrm{P}(\mathrm{X}=3) = 0.0036 \quad \mathrm{P}(\mathrm{X}=4) = 0.0001$$

Although X never assumes the value 0.4, the weighted average of the possible values is 0.4. To calculate $V(X)$, a table is convenient.

x	$x-0.4$	$(x-0.4)^2$	$p(x)$	$p(x)(x-0.4)^2$
0	-0.4	0.16	0.6561	0.104976
1	0.6	0.36	0.2916	0.104976
2	1.6	2.56	0.0486	0.124416
3	2.6	6.76	0.0036	0.024336
4	3.6	12.96	0.001	0.001296

$V(X)=\sigma^2=\sum_{i=1}^{5} p(x_i)(x_i-0.4)^2=0.36$ (The alternative formula for variance could also be used to obtain the same result.)

Example 9: Two new product designs are to be compared on the basis of revenue (kazanç, hasılat) potential. Marketing feels that the revenue from design A can be predicted quite accurately to be \$3 million. The revenue potential of design B is more difficult to assess. Marketing concludes that there is a probability of 0.3 that the revenue from design B will be \$7 million, but there is a 0.7 probability that the revenue will be only \$2 million. Which design do you prefer?

Example 10: The number of messages sent per hour over a computer network has the following distribution:

x=number of messages	10	11	12	13	14	15
$p(x)$	0.08	0.15	0.30	0.20	0.20	0.07

Determine the mean and standard deviation of the number of messages sent per hour.

<u>Expected Value of a Function of a Discrete Random Variable</u>

If X is a discrete random variable with probability mass function $f(x)$

$$E[g(x)]=\sum_x x\,g(x)\,p(x) \qquad (1)$$

Example 11: In Example 8, X is the number of bits in error in the next four bits transmitted. What is the expected value of the square of the number of the number of bits in error? Now, $g(X)=X^2$. Therefore,

$$E[g(X)]=0^2\times 0.6561+1^2\times 0.2916+2^2\times 0.0486+3^2\times 0.0036+4^2\times 0.0001=0.52$$

In the previous example, the expected value of X^2 does not equal $E(X)$ squared (That means $E(X^2)\neq[E(X)]^2$. However, in the special case that $g(X)=aX+b$ for any constant a an b, $E[g(X)]=aE(X)+b$. This can be shown from the properties of sums in the definition in Eq. 1.

5.1.4. Expected Value of a Function of a Discrete Random Variable

Theorem 1. *Let X be a discrete random variable with probability mass function p(x) and g(X) be a realvalued function of X. Then the expected value of g(X) is given by*

$$E[g(X)]=\sum_x g(x)p(x)$$

5.1.5. Properties of Mathematical Expectation

5.1.5.1. Constants

Theorem 2. *Let X be a discrete random variable with probability function p(x) and c be a constant. Then* $E(c) = c$.

5.1.5.2. Constants Multiplied by Functions of Random Variables

Theorem 3. *Let X be a discrete random variable with probability function p(x), g(X) be a function of X, and let c be a constant. Then* $E\left[c\, g(X)\right] = c\, E\left[g(X)\right]$.

5.1.5.3. Sums of Functions of Random Variables

Theorem 4. *Let X be a discrete random variable with probability function p(x),* $g_1(X)$, $g_2(X)$, $g_3(X)$, ..., $g_k(X)$ *be k functions of X. Then*

$$E\left[g_1(X)+g_2(X)+g_3(X)+\cdots+g_k(X)\right]=E\left[g_1(X)\right]+E\left[g_2(X)\right]+E\left[g_3(X)\right]+\cdots+E\left[g_k(X)\right]$$

EXERCISE 4

Exercise 4.1: The random variable X has the following probability distribution:

x	2	3	5	8
	0.2	0.4	0.3	0.1

Determine the following:

a) $P(X \le 3)$ **b)** $P(X > 2.5)$ **c)** $P(2.7 < X < 5.1)$ **d)** $E(X)$ **e)** $V(X)$

Exercise 4.2: A department supervisor is considering purchasing a photocopy machine. One consideration is how often the machine will need repairs. Let *X* denote the number of repairs during a year. Based on past performance, the distribution of *X* is shown below:

Number of repairs, *x*	0	1	2	3
$p(x)$	0.2	0.3	0.4	0.1

a) What is the expected number of repairs during a year?

b) What is the variance of the number of repairs during a year?

Exercise 4.3:X discrete random variable has probability mass function as follows:

$$\begin{aligned} p(x) &= kx, \qquad x = 1,2,3,4,5 \\ &= 0, \qquad \textit{otherwise} \end{aligned}$$

a) Find constant k value.
b) Find the expected value of X.
c) Find the variance of X.
d) Find the cumulative distribution function of X.
e) Find probabilities: P(X>2), P(2<X≤4), P(2≤X<4).

EXERCISES FOR SECTION 1 AND SECTION 2

1) Find the number of ways in which one A, three B's, two C's, and one F can be distributed among seven students taking course in statistics.

   Solution: $\frac{7!}{3!2!} = \frac{4.5.6.7}{2} = 420$

2) If someone takes three shots at a target and we care only whether each shot is a hit or a miss, describe a suitable sample spaces that constitute event M that the person will miss the target three times in a row, and the elements of event N that the person will hit the target once and miss it twice. Find probabilities of M, N events if he hits the target with p (0<p<1) probability.

   Solution: S={ (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1) }

   M={ (0,0,0) } and N={ (1,0,0), (0,1,0), (0,0,1) }

   $P(M)= (1-p)^3$ $\quad$ $P(N)=3(1-p)^2p$

3) The bowl contains 3 Blue, 5 White, 4 Red idential balls.
   a) Construct the sample space when 3 balls drawn without replacement. Find the probability of an event that 2 Red, 1 White balls drawn.
   b) Construct the sample space when 3 balls drawn consecutively without replacement. Find the probability of an event that 2 Red, 1 White balls drawn.

   Solution:

   a) S {(3 Red Balls), (3 White Balls), (3 Blue Balls),
   (1 Blue and 2 White Balls), (1 Blue and 2 Red Balls), (1 Red and 2 White Balls), (1 Red and 2 Blue Balls), (1 White and 2 Red Balls),
   (1 White and 2 Blue Balls) (1 Blue 1 White 1 Red Balls) }

$$P(\{2Red, 1White\}) = \frac{4}{12}.\frac{3}{11}.\frac{5}{10}.\frac{3!}{2!} = \frac{3}{22}$$

or

$$P(\{2Red, 1White\}) = \frac{\binom{4}{2}\binom{5}{1}\binom{3}{0}}{\binom{12}{3}} = \frac{\frac{3\times4}{2!}(5)}{\frac{12!}{3!9!}} = (3)\frac{3\times4\times5}{10\times11\times12} = \frac{3}{22}$$

   b)

S={ (RRR), (RRB), (RRW), (RBR), (RWR), (BRR), (WRR), (RBW), (RWB)

(BBB), (BBR), (BBW), (BRB), (BWB), (RBB), (WBB), (BRW), (BWR)

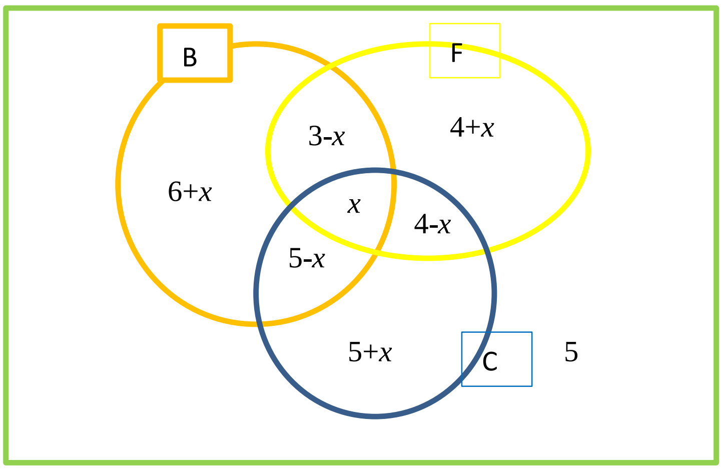
(WWW), (WWR), (WWB), (WRW), (WBW), (RWW), (BWW), (WRB), (WBR) }

$$P(\{2Red,1White\}) = P((RRW)\,or\ (RWR)\,or\ (WRR))$$
$$= \frac{4}{12}.\frac{3}{11}.\frac{5}{10} + \frac{4}{12}.\frac{5}{11}.\frac{3}{10} + \frac{5}{12}.\frac{4}{11}.\frac{3}{10}$$
$$= \frac{4}{12}.\frac{3}{11}.\frac{5}{10}.3 = \frac{3}{22}$$

4) If Ms.Brown buys one of the 35 houses advertised for sale in a Seattle newspaper, B is the event that the house has three or more baths, F is the event that it has a fireplace, C is the event that it costs more than \$ 100,000. Find the value of intersection of these three events. $B \cap F \cap C$.

B: the house has three or more baths	14
F: the house has a fireplace	11
C: the house cost's is more than \$ 100,000	14
$B \cap F$	3
$B \cap C$	5
$F \cap C$	4
$(B \cup F \cup C)'$	5

Solution:



$$s(B \cup F \cup C) = s(B) + s(F) + s(C) - s(B \cap F) - s(B \cap C) - s(F \cap C) + s(B \cap F \cap C)$$
$$30 = 14 + 11 + 14 - 3 - 5 - 4 + x \Rightarrow x = 30 - 27 = 3$$

5) A hat contains twenty white slips of paper numbered from 1 through 20, ten red slips of paper numbered from 1 through 10, forty yellow slips of paper numbered from 1 through 40, and ten blue slips of paper numbered 1 through 10. If these 80 slips of paper are thoroughly shuffled so that each slip has the same probability of being draw, find the probabilities of drawing a slip of paper that is

a) Blue or white;
b) Numbered 1,2,3,4, or 5;
c) Red or yellow and numbered 1, 2, 3, or 4;
d) Numbered 5, 15, 25, or 35;
e) White and numbered higher than 12 or yellow and numbered higher than 26.

Solution:

White 20, Red 10, Yellow 40, Blue 10 and total 80 slips of paper

a) $A = \{drawn\, a\ paper\, is\, "Blue"\}$ $B = \{drawn\, a\ paper\, is\, "White"\}$

$$P(A \cup B) = P(A) + P(B) = \frac{10}{80} + \frac{20}{80} = \frac{3}{8}$$

b) $C = \{numbered\, 1, 2, 3, 4\, or\, 5\}$

$$P(C) = P("numbered\, 1") + P("numbered\, 2") + P("numbered\, 3") + P("numbered\, 4") + P("numbered\, 5")$$

$$= \frac{4}{80} + \frac{4}{80} + \frac{4}{80} + \frac{4}{80} + \frac{4}{80} = \frac{20}{80} = \frac{1}{4}$$

c) $D = \{Red\ or\ yellow\ and\ numbered\ 1,\ 2,\ 3,\ or\ 4\}$

$$P(D) = P(\mathrm{Re}d \cap "1") + P(\mathrm{Re}d \cap "2") + P(\mathrm{Re}d \cap "3") + P(\mathrm{Re}d \cap "4") + P(Yellow \cap "1") + P(Yellow \cap "2") + P(Yellow \cap "3") + P(Yellow \cap "4")$$

$$= \frac{1}{80} + \frac{1}{80} + \frac{1}{80} + \frac{1}{80} + \frac{1}{80} + \frac{1}{80} + \frac{1}{80} + \frac{1}{80} = \frac{1}{10}$$

d)

$$P(E) = P("5") + P("15") + P("25") + P("35")$$

$$= \frac{4}{80} + \frac{2}{80} + \frac{1}{80} + \frac{1}{80} = \frac{1}{10}$$

e)

$$P(F) = P("White\, and\ higher\ than\, 12") + P("Yellow\, and\ higher\ than\, 26")$$

$$= \frac{8}{80} + \frac{14}{80} = \frac{22}{80} = \frac{11}{40}$$

EXERCISES FOR SECTION 2 AND SECTION 3

1. A digital scale is used that provides weights to the nearest gram. This scale rounds the weights to the nearest integer.

a) What is the sample space for this experiment?

Let A denote the event that a weight exceeds 11 grams, let B denote the event that a weight is less than or equal to 15 grams, and let C denote the event that a weight is greater than or equal to 8 grams and less than 12 grams.

b) $A \cup B$ **c)** $A \cap B$ **d)** A' **e)** $A \cup B \cup C$ **f)** $(A \cup C)'$ **g)** $A \cap B \cap C$ **h)** $B' \cap C$
i) $A \cup (B \cap C)$

Solution:

a) $S = \{0,1,2,3,\cdots\}$ **b)** $A \cup B = S = \{0,1,2,3,\cdots\}$ **c)** $A \cap B = \{12,13,14,15\}$
d) $A' = \{0,1,2,\cdots,11\}$ **e)** $A \cup B \cup C = S = \{0,1,2,3,\cdots\}$ **f)** $(A \cup C)' = \{0,1,2,\cdots,7\}$
g) $A \cap B \cap C = \{\varnothing\}$ **h)** $B' \cap C = \{\varnothing\}$ **i)** $A \cup (B \cap C) = \{8,9,10,\cdots\}$

2. The rise time of a reactor is measured in minutes (and fractions of minutes). Let the sample space be positive, real numbers. Define the events A and B as follows:

$$A = \{x | x < 72.5\} \quad and \quad B = \{x | x > 52.5\}$$

Describe each of the following events:

a) A' **b)** B' **c)** $A \cap B$ **d)** $A \cup B$

Solution:

a) $A' = \{x | x \geq 72.5\}$ **b)** $B' = \{x | x \leq 52.5\}$ **c)** $A \cap B = \{x | 52.5 < x < 72.5\}$
d) $A \cup B = \{x | x > 0\}$

3. Orders for a computer are summarized by the optional features that are requested as follows:

	proportion of orders
no optional feature	0.3
one optional feature	0.5
more than one optional feature	0.2

a) What is the probability that an order requests at least one optional feature?
b) What is the probability that an order does not request more than one optional feature?

Solution:

a) $A = \{an\ order\ requests at\ least\ one\ optional\ feature\} \ \Rightarrow P(A) = 0.5 + 0.2 = 0.7$

b) $B = \{an\ order\ does\ not\ request\ more\ than\ one\ optional\ feature\}$

$P(B) = 0.3 + 0.5 = 0.8$

4. Disk of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		shock resistance	
		high	low
scratch resistance	high	70	9
	low	16	5

Let A denote the event that a disk has high shock resistance, and let B denote the event that a disk has high scratch resistance. If a disk is selected at random, determine the following probabilities.

a) $P(A)$ **b)** $P(B)$ **c)** $P(A')$ **d)** $P(A \cap B)$ **e)** $P(A \cup B)$ **f)** $P(A' \cup B)$ **g)** $P(A|B)$ **h)** $P(B|A)$

Solution:

a) $P(A) = 86/100 = 0.86$ **b)** $P(B) = 79/100 = 0.79$ **c)** $P(A') = 1 - (86/100) = 0.14$

d) $P(A \cap B) = 70/100 = 0.70$

e) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.86 + 0.79 - 0.70 = 0.95$

f)
$$\begin{aligned} P(A' \cup B) &= P(A') + P(B) - P(A' \cap B) = P(A') + P(B) - [P(B) - P(A \cap B)] \\ &= P(A') + P(A \cap B) \\ &= 0.14 + 0.70 \\ &= 0.84 \end{aligned}$$

g) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.70}{0.79} = \dfrac{70}{79}$

h) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.70}{0.86} = \dfrac{70}{86}$

5. If $P(A) = 0.3$, $P(B) = 0.2$ and $P(A \cap B) = 0.1$, determine the following probabilities:

a) $P(A')$ **b)** $P(A \cup B)$ **c)** $P(A' \cap B)$ **d)** $P(A \cap B')$ **e)** $P[(A \cup B)]'$ **f)** $P(A' \cup B)$

Solution:

a) $P(A') = 1 - 0.3 = 0.7$ **b)** $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.2 - 0.1 = 0.4$

c) $P(A' \cap B) = P(B) - P(A \cap B) = 0.2 - 0.1 = 0.1$

d) $P(A \cap B') = P(A) - P(A \cap B) = 0.3 - 0.1 = 0.2$

e) $P\left[(A \cup B)\right]' = 1 - P(A \cup B) = 1 - 0.4 = 0.6$ (from part b)

f)
$$\begin{aligned} P(A' \cup B) &= P(A') + P(B) - P(A' \cap B) = P(A') + P(B) - \left[P(B) - P(A \cap B)\right] \\ &= P(A') + P(A \cap B) \\ &= 0.7 + 0.1 \\ &= 0.8 \end{aligned}$$

6. The analysis of shafts (mil, dingil, aks) for a compressor (sıkıştırıcı) is summarized by conformance to specifications:

		roundness conforms	
		yes	no
surface finish	yes	345	5
conforms	no	12	8

Solution:

a) If we know that a shaft conforms to roundness requirements, what is the probability that it conforms to surface finish requirements?

b) If we know that a shaft does not conform to roundness requirements, what is the probability that it conforms to surface finish requirements?

Solution:

$R = \{conforming\ to\ roundness\ requirements\}$

$S = \{conforming\ to\ surface\ finish\ requirements\}$

a) $P(S|R) = \dfrac{P(R \cap S)}{P(R)} = \dfrac{345}{357}$ **b)** $P(S|R') = \dfrac{P(R' \cap S)}{P(R')} = \dfrac{5}{13}$

7. Suppose that $P(A|B) = 0.2$, $P(A|B') = 0.3$, and $P(B) = 0.8$. What is $P(A)$?

Solution:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(B)P(A|B) \Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(B)P(A|B) \Rightarrow P(A \cap B) = 0.8 \times 0.2 = 0.16$$

$$P(A|B') = \frac{P(A\cap B')}{P(B')} \Rightarrow P(A\cap B') = P(B')P(A|B')$$
$$\Rightarrow P(A\cap B') = 0.2\times 0.3 = 0.06$$

$$P(A) = P(A\cap B) + P(A\cap B') = 0.16 + 0.06 = 0.22$$

8. Suppose 2% of cotton fabric rolls and 3% of nylon fabric rolls contain flaws (kusur). Of the rolls used by a manufacturer, 70 % are cotton and 30 % are nylon. What is the probability that a randomly selected roll used by the manufacturer contains flaws?

Solution:

$$C = \{cotton\ fabric\ rolls\} \quad N = \{nylon\ \ fabric\ rolls\}\ F = \{containing\ flaws\}$$

$$P(F|C) = 0.02 \quad P(F|N) = 0.03 \quad P(C) = 0.70 \quad P(N) = 0.30$$

$$P(F) = P(C)P(F|C) + P(N)P(F|N) = 0.70\times 0.02 + 0.30\times 0.03 = 0.023$$

9. If $P(A|B) = 0.4$, $P(B) = 0.8$ and $P(A) = 0.5$ are the events A and B independent?

Solution:

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow 0.4 = \frac{P(A\cap B)}{0.8} \Rightarrow P(A\cap B) = 0.32$$

If only if $P(A\cap B) = P(A).P(B)$ then A and B are independent events.

So that $0.32 \neq 0.8\times 0.5$ then A and B are not independent events.

10. Because a new medical procedure has been shown to be effective in the early detection of an illness, a medical screening of the population is proposed. The probability that the test correctly identifies someone with the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence (raslantı, etki alanı) of the illness in the general population is 0.0001. You take the test, and the result is positive. What is the probability that you have the illness?

Solution:

Let D denote the event that you have the illness, and let S the event that the test signals positive. The probability requested can be denoted as $P(D|S)$. The probability that the test correctly signals someone without the illness as negative is 0.95. Consequently, the probability of a positive test without the illness is $P(S|D') = 0.05$. From Bayes' Theorem,

$$P(D|S)=\frac{P(S|D)P(D)}{P(S|D)P(D)+P(S|D')P(D')}=\frac{0.99(0.0001)}{0.99(0.0001)+0.05(1-0.0001)}=\frac{1}{506}=0.002$$

Surprisingly, even though the test is effective, in the sense that $P(S|D)$ is high and $P(S|D')$ is low, because the incidence of the illness in the general population is low, the chances are quite small that you actually have the disease even if the test is positive.