INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #1

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Exercise 1.

1. Let $a \in \mathbb{R}$. The function $x \mapsto \mathrm{e}^{-at}$ is continuous on $[0, +\infty)$, hence the improper integral $I(a)$ is improper at $+\infty$. First observe that if $a = 0$ then

$$\lim_{t \to +\infty} \mathrm{e}^{-at} = 1 \neq 0,$$

hence $I(0)$ diverges. So, from now on, we assume that $a \neq 0$. Let $X > 0$. Then:

$$\int_0^X \mathrm{e}^{-at}\,\mathrm{d}t = \left[-\frac{\mathrm{e}^{-at}}{a}\right]_{t=0}^{t=X} = \frac{1}{a}\bigl(1 - \mathrm{e}^{-aX}\bigr).$$

Now,

$$\lim_{X \to +\infty} \mathrm{e}^{-aX} = \begin{cases} +\infty & \text{if } a < 0 \\ 0 & \text{if } a > 0, \end{cases}$$

hence $I(a)$ diverges when $a < 0$ and $I(a)$ converges if $a > 0$; in the case $a > 0$ we conclude that $I(a) = 1/a$.

Conclusion: $I(a)$ converges if and only if $a > 0$, and in this case, $I(a) = 1/a$.

2. Let $a \in \mathbb{R}$. The function $t \mapsto t^a/\bigl(1 + t^2\bigr)$ is continuous on $\mathbb{R}_+^*$ (and not $\mathbb{R}_+$ in the case $a < 0$). Hence the improper integral $J(a)$ is improper at 0 and at $+\infty$.

   - Convergence at 0: from the equivalent

   $$\frac{t^a}{1 + t^2} \underset{t \to 0^+}{\sim} t^a = \frac{1}{t^{-a}} > 0,$$

   and the well-known Riemann integrals at 0, the improper integral

   $$\int_0^1 \frac{\mathrm{d}t}{t^{-a}}$$

   converges at 0 if and only if $-a < 1$, i.e., $a > -1$. Hence, by the equivalent test, $J(a)$ converges at 0 if and only if $a > -1$.

   - Convergence at $+\infty$: from the equivalent

   $$\frac{t^a}{1 + t^2} \underset{t \to +\infty}{\sim} \frac{t^a}{t^2} = \frac{1}{t^{2-a}} > 0,$$

   and the Riemann integrals at $+\infty$, the improper integral

   $$\int_1^{+\infty} \frac{\mathrm{d}t}{t^{2-a}}$$

   converges at $+\infty$ if and only if $2 - a > 1$, i.e., if and only if $a < 1$. Hence, by the equivalent test, $J(a)$ converges at $+\infty$ if and only if $a < 1$.

Conclusion: the improper integral $J(a)$ converges if and only if $a \in (-1, 1)$.

Exercise 2.

1. Let $f \in E$. The function $t \mapsto \bigl|f(t)\bigr|/\sqrt{t}$ is continuous on $(0, 1]$, hence the improper integral

$$\int_0^1 \bigl|f(t)\bigr| \frac{\mathrm{d}t}{\sqrt{t}}$$

is improper at 0. Since $|f|$ is continuous on the closed and bounded interval $[0, 1]$, the Extreme Value Theorem guarantees the existence of $M \in \mathbb{R}$ such that

$$\forall t \in [0, 1],\ 0 \leq \bigl|f(t)\bigr| \leq M.$$

Hence,

$$\forall t \in (0,1],\ 0 \le \frac{|f(t)|}{\sqrt{t}} \le \frac{M}{\sqrt{t}}.$$

Now, the improper integral

$$\int_0^1 \frac{M}{\sqrt{t}}\,\mathrm{d}t$$

converges (by Riemann at 0 with $\alpha = 1/2 < 1$), hence, by the comparison test (observing that its application is valid since we're dealing with a non-negative function), the improper integral

$$\int_0^1 |f(t)|\frac{\mathrm{d}t}{\sqrt{t}}$$

converges.

2. - Let $f \in E$ such that $N(f) = 0$. Since the function $t \mapsto |f(t)|/\sqrt{t}$ is continuous and non-negative on $(0,1]$, we must have:

   $$\forall t \in (0,1],\ \frac{|f(t)|}{\sqrt{t}} = 0,$$

   hence

   $$\forall t \in (0,1],\ f(t) = 0,$$

   and since f is continuous at 0 we also have

   $$f(0) = \lim_{t\to 0^+} f(t) = 0,$$

   hence $f = 0_E$.
   - Let $f \in E$ and $\lambda \in \mathbb{R}$. Then:

   $$N(\lambda f) = \int_0^1 |\lambda f(t)|\frac{\mathrm{d}t}{\sqrt{t}} = |\lambda|\int_0^1 |f(t)|\frac{\mathrm{d}t}{\sqrt{t}} = |\lambda| N(f).$$

   - Let $f, g \in E$. Then:

   $$\forall t \in (0,1],\ |f(t) + g(t)| \le |f(t)| + |g(t)|$$

   hence

   $$\forall t \in (0,1],\ \frac{|f(t) + g(t)|}{\sqrt{t}} \le \frac{|f(t)|}{\sqrt{t}} + \frac{|g(t)|}{\sqrt{t}},$$

   hence

   $$N(f+g) = \int_0^1 |f(t)+g(t)|\,\mathrm{d}t \le \int_0^1 \left(\frac{|f(t)|}{\sqrt{t}} + \frac{|g(t)|}{\sqrt{t}}\right)\mathrm{d}t = \int_0^1 |f(t)|\frac{\mathrm{d}t}{\sqrt{t}} + \int_0^1 |g(t)|\frac{\mathrm{d}t}{\sqrt{t}} = N(f)+N(g).$$

3. a) The function $s \mapsto \mathrm{e}^{-s}/\sqrt{s}$ is continuous on $\mathbb{R}_+^*$. Now,

   $$\frac{\mathrm{e}^{-s}}{\sqrt{s}} \underset{s\to 0^+}{\sim} \frac{1}{\sqrt{s}} > 0,$$

   and we know (Riemann at 0 with $\alpha = 1/2 < 1$) that the improper integral

   $$\int_0^1 \frac{\mathrm{d}s}{\sqrt{s}}$$

   converges at 0 hence, by the equivalent test, our improper integral converges at 0. Moreover,

   $$\forall s \ge 1,\ 0 \le \frac{\mathrm{e}^{-s}}{\sqrt{s}} \le \mathrm{e}^{-s}$$

   and we know that the improper integral

   $$\int_1^{+\infty} \mathrm{e}^{-s}\,\mathrm{d}s$$

converges (see, e.g., Exercise 1, Question 1) hence, by the comparison test (which is valid since we're dealing with non-negative functions), the improper integral

$$\int_1^{+\infty} \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{\sqrt{s}}$$

converges at $+\infty$. Hence the improper integral

$$\int_0^{+\infty} \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{\sqrt{s}}$$

converges.

b) Let $n \in \mathbb{N}^*$. Then

$$N(f_n - 0_E) = \int_0^1 \bigl|f_n(t)\bigr|\,\frac{\mathrm{d}t}{\sqrt{t}} = \int_0^1 \mathrm{e}^{-nt}\,\frac{\mathrm{d}t}{\sqrt{n}} = \int_0^n \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{n\sqrt{s/n}} = \frac{1}{\sqrt{n}}\int_0^n \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{\sqrt{s}}.$$

Now, since the function

$$s \longmapsto \frac{\mathrm{e}^{-s}}{\sqrt{s}}$$

is non-negative and since the improper integral

$$\int_0^{+\infty} \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{\sqrt{s}}$$

converges, we have:

$$\int_0^n \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{\sqrt{s}} \le \int_0^{+\infty} \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{\sqrt{s}}.$$

Hence,

$$N(f_n - 0_E) \le \frac{1}{\sqrt{n}}\int_0^1 \mathrm{e}^{-s}\,\frac{\mathrm{d}s}{\sqrt{s}} \xrightarrow[n\to+\infty]{} 0.$$

Conclusion: the sequence $(f_n)_{n\in\mathbb{N}^*}$ converges to 0_E for the norm N.

c) For $n \in \mathbb{N}^*$,

$$\bigl|f_n(0)\bigr| = 1 \le \|f_n\|_\infty = \|f_n - 0_E\|_\infty,$$

hence

$$\|f_n - 0_E\|_\infty \underset{n\to+\infty}{\not\longrightarrow} 0,$$

hence the sequence $(f_n)_{n\in\mathbb{N}^*}$ doesn't converge to 0_E for the norm N.

4. The norms N and $\|\cdot\|_\infty$ are not equivalent since the sequence $(f_n)_{n\in\mathbb{N}^*}$ converges to 0_E for N but not for $\|\cdot\|_\infty$.

Exercise 3. Let $A_0 \in E$ and $H \in E$. Then:

$$f(A_0 + H) = (A_0 + H)^2 = (A_0 + H)(A_0 + H) = A_0^2 + A_0H + HA_0 + H^2 = f(A_0) \;+\; (A_0H + HA_0) \;+\; H^2.$$

Clearly, the mapping

$$\begin{aligned} E &\longrightarrow E \\ H &\longmapsto A_0H + HA_0 \end{aligned}$$

is linear (and continuous since we're in a finite-dimensional vector space). We identify our remainder as being the term H^2. We need to choose a norm on E: we'll use the following one:

$$\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\| = |a| + |b| + |c| + |d|$$

(which is directly obtained from the 1-norm of $\mathbb{R}^4$). Then, for $H \in E$, say $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$,

$$H^2 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} h_{11}^2 + h_{12}h_{21} & h_{11}h_{12} + h_{12}h_{22} \\ h_{21}h_{11} + h_{22}h_{21} & h_{21}h_{12} + h_{22}^2 \end{pmatrix},$$

hence

$$\begin{aligned}\left\|H^2\right\| &= \left|h_{11}^2 + h_{12}h_{21}\right| + \left|h_{11}h_{12} + h_{12}h_{22}\right| + \left|h_{21}h_{11} + h_{22}h_{21}\right| + \left|h_{21}h_{12} + h_{22}^2\right| \\ &\leq \left|h_{11}^2\right| + \left|h_{12}h_{21}\right| + \left|h_{11}h_{12}\right| + \left|h_{12}h_{22}\right| + \left|h_{21}h_{11}\right| + \left|h_{22}h_{21}\right| + \left|h_{21}h_{12}\right| + \left|h_{22}^2\right| \\ &\leq 8\|H\|^2.\end{aligned}$$

Hence, for $H \in E \setminus \{0_E\}$,

$$\frac{\left\|H^2\right\|}{\|H\|} \leq 8\|H\| \underset{\|H\|\to 0}{\longrightarrow} 0.$$

This shows that f is differentiable at A_0 and that

$$\begin{array}{rrcl} D_{A_0}f : & E & \longrightarrow & E \\ & H & \longmapsto & A_0H + HA_0. \end{array}$$

Exercise 4.

1. Let $f \in F$ and $g \in G$. We know that
$$\forall t \in \mathbb{R}_+,\ 0 \leq \left|f(t)\right| \leq \|f\|_\infty,$$
hence
$$\forall t \in \mathbb{R}_+,\ 0 \leq \left|f(t)g(t)\right| \leq \|f\|_\infty g(t),$$
and since $g \in G$, we can apply the Comparison Test to conclude that the improper integral
$$\int_0^{+\infty} \left|f(t)g(t)\right| \mathrm{d}t$$
converges, and that
$$\int_0^{+\infty} \left|f(t)(g(t)\right| \mathrm{d}t \leq \|f\|_\infty \int_0^{+\infty} \left|g(t)\right| \mathrm{d}t \leq \|f\|_\infty \|g\|_1.$$
2. Clearly, the mapping φ_{g_0} is linear, hence we only need to check continuity at 0_F: let $f \in F$. Then,
$$\left|\varphi_{g_0}(f)\right| = \left|\int_0^{+\infty} f(t)g_0(t)\,\mathrm{d}t\right| \leq \int_0^{+\infty} \left|f(t)g_0(t)\right| \mathrm{d}t \leq \|g_0\|_1 \|f\|_\infty \underset{\|f\|_\infty \to 0}{\longrightarrow} 0,$$
hence
$$\lim_{\|f-0_F\|_\infty \to 0} \left|\varphi_{g_0}(f) - \varphi_{g_0}(0_F)\right| = 0,$$
hence φ_{g_0} is continuous at 0_F.
3. Similarly, we recognize that ψ_{f_0} is linear, hence we only check that ψ_{f_0} is continuous at 0_G: let $g \in G$, then:
$$\left|\psi_{f_0}(f)\right| = \left|\int_0^{+\infty} f_0(t)g(t)\,\mathrm{d}t\right| \leq \int_0^{+\infty} \left|f_0(t)g(t)\right| \mathrm{d}t \leq \|f_0\|_\infty \|g\|_1,$$
hence ψ_{f_0} is continuous at 0_G.
4. a) Let $n \in \mathbb{N}$. Then
$$\|f_n - \cos\|_\infty = \sup_{t \in \mathbb{R}_+} \frac{1}{t^2 + 1 + n} = \frac{1}{n+1} \underset{n \to +\infty}{\longrightarrow} 0,$$
hence the sequence of functions converges to cos in $(F, \|\cdot\|_\infty)$ (here, by cos, we mean the restriction of the cosine function to $\mathbb{R}_+$, and noticing that $\cos \in F$).

   b) Let g_0 be the function defined by
$$\begin{array}{rrcl} g_0 : & \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \mathrm{e}^{-t}. \end{array}$$
Clearly, the function g_0 belongs to G. We notice that the limit we want to compute is:
$$\lim_{n \to +\infty} \varphi_{g_0}(f_n).$$
Since φ_{g_0} is continuous, and since the sequence $(f_n)_{n \in \mathbb{N}}$ converges to cos in $(F, \|\cdot\|_\infty)$ we have:
$$\lim_{n \to +\infty} \varphi_{g_0}(f_n) = \varphi_{g_0}(\cos) = \int_0^{+\infty} \cos(t)\mathrm{e}^{-t}\,\mathrm{d}t = \frac{1}{2}.$$
(The details of this last integral are left to the reader: hint, use complex numbers).