INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #1 — 1h30

October 17, 2022

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 1 and 3 are common with PCC2.

Exercise 1 (*5.5 points*)**.** Study the nature of the following improper integrals:

$$I = \int_1^{+\infty} \frac{t^2}{(t-1)\left(1+t^5\right)}\,\mathrm{d}t \qquad J = \int_0^{+\infty} \frac{t^2}{\left(1+t^4\right)\ln(2+t)}\,\mathrm{d}t \qquad K = \int_0^{+\infty} \sin\left(\frac{1}{t^2}\right)\mathrm{d}t.$$

Exercise 2 (*6 points*)**.** The three questions of this exercise are independent of each other.

1. Let $\alpha \in \mathbb{R}$ and define:
$$\begin{array}{rccc} N: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}_+ \\ & (x,y) & \longmapsto & \max\left\{|x+\alpha y|, |y|\right\} \end{array}$$
Show that N is a norm on $\mathbb{R}^2$, and plot its unit ball in the three different cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.
2. Let $E = C\left([0,\pi/2]\right)$, let $\|\cdot\|_1$ be the 1-norm on E.
   a) Recall the definition of $\|\cdot\|_1$.
   b) Define the following elements of E:
$$\begin{array}{rccc} f: & [0,\pi/2] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x \end{array} \qquad \text{and} \qquad \begin{array}{rccc} s: & [0,\pi/2] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sin(x) \end{array}$$
   Compute the distance between f and s with respect to $\|\cdot\|_1$.
3. Let E be a vector space and let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on E such that
$$\forall u \in E,\ \|u\| \leq 2\|u\|'$$
For $r > 0$ we define:
$$\overline{B_r} = \left\{u \in E \,\middle|\, \|u\| \leq r\right\} \qquad \text{and} \qquad \overline{B'_r} = \left\{u \in E \,\middle|\, \|u\|' \leq r\right\}.$$
(these are closed balls centered at 0_E of radius r). One of the following four relations is correct. Which one is it? (and prove it is correct):
$$\begin{gathered} \forall r > 0,\ \overline{B_r} \subset \overline{B'_{2r}} \\ \forall r > 0,\ \overline{B_{2r}} \subset \overline{B'_r} \\ \forall r > 0,\ \overline{B'_r} \subset \overline{B_{2r}} \\ \forall r > 0,\ \overline{B'_{2r}} \subset \overline{B_r} \end{gathered}$$

Exercise 3 (*8.5 points*)**.** The results of Part I can be used in Part II.

Part I – *Preliminary questions*

1. Let $\lambda \in \mathbb{R}_+^*$. Show that

$$I = \int_0^{+\infty} x\mathrm{e}^{-\lambda x}\,\mathrm{d}x$$

is convergent and determine its value.

2. a) Show that the improper integral

$$G = \int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$$

is convergent.

You're given that $G = \dfrac{\sqrt{\pi}}{2}$.

b) Let $\sigma \in \mathbb{R}_+^*$. Determine the value of

$$J = \int_0^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x$$

c) Use an integration by parts to show that

$$\int_0^{+\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x = \frac{\sigma^3\sqrt{2\pi}}{2}.$$

d) Deduce, without any further computations (but justify your answer) the value of

$$K = \int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x.$$

Part II – *Applications*

1. Let $\lambda \in \mathbb{R}_+^*$ and define

$$\begin{array}{rccc} \phi : & \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \lambda\mathrm{e}^{-\lambda x} \end{array}$$

Show that the improper integral

$$H_1 = -\int_0^{+\infty} \phi(x)\ln\big(\phi(x)\big)\,\mathrm{d}x$$

is convergent, and compute its value.

2. Let $\sigma \in \mathbb{R}_+^*$ and define

$$\begin{array}{rccc} \phi : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\dfrac{x^2}{2\sigma^2}\right) \end{array}$$

Show that the improper integral

$$H_2 = -\int_{-\infty}^{+\infty} \phi(x)\ln\big(\phi(x)\big)\,\mathrm{d}x$$

is convergent, and compute its value.

Note. The numbers H computed correspond to the differential entropy of the random variables with density ϕ, a notion used in probability and information theory.