Physics test - Electromagnetism
June 14th 2021
correction

Exercise 1: Maps of electric field ($\sim$ 6 pts)

Identifiying sysmmetry/Antisymmetry elements with respect to the charge distribution:

- (yOz) is a symmetry plane (and/or) (Oy) is a symmetry axis — 0.5 Pt
  - $\vec{E}$ belongs to (Oy) along (Oy) — 0.5 Pt
  - field lines are symmetric with respect to (Oy) — 0.5 Pt
- (xOz) is an antisymmetry plane (and/or) (Ox) is an antisymmetry axis — 0.5 Pt
  - $\vec{E}$ is perpendicular to (Ox) along (Ox) — 0.5 Pt
  - field lines are antisymmetric with respect to (Ox) — 0.5 Pt

There are no other symmetry/antisymmetry elements in the charge distribution.

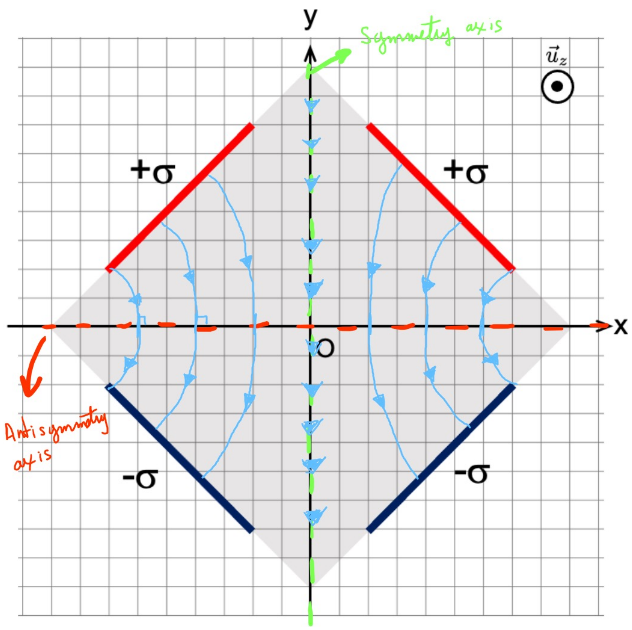

Figure 1

Drawing of the field lines (see Figure 1) including:

- Field line along (Oy) — 0.5 Pt
- 2 or 3 field lines at each side of (yOz) plane — 1.5 Pt
- Field line going from positive to negative charges — 0.5 Pt
  - clear indication of the orientation of the field lines

The electric field at O is directed along $-\vec{u}_y$. $\vec{E}(O)$ is definititely not nil since the neat contribution of the positive charges points in the same direction as the one from the negative charges (statement on superposition theorem at O). It could be eventually shown (not expected here) that the intensity of $\vec{E}$ is minimal at O. — 0.5 Pt

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Total Ex. 1 6 Pts

Exercise 2: On magnetic fluxes (~ 6 pts)

1. Determine the current intensity i_1 running in the loop, justifying precisely its orientation.

The current intensity running in the loop can be deduced by computing the e.m.f. induced in the loop. Since there is no predefined orientation of the surface of the loop, let us choose $\vec{n_1} = \vec{u}_z$ as the normal unit vector. 0.5 Pt orientation

The loop is subjected to Neumann induction due to the increasing intensity of $\vec{B}$ field. The induced e.m.f. (e_1) can be computed from the magnetic flux (ϕ_1) using Lenz's law:

$$e_1 = -\frac{d\phi_1}{dt}$$

0.5 Pt Lenz's law

with:

$$\phi_1 = \iint_{loop\,1} \vec{B_1}(M,t) \cdot \vec{n}_1 \, dS$$

Given $\vec{B_1}$ is uniform, directed along $\vec{u}_z$ and restricted to the cross section of the solenoid, the magnetic flux writes:

$$\phi_1 = \iint_{solenoid\,1} \vec{B_1}(M,t) \cdot \vec{n}_1 \, dS = ||\overrightarrow{B_1}||(t) \cdot \pi \frac{d_1^2}{4} = \frac{\pi\, At\, d_1^2}{4}$$

Therefore:

$$e_1 = -\frac{A\,\pi\, d_1^2}{4}$$

0.5 Pt finding e_1

The orientation of e_1 (and I_1) being determined by $\vec{n}_1$, the equivalent circuit is the following: 0.5 Pt orientation (figure or statement)

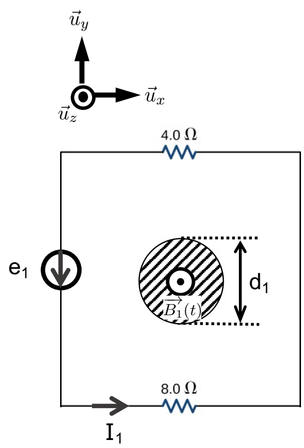

Figure 2

I_1 can be then deduced using Kirchhoff's circuit law: 0.5 Pt KCL

$$e_1 = (4\,\Omega + 8\,\Omega)\; I_1$$

N.A.: $e_1 \simeq -0.4\,V$(optional)

hence: 0.5 Pt I_1

$$I_1 = -\frac{A\,\pi\, d_1^2}{48\,\Omega} \simeq -33\,\text{mA}$$

TOTAL Q1 3 Pts

2. Determine the different current intensities running through the resistances, justifiying their orientation.

Adding a second loop and a solenoid adds another emf in the circuit as depicted in Fig.3 0.5 Pt Figure or eq.

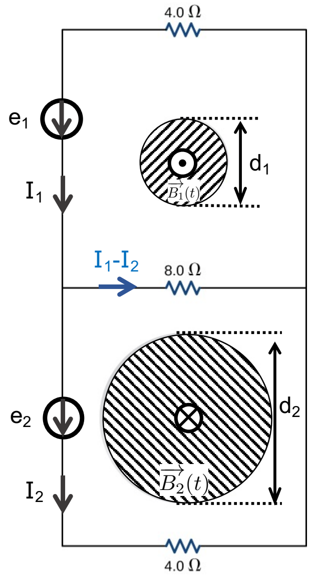

Figure 3

Keeping the same normal unit vector $\vec{n}_1 = \vec{u}_z$ (or $\vec{n}_2 = -\vec{u}_z$ w/ a consistent orientation of corresponding emfs/currents...), e_2 is to be computed with the same methodology than e_1. 0.5 Pt finding e_2

$$e_2 = -\frac{d\phi_2}{dt}$$

given the orientation of $\vec{B}_2$, the magnetic flux is negative:

$$\phi_2 = \iint_{solenoid\,2} \vec{B}_2(M,t) \cdot \vec{n}_1\, dS = -\frac{\pi\, At\, d_2^2}{4}$$

then:

$$e_2 = +\frac{A\,\pi\, d_2^2}{4}$$

N.A.: $e_2 \simeq +1.6\,V$ (optional)

I_1 and I_2 can be then found using Kirchhoff's circuit laws. Note than the current in the branch shared by both loops can be determined by Kirchhoff's current law (see Fig.3). 1 Pt KCL + eq.

in loop 1 : $e_1 = 4\,\Omega\, I_1 + 8\,\Omega\,(I_1 - I_2)$
in loop 2 : $e_2 = 4\,\Omega\, I_2 - 8\,\Omega\,(I_1 - I_2)$

Solving this linear system brings:

$$\begin{cases} I_1 &= \frac{3\,e_1 + 2\,e_2}{20\,\Omega} \\ I_2 &= \frac{2\,e_1 + 3,e_2}{20\,\Omega} \end{cases}$$

Finally: 1 Pt finding I_1,I_2

$$\begin{cases} I_1 & \simeq +98\,\text{mA} \\ I_2 & \simeq +196\,\text{mA} \end{cases}$$

TOTAL Q2 3 Pts

Total Ex. 2 6 Pts

Exercise 3: Barlow's wheel (~ 12 pts)

Qualitative interpretation of the physical phenomena

1. Within few sentences, could you explain the observed phenomena? For instance you might identify the different successive physical phenomenon occuring on the wheel.

As K is closed, a current $i(t)$ runs in the circuit including the 4 spokes.
Given the presence of a $\vec{B}$ field, the spokes will be subjected to Laplace force. Laplace force will drive the rotation of the wheel under a constant acceleration. 0.5 Pt Laplace

The wheel, including the spokes, can be considered as a moving circuit in a $\vec{B}$ field and will be then subjected to motional induction.

0.5 Pt motional induction

Given induction, an emf will be induced along the spokes, reducing the current intensity and therefore the intensity of Laplace Force (induction opposes the change that induced it). 0.5 Pt emf+i decrea

Total Q1 1.5 Pts

Mechanical equation

2. Let's study a small element of a spoke (*e.g.* OA), of length dr, located at a distance r from the O. Calculate $\overrightarrow{dF}$, the Laplace force exerted on this element in terms of $i(t)$, N, dr, B_0 and one of the unit vector of the local frame.

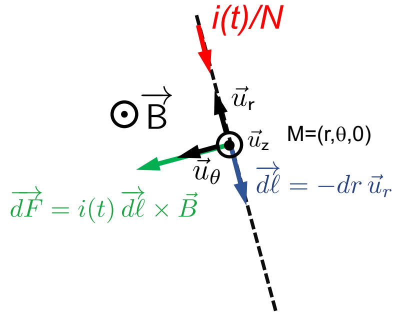

Figure 4: Computing $\vec{dF}$

The elementary Laplace force exerted on a element of spoke at $M = (r, \theta, 0)$ writes: 0.5 Pt correct $\overrightarrow{d\ell}$

$$\overrightarrow{dF} = \frac{i(t)}{N}\,\overrightarrow{d\ell} \times \vec{B}$$

where $\overrightarrow{d\ell}$ should be directed as $i(t)$ (see Figure 4) : $\overrightarrow{d\ell} = -dr\,\vec{u}_r$.
With $\vec{B}$ along $\vec{u}_z$, computing the cross-product (or using the right hand rule) brings: 1 Pt $\overrightarrow{dF}$ expr.

$$\boxed{\overrightarrow{dF} = \frac{i(t)}{N}\,B_0\,dr\,\vec{u}_\theta}$$

-0.5 Pt wrong sign

Total Q2 1.5 Pts

3. Let us find the moment exerted by Laplace for on this spoke.

a) Calculate the elementary moment of $\overrightarrow{dF}$ about (Oz) , namely $d\Gamma_z$, in terms of $i(t)$, N, r, B_0 and dr.

Given the definition of the moment of a force about an axis:

$$d\Gamma_z = \left(\overrightarrow{OM} \times \overrightarrow{dF}\right) \cdot \vec{u}_z$$

with $\overrightarrow{OM} = r\,\vec{u_r}$: 1 Pt $d\Gamma_z$ expr.

$$\boxed{d\Gamma_z = \frac{i(t)}{N} B_0\, r dr}$$

b) Calculate the moment Γ_z due to Laplace force exerted on the whole spoke.

The moment exerted on the whole spoke can be determined by integrating $d\Gamma_z$ over the spoke: 1 Pt Γ_z expr.

$$\Gamma_z = \int_0^a d\Gamma_z = \frac{i(t)}{N} B_0 \int_0^a r\,dr$$

$$\boxed{\Gamma_z = \frac{1}{2}\frac{i(t)}{N} B_0\, a^2}$$

Total Q3 2 Pts

4. Deduce from the previous questions the *total moment* $\Gamma_z^{\mathtt{tot}}$ about (Oz) axis due to the Laplace force of the N spokes. Show that $\Gamma_z^{\mathtt{tot}} = K \cdot i$ where K is a constant to be determined.

Since each spoke is subjected to the same moment: 0.25 Pt

$$\Gamma_z^{\mathtt{tot}} = N \cdot \Gamma_z = \frac{1}{2} i(t)\, B_0\, a^2$$

Finally: 0.25 Pt

$$\boxed{\Gamma_z^{\mathtt{tot}} = K \cdot i(t) \quad \text{with} \quad K = \frac{1}{2} B_0\, a^2}$$

Total Q4 0.5 Pt

5. Establish (but do not solve) the differential equation ruling $\omega(t)$.

Considering the wheel of moment of inertia J is only subjected to the total moment of Laplace force and using Using Newton's second law for rotation: 0.5 Pt +0.5 Pt

$$\boxed{J\dot{\omega} = K \cdot i(t)}$$

Total Q5 1 Pt

Electrical equation

6. Which type of induction phenomenon are we facing? Find the expression of e, the induced electromotive force in each spoke, by computing the circulation of a vector (*i.e.* doing a line integral). Show that e can be written $e = \pm K \cdot \omega$, the $\pm$ sign depending on your orientation choice you will have to detail.

Given each spokes, crossed by a current $i(t)$, can be considered as moving circuit in a static $\vec{B}$ field, we are clearly facing motional induction. The induced emf can be computed through the circulation of the

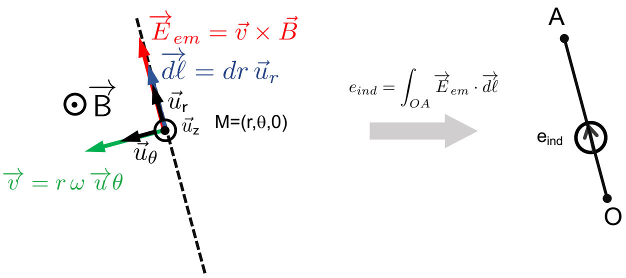

Figure 5: Computing the induced emf

electromotive field $\overrightarrow{E}_{em}$ along the spoke (see Figure 5).

Let $M = (r, \theta, 0)$ be a point along a spoke, using the definition of the electromotive field:

$$\overrightarrow{E}_{em}(M) = \vec{v}(M) \times \vec{B}(M)$$

where $\vec{M}$ is the velocity of M. For a rotation about (Oz) at an angular velocity ω: 0.5 Pt $\vec{v}$ expr.

$$\vec{v}(M) = r\,\omega \vec{u_\theta}$$

hence: 0.5 Pt $\overrightarrow{E}_{em}$ expr.

$$\overrightarrow{E}_{em}(M) = r\,\omega\,B_0\,\vec{u_r}$$

Finally the induced emf can be found from the line integral of $\overrightarrow{E}_{em}$ along the spoke. Choosing $\overrightarrow{d\ell}$ along $\vec{u_r}$:

$$e = \int_0^a \overrightarrow{E}_{em}(M) \cdot \overrightarrow{d\ell} = \int_0^a \overrightarrow{E}_{em}(M) \cdot \vec{u_r}\,dr = \int_0^a r\,\omega\,B_0\,dr$$

leading to: 0.5 Pt e expr.

$$\boxed{e = \frac{1}{2} a^2\,B_0\,\omega = K \cdot \omega}$$

Given $\overrightarrow{d\ell}$ points from O to A, e is also directed from O to A

Note: Choosing $\overrightarrow{d\ell}$ along $-\vec{u_r}$ would have given $e_{ind} = -K \cdot \omega$ with e pointing from A to O...

Total Q6 1.5 Pts

7. Draw the equivalent electrical scheme of the circuit, accounting for all resistances in the circuit and the induced electromotive forces. Deduce the electrical equation associated to this circuit in terms of $i(t)$, E, e, R and R_0. Finally, infer the expression of $i(t)$ in terms of E, R, R_0, K and $\omega(t)$. Provide then a new expression of for the differential equation ruling $\omega(t)$.

Given the each spoke is equivalent to a resistor R_0 with an emf $K\omega$ and that the 4 spokes are in parallel, the equivalent circuit is the following: 1 Pt scheme

Using Kirchhoff voltage law in the loop comprising the voltage source and one of the spokes:

$$E - R\,i = K\omega + R_0 \frac{i}{4}$$

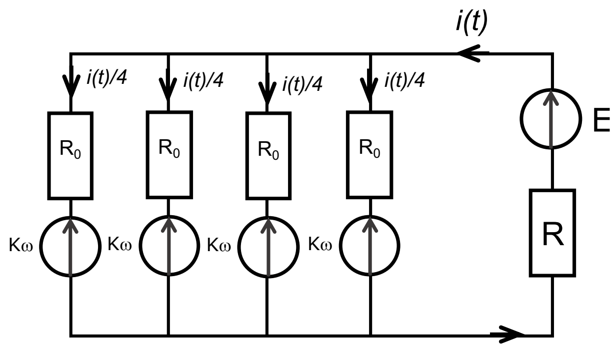

Figure 6

leading to:

$$\boxed{i(t) = \frac{E - K\,\omega}{R + R_0/4}}$$

Injecting this expression of $i(t)$ in the differential equation established in Q5 and upon simplification: 1 Pt diff. Eq.

$$\boxed{J\dot{\omega} = -\frac{K^2}{R + R_0/4}\omega + \frac{KE}{R + R_0/4}}$$

TOTAL Q7 2 Pts

Solving the coupled equations

Solving the mechanical equation would lead to the following expression for the angular velocity:

$$\omega(t) = \omega_0\,(1 - e^{-\frac{t}{\tau}})$$

8. By exploiting the previous question, find the expression of ω_0, the angular velocity in steady-state in terms of E and K.
BONUS: Determine τ from Q7 (you may for instance use a dimensional analysis or exploit the expression provided for $\omega(t)$...). What is τ worth as $R_0 \ll R$?

Exploiting the previous expression in steady-state ($\omega = \omega_0 \Rightarrow \dot{\omega} = 0$): 0.5 Pts

$$0 = -\frac{K^2}{R + R_0/4}\omega_0 + \frac{KE}{R + R_0/4}$$

therefore: 0.5 Pts

$$\boxed{\omega_0 = \frac{E}{K}}$$

BONUS: 1) The differential equation can be rewritten as follows:

$$\dot{\omega} + \frac{K^2}{J\,(R + R_0/4)}\omega = \frac{KE}{J\,(R + R_0/4)}$$

Such an equation is a first order differential equation, with constant coefficients and second member. The homogeneous equation writes:

$$\dot{\omega_h} + \frac{K^2}{J\,(R + R_0/4)}\omega_h = 0$$

whith a general solution writing:

$$\omega_h(t) = A\, e^{-t/\tau} \quad \text{with} \quad \tau = \frac{J\,(R + R_0/4)}{K^2}$$

τ can be inferred considering that $\frac{K^2}{J(R+R_0/4)}\omega_h$ should have the same dimension as $\dot{\omega} = \frac{d\omega}{dt}$......

2) τ can also be retrieved by injecting the provided solution $\omega(t) = \omega_0\,(1 - e^{-\frac{t}{\tau}})$ into the differential equation:

$$J\frac{\omega_0}{\tau}e^{-t/\tau} = -\frac{K^2}{R + R_0/4}\omega_0(1 - e^{-t/\tau}) + \frac{KE}{R + R_0/4}$$

given $w_0 = \frac{E}{K}$:

$$\frac{JE}{K\tau}e^{-t/\tau} = -\frac{KE}{R + R_0/4}(1 - e^{-t/\tau}) + \frac{KE}{R + R_0/4}$$

$$\frac{JE}{K\tau}e^{-t/\tau} = \frac{KE}{R + R_0/4}e^{-t/\tau}$$

$$\tau = \frac{J\,(R + R_0/4)}{K^2}$$

if $R_0 \ll R$ then:

$$\tau \overset{R_0 \ll R}{\simeq} \frac{JR}{K^2}$$

Total Q8 1 Pt
Bonus 1 Pt

9. Show that $i(t)$ can be expressed as $i(t) = i_0\, e^{-t/\tau'}$ and determine i_0 and τ' in terms of E, R and K and J; note that we will assume that $R_0 \ll R$. Plot the qualitative graph of $i(t)$. Comment on the value of the intensity once the steady state has been reached.

Given $i(t) = \frac{E - K\,\omega}{R+R_0/4}$ and $\omega(t) = \omega_0\,(1 - e^{-\frac{t}{\tau}})$:

$$i(t) = i_0\, e^{-t/\tau'}$$

with:

$$\begin{cases} i_0 & = \frac{E}{R+R_0/4} \simeq \frac{E}{R} \\ \tau' & = \tau = \frac{JR}{K^2} \end{cases}$$

In steady state we find $i(t \to \infty) \to 0$; this can be understood as the steady-state value of the induced emf is $e(t \to \infty) = K\omega_0 = K\frac{E}{R} = E$ opposes the emf of the voltage source.

Total Q9 1 Pt

Total Ex. 3 12 Pts
+Bonus 1 Pt

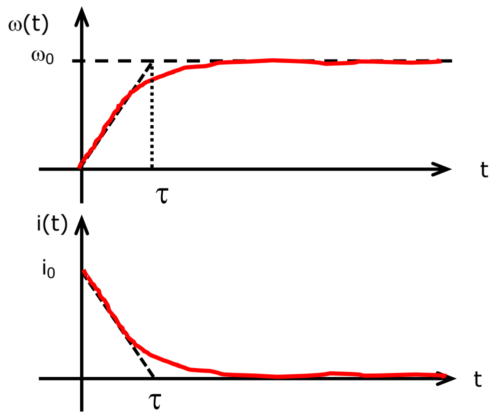

Figure 7