INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of DS #1

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Exercise 1.

1. a) Let $t \in \mathbb{R}_+$. Then for all $n \in \mathbb{N}$ such that $n \geq t$ one has

$$f_n(t) = \frac{1}{1+t} \underset{n\to+\infty}{\longrightarrow} \frac{1}{1+t}.$$

Hence the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}_+$ to the function f defined by

$$f : \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{1}{1+t}. \end{array}$$

Clearly, the function f is continuous and bounded, hence $f \in E$.

b) Let $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

$$\bigl|f_n(t) - f(t)\bigr| = \begin{cases} 0 & \text{if } t \leq n \\ \dfrac{1}{1+t} & \text{if } t > n, \end{cases}$$

hence

$$\|f_n - f\|_\infty = \frac{1}{1+n} \underset{n\to+\infty}{\longrightarrow} 0.$$

c) For all $n \in \mathbb{N}$, the function f_n belongs to F since, for $X > n$,

$$\int_0^X f_n(t)\,\mathrm{d}t = \int_0^n f_n(t)\,\mathrm{d}t \underset{X\to+\infty}{\longrightarrow} \int_0^n f_n(t)\,\mathrm{d}t \in \mathbb{R}.$$

Now, $f \notin F$ since, for $X > 0$,

$$\int_0^X f(t)\,\mathrm{d}t = \int_0^X \frac{1}{1+t}\,\mathrm{d}t = \ln(1+X) \underset{X\to+\infty}{\longrightarrow} +\infty.$$

Hence, $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of F that converges to $f \notin F$ for the norm $\|\cdot\|_\infty$, and we can conclude that F is not closed in $\bigl(E, \|\cdot\|_\infty\bigr)$.

2. a) We show that the sequence $(g_n)_{n\in\mathbb{N}^*}$ converges to $g = 0_E$ for the norm $\|\cdot\|_\infty$: for $n \in \mathbb{N}^*$, the function g_n is clearly positive and decreasing, hence

$$\|g_n - 0_E\|_\infty = \|g_n\|_\infty = g_n(0) = \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0.$$

b) We show that $E \setminus F$ is not closed in $\bigl(E, \|\cdot\|_\infty\bigr)$: first observe that for $n \in \mathbb{N}^*$, $g_n \in E \setminus F$; indeed, for $X > 0$,

$$\int_0^X g_n(t)\,\mathrm{d}t = \int_0^X \frac{\mathrm{d}t}{n(1+t)} = \frac{1}{n}\ln(1+X) \underset{X\to+\infty}{\longrightarrow} +\infty.$$

Hence, for all $n \in \mathbb{N}^*$, $g_n \in E \setminus F$. Now, it is clear that $g = 0_E$ belongs to F, hence $E \setminus F$ can't be closed, hence F can't be open.

Exercise 2.

1. We define the function u as

$$u : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x\sin y - y\sin x. \end{array}$$

Clearly, u is of class C^2 and for all $(x,y) \in \mathbb{R}^2$,

$$\partial_1 u(x,y) = \sin y - y\cos x, \qquad \partial_2 u(x,y) = x\cos y - \sin x,$$
$$\partial^2_{1,1} u(x,y) = y\sin x, \qquad \partial^2_{1,2} u(x,y) = \cos y - \cos x, \qquad \partial^2_{2,2} u(x,y) = -x\sin y.$$

Hence,

$$\partial_1 u(0,0)=0,\qquad \partial_2 u(0,0)=0,\qquad \partial^2_{1,1}u(0,0)=0,\qquad \partial^2_{1,2}u(0,0)=0,\qquad \partial^2_{2,2}u(0,0)=0.$$

Hence, we conclude that

$$u(x,y)\underset{(x,y)\to(0,0)}{=}o\big(x^2+y^2\big).$$

This shows that

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)}\frac{u(x,y)}{x^2+y^2}=0=f(0,0).$$

Hence f is continuous at $(0,0)$. Clearly, f is continuous on $\mathbb{R}^2\setminus\{(0,0)\}$, hence f is continuous on $\mathbb{R}^2$.

2. f is clearly of class C^∞ on $\mathbb{R}^2\setminus\{(0,0)\}$, and for $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$,

$$\begin{aligned}\partial_1 f(x,y)&=\frac{\sin(y)-y\cos(x)}{x^2+y^2}-2x\frac{x\sin(y)-y\sin(x)}{\big(x^2+y^2\big)^2},\\ \partial_2 f(x,y)&=\frac{x\cos(y)-\sin(x)}{x^2+y^2}-2y\frac{x\sin(y)-y\sin(x)}{\big(x^2+y^2\big)^2}.\end{aligned}$$

We now compute the partial derivatives of f at $(0,0)$: for $t\in\mathbb{R}^*$,

$$\frac{f(t,0)-f(0,0)}{t}=0\underset{t\to0}{\longrightarrow}0,\qquad\qquad \frac{f(0,t)-f(0,0)}{t}=0\underset{t\to0}{\longrightarrow}0,$$

hence

$$\partial_1 f(0,0)=0,\qquad\qquad \partial_2 f(0,0)=0.$$

Exercise 3. Define the function f as

$$f:\begin{array}{ccc}\mathbb{R}^2&\longrightarrow&\mathbb{R}\\(x,y)&\longmapsto&x\mathrm{e}^y-y^2\mathrm{e}^x.\end{array}$$

We apply the Implicit Function Theorem to f at $(1,1)$:

- $f(1,1)=\mathrm{e}-\mathrm{e}=0$.
- Clearly, the function f is of class C^∞.
- For $(x,y)\in\mathbb{R}^2$,

  $$\partial_2 f(x,y)=x\mathrm{e}^y-2y\mathrm{e}^x,$$

  hence $\partial_2 f(1,1)=\mathrm{e}-2\mathrm{e}=-\mathrm{e}\neq0$.

Hence, by the Implicit Function Theorem, there exists a neighborhood I of 1, a neighborhood J of 1 and a function $\varphi:I\longrightarrow J$ of class C^∞ such that $\varphi(1)=1$ and

$$\forall(x,y)\in I\times J,\ f(x,y)=0\iff y=\varphi(x).$$

Moreover,

$$\forall x\in I,\ \varphi'(x)=-\frac{\partial_1 f\big(x,\varphi(x)\big)}{\partial_2 f\big(x,\varphi(x)\big)}=-\frac{\mathrm{e}^{\varphi(x)}-\varphi(x)^2\mathrm{e}^x}{x\mathrm{e}^{\varphi(x)}-2\varphi(x)\mathrm{e}^x}.$$

Since $\varphi(1)=1$ we conclude that

$$\varphi'(1)=-\frac{\mathrm{e}-\mathrm{e}}{-\mathrm{e}}=0,$$

hence an equation of the tangent line to $(\mathscr{C})$ at $(1,1)$ is

$$\Delta:\qquad y=1.$$

Define the functions N and D as

$$N:\begin{array}{ccc}I&\longrightarrow&\mathbb{R}\\x&\longmapsto&\mathrm{e}^{\varphi(x)}-\varphi(x)^2\mathrm{e}^x\end{array}\qquad\text{and}\qquad D:\begin{array}{ccc}I&\longrightarrow&\mathbb{R}\\x&\longmapsto&x\mathrm{e}^{\varphi(x)}-2\varphi(x)\mathrm{e}^x,\end{array}$$

so that

$$\forall x \in I,\ \varphi'(x) = -\frac{N(x)}{D(x)}.$$

Then,

$$\forall x \in I,\ \varphi''(x) = -\frac{N'(x)D(x) - N(x)D'(x)}{D(x)^2},$$

and since $N(1) = 0$ we have

$$\varphi''(1) = -\frac{N'(1)}{D(1)}.$$

Now, for $x \in I$,

$$N'(x) = \varphi'(x)\mathrm{e}^{\varphi(x)} - 2\varphi'(x)\varphi(x)\mathrm{e}^x - \varphi(x)^2\mathrm{e}^x,$$

hence (using the fact that $\varphi(1) = 1$ and $\varphi'(1) = 0$),

$$N'(1) = -\mathrm{e},$$

hence

$$\varphi''(1) = -\frac{-\mathrm{e}}{-\mathrm{e}} = -1 < 0.$$

We hence conclude that $(\mathscr{C})$ lies below Δ in a neighborhood of $(1,1)$. The graph of $(\mathscr{C})$ is shown on Figure 1.

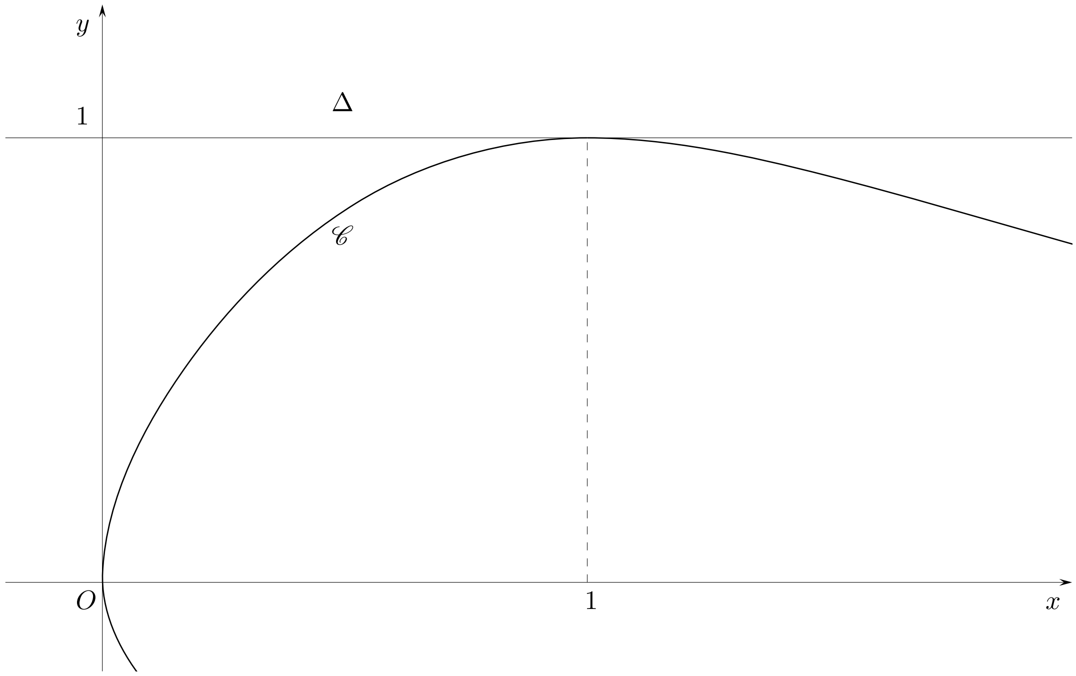

Figure 1. Curve $(\mathscr{C})$ of Exercise 3.

Exercise 4.

1. We first show that φ is a bijection: let $(x,y) \in \mathbb{R}^2$ and let $(u,v) \in \mathbb{R}_+^* \times \mathbb{R}$. Then:

$$\begin{aligned}
\varphi(x,y) = (u,v) &\iff \begin{cases} \mathrm{e}^x + \mathrm{e}^y = u \\ y - x = v \end{cases} \iff \begin{cases} y = v + x \\ \mathrm{e}^x + \mathrm{e}^{v+x} = u \end{cases} \iff \begin{cases} y = v + x \\ \mathrm{e}^x\bigl(1 + \mathrm{e}^v\bigr) = u \end{cases} \\
&\iff \begin{cases} y = v + x \\ \mathrm{e}^x = \dfrac{u}{1 + \mathrm{e}^v} \quad \textit{since } 1 + \mathrm{e}^v \neq 0 \end{cases} \iff \begin{cases} y = v + x \\ x = \ln(u) - \ln\bigl(1 + \mathrm{e}^v\bigr) \quad \textit{since } u > 0 \end{cases} \\
&\iff \begin{cases} y = v + \ln(u) - \ln\bigl(1 + \mathrm{e}^v\bigr) \\ x = \ln(u) - \ln\bigl(1 + \mathrm{e}^v\bigr). \end{cases}
\end{aligned}$$

Since the equation $\varphi(x,y)=(u,v)$ possesses a unique solution, we conclude that φ is a bijection. As a byproduct, we also have φ^{-1} explicitly:

$$\begin{array}{rccc}\varphi^{-1}: & \mathbb{R}_+^*\times\mathbb{R} & \longrightarrow & \mathbb{R}^2\\ & (u,v) & \longmapsto & \big(\ln(u)-\ln(1+\mathrm{e}^v),v+\ln(u)-\ln(1+\mathrm{e}^v)\big).\end{array}$$

We now show that φ is a C^∞-diffeomorphism: clearly, φ is of class C^∞. To show that φ^{-1} is of class C^∞ we have two possibilities: the first one is to notice, from the explicit form of φ^{-1} that φ^{-1} is clearly of class C^∞; the other possibility is to show that the Jacobian matrix of φ is invertible throughout $\mathbb{R}^2$: for $(x,y)\in\mathbb{R}^2$,

$$J_{(x,y)}\varphi=\begin{pmatrix}\mathrm{e}^x & \mathrm{e}^y\\ -1 & 1\end{pmatrix}$$

hence $\det\big(J_{(x,y)}\varphi\big)=\mathrm{e}^x+\mathrm{e}^y\neq 0$.

2. Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a function and set $g=f\circ\varphi^{-1}$. Since φ is a C^1-diffeomorphism, we have the equivalence:

$$f\text{ is of class }C^1\text{ on }\mathbb{R}^2\iff g\text{ is of class }C^1\text{ on }\mathbb{R}_+^*\times\mathbb{R}.$$

From now on, we assume that f (and hence g) is of class C^1. Since $f=g\circ\varphi$ we can easily express the partial derivatives of f in terms of the partial derivatives of g: let $(x,y)\in\mathbb{R}^2$ and set $(u,v)=\varphi(x,y)$. Then:

$$\begin{aligned}\partial_1 f(x,y)&=\mathrm{e}^x\partial_1 g(u,v)-\partial_2 g(u,v)\\ \partial_2 f(x,y)&=\mathrm{e}^y\partial_1 g(u,v)+\partial_2 g(u,v).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{e}^y\partial_1 f(x,y)-\mathrm{e}^x\partial_2 f(x,y)+\big(\mathrm{e}^x+\mathrm{e}^y\big)f(x,y)&=\mathrm{e}^{x+y}\partial_1 g(u,v)-\mathrm{e}^y\partial_2 g(u,v)\\ &\quad-\mathrm{e}^{x+y}\partial_1 g(u,v)-\mathrm{e}^x\partial_2 g(u,v)\\ &\quad+\big(\mathrm{e}^x+\mathrm{e}^y\big)g(u,v)\\ &=\big(\mathrm{e}^x+\mathrm{e}^y\big)\big(-\partial_2 g(u,v)+g(u,v)\big).\end{aligned}$$

Hence,

$$f\text{ is a solution of }(E_1)\iff\forall(u,v)\in\mathbb{R}_+^*\times\mathbb{R},\ -\partial_2 g(u,v)+g(u,v)=0.$$

To solve the partial differential equation in g, we can use the following auxilliary ordinary differential equation on $\mathbb{R}$:

$$\forall v\in\mathbb{R},\ z'(v)-z(v)=0,$$

the general solution of which is:

$$\exists A\in\mathbb{R},\ \forall v\in\mathbb{R}_+^*,\ z(v)=A\mathrm{e}^v.$$

We conclude that

$$\begin{aligned}f\text{ is a solution of }(E_1)&\iff\exists A:\mathbb{R}_+^*\longrightarrow\mathbb{R}\text{ of class }C^1,\ \forall(u,v)\in\mathbb{R}_+\times\mathbb{R},\ g(u,v)=A(u)\mathrm{e}^v\\ &\iff\exists A:\mathbb{R}_+^*\longrightarrow\mathbb{R}\text{ of class }C^1,\ \forall(u,v)\in\mathbb{R}_+\times\mathbb{R},\ f(x,y)=A\big(\mathrm{e}^x+\mathrm{e}^y\big)\mathrm{e}^{y-x}.\end{aligned}$$

3. Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a function and set $g=f\circ\varphi^{-1}$. Since φ is a C^2-diffeomorphism, we have the equivalence:

$$f\text{ is of class }C^2\text{ on }\mathbb{R}^2\iff g\text{ is of class }C^2\text{ on }\mathbb{R}_+^*\times\mathbb{R}.$$

From now on, we assume that f (and hence g) is of class C^2. Since $f=g\circ\varphi$ we can easily express the second order partial derivatives of f in terms of the partial derivatives of g (notice that the first order partial derivatives were already obtained in the previous question): let $(x,y)\in\mathbb{R}^2$ and set $(u,v)=\varphi(x,y)$. Then:

$$\begin{aligned}\partial_{1,1}^2 f(x,y)&=\mathrm{e}^x\partial_1 g(u,v)+\mathrm{e}^{2x}\partial_{1,1}^2 g(u,v)-\mathrm{e}^x\partial_{2,1}^2 g(u,v)-\mathrm{e}^x\partial_{1,2}^2 g(u,v)+\partial_{2,2}^2 g(u,v)\\ &=\mathrm{e}^x\partial_1 g(u,v)+\mathrm{e}^{2x}\partial_{1,1}^2 g(u,v)-2\mathrm{e}^x\partial_{1,2}^2 g(u,v)+\partial_{2,2}^2 g(u,v)\qquad\textit{(by Schwarz' Theorem)}\\ \partial_{1,2}^2 f(x,y)&=\mathrm{e}^{x+y}\partial_{1,1}^2 g(u,v)-\mathrm{e}^y\partial_{2,1}^2 g(u,v)+\mathrm{e}^x\partial_{1,2}^2 g(u,v)-\partial_{2,2}^2 g(u,v)\\ &=\mathrm{e}^{x+y}\partial_{1,1}^2 g(u,v)+\big(\mathrm{e}^x-\mathrm{e}^y\big)\partial_{1,2}^2 g(u,v)-\partial_{2,2}^2 g(u,v)\qquad\textit{(by Schwarz' Theorem)}\\ \partial_{2,2}^2 f(x,y)&=\mathrm{e}^y\partial_1 g(u,v)+\mathrm{e}^{2y}\partial_{1,1}^2 g(u,v)+\mathrm{e}^y\partial_{2,1}^2 g(u,v)+\mathrm{e}^y\partial_{1,2}^2 g(u,v)+\partial_{2,2}^2 g(u,v)\end{aligned}$$

$$= \mathrm{e}^y \partial_1 g(u,v) + \mathrm{e}^{2y} \partial^2_{1,1} g(u,v) + 2\mathrm{e}^y \partial^2_{1,2} g(u,v) + \partial^2_{2,2} g(u,v) \qquad \textit{(by Schwarz' Theorem)}.$$

Hence,

$$\begin{aligned}
&\mathrm{e}^y \partial^2_{1,1} f(x,y) + \left(\mathrm{e}^y - \mathrm{e}^x\right) \partial^2_{1,2} f(x,y) - \mathrm{e}^x \partial^2_{2,2} f(x,y)\\
&\qquad = \mathrm{e}^y \Big(\mathrm{e}^x \partial_1 g(u,v) + \mathrm{e}^{2x} \partial^2_{1,1} g(u,v) - 2\mathrm{e}^x \partial^2_{1,2} g(u,v) + \partial^2_{2,2} g(u,v)\Big)\\
&\qquad\quad + \left(\mathrm{e}^y - \mathrm{e}^x\right)\Big(\mathrm{e}^{x+y} \partial^2_{1,1} g(u,v) + \left(\mathrm{e}^x - \mathrm{e}^y\right) \partial^2_{1,2} g(u,v) - \partial^2_{2,2} g(u,v)\Big)\\
&\qquad\quad - \mathrm{e}^x \Big(\mathrm{e}^y \partial_1 g(u,v) + \mathrm{e}^{2y} \partial^2_{1,1} g(u,v) + 2\mathrm{e}^y \partial^2_{1,2} g(u,v) + \partial^2_{2,2} g(u,v)\Big)\\
&\qquad = \mathrm{e}^{2x+y} \partial^2_{1,1} g(u,v) - 2\mathrm{e}^{x+y} \partial^2_{1,2} g(u,v) + \mathrm{e}^y \partial^2_{2,2} g(u,v)\\
&\qquad\quad \left(\mathrm{e}^{x+2y} - \mathrm{e}^{2x+y}\right) \partial^2_{1,1} g(u,v) - \left(\mathrm{e}^x - \mathrm{e}^y\right)^2 \partial^2_{1,2} g(u,v) - \left(\mathrm{e}^y - \mathrm{e}^x\right) \partial^2_{2,2} g(u,v)\\
&\qquad\quad - \mathrm{e}^{x+2y} \partial^2_{1,1} g(u,v) - 2\mathrm{e}^{x+y} \partial^2_{1,2} g(u,v) - \mathrm{e}^x \partial^2_{2,2} g(u,v)\\
&\qquad = -4\mathrm{e}^{x+y} \partial^2_{1,2} g(u,v) - \left(\mathrm{e}^x - \mathrm{e}^y\right)^2 \partial^2_{1,2} g(u,v)\\
&\qquad = -\left(\mathrm{e}^x + \mathrm{e}^y\right)^2 \partial^2_{1,2} g(u,v)\\
&\qquad = -u^2 \partial^2_{1,2} g(u,v).
\end{aligned}$$

Hence,

$$f \text{ is a solution of } (E_2) = \iff \forall (u,v) \in \mathbb{R}_+^* \times \mathbb{R},\ \partial^2_{1,2} g(u,v) = 0.$$

We now solve the partial differential equation

$$\partial^2_{1,2} g = 0$$

on $\mathbb{R}_+^* \times \mathbb{R}$:

$$\begin{aligned}
\partial^2_{1,2} g = 0 &\iff \exists A : \mathbb{R} \longrightarrow \mathbb{R} \text{ of class } C^1,\ \forall (u,v) \in \mathbb{R}_+^* \times \mathbb{R},\ \partial_2 g(u,v) = A(v)\\
&\iff \exists A : \mathbb{R} \longrightarrow \mathbb{R} \text{ of class } C^1,\ \exists B : \mathbb{R} \longrightarrow \mathbb{R} \text{ of class } C^2,\\
&\qquad \forall (u,v) \in \mathbb{R}_+^* \times \mathbb{R},\ g(u,v) = \int A(v)\,\mathrm{d}v + B(u)\\
&\iff \exists C : \mathbb{R} \longrightarrow \mathbb{R} \text{ of class } C^2,\ \exists B : \mathbb{R} \longrightarrow \mathbb{R} \text{ of class } C^2,\\
&\qquad \forall (u,v) \in \mathbb{R}_+^* \times \mathbb{R},\ g(u,v) = B(u) + C(v)
\end{aligned}$$

where the function C that appears in the last step is an antiderivative of the function A that appears in the previous step. The class of differentiability of the functions A, B and C that appear are obtained from the fact that g is of class C^2. We hence conclude:

$$\begin{aligned}
f \text{ is a solution of } (E_2) \iff{}& \exists C : \mathbb{R} \longrightarrow \mathbb{R} \text{ of class } C^2,\ \exists B : \mathbb{R} \longrightarrow \mathbb{R} \text{ of class } C^2,\\
& \forall (x,y) \in \mathbb{R}^2,\ f(x,y) = B\left(\mathrm{e}^x + \mathrm{e}^y\right) + C(y-x).
\end{aligned}$$

Exercise 5.

1. From the sequence $(u_n)_{n\in\mathbb{N}}$ we define the *sequence of the partial sums* $(S_N)_{N\in\mathbb{N}}$ as follows:

$$\forall N \in \mathbb{N},\ S_N = \sum_{n=0}^{N} u_n.$$

   The series $\sum_{n\geq 0} u_n$ *converges* means: *the sequence* $(S_N)_{N\in\mathbb{N}}$ *converges.*

2. Let $q \in \mathbb{C}$.

$$\text{the series } \sum_n q^n \text{ converges} \iff |q| < 1.$$

   If $|q| < 1$,

$$\sum_{n=0}^{+\infty} q^n = \frac{1}{1-q}.$$

3. (1) For all $n \in \mathbb{N}^*$ one has:

$$0 \leq \frac{1}{n2^n} \leq \frac{1}{2^n}.$$

Since we know that the series

$$\sum_{n\geq 1} \frac{1}{2^n}$$

converges (as a geometric series of ratio $1/2$), we conclude, by the comparison test, that the series (1) converges.

(2) For $n \in \mathbb{N}$ set

$$u_n = (-1)^n \frac{n}{\sqrt{n^2+1}}.$$

We have:

$$\lim_{n\to+\infty} |u_n| = 1 \neq 0$$

hence

$$u_n \underset{n\to+\infty}{\not\longrightarrow} 0$$

hence we conclude that the series (2) diverges.

(3)

$$\left(1 + \frac{(-1)^n}{n}\right) \ln\left(1 + \frac{1}{n^4}\right) \underset{n\to+\infty}{\sim} \frac{1}{n^4} > 0.$$

Now, the series $\sum_{n\geq 1} 1/n^4$ is a Riemann series with $\alpha = 4 > 1$, hence it is a convergent series. By the equivalent test, we conclude that the series (3) converges too.

Exercise 6.

1. Let $A, B \in \mathbb{R}_+^*$. The function

$$t \mapsto \frac{1}{\sqrt{\left(A^2+t^2\right)\left(B^2+t^2\right)}}$$

is (well-defined) and continuous on $\mathbb{R}$, hence the integral $I(A, B)$ is improper at $-\infty$ and at $+\infty$. Now,

$$\frac{1}{\sqrt{\left(A^2+t^2\right)\left(B^2+t^2\right)}} \underset{t\to+\infty}{\sim} \frac{1}{t^2} > 0$$

and

$$\frac{1}{\sqrt{\left(A^2+t^2\right)\left(B^2+t^2\right)}} \underset{t\to-\infty}{\sim} \frac{1}{t^2} > 0.$$

Since the improper integrals

$$\int_{-\infty}^{-1} \frac{\mathrm{d}t}{t^2} \qquad \text{and} \qquad \int_{1}^{+\infty} \frac{\mathrm{d}t}{t^2}$$

converge (by Riemann at $\pm\infty$ with $\alpha = 2 > 1$) we conclude, by the equivalent test, that $I(A, B)$ converges.

2. The value of $f(1)$ is given by the following improper integral:

$$f(1) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{1+t^2} = 2\int_{0}^{+\infty} \frac{\mathrm{d}t}{1+t^2}$$

since the function

$$t \mapsto \frac{1}{1+t^2}$$

is even. Let $X \in \mathbb{R}_+^*$. Then:

$$\int_0^X \frac{\mathrm{d}t}{1+t^2} = \arctan(X) \underset{X\to+\infty}{\longrightarrow} \frac{\pi}{2}.$$

Hence $f(1) = \pi$.

3. Let $A, B \in \mathbb{R}_+^*$ such that $A < B$. Define the function u as

$$u : \begin{array}{ccc} \mathbb{R} \times [A, B] & \longrightarrow & \mathbb{R} \\ (t, x) & \longmapsto & \dfrac{1}{\sqrt{\left(1 + t^2\right)\left(x^2 + t^2\right)}}. \end{array}$$

We apply the differentiation theorem to u:

- Clearly, the partial derivative $\partial_2 u$ exists throughout $\mathbb{R} \times [A, B]$ and

$$\forall (t, x) \in \mathbb{R} \times [A, B],\ \partial_2 u(t, x) = -\frac{x}{\sqrt{\left(1 + t^2\right)\left(x^2 + t^2\right)^3}}.$$

- Clearly, for all $x \in [A, B]$, the functions

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & u(t, x) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \partial_2 u(t, x) \end{array}$$

  are continuous.
- For all $x \in [A, B]$, we already know that the improper integral

$$\int_{-\infty}^{+\infty} |u(t, x)| \,\mathrm{d}t$$

  converges.
- For the domination function, we choose:

$$g : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{B}{\sqrt{\left(1 + t^2\right)\left(A^2 + t^2\right)^3}}. \end{array}$$

  Indeed, we have:

$$\forall (t, x) \in \mathbb{R} \times [A, B],\ \left|\partial_2 u(t, x)\right| = \frac{x}{\sqrt{\left(1 + t^2\right)\left(x^2 + t^2\right)^3}} \le \frac{B}{\sqrt{\left(1 + t^2\right)\left(A^2 + t^2\right)^3}} = g(t).$$

  Clearly, g is continuous; the improper integral

$$\int_{-\infty}^{+\infty} g(t) \,\mathrm{d}t$$

  is improper at $-\infty$ and at $+\infty$; now,

$$g(t) \underset{t \to +\infty}{\sim} \frac{B}{t^4} > 0 \qquad \text{and} \qquad g(t) \underset{t \to -\infty}{\sim} \frac{B}{t^4} > 0,$$

  and we know (by Riemann at $\pm\infty$ with $\alpha = 4 > 1$) that the improper integrals

$$\int_{-\infty}^{-1} \frac{\mathrm{d}t}{t^4} \qquad \text{and} \qquad \int_{1}^{+\infty} \frac{\mathrm{d}t}{t^4}$$

  converge, hence, by the equivalent test, the improper integral

$$\int_{-\infty}^{+\infty} g(t) \,\mathrm{d}t$$

  converges.

Hence we conclude that f is differentiable on $[A, B]$ and that

$$\forall x \in [A, B],\ f'(x) = -x \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{\sqrt{\left(1 + t^2\right)\left(x^2 + t^2\right)^3}}.$$

Since this is true for all $A, B \in \mathbb{R}_+^*$ such that $A < B$ we conclude that f is differentiable on $\mathbb{R}_+^*$ and that

$$\forall x \in \mathbb{R}_+^*,\ f'(x) = -x \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{\sqrt{\left(1 + t^2\right)\left(x^2 + t^2\right)^3}}.$$

4. Let $A, B \in \mathbb{R}_+^*$ and $t \in \mathbb{R}$. Then:

$$\frac{1}{\sqrt{\left(A^2+t^2\right)\left(B^2+t^2\right)}} = \frac{1}{A^2}\frac{1}{\sqrt{1+(t/A)^2)\left((B/A)^2+(t/A)^2\right)}}.$$

Hence,

$$I(A,B) = \frac{1}{A^2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{\sqrt{\left(1+(t/A)^2\right)\left((B/A)^2+(t/A)^2\right)}}$$

and using the linear substitution $s = t/A$ (hence $\mathrm{d}t = A\,\mathrm{d}s$) yields

$$I(A,B) = \frac{1}{A}\int_{-\infty}^{+\infty}\frac{\mathrm{d}s}{\sqrt{\left(1+s^2\right)\left((B/A)^2+s^2\right)}} = \frac{1}{A}f\left(\frac{B}{A}\right).$$

5. Let $A, B \in \mathbb{R}_+^*$. Notice that the function

$$s \longmapsto \frac{1}{2}\left(s - \frac{AB}{s}\right)$$

is an increasing bijection from $\mathbb{R}_+^*$ to $\mathbb{R}$. Let $t \in \mathbb{R}$ and let $s \in \mathbb{R}_+^*$ such that

$$t = \frac{1}{2}\left(s - \frac{AB}{s}\right).$$

Then:

$$\begin{aligned}
\left(\frac{A+B}{2}\right)^2 + t^2 &= \frac{1}{4}\left((A+B)^2 + \left(s-\frac{AB}{s}\right)^2\right) = \frac{1}{4}\left((A+B)^2 + s^2 - 2AB + \frac{AB}{s^2}\right)\\
&= \frac{1}{4}\left(A^2+B^2+s^2+\frac{AB}{s^2}\right) = \frac{1}{4s^2}\left(A^2s^2+B^2s^2+s^4+AB\right)\\
&= \frac{1}{4s^2}\left(A^2+s^2\right)\left(B^2+s^2\right)\\
AB + t^2 &= \left(AB + \frac{1}{4}\left(s^2 - 2AB + \frac{A^2B^2}{s^2}\right)\right) = \frac{1}{4}\left(s^2+2AB+\frac{A^2B^2}{s^2}\right) = \frac{1}{4}\left(s+\frac{AB}{s}\right)^2.
\end{aligned}$$

We now compute the improper integral defining $I\left((A+B)/2, \sqrt{AB}\right)$, using the given substitution. Notice that

$$\mathrm{d}t = \frac{1}{2}\left(1+\frac{AB}{s^2}\right)\mathrm{d}s.$$

Let $X, Y \in \mathbb{R}$ such that $X < Y$ and let $X', Y' \in \mathbb{R}_+^*$ such that

$$X = \frac{1}{2}\left(X' - \frac{AB}{X'}\right) \qquad \text{and} \qquad Y = \frac{1}{2}\left(Y' - \frac{AB}{Y'}\right).$$

Then:

$$\begin{aligned}
\int_X^Y \frac{\mathrm{d}t}{\sqrt{\left(\left(\frac{A+B}{2}\right)^2+t^2\right)\left(AB+t^2\right)}} &= \int_{X'}^{Y'}\frac{\frac{1}{2}\left(1+\frac{AB}{s^2}\right)\mathrm{d}s}{\sqrt{\frac{1}{4s^2}\left(A^2+s^2\right)\left(B^2+s^2\right)\frac{1}{4}\left(s+\frac{AB}{s}\right)^2}}\\
&= 2\int_{X'}^{Y'}\frac{\left(1+\frac{AB}{s^2}\right)\mathrm{d}s}{\sqrt{\left(A^2+s^2\right)\left(B^2+s^2\right)\left(1+\frac{AB}{s^2}\right)^2}}\\
&= 2\int_{X'}^{Y'}\frac{\mathrm{d}s}{\sqrt{\left(A^2+s^2\right)\left(B^2+s^2\right)}}.
\end{aligned}$$

Now, $X' \underset{X \to -\infty}{\longrightarrow} 0$ and $Y' \underset{Y \to +\infty}{\longrightarrow} +\infty$, hence

$$I\left(\frac{A+B}{2}, \sqrt{AB}\right) = 2\int_0^{+\infty} \frac{\mathrm{d}s}{\sqrt{\left(A^2+s^2\right)\left(B^2+s^2\right)}}.$$

Since the function

$$s \longmapsto \frac{1}{\sqrt{\left(A^2+s^2\right)\left(B^2+s^2\right)}}$$

is even, we have:

$$2\int_0^{+\infty} \frac{\mathrm{d}s}{\sqrt{\left(A^2+s^2\right)\left(B^2+s^2\right)}} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}s}{\sqrt{\left(A^2+s^2\right)\left(B^2+s^2\right)}} = I(A,B),$$

hence the result.

6. First notice that, by Question 3,
$$\forall n \in \mathbb{N},\ w_n = I(u_n, v_n).$$
Now, let $n \in \mathbb{N}$. Then, by Question 5, and the definition of $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$,
$$w_{n+1} = I(u_{n+1}, v_{n+1}) = I\left(\frac{u_n+v_n}{2}, \sqrt{u_n v_n}\right) = I(u_n, v_n) = w_n.$$
Hence the sequence $(w_n)_{n\in\mathbb{N}}$ is constant. Now, since $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to the same (non-nil) limit $\mu(a,b)$, we have
$$\lim_{n\to+\infty} \frac{v_n}{u_n} = \frac{\mu(a,b)}{\mu(a,b)} = 1$$
and *since f is continuous at* 1 *since f is differentiable on* $\mathbb{R}_+^*$,
$$\lim_{n\to+\infty} w_n = \frac{1}{\mu(a,b)} f(1) = \frac{\pi}{\mu(a,b)}.$$
Hence, the value of the constant sequence $(w_n)_{n\in\mathbb{N}}$ is $\pi/\mu(a,b)$.

7. We conclude that for all $a, b \in \mathbb{R}_+^*$,
$$\frac{\pi}{\mu(a,b)} = w_0 = I(a,b),$$
hence the result.