SCAN 2 — Solution of Math Test #2

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Exercise 1.

1. Let $f \in E$. Since f is continuous, the function

$$F : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\int_0^x f(t)^2 \,\mathrm{d}t \end{array}$$

is well defined, and it is continuous as it is an antiderivative of f^2. Hence $F \in E$ and we conclude that Φ is well defined.

2. Let $f_0 \in E$ and $f \in E$. Let $x \in [0,1]$. Then:

$$\begin{aligned} \big|\Phi(f)(x) - \Phi(f_0)(x)\big| &= \left|\int_0^x \big(f(x)^2 - f_0(x)^2\big)\,\mathrm{d}x\right| \\ &\leq \int_0^x \big|f(x)^2 - f_0(x)^2\big|\,\mathrm{d}x \\ &= \int_0^x \big|f(x) - f_0(x)\big|\big|f(x) + f_0(x)\big|\,\mathrm{d}x \\ &\leq x\|f - f_0\|_\infty\|f + f_0\|_\infty \\ &\leq \|f - f_0\|_\infty\|f + f_0\|_\infty. \end{aligned}$$

Hence $K = 1$ satisfies the property.

3. Let $f_0 \in E$. We show that Φ is continuous at f_0: let $f \in E$. Then:

$$\big\|\Phi(f)-\Phi(f_0)\big\|_\infty \leq \|f-f_0\|_\infty\|f+f_0\|_\infty = \|f-f_0\|_\infty\|f-f_0+2f_0\|_\infty \leq \|f-f_0\|_\infty\big(\|f-f_0\|_\infty+2\|f_0\|_\infty\big) \underset{\|f-f_0\|_\infty\to0}{\longrightarrow} 0$$

Hence Φ is continuous at f_0.

4. Yes, Ψ_{f_0} is linear: let $h_1, h_2 \in E$ and $\lambda \in \mathbb{R}$. For $x \in [0,1]$:

$$\begin{aligned} \Psi_{f_0}(h_1 + \lambda h_2)(x) &= \int_0^x f_0(t)\big(h_1(t) + \lambda h_2(t)\big)\,\mathrm{d}t = \int_0^x f_0(t)h_1(t)\big)\,\mathrm{d}t + \lambda\int_0^x f_0(t)h_2(t)\,\mathrm{d}t \\ &= \Psi_{f_0}(h_1)(x) + \lambda\Psi_{f_0}(h_2)(x) \end{aligned}$$

hence $\Psi_{f_0}(h_1 + \lambda h_2) = \Psi_{f_0}(h_1) + \lambda\Psi_{f_0}(h_2)$.
Since Ψ_{f_0} is linear, we only need to show that it is continuous at 0_E: for $h \in E$ and $x \in [0,1]$,

$$\big|\Psi_{f_0}(h)(x)\big| \leq \int_0^x \big|f_0(t)h(t)\big|\,\mathrm{d}t \leq x\|f_0\|_\infty\|h\|_\infty \leq \|f_0\|_\infty\|h\|_\infty.$$

Hence

$$\|\Psi_{f_0}(h)\|_\infty \leq \|f_0\|_\infty\|h\|_\infty \underset{\|h\|_\infty\to0}{\longrightarrow} 0,$$

hence Ψ_{f_0} is continuous at 0_E, hence Ψ is continuous.

5. Let $f_0 \in E$. We show that Ψ is differentiable at f_0: for $h \in E$ and $x \in [0,1]$,

$$\begin{aligned} \Phi(f_0 + h)(x) &= \int_0^x \big(f_0(t) + h(t)\big)^2\,\mathrm{d}t \\ &= \int_0^x f_0(t)^2\,\mathrm{d}t + 2\int_0^x f_0(t)h(t)\,\mathrm{d}t + \int_0^x h(t)^2\,\mathrm{d}t \\ &= \Phi(f_0)(x) + 2\Psi_{f_0}(h)(x) + \Phi(h)(x) \end{aligned}$$

hence

$$\Phi(f_0 + h) = \Phi(f_0) + 2\Psi_{f_0}(h) + \Phi(h)$$

We already know that Ψ_{f_0} is linear and continuous, so we only need to show that

$$\Phi(h) \underset{\|h\|_\infty\to 0}{=} o\big(\|h\|_\infty\big).$$

From Question 1 (and since $\Phi(0_E)=0_E$) we know that for $h\in E$,

$$\big\|\Phi(h)\big\|_\infty = \big\|\Phi(h)-\Phi(0_E)\big\|_\infty \leq \big\|h\big\|_\infty^2$$

hence

$$\frac{\big\|\Phi(h)\big\|_\infty}{\|h\|_\infty} \leq \big\|h\big\|_\infty \underset{\|h\|_\infty\to 0}{\longrightarrow}.$$

We can now conclude that Φ is differentiable at f_0 and that $D_{f_0}\Phi = 2\Psi_{f_0}$.

Exercise 2.

1. Let $U=\mathbb{R}^2\setminus\big\{(0,0)\big\}$. Since U is an open set and the expression of f on U is a rational function, f is continuous on U. We now show that f is continuous at $(0,0)$: let $(x,y)\in U$ (notice that U is a punctured neighborhood of $(0,0)$):

$$\big|f(x,y)-f(0,0)\big| = \left|\frac{xy^2}{x^2+2y^2}\right| \leq \frac{\big\|(x,y)\big\|_2^3}{\big\|(x,y)\big\|_2^2} = \big\|(x,y)\big\|_2^2 \underset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

since $x^2+2y^2\geq x^2+y^2=\big\|(x,y)\big\|_2^2$. Hence f is continuous at $(0,0)$.

2. let $v=(v_1,v_2)\in\mathbb{R}^2$. If $v=(0,0)$ we know that $\nabla_v f(0,0)=0$, so we assume $v\neq(0,0)$: for $t\in\mathbb{R}^*$,

$$f(tv) = \frac{t^3v_1v_2^2}{t^2(v_1^2+2v_2^2)} = tf(v)$$

hence

$$\frac{f(tv)-f(0)}{t} = f(v) \underset{t\to 0}{\longrightarrow} f(v).$$

We conclude that

$$\forall v\in\mathbb{R}^2, \nabla_v f(0,0) = f(v)$$

(also true for $v=(0,0)$ since $f(0,0)=0$).

3. By contradiction: assume that f is differentiable at $(0,0)$. Then:

$$\forall v\in\mathbb{R}^2,\ \mathrm{d}_{(0,0)}f(v) = \nabla_v f(0,0) = f(v),$$

i.e., $\mathrm{d}_{(0,0)}f=f$. Since $\mathrm{d}_{(0,0)}f$ is linear, we must have f linear, which is not since:

$$f(1,1) = \frac{1}{3} \neq f(1,0)+f(0,1) = 0$$

Hence f is not differentiable at $(0,0)$.

4. Two cases:
   - At $(x,y)\in U$:

$$\partial_1 f(x,y) = \frac{y^2(x^2+2y^2)-2x^2y^2}{(x^2+2y^2)^2} = \frac{y^2(-x^2+2y^2)}{(x^2+2y^2)^2}, \qquad \partial_2 f(x,y) = \frac{2x^3y}{(x^2+2y^2)^2}.$$

   - At $(0,0)$: we know that $\partial_1 f(0,0)=\nabla_{(1,0)}f(0,0)=f(1,0)=0$ and $\partial_2 f(0,0)=f(0,1)=0$.

5. No: define

$$\begin{aligned} p\ :\ \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (t,t). \end{aligned}$$

Notice that $p(t)\underset{t\to 0}{\longrightarrow}(0,0)$. Then for $t\in\mathbb{R}^*$,

$$\partial_1 f\big(p(t)\big) = \partial_1 f(t,t) = \frac{t^4}{9t^4} = \frac{1}{9} \underset{t\to 0}{\longrightarrow} \frac{1}{9} \neq \partial_1 f(0,0) = 0$$

Exercise 3.

1. P_2 is correct: assume that $(u_n)_{n\in\mathbb{N}}$ converges to ℓ for N_2 then for $n \in \mathbb{N}$

$$N_1(u_n - \ell) \leq \frac{1}{\alpha} N_2(u_n - \ell) \underset{n\to+\infty}{\longrightarrow} 0,$$

hence $(u_n)_{n\in\mathbb{N}}$ also converges for N_1.

2. a)

$$\forall \varepsilon > 0,\ \exists r > 0,\ \forall x \in E,\ \Big(N(x - x_0) < r \implies N'\big(\varphi(x) - \varphi(x_0)\big) < \varepsilon\Big).$$

b) Q_3 is correct: assume that f is continuous from (E, N_1) to (E, N_2). We show that f is continuous from (E, N_2) to (E, N_1): let $\varepsilon > 0$. Using the definition of continuity of f from (E, N_1) to (E, N_2) with $\alpha\varepsilon > 0$, we obtain $r' > 0$ such that

$$\forall x \in E,\ \Big(N_1(x - x_0) < r' \implies N_2\big(f(x) - f(x_0)\big) \leq \alpha\varepsilon\Big).$$

We set $r = r'/\alpha$ and show that

$$\forall x \in E,\ \Big(N_2(x - x_0) < r \implies N_1\big(f(x) - f(x_0)\big) \leq \varepsilon\Big).$$

Let $x \in E$ such that $N_2(x - x_0) < r$. Then $\alpha N_1(x - x_0) \leq N_2(x - x_0) < r = \alpha r'$ hence $N_1(x - x_0) < r'$, from which we conclude that $N_2\big(f(x) - f(x_0)\big) < \alpha\varepsilon$. Hence $\alpha N_1\big(f(x) - f(x_0)\big) < \alpha\varepsilon$ and we conclude that $N_1\big(f(x) - f(x_0)\big) < \varepsilon$.

Exercise 4.

1. Let $n \in \mathbb{N}$. The function $t \mapsto t^n/\sqrt{t}$ is continuous on $(0, 1]$ (and even on $[0, 1]$ when $n > 0$) hence integral I_n is improper at 0^+ (and even definite if $n > 0$). Let $X \in (0, 1]$. Then

$$\int_X^1 \frac{t^n}{\sqrt{t}}\,\mathrm{d}t = \int_X^1 t^{n-1/2}\,\mathrm{d}t = \left[\frac{1}{n + 1/2} t^{n+1/2}\right]_{t=X}^{t=1} = \frac{1}{n + 1/2}\big(X^{n+1/2} - 1\big) \underset{X\to 0^+}{\longrightarrow} \frac{1}{n + 1/2} = \frac{2}{2n + 1}.$$

2. N takes values in $\mathbb{R}_+$ and:

- Separation property: let $P \in E$ such that $N(P) = 0$. Since $t \mapsto \big|P(t)\big|$ is continuous and non-negative, we conclude that
$$\forall t \in [0, 1],\ P(t) = 0,$$
hence P has an infinite number of roots, hence $P = 0_E$.
- Triangle inequality: let $P, Q \in E$. Then
$$N(P + Q) = \int_0^1 \big|P(t) + Q(t)\big| \leq \int_0^1 \big|P(t)\big| + \big|Q(t)\big|\,\mathrm{d}t = N(P) + N(Q).$$
- Absolute homogeneity: let $P \in E$ and $\lambda \in \mathbb{R}$. Then
$$N(\lambda P) = \int_0^1 |\lambda P(t)|\,\mathrm{d}t = |\lambda| \int_0^1 |P(t)|\,\mathrm{d}t = |\lambda| N(P).$$

3. Let $a, b \in \mathbb{R}_+^*$ and define $P = aX + b$. Then $N_\infty(P) = a + b$ and by Question 1:

$$N(P) = \int_0^1 \frac{|at + b|}{\sqrt{t}}\,\mathrm{d}t = \int_0^1 \frac{at + b}{\sqrt{t}}\,\mathrm{d}t = \frac{2}{3}a + 2b$$

(since for all $t \in [0, 1]$, $at + b > 0$). We now solve the system

$$\begin{cases} a + b = 1 \\ \dfrac{2}{3}a + 2b = 1 \end{cases} \iff \begin{cases} a = \dfrac{3}{4} \\ b = \dfrac{1}{4} \end{cases}$$

4. Let $P \in E$. Then
$$\forall t \in (0,1], \left|P(t)\right| \leq \frac{\left|P(t)\right|}{\sqrt{t}}$$
hence $N_1(P) \leq N(P)$. Also,
$$N_1(P) = \int_0^1 \frac{\left|P(t)\right|}{\sqrt{t}} \leq N_\infty(P) \int_0^1 \frac{\mathrm{d}t}{\sqrt{t}} = 2N_\infty(P).$$

5.
   - $(X^n)_{n\in\mathbb{N}}$ converges to 0_E for N:
$$N\big(X^n\big) = \int_0^1 \frac{t^n}{\sqrt{n}}\,\mathrm{d}t = \frac{2}{2n+1} \underset{n\to+\infty}{\longrightarrow} 0.$$
   - By Question 4, $(X^n)_{n\in\mathbb{N}}$ also converges to 0_E for N_1.
   - $(X^n)_{n\in\mathbb{N}}$ doesn't converges to 0_E for N_∞:
$$N_\infty(X^n) = \max_{t\in[0,1]} t^n = 1 \underset{n\to+\infty}{\not\longrightarrow} 0.$$

6. a) Yes: $(X^n)_{n\in\mathbb{N}}$ converges to 0_E for N but not for N_∞.

   b) No: so far we only have a sequence that converges to 0_E for both norms (which is a necessary but not sufficient condition for the norms to be equivalent).

7. a) We can notice that φ is linear so it is sufficient to prove its continuity at 0_E, but the continuity at a point $P_0 \in E$ can be proven without too much extra effort: let $P \in E$. Then
$$\left|\varphi(P) - \varphi(P_0)\right| = \left|P(1) - P_0(1)\right| \leq \max_{t\in[0,1]}\left|P(t) - P_0(t)\right| = N_\infty(P - P_0) \underset{N_\infty P-P_0\to 0}{\longrightarrow} 0.$$

   b) We know that $(X^n)_{n\in\mathbb{N}}$ converges to 0_E for N. If φ we continuous from (E,N) to $\mathbb{R}$ we would expect:
$$\lim_{n\to+\infty} \varphi\big(X^n\big) = \varphi(0_E) = 0.$$
   But instead we have:
$$\varphi\big(X^n\big) = 1 \underset{n\to+\infty}{\longrightarrow} 1.$$
   Hence φ is not continuous from (E,N) to $\mathbb{R}$.