No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercises 4 is common with PCC2.

Exercise 1 (*5 points*)**.** Let $E = C\big([0,1]\big)$ and define

$$\begin{array}{rccl} \Phi : & E & \longrightarrow & E \\ & f & \longmapsto & \left(x \longmapsto \displaystyle\int_0^x f(t)^2 \,\mathrm{d}t\right) \end{array}$$

and for $f_0 \in E$ define

$$\begin{array}{rccl} \Psi_{f_0} : & E & \longrightarrow & E \\ & h & \longmapsto & \left(x \longmapsto \displaystyle\int_0^x f_0(t)h(t) \,\mathrm{d}t\right). \end{array}$$

We use the norm $\|\cdot\|_\infty$ on E.

1. Briefly explain why Φ is well defined.

   You're given that for $f_0 \in E$, Ψ_{f_0} is well defined.
2. Let $f_0 \in E$. Show that there exists $K \in \mathbb{R}_+$ such that
$$\forall f \in E,\ \big\|\Phi(f) - \Phi(f_0)\big\|_\infty \le K\|f - f_0\|_\infty \|f + f_0\|_\infty.$$
3. Deduce that Φ is continuous.
4. Let $f_0 \in E$. Is Ψ_{f_0} a linear map? Show that Ψ_{f_0} is continuous.
5. Is Φ differentiable? if yes, determine the differential $D_{f_0}\Phi$ of Φ at $f_0 \in E$.

Exercise 2 (*5 points*)**.** Let

$$\begin{array}{rccl} f : & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \begin{cases} \dfrac{xy^2}{x^2+2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} \end{array}$$

1. Show that f is continuous.
2. Show that all the directional derivatives at $(0,0)$ of f in any direction exist.
3. Show that f is not differentiable at $(0,0)$.
4. Determine the first order partial derivatives of f at a point $(x,y) \in \mathbb{R}^2$.
5. Is $\partial_1 f$ continuous at $(0,0)$?

Exercise 3 (*3 points*). Let $\alpha \in \mathbb{R}_+^*$ and let E be a vector space and let N_1 and N_2 be two norms on E such that

$$\forall x \in E,\ \alpha N_1(x) \le N_2(x).$$

1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E and let ℓ in E. Which of the following two implications is (are) correct?

$$\begin{aligned} P_1 :&\quad (u_n)_{n\in\mathbb{N}} \text{ converges to } \ell \text{ for } N_1 \implies (u_n)_{n\in\mathbb{N}} \text{ converges to } \ell \text{ for } N_2 \\ P_2 :&\quad (u_n)_{n\in\mathbb{N}} \text{ converges to } \ell \text{ for } N_2 \implies (u_n)_{n\in\mathbb{N}} \text{ converges to } \ell \text{ for } N_1 \end{aligned}$$

   Justify your answer.

2. a) In this question only (E, N) and (F, N') are two normed vector spaces, and $\varphi\colon E \to F$. Let $x_0 \in E$. Recall the definition of "φ is continuous at x_0."

   b) Let $f\colon E \to E$ and let $x_0 \in E$. Which of the following four implications is (are) correct?

$$\begin{aligned} Q_1 :&\quad f \text{ is continuous from } (E,N_1) \text{ to } (E,N_1) \implies f \text{ is continuous from } (E,N_2) \text{ to } (E,N_2) \\ Q_2 :&\quad f \text{ is continuous from } (E,N_2) \text{ to } (E,N_1) \implies f \text{ is continuous from } (E,N_1) \text{ to } (E,N_2) \\ Q_3 :&\quad f \text{ is continuous from } (E,N_1) \text{ to } (E,N_2) \implies f \text{ is continuous from } (E,N_2) \text{ to } (E,N_1) \\ Q_4 :&\quad f \text{ is continuous from } (E,N_2) \text{ to } (E,N_2) \implies f \text{ is continuous from } (E,N_1) \text{ to } (E,N_1) \end{aligned}$$

   Justify your answer.

Exercise 4 (*7 points*). Let $E = \mathbb{R}[X]$ be the vector space of formal polynomials with real coefficients and indeterminate X, and for $P \in E$ define:

$$N_1(P) = \int_0^1 \left|P(t)\right| \mathrm{d}t, \qquad N(P) = \int_0^1 \frac{\left|P(t)\right|}{\sqrt{t}}\, \mathrm{d}t, \qquad N_\infty(P) = \max_{t\in[0,1]} \left|P(t)\right|.$$

1. Let $n \in \mathbb{N}$. Briefly justify that the improper integral

$$I_n = \int_0^1 \frac{t^n}{\sqrt{t}}\, \mathrm{d}t$$

   converges and that $I_n = \dfrac{2}{2n+1}$.

2. Show that N_1 is a norm on E. You're given that N and N_∞ are also norms on E.

3. Let S_1, S and S_∞ be the unit spheres of these norms.

   Determine $a > 0$ and $b > 0$ such that $aX + b \in S \cap S_\infty$.

4. Show that:

$$\forall P \in E,\ N_1(P) \le N(P) \le 2N_\infty(P).$$

5. Study the convergence of the sequence $(X^n)_{n\in\mathbb{N}}$ to 0_E for the norms N_1, N and N_∞.

6. a) Can we deduce that N and N_∞ are not equivalent? justify your answer.

   b) Can we deduce that N and N_1 are equivalent? justify your answer.

7. Define

$$\begin{aligned} \varphi :\ E &\longrightarrow \mathbb{R} \\ P &\longmapsto P(1). \end{aligned}$$

   a) Show that the function φ is continuous from (E, N_∞) to $\mathbb{R}$.

   b) Using the sequence $(X^n)_{n\in\mathbb{N}}$, show that the function φ is not continuous at 0_E from (E, N) to $\mathbb{R}$.