INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #1 — 1h30

October 23, 2020

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Prove, by induction on n, that:

$$\forall n \in \mathbb{N}^*, \ \sum_{k=1}^{n} (k+1)2^k = n2^{n+1}.$$

Exercise 2. Find all the elements $x \in \mathbb{R}$ such that

$$\cos(x) + \sin(x) = \sqrt{2}.$$

Exercise 3. Throughout this exercise, $N \in \mathbb{N}$ is a fixed value.
For $p \in \mathbb{N}$ we define:

$$S_p = \sum_{k=0}^{N} k^p.$$

1. Give (without any justifications) the value of S_0 and of S_1.
2. We now compute the value of S_2: Define:

$$T_2 = \sum_{k=0}^{N} (k+1)^3 - k^3.$$

   a) By evaluating the sum T_2 in two different ways[1], show that:

$$T_2 = (N+1)^3 = 3S_2 + 3S_1 + S_0.$$

   b) Deduce the value of S_2.

3. Adapt the previous method to determine the value of S_3.

[1] one way is to observe that T_2 is a telescopic sum, another way is to expand and simplify the term $(k+1)^3 - k^3$

Exercise 4. Let

$$x : \mathbb{R} \longrightarrow (-\infty, 1]$$
$$f \longmapsto \begin{cases} \dfrac{1}{1+f} & \text{if } f > 0 \\ 1 - f^2 & \text{if } f < 0 \\ 0 & \text{if } f = 0. \end{cases}$$

You're given that the function x is well-defined.

1. Sketch the graph of the function x.
2. What is the range of x? (no justifications required).
3. Is the function x injective? surjective? bijective? justify your answer.
4. Determine, using the graph of x whether:
   - the function x is bounded from above;
   - the function x bounded from below.
5. Determine the following sets (no justifications required):

$$x(\mathbb{R}_+), \qquad x(\mathbb{R}_-), \qquad x(\mathbb{R}_+^*), \qquad x\big((-1, 1]\big),$$

$$x^{[-1]}(\mathbb{R}_-), \qquad x^{[-1]}(\mathbb{R}_-^*), \qquad x^{[-1]}\big([0, 1]\big), \qquad x^{[-1]}\big((0, 1]\big).$$

6. Prove that the function x is increasing on $\mathbb{R}_-^*$.

Exercise 5. Let A, B, C be three non-empty sets and let $f : A \to B$, $g : B \to C$ and $h : A \to C$ be three functions such that $h = g \circ f$.

1. Show that if h is injective then f is injective.
2. Show that if h is surjective then g is surjective.
3. Assume that f is injective and g is surjective. Is h necessarily bijective? justify your answer.

Exercise 6. Let p be the polynomial function defined by:

$$p : \mathbb{C} \longrightarrow \mathbb{C}$$
$$x \longmapsto 2x^4 - 4x^3 + 12x^2 - 4x + 10.$$

Show that i is a root of p and give the factorization of p in $\mathbb{R}$ and in $\mathbb{C}$.