INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #1 — 1h30

October 25, 2019

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1.

1. Recall (without any justifications) the following transformation of sum into product formulas:

$$\cos(p) + \cos(q) \qquad \text{and} \qquad \sin(p) + \sin(q),$$

where $p, q \in \mathbb{R}$.

2. Use these formulas to solve the following equation:

$$\text{(E)} \qquad \cos(2x) + \cos(3x) = \sin(2x) + \sin(3x).$$

Exercise 2. Let f be the function defined by:

$$f : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} x^2 & \text{if } x \leq 0 \\ \dfrac{1}{x} & \text{if } x > 0 \end{cases} \end{array}$$

1. Sketch the graph of f.
2. Is f injective? surjective? bijective? justify your answer.
3. Determine the following sets (no justifications required):

$$f\big([0,1]\big), \qquad f\big((-1,1)\big), \qquad f(\mathbb{R}), \qquad f(\mathbb{R}_+),$$
$$f^{[-1]}(\mathbb{R}_+), \qquad f^{[-1]}(\mathbb{R}_+^*), \qquad f^{[-1]}(\mathbb{R}_-), \qquad f^{[-1]}\big([1,+\infty)\big).$$

Exercise 3. Let f be the polynomial function defined by:

$$f : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^6 - 4x^5 + 6x^4 - 8x^3 + 9x^2 - 4x + 4. \end{array}$$

Show that i is a root of f of multiplicity 2, and give the factored form of f in $\mathbb{C}$ and in $\mathbb{R}$.

Exercise 4.

1. Find the largest subset D of $\mathbb{R}$ such that

$$\forall x \in D,\ \frac{1}{\sqrt{2-\mathrm{e}^{-x}}}$$

is defined.

We hence define the following function:

$$\begin{array}{rccc} f: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{1}{\sqrt{2-\mathrm{e}^{-x}}}. \end{array}$$

2. Determine the variations of f.

3. Give (without any justifications) the range J of f.

We hence define the following surjective function:

$$\begin{array}{rccc} g: & D & \longrightarrow & J \\ & x & \longmapsto & \dfrac{1}{\sqrt{2-\mathrm{e}^{-x}}}. \end{array}$$

4. Briefly explain why g is a bijection.

5. Determine g^{-1} explicitly.

Exercise 5. Let $n \in \mathbb{N}^*$.

1. a) Let $a, b \in \mathbb{R}$. Recall the Binomial Theorem for $(a+b)^{2n}$.
   b) Deduce that $4^n \geq \binom{2n}{n}$.

2. a) Show that

$$\forall k \in \{1, \ldots, n\},\ \binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}.$$

   b) Deduce that for $k \in \mathbb{N}^*$ such that $k \leq n/2$ one has:

$$\binom{n}{k-1} < \binom{n}{k}.$$

   c) Show that for $k \in \mathbb{N}^*$ such that $n/2 \leq k \leq n-1$ one has:

$$\binom{n}{k} > \binom{n}{k+1}.$$

3. For $k \in \{0, \ldots, 2n\}$, compare the values of $\binom{2n}{k}$ and $\binom{2n}{n}$ and deduce that

$$\binom{2n}{n} \geq \frac{4^n}{2n+1}.$$

We have hence proved that

$$\forall n \in \mathbb{N}^*,\ \frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n.$$

Exercise 6. Let A and B be two non-empty sets, and let $f: A \longrightarrow B$ and $g: B \longrightarrow A$ be two functions such that

$$\forall y \in B,\ (f \circ g)(y) = y.$$

Prove that f is surjective and that g is injective.