INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #1

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Exercise 1.

1. The function $t \mapsto \dfrac{t^2}{(t-1)\big(1+t^5\big)}$ is continuous on $(1,+\infty)$ hence I is improper at 1^+ and at $+\infty$. We show that I diverges at 1^+:

$$\frac{t^2}{(t-1)\big(1+t^5\big)} \underset{t\to1^+}{\sim} \frac{1}{2(t-1)} > 0$$

We know that

$$\int_1^{42} \frac{\mathrm{d}t}{t-1}$$

diverges (Riemann at a finite point with $\alpha = 1$) hence by the equivalent test, I diverges at 1^+. Hence I diverges.

2. The function $t \mapsto \dfrac{t^2}{\big(1+t^4\big)\ln(2+t)}$ is continuous on $[0,+\infty)$, hence J is improper at $+\infty$ only. Notice that:

$$\forall t \in \mathbb{R}_+,\ 0 \le \frac{t^2}{\big(1+t^4\big)\ln(2+t)} \le \frac{t^2}{t^4\ln(2+t)} \le \frac{1}{t^2\ln(2+t)} \le \frac{1}{t^2\ln(2)}.$$

We know that $\int_{42}^{+\infty} \frac{\mathrm{d}t}{t^2}$ (Riemann at $+\infty$ with $\alpha = 2 > 1$) hence, by the Comparison Test, J converges.

3. The function $t \mapsto \sin\big(1/t^2\big)$ is continuous on $(0,+\infty)$ hence K is improper at 0^+ and $+\infty$.

   - Convergence at $+\infty$:

   $$\sin\left(\frac{1}{t^2}\right) \underset{t\to+\infty}{\sim} \frac{1}{t^2} > 0$$

   and since $\int_{42} +\infty \dfrac{\mathrm{d}t}{t^2}$ converges at $+\infty$ (Riemann at $+\infty$ with $\alpha = 2 > 1$) we conclude, by the Equivalent Test, that K converges at $+\infty$.

   - Convergence at 0^+: observe that

   $$\forall t \in \mathbb{R}_+^*,\ 0 \le \left|\sin\left(\frac{1}{t^2}\right)\right| \le 1$$

   and since the integral $\int_0^{42} 1\mathrm{d}t$ converges at 0^+ we conclude, by the Comparison Test, that K converges absolutely at 0^+, hence K converges at 0^+.

   We conclude that K converges.

Exercise 2.

1. Define

$$\begin{array}{rccc} \varphi: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (x+\alpha y, y). \end{array}$$

Clearly φ is a linear map, with matrix

$$[\varphi]_{\mathrm{std}} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

and we have

$$\forall (x,y) \in \mathbb{R}^2,\ N(x,y) = \big\|\varphi(x,y)\big\|_\infty.$$

Moreover, φ is invertible since $\det\varphi = 1 \neq 0$. We hence conclude that N is a norm on $\mathbb{R}^2$. Let B_N be the closed unit ball of N and B_∞ be the closed unit ball of $\|\cdot\|_\infty$. We know that

$$B_N = \varphi^{-1}(B_\infty)$$

and we'll use this relation to plot B_N:

$$[\varphi^{-1}]_{\text{std}} = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$$

We need the image of two non-collinear vertices of B_∞ by φ^{-1} to recover B_N:

$$\varphi^{-1}(1,1) = (1-\alpha, 1), \qquad \varphi^{-1}(1,-1) = (1+\alpha, -1),$$

See Figure 1 for the representation B_N in the three cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.

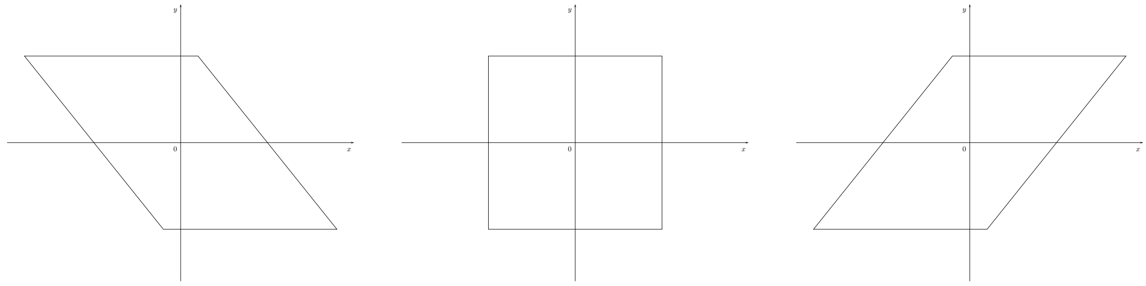

Figure 1. Representation of the closed unit ball of the norm N of Exercise 2 ($\alpha > 0$ on the left, $\alpha = 0$ in the middle, and $\alpha < 0$ on the right)

2. a) The norm $\|\cdot\|_1$ is defined by:

$$\begin{aligned} \|\cdot\|_1 \ : \ E &\longrightarrow \mathbb{R}_+ \\ f &\longmapsto \int_0^{\pi/2} \big|f(t)\big| \,\mathrm{d}t. \end{aligned}$$

   b) The distance between f and s is given by:

$$d = \|f - s\|_1$$

   and we compute it as follows:

$$\begin{aligned} d &= \int_0^{\pi/2} \big|t - \sin(t)\big| \,\mathrm{d}t \\ &= \int_0^{\pi/2} t - \sin(t) \,\mathrm{d}t && \textit{since } \forall t \in \mathbb{R}_+,\ \sin(t) \le t \\ &= \frac{\pi^2}{8} - 1. \end{aligned}$$

3. The correct statement is:

$$\forall r > 0,\ \overline{B'_r} \subset \overline{B_{2r}}.$$

   Proof. Let $r > 0$.
   Let $u \in \overline{B'_r}$, i.e., $u \in E$ and $\|u\|' \le r$. Hence $2\|u\|' \le 2r$, and since $\|u\| \le 2\|u\|'$ we conclude that $\|u\| \le 2r$, hence $u \in \overline{B_{2r}}$. Hence $\overline{B'_r} \subset \overline{B_{2r}}$.

Exercise 3.

Part I

1. The function $x \mapsto x\mathrm{e}^{-\lambda x}$ is continuous on $\mathbb{R}_+$ hence I is improper at $+\infty$. Let $A \in \mathbb{R}_+^*$. Then:

$$\begin{aligned} \int_0^A x\mathrm{e}^{-\lambda x} \,\mathrm{d}x &= \left[-\frac{1}{\lambda} x\mathrm{e}^{-\lambda x}\right]_{x=0}^{x=A} + \frac{1}{\lambda}\int_0^A \mathrm{e}^{-\lambda x} \,\mathrm{d}x && \textit{by an integration by parts} \\ &= -\frac{A}{\lambda}\mathrm{e}^{-\lambda A} + \frac{1}{\lambda}\left[-\frac{1}{\lambda}\mathrm{e}^{-\lambda x}\right]_{x=0}^{x=A} \\ &= -\frac{A}{\lambda}\mathrm{e}^{-\lambda A} - \frac{1}{\lambda^2}\mathrm{e}^{-\lambda A} + \frac{1}{\lambda^2} \underset{A\to+\infty}{\longrightarrow} 0 + 0 + \frac{1}{\lambda^2} \end{aligned}$$

   Hence I converges and $I = 1/\lambda^2$.

2. a) For $x \geq 1$ one has $x^2 \geq x$ hence $-x^2 \leq -x$ and we hence conclude:

$$\forall x \in [1, +\infty),\ 0 \leq \mathrm{e}^{-x^2} \leq \mathrm{e}^{-x}$$

We know that $\int_0^{+\infty} \mathrm{e}^{-x}\,\mathrm{d}x$ converges at $+\infty$ hence we conclude, by the Comparison Test, that G converges.

b) The integral J is only improper at $+\infty$. Let $A \in \mathbb{R}_+^*$. Then, using the substitution $u = \dfrac{x}{\sqrt{2}\,\sigma}$ yields:

$$\begin{aligned}\int_0^A \exp\left(-\frac{x^2}{2\sigma^2}\right)\,\mathrm{d}x &= \int_0^{A/\sqrt{2}\,\sigma} \mathrm{e}^{-u^2}\sqrt{2}\,\sigma\,\mathrm{d}u\\ &\underset{A\to+\infty}{\longrightarrow} \sqrt{2}\,\sigma G = \frac{\sqrt{2\pi}\sigma}{2}\end{aligned}$$

c) The improper integral is only improper at $+\infty$. Now for $X \in \mathbb{R}_+^*$,

$$\begin{aligned}\int_0^X x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\,\mathrm{d}x &= \int_0^X x \times x \exp\left(-\frac{x^2}{2\sigma^2}\right)\,\mathrm{d}x\\ &= \left[-x\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_{x=0}^{x=X} + \sigma^2 \int_0^X \exp\left(-\frac{x^2}{2\sigma^2}\right)\\ &= -X\sigma^2 \exp\left(-\frac{X^2}{2\sigma^2}\right) + \sigma^2 \int_0^X \exp\left(-\frac{x^2}{2\sigma^2}\right)\\ &\underset{X\to+\infty}{\longrightarrow} \sigma^2 \int_0^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) = \frac{\sigma^3\sqrt{2\pi}}{2}\end{aligned}$$

d) The improper integral K is improper at $+\infty$ and at $-\infty$. We already showed that K is convergent at $+\infty$. Since the function $x \mapsto x^2 \exp\left(-\dfrac{x^2}{2\sigma^2}\right)$ is even, we conclude that K is also convergent at $-\infty$ and that

$$K = 2\int_0^{+\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\,\mathrm{d}x = \sigma^3\sqrt{2\pi}.$$

Part II

1. Since ϕ is continuous and positive on $\mathbb{R}_+$, the function $x \mapsto \phi(x)\ln\bigl(\phi(x)\bigr)$ is well-defined and continuous on $\mathbb{R}_+$, hence the improper integral H_1 is improper at $+\infty$. For $x \in \mathbb{R}_+$,

$$\phi(x)\ln\bigl(\phi(x)\bigr) = \lambda\mathrm{e}^{-\lambda x}\bigl(\ln(\lambda) - \lambda x\bigr) = \lambda\ln(\lambda)\mathrm{e}^{-\lambda x} - \lambda^2 x\mathrm{e}^{-\lambda x}.$$

We know that the following improper integrals converge (and we even know their values):

$$\int_0^{+\infty} \lambda\ln(\lambda)\mathrm{e}^{-\lambda x}\,\mathrm{d}x = \ln(\lambda) \qquad \text{and} \qquad \int_0^{+\infty} \lambda^2 x\mathrm{e}^{-\lambda x} = \lambda^2 I = 1.$$

Hence H_1 is convergent and

$$H_1 = 1 - \ln(\lambda).$$

2. Since ϕ is continuous and positive on $\mathbb{R}$, the function $x \mapsto \phi(x)\ln\bigl(\phi(x)\bigr)$ is well-defined and continuous on $\mathbb{R}$, hence the improper integral H_2 is improper at $-\infty$ and $+\infty$. For $x \in \mathbb{R}$,

$$\phi(x)\ln\bigl(\phi(x)\bigr) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)\left(-\frac{1}{2}\ln\bigl(2\pi\sigma^2\bigr) - \frac{x^2}{2\sigma^2}\right)$$

We know that the following improper integrals converge (and we even know their values):

$$J = \int_0^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right)\,\mathrm{d}x = \frac{\sqrt{2\pi}\,\sigma}{2} \qquad \text{and} \qquad K = \int_0^{+\infty} x^2\exp\left(-\frac{x^2}{2\sigma^2}\right)\,\mathrm{d}x = \frac{\sigma^3\sqrt{2\pi}}{2}$$

hence K converges at $+\infty$. Since the function we're integrating is even, K is also convergent at $-\infty$ and:

$$H_2 = \frac{\ln\bigl(2\pi\sigma^2\bigr)}{2\sqrt{2\pi\sigma^2}}2J\frac{1}{2\sigma^2\sqrt{2\pi\sigma^2}}K = \frac{1}{2}\ln\bigl(2\pi\sigma^2\bigr) + \frac{1}{2} = \frac{\ln\bigl(2\pi\sigma^2\bigr) + 1}{2}.$$