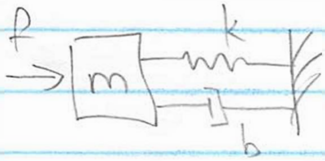
Root Locus Plotting - Introduction

e.g.



$$(mD^2 + bD + k)x = f$$

$$T_{fx} = x/f = \frac{1}{mD^2 + bD + k}$$

$$T_{fx}(s) = \frac{1}{ms^2 + bs + k}$$

Poles of T_{fx} are the roots (sol'ns) s of

$$ms^2 + bs + k = 0$$

or $$s^2 + \frac{b}{m}s + \frac{k}{m} = 0 \quad (1)$$

or $$s^2 + 2\sigma s + \omega_n^2 = 0 \quad (2), \quad \sigma = \frac{b}{2m}, \quad \omega_n^2 = \frac{k}{m}$$

a) Plot s in (2) as k increases from 0 to ∞, or equivalently, as ω_n increases from 0 to ∞.

Solution: $\sigma = \frac{b}{2m}$ is constant since b and m are constant.

(2) $\Rightarrow$ $$s = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2} = -\sigma \pm d \quad (3), \text{ where } d = \sqrt{\sigma^2 - \omega_n^2}$$

$$s_1 = -\sigma + d, \quad s_2 = -\sigma - d$$

$\therefore$ The midpoint of the 2 poles is

$$\frac{1}{2}(s_1 + s_2) = -\sigma$$

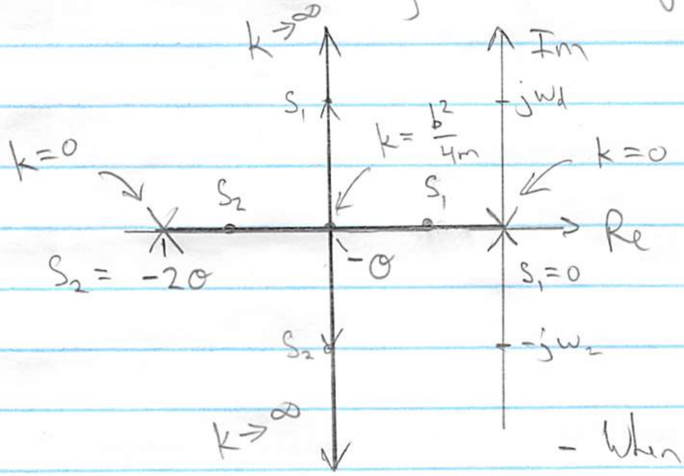
$\therefore$ The midpoint of the poles is constant $= -\sigma$.

When $k = 0 = \omega_n$, (2) gives

$$s(s+2\sigma) = 0 \Rightarrow s_1 = 0 \text{ and } s_2 = -2\sigma$$

as the 2 sol'ns when $k=0$.

Plot these starting values of s as X :



- As k increases, d in (2) gets smaller, so s_1 moves left and s_2 moves right, maintaining the constant midpoint $-\sigma$.

- When $k = \frac{b^2}{4m}$, $\omega_n^2 = \sigma^2$ and $d = 0$, so $s_1 = s_2 = -\sigma$.

For $k > \frac{b^2}{4m}$, $\omega_n^2 > \sigma^2$ and $d = j\omega_d$, $\omega_d = \sqrt{\omega_n^2 - \sigma^2}$

So $s_1 = -\sigma + j\omega_d$ and $s_2 = -\sigma - j\omega_d$ as shown.

As $k \to 0$, $s = -\sigma \pm j\omega_d$ where $\omega_d \to \infty$.

b) Now consider instead m and k to be constant while b varies from 0 to ∞.

Then $\omega_n = \sqrt{\frac{k}{m}}$ is constant, and

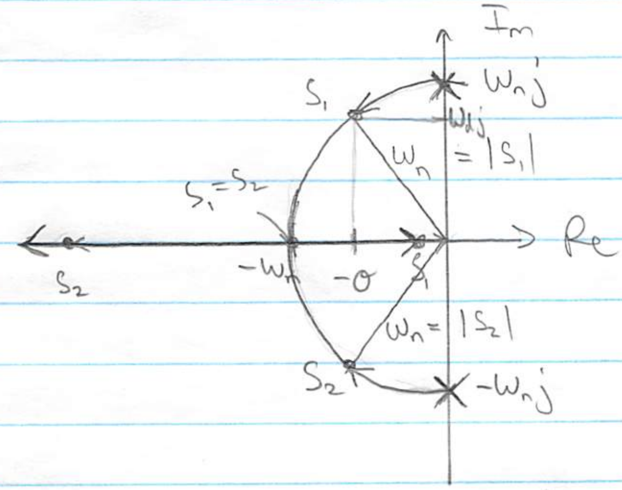
$\sigma = \frac{b}{2m}$ varies from 0 to ∞.

Plot sol'ns s of (2): $s^2 + 2\sigma s + \omega_n^2 = 0$ as σ increases.

Start with $b = 0 = \sigma$ in (2):

$$s^2 + \omega_n^2 = 0 \Rightarrow s_{1,2} = \pm \omega_n j$$

Mark these starting values as X:



When $s_{1,2}$ are complex,

$$s_{1,2} = -\sigma \pm j\omega_d$$

And $|s_1| = |s_2| = \sqrt{\sigma^2 + \omega_d^2}$
$= \sqrt{\omega_n^2} = \omega_n$

Since ω_n is constant, complex sol'ns move with constant magnitude ω_n

$\therefore$ Complex sol'ns s_1, s_2 move on a circle.

When $\sigma = \omega_n$, $s_1 = s_2 = -\omega_n$.

Since $\omega_n^2 = s_1 s_2$ (whether s_1, s_2 are complex or real) and since ω_n^2 is constant, $s_2 = 1/s_1$, and so

$s_2 \to \bar{\infty}$ as $s_1 \to \bar{0}$.

Root Locus Plotting Techniques

Recall that the poles of $T(s) = \frac{z(s)}{p(s)}$ are the roots of $p(s)$, i.e. the solutions s of the characteristic equation $p(s) = 0$ (1)

An R-L plot shows how these roots move in the complex plane as some parameter k in $p(s)$ varies

R-L Step 1) rewrite (1) in the form

$$1 + k\,L(s) = 0 \qquad (2) \qquad \text{(R-L form)}$$

where $L(s) = \frac{b(s)}{a(s)}$ is a ratio of monic polynomials
↳ (leading coefficients = 1).

e.g. 1) mass-spring-damper with $m=1$, $b=2$ and variable stiffness k:

$$(1) \rightarrow \quad p(s) = s^2 + 2s + k = 0 \qquad (3)$$

To write (3) in R-L form (2), $\div$ by s^2+2s:

$$1 + k\frac{1}{s^2+2s} = 0, \quad \text{so } L(s) = \frac{1}{s^2+2s}.$$

We will learn how to plot root loci from this form.

In the previous lecture, we plotted the loci directly from (3) using the quadratic equation. However, higher-order systems require R-L plotting techniques.

e.g. 2) Suppose proportional control $u = ke = k(r-y)$ is applied to the plant $P(D) = \dfrac{1}{D^2+2D}$, which might be a mass $m=1$ with damping $b=2$, position y, and control force u. A closed-loop transfer function is (with $C(D) = k$):

$$T_{ry} = \frac{y}{r} = \frac{PC}{1+PC} = \frac{\frac{k}{D^2+2D}}{1+\frac{k}{D^2+2D}} = \frac{k}{D^2+2D+k}$$

This has the same char. eq'n (3) as in e.g. 1. However, notice that T_{ry} gives the R-L form directly by setting $1 + P(s)C(s) = 0$, where $C(s) = k$:

$$1 + k\,P(s) = 0$$

$$1 + k\frac{1}{s^2+s} = 0$$

In summary, the poles of $T_{ry}(D)$ are the values of s that make $|T_{ry}(s)| = \infty$, which is equivalent to

$1 + P(s)C(s) = 0$ and to $s^2 + 2s + k = 0$.

e.g. 3) Suppose P-I control $C(D) = \dfrac{k_P D + k_I}{D}$ is used in the previous example, with $\dfrac{k_I}{k_P} = 1$ and variable k_P

Then $C(D) = k_P\left(\dfrac{D + \frac{k_I}{k_P}}{D}\right) = k_P \dfrac{D+1}{D}$ and

$$P(D)\,C(D) = k_P \frac{D+1}{D^2(D+2)} \equiv k_P\, L(D)$$

This gives R-L form of char eq'n: $1 + k_P L(s) = 0$.

R-L Plotting Rules

Char. eq'n: $1 + k\,L(s) = 0$ (2) where

$$L(s) = \frac{b(s)}{a(s)} = \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} \quad (4)$$

and we assume that $n \geq m$ ($L(s)$ is proper).

(4) in (2): $(s-p_1)(s-p_2)\cdots(s-p_n) + k\,(s-z_1)(s-z_2)\cdots(s-z_m) = 0$ (5)

This is the char eq'n in polynomial form. Since the l.h.s. is an n^{th} order polynomial, (5) has n (complex) solutions. This follows from the Fundamental Theorem of Algebra.

i.e. (5) may by written (for a given value of k) as:

$$(s-s_1)(s-s_2)\dots(s-s_n) = 0 \quad (6) \quad \text{where}$$

the s_i are the n complex sol'ns of (5) (and (6)). We want to plot the s_i as k varies!

If $k=0$, (5) gives $(s-p_1)(s-p_2)\dots(s-p_n) = 0$, so the solutions are $s_1 = p_1$, $s_2 = p_2$ ... $s_n = p_n$.

⇒ Rule 1a) There are n roots of (2) (c.L. poles) and they start (at $k=0$) at the poles of $L(s)$ (the open-loop poles).

(2) gives $L(s) = -\frac{1}{k}$ (7), so as

$$k \to \infty, \quad L(s) \to 0$$

(4) $\Rightarrow$

There are n ways that $L(s)$ can approach 0:

Rule 1b) m of the roots approach the m zeros $z_1, z_2 \dots z_m$ of $L(s)$

Rule 1c) the other $n-m$ roots become infinite in magnitude : $|s| \to \infty$ (with angles that satisfy (7))

Angle Condition:

(7) $\Rightarrow$ $\angle L(s) = \angle -\frac{1}{k} = 180^\circ$ (since $k > 0$)

Also $\angle L(s) = \angle \frac{b(s)}{a(s)}$

$$= \angle b(s) - \angle a(s)$$

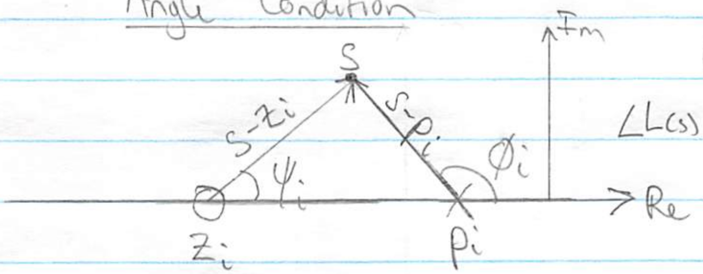
$$= \angle (s-z_1)(s-z_2) \dots (s-z_m) - \angle (s-p_1)(s-p_2) \dots (s-p_n)$$

$$= \angle (s-z_1) + \angle (s-z_2) + \dots \angle s-z_m - \left(\angle (s-p_1) + \angle (s-p_2) + \dots \angle (s-p_n)\right)$$

$$= \sum_{i=1}^{m} \psi_i - \sum_{i=1}^{n} \phi_i$$

where $\psi_i = \angle (s-z_i)$ and $\phi_i = \angle (s-p_i)$

Angle Condition



(7) ⇒ Angle Condition:

$$\angle L(s) = \sum^{m} \psi_i - \sum^{n} \phi_i = 180 \qquad (8)$$

s must satisfy (8) to be on the root locus.

Also,

$$(7) \Rightarrow \quad k = \frac{-1}{L(s)} = -\frac{a(s)}{b(s)}$$

For $k > 0$, $\quad k = |k| = \dfrac{|a(s)|}{|b(s)|}$

$$\therefore \quad k = \frac{|s-p_1||s-p_2| \cdots |s-p_n|}{|s-z_1||s-z_2| \cdots |s-z_m|} \qquad (9)$$

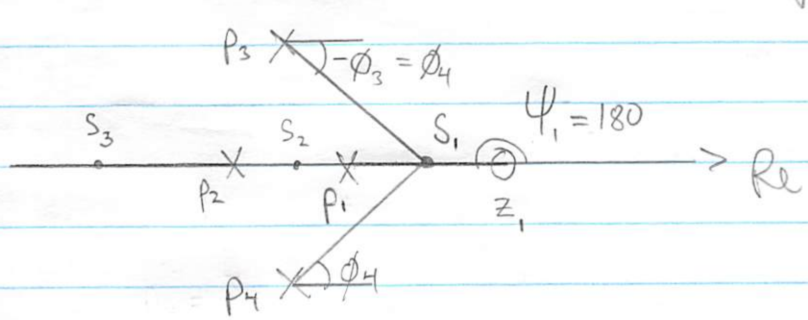
This magnitude condition gives k for a given s on the locus.

∴ We can use (8) to draw the locus, and (9) to find k.

R-L Plotting Rules (based on Angle Condition)

Rule 2) The loci include the real axis to the left of an odd number of poles + zeros:

e.g.



$\angle L(s_2) = 0$
so $s_2 \notin$ locus

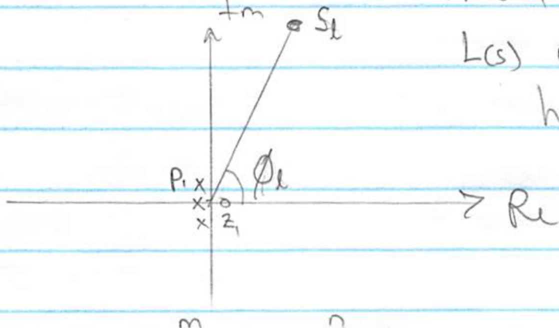
$\angle L(s_3) = 180$
so $s_3 \in$ locus.

At s_1, $\psi_1 = 180$, $\phi_1 = \phi_2 = 0$, and $\phi_3 + \phi_4 = 0$

$\therefore \angle L(s_1) = \sum \psi_i - \sum \phi_i = \psi_1 = 180$, so s_1 is on the locus

Rule 1c says $n-m$ roots approach ∞ in magnitude.

Consider an $s = s_\ell$, where $|s_\ell|$ is very large:



- Viewed from ∞, all poles and zeros of $L(s)$ are near the origin and have the same angle ϕ_ℓ:

$$\phi_\ell \simeq \angle(s_\ell - p_1) \simeq \angle(s_\ell - z_1)$$
$$\simeq \angle(s_\ell - p_i) \text{ etc.}$$

$$\therefore \quad \angle L(s_\ell) = \sum^{m} \psi_i - \sum^{n} \phi_i \quad \simeq \quad (m-n)\,\phi_\ell$$

Since $180 \equiv -180$, the angle condition gives

$$(m-n)\,\phi_\ell \equiv {}^{+}180$$
$$\equiv -180$$
$$(n-m)\,\phi_\ell \equiv 180 \qquad (10)$$

(10) has $n-m$ solutions for ϕ, which give the angles of $n-m$ asymptotes as $|s_\ell| \to \infty$.

The $n-m$ sol'ns of (10) are

$$\phi_\ell = \frac{180 + 360(\ell-1)}{n-m}, \quad \ell = 1, 2 \ldots, n-m \qquad (11)$$

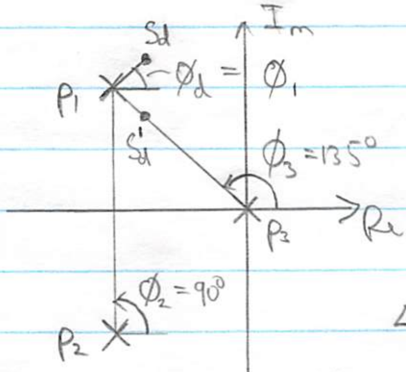
It can be shown that the asymptotes cut the real axis at

$$\alpha = \frac{\sum^{n} p_i - \sum^{m} z_i}{n-m} \qquad (12)$$

Departure and Arrival Angles

As k increases from zero, roots s depart from the poles of $L(s)$. As $k \to \infty$, roots approach asymptotes or arrive at the zeros of $L(s)$. The departure and arrival angles must satisfy the angle condition.

For example, suppose $L(s) = \dfrac{1}{s(s^2+2s+2)} = \dfrac{1}{s((s+1)^2+1)}$, so $p_1 = 1+j$, $p_2 = \bar{p}_1 = 1-j$, $p_3 = 0$:



Find the angle of departure ϕ_d from p_1 of a point s_d on the locus.

The angle condition gives

$$\angle(s_d - p_1) + \angle(s_d - p_2) + \angle(s_d - p_3) = 180^\circ$$

or

$$\phi_1 + \phi_2 + \phi_3 = 180^\circ$$

As $s_d \to p_1$, this gives

$$\phi_d + 90^\circ + 135^\circ = 180^\circ$$

$$\therefore \quad \phi_d = \underline{\underline{-45^\circ}}$$

Therefore, the s_d shown in the figure is <u>not</u> on the root locus: the correct position is s_d'.

Also, the root departing from p_2 is the complex conjugate of the one departing from p_1, so its departure angle must be $+45^\circ$.

exercise: check this directly from the angle condition.

Generalizing this example gives:

Rule 4. The locus departs from a single pole p_i at an angle of

$$\phi_d = \sum_{j=1}^{m} \psi_i - \sum_{j \neq i}^{n} \phi_j + 180 \qquad (13)$$

If the pole p_i has a multiplicity of q, then q root locus branches depart from it at the q angles ϕ_d found by solving

$$q\,\phi_d = \sum_{i=1}^{m} \psi_i - \sum \phi_i + 180 \qquad (14)$$

where the sum $\sum \phi_i$ excludes the q angles $q\,\phi_d$ from p_i to s_d already accounted for on the l.h.s of (14).

Similarly, if a zero z_i has multiplicity q, then q root locus branches approach it at the q angles ϕ_a found by solving

$$q\,\psi_a = \sum_{i=1}^{n} \phi_i - \sum \psi_i - 180 \qquad (15)$$

where $\sum \psi_i$ excludes those already in the l.h.s of (15)

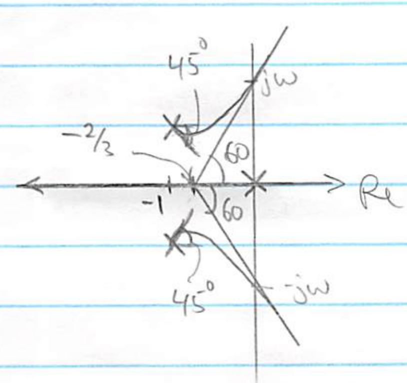
Let's use Rules 1-4 to complete the previous example, assuming $p_1 = -1+j$, $p_2 = -1-j$ and $p_3 = 0$.

Rule 3: $n=3$, $m=0$, so $(n-m)\phi_l = 3\phi_l = 180$

$$\Rightarrow \phi_l = \pm 60,\ 180$$

These 3 asymptotes cut the real axis at

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{-1+-1}{3} = -2/3$$



Rule 2 says that the real axis to the left of the pole at 0 is on the locus.

In this example, $L(s) = \dfrac{1}{(s-p_3)(s-p_1)(s-p_2)}$

$$= \frac{1}{s\,(s+1-j)(s+1+j)} = \frac{1}{s\,(s^2+2s+2)}$$

Find the range of $k>0$ such that $(1+k\,L(s))^{-1}$ is a stable transfer f'n, i.e. $1+k\,L(s)=0$ has all roots s in the L.H.P.

Sol'n: Instability occurs when s crosses the imaginary axis at $s=\pm jw$ for some w.

$\therefore$ find this w and find $k = -\dfrac{1}{L(jw)}$.

$$1+k\,L(s)=0 \Rightarrow s\,(s^2+2s+2)+k=0$$

$$s^3+2s^2+2s+k=0$$

At $s=jw$,

$$(jw)^3+2(jw)^2+2jw+k=0$$

$$jw\,(2-w^2)+k-2w^2=0 \qquad (16)$$

(16) $\Rightarrow$ $w=0$ and $k=0$ <u>or</u>

$w^2=2$ and $k=2w^2=4$

Therefore, the system is stable for

$\underline{\underline{0<k<4}}$, and $\underline{s=\pm\sqrt{2}\,j \text{ when } k=4}$

e.g. Suppose proportional control $u = ke$ is applied to the plant $P = \frac{1}{(D+1)^2(D+4)}$

Plot the root loci of the closed-loop poles for $k > 0$ and find the range of k for stability. Label any imaginary axis crossings with their values.

Asymptotes: $3\phi_l = 180 \Rightarrow \phi_l = \pm 60, 180$

These cut $\mathbb{R}$ at $\alpha = \frac{-1 + -1 + -4}{3} = \frac{-6}{3} = -2$

char eq'n: $(s+1)^2(s+4) + k = 0$

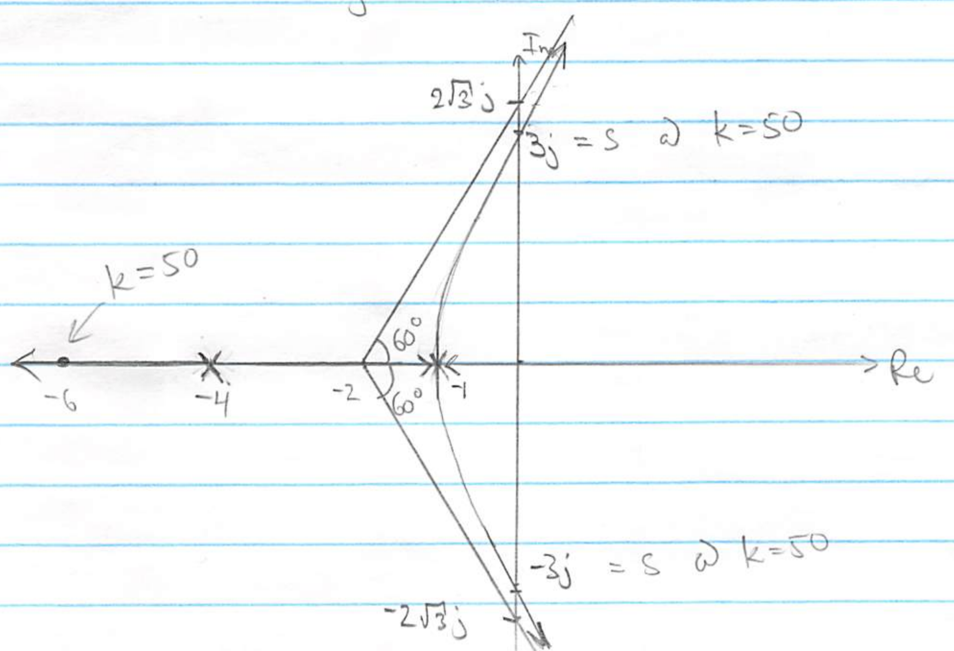
$$(s^2+2s+1)(s+4) + k = 0$$

$$s^3 + 6s^2 + 9s + 4 + k = 0$$

$$j\omega(9-\omega^2) + k + 4 - 6\omega^2 = 0$$

$\omega = 0$ gives $k = -4 \not> 0$

$\omega = 3$ gives $k = 6(3)^2 - 4 = 50$



Additional Root Locus Properties

Lemma 1: If the pole-zero pattern of $L(s)$ shifts to the left or right, then so does the root locus.

Proof: If all poles and zeros of $L(s) = \dfrac{\prod_{k=1}^{m}(s-z_k)}{\prod_{k=1}^{n}(s-p_k)}$ are shifted

right by $r \in \mathbb{R}$, then $L(s)$ shifts to $L^*(s) = \dfrac{\prod_{k=1}^{m}(s-(z_k+r))}{\prod_{k=1}^{n}(s-(p_k+r))}$

which gives $L^*(s) = L(s-r)$

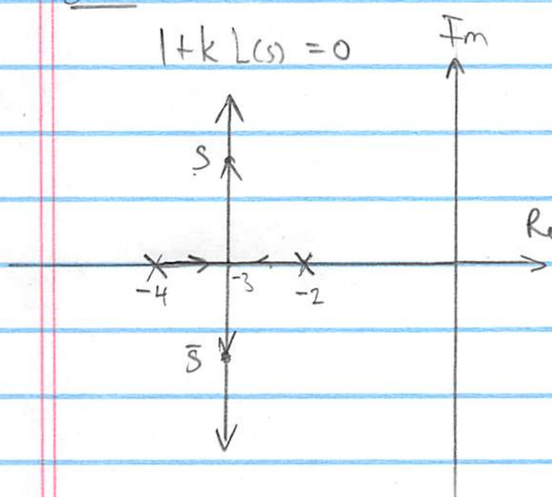
and $L^*(s+r) = L(s)$

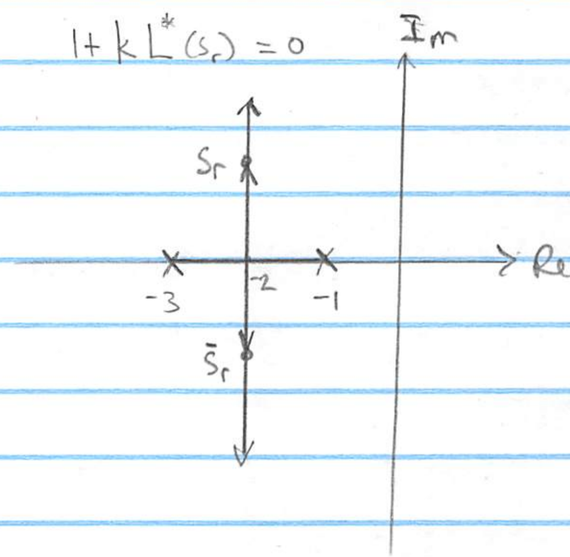
$\therefore$ If s solves $1 + kL(s) = 0$, then

$s_r = s+r$ solves $1 + kL^*(s_r) = 0$.

example: Plot the root loci for poles of $L(s)$ at -4 and -2 (and no zeros) and for poles of $L^*(s)$ at -3 and -1.

sol'n





<u>Lemma 2</u> : If $L(s)$ has one zero z and two poles, then any complex solutions of $1 + kL(s) = 0$ lie on a circle centred at z.

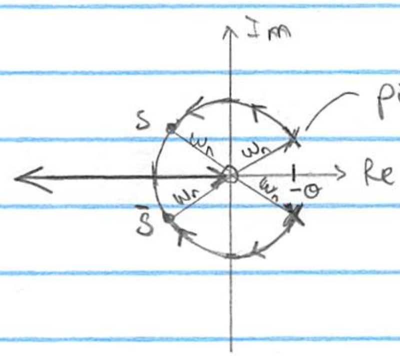
<u>Proof</u>: Start with the case $z = 0$.

Then $L(s) = \dfrac{s}{s^2 + 2\sigma s + \omega_n^2}$ and

$1 + kL(s) = 0$ gives $s^2 + (2\sigma + k)s + \omega_n^2 = 0$ (1)

and $s_1 s_2 = \omega_n^2$, where $s = s_1, s_2$ solve (1).

If a solution s of (1) is complex, then $s_1 s_2 = s\bar{s} = \omega_n^2$ and hence $|s| = \omega_n$. This gives the root locus:



If the zero of $L(s)$ is not at $z = 0$ but at some general $z \in \mathbb{R}$, then the above locus shifts by z, but is still a circle wherever it is complex.