Block Diagrams and Feedback Control

Given $u\, z(D) = y\, p(D)$, we may write

$u\, T_{uy}(D) = y$, where $T_{uy}(D) = \dfrac{z(D)}{p(D)}$ is the transfer function from u to y.

$$u \longrightarrow \boxed{T_{uy}(D)} \longrightarrow y$$

(block diagram)

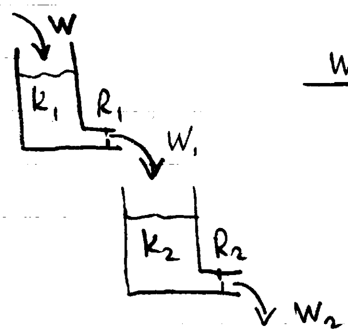
Larger systems may be formed from interconnections of smaller subsystems.

1. **Series Connection** (Multiplication of Transfer Functions)

$$u \longrightarrow \boxed{T_{uv}(D)} \xrightarrow{\;v\;} \boxed{T_{vy}(D)} \longrightarrow y \quad \equiv \quad u \longrightarrow \underbrace{\boxed{T_{uv}(D)\, T_{vy}(D)}}_{T_{uy}(D)} \longrightarrow y$$

i.e. $T_{uv} T_{vy} = T_{uy}$ since $\frac{v}{u} \cdot \frac{y}{v} = \frac{y}{u}$

e.g.

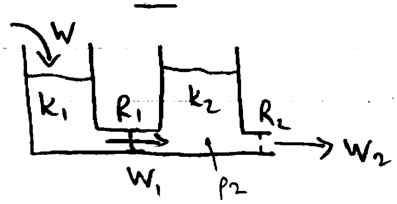


$$W \longrightarrow \boxed{\frac{k_1}{R_1 D + k_1}} \longrightarrow W_1$$

$$W_1 \longrightarrow \boxed{\frac{k_2}{R_2 D + k_2}} \longrightarrow W_2$$

$$\Longrightarrow \quad W \longrightarrow \boxed{\frac{k_1 k_2}{(R_1 D + k_1)(R_2 D + k_2)}} \longrightarrow W_2$$

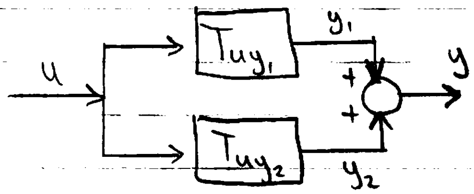
Note: This is <u>not</u> the same system as



Here, W_1 depends on W and W_2 whereas, in the above system, W_1 depends only on W.

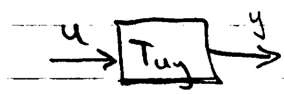
W_1 depends on back pressure p_2

2. Parallel Connection (Addition of TFs)



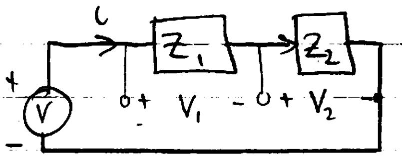
$$y = y_1 + y_2$$
$$= u(T_{uy_1}) + u(T_{uy_2})$$
$$= u\,(T_{uy_1} + T_{uy_2})$$

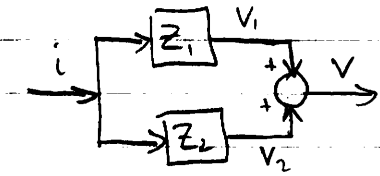
$\equiv$



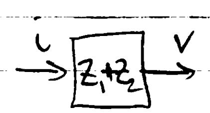
where
$T_{uy} = T_{uy_1} + T_{uy_2}$

e.g. 1)

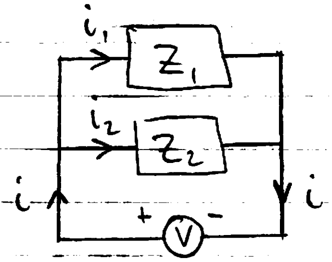


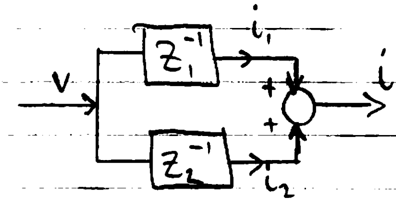


$\equiv$

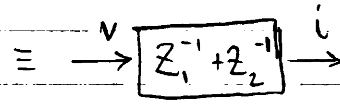


e.g. 2)

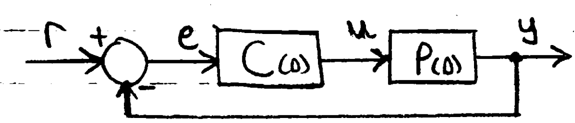




$\equiv$



3. Feedback Connection



This block diagram implies:

$$e = r - y \quad (1)$$
$$eC = u \quad (2)$$
$$uP = y \quad (3)$$

where $y(t)$ is the output of the 'plant' $P(s)$
$r(t)$ is a reference that we want $y(t)$ to follow.
$e(t) = r(t) - y(t)$ is the tracking error.
$u(t)$ is the control input to the plant $P(s)$
$u(t)$ is computed by the controller $C(s)$ in response to the measured error $e(t)$.

The performance of the feedback system is determined by the closed-loop transfer functions

$$T_{re}(s), \quad T_{ru}(s), \quad \text{and} \quad T_{ry}(s).$$

a) Find $T_{re} = \frac{e}{r}$:

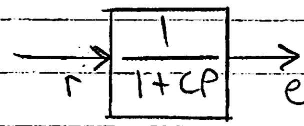
The series connection CP (eq'ns (2) & (3)) gives

$$e\,CP = y$$

add (1) : $e = r - y$

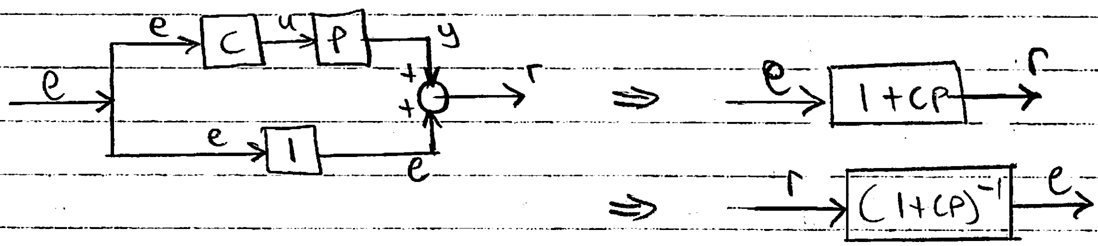
$$e(1+CP) = r \quad \Rightarrow \quad \frac{e}{r} = \frac{1}{1+CP}$$

∴ The feedback connection reduces to

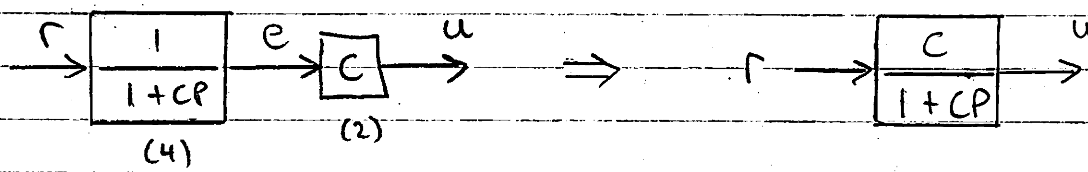


(4)

Alternately, we can get (4) by recognizing that the feedback connection (1), (2), (3) is just a parallel connection in disguise:

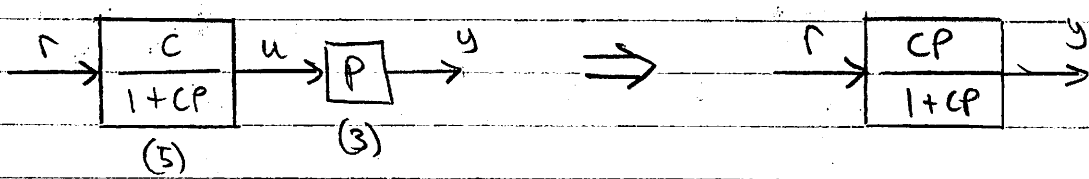


b) Find $T_{ru} = \frac{u}{r}$:



(5)

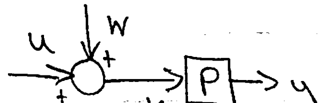
c) Find $T_{ry} = \frac{y}{r}$:



(6)

Disturbance Effects

Typically the plant P is influenced by a disturbance $w(t)$, in addition to the control input $u(t)$:

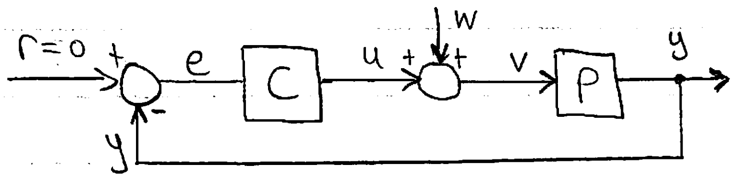


$= (u+w)P$

We can control u, but w is unknown.

We can analyse the effect of w on the closed-loop system (CLS), by setting $r=0$:

CLS:



$V = u + w$ (1)

$vP = y$ (2)

$y(-1) = e$ (3) since $r=0$

$eC = u$ (4)

Find: a) $T_{wv} = \frac{v}{w}$ b) T_{wy} c) T_{we} d) T_{wu}

Sol'n: a) $v = w + u$, where $u = vP(-1)C = -vPC$ from (2),(3),(4)

$= w - vPC$

$\Rightarrow v(1+PC) = w \Rightarrow T_{wv} = \frac{v}{w} = \boxed{\frac{1}{1+PC}}$ (5)

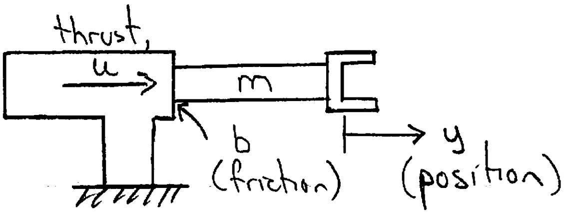
b) $T_{wy} = T_{wv}\,T_{vy} = \underset{(5)}{\frac{1}{1+PC}}\ \underset{(4)}{P} = \boxed{\frac{P}{1+PC}}$ (6)

c) $T_{we} = T_{wy}\,T_{ye} = \underset{(6)}{\frac{P}{1+PC}}\ \underset{(3)}{(-1)} = \boxed{\frac{-P}{1+PC}}$ (7)

d) $T_{wu} = T_{we}\,T_{eu} = \underset{(7)}{\frac{-P}{1+PC}}\ \underset{(4)}{C} = \boxed{\frac{-PC}{1+PC}}$ (8)

TFs (5)-(8) give the effect of w acting alone, with $r=0$.
If $r \neq 0$ and $w \neq 0$, then $e = r\,T_{re} + w\,T_{we}$, etc.
ie. The CLS has 2 inputs (r, w) and 4 outputs ...

e.g. 1) <u>Robot Control</u>



Eq'n of motion (plant):

$$u = m\ddot{y} + b\dot{y} = y\,(mD^2 + bD)$$

$$\therefore \quad y = u\,P(D), \quad P(D) = \frac{1}{mD^2 + bD} = \frac{1}{D(mD+b)} \qquad (1)$$

Design $C(D)$ so $y(t)$ follows ref $r(t)$ with small error $e(t) = r(t) - y(t)$

Try <u>Proportional Control</u>: $C(D) = k > 0$, i.e., $\underline{u = ke}$ (2)

a) Find $T_{re}(D) = \frac{e}{r}$

b) Is the C.L.S. stable for all $k > 0$?

c) If $r(t) = 2$ (constant), find the steady-state error
$e_{ss} = e_{\infty} = \lim_{t \to \infty} e(t)$.

<u>Sol'n:</u> a) $$T_{re}(D) = \frac{1}{1+CP} = \frac{1}{1+kP} = \frac{1}{1 + \frac{k}{mD^2 + bD}} = \frac{D(mD+b)}{mD^2 + bD + k} \qquad (3)$$

b) poles of $T_{re} = -\frac{b}{2m} \pm \sqrt{b^2 - 4mk}$. Since $m > 0$, $b > 0$, and $k > 0$, real part of poles < 0 (stable).

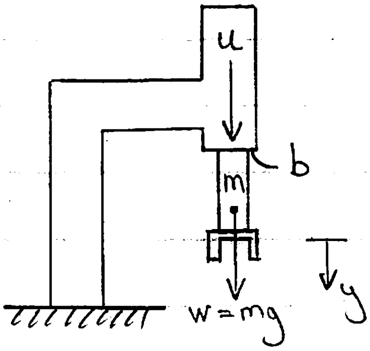
c) $e_{ss} = 2\,T_{re}(0) = \underline{\underline{0}}$

This result can be found 2 ways:

i) $r(t) = 2 = 2e^{0t} \Rightarrow s = 0$ and $T_{re}(s)\big|_{s=0} = 0$

ii) Consider step input $r = 2h = \frac{2}{D}$.
Then, $e = \frac{2}{D}\,T_{re}(D)$ and $e_{\infty} = \lim_{D \to 0} D\,e(D) = 0$.

e.g. 2) <u>Robot Control with Disturbance, w</u>



Now, gravity adds a disturbance force $w = mg$. So $y = (u+w)P(D)$ (4)

Assume $r = 0$.

Want small $e = r - y = -y$ despite w.

d) Find $T_{we} = \left.\dfrac{e}{w}\right|_{r=0}$ with proportional control $u = ke$

e) Find e_{ss}.

Sol'n d) From (7) on p.4,

$$T_{we}(D) = \frac{-P}{1+PC} = \frac{-P}{1+kP} = -\frac{1}{D(mD+b)} \cdot \frac{D(mD+b)}{mD^2+bD+k} = \frac{-1}{mD^2+bD+k} \quad (5)$$

e) $w = mge^{0t} \Rightarrow e_{ss} = w\,T_{we}(0) = \dfrac{-mg}{k}$ (6)

$\therefore$ The bigger the control gain k, the smaller the error e_{ss} due to mg.

e.g. 3) Try <u>Integral Control</u>: $u(t) = k_I \int_{-\infty}^{t} e(\tau)\,d\tau$ (7)

(7) $\Rightarrow u = k_I h e = \dfrac{k_I}{D} e \Rightarrow C(D) = \dfrac{k_I}{D}$ (8)

$$\text{Now } T_{we}(D) = \frac{-P}{1+PC} = \frac{-P}{1+\frac{k_I}{D}P} = \frac{-\frac{1}{D(mD+b)}}{1+\frac{k_I}{D^2(mD+b)}} = \frac{-D}{mD^3+bD^2+k_I} \quad (9)$$

$\therefore$ If $T_{we}(D)$ is stable, then $e_{ss} = T_{we}(0) = \underline{0}$!