Step Response of 1^{st} + 2^{nd} Order Systems

1. First order system : $y = T_1(D)\, u$

where $T_1(D) = \frac{c}{\tau D + 1}$ where c is the dc gain and τ is the time constant.

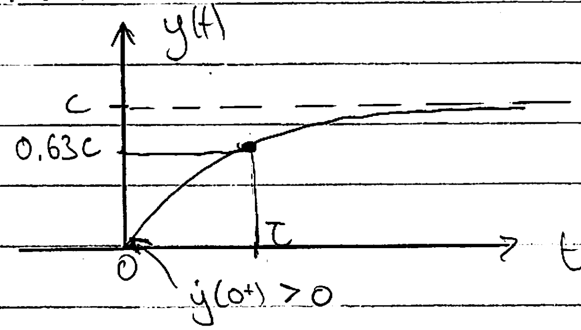
or $T_1(D) = \frac{ca}{D+a}$ where $-a = -\frac{1}{\tau}$ is the pole.
If $a > 0$, then $T_1(D)$ is stable.

Unit Step response : $y = \frac{ca}{D+a} \cdot \frac{1}{D}$
$(u = \frac{1}{D})$

$$= c\left(\frac{1}{D} - \frac{1}{D+a}\right)$$

$$= c\left[1 - e^{-at}\right] = c\left[1 - e^{-t/\tau}\right] \qquad (1)$$

Assume $a > 0$:
(stable)



$y_{ss} = c$ (dc gain)

2. Second-Order System (with no zeros)

$$y = T_2(D)\,u \quad , \quad T_2(D) = \frac{c\,\omega_n^2}{D^2 + 2\sigma D + \omega_n^2} \qquad (2)$$

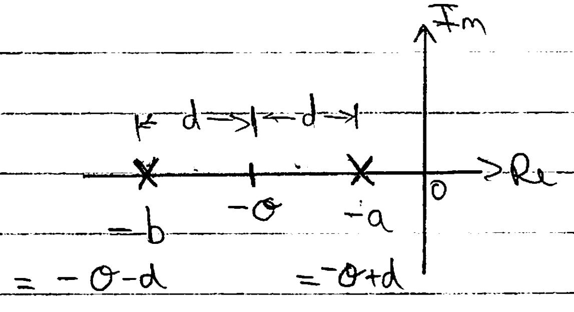
where c is the dc gain
σ is the relative damping
ω_n is the undamped natural frequency

poles : $\lambda_{1,2} = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2}$ (3)

If $\sigma > 0$ and $\omega_n \neq 0$, then $Re(\lambda_{1,2}) < 0$,
and thus $T_2(D)$ is stable.
Assume this.

Case a) Overdamped System : $\sigma > \omega_n$

Then $\lambda_{1,2} = -\sigma \pm d$, where $d = \sqrt{\sigma^2 - \omega_n^2}$
$= -a, -b$ $\qquad 0 < d < \sigma$



- we use X to mark poles in $\mathbb{C}$

$$\therefore \; T_2(D) = \frac{c\,ab}{(D+a)(D+b)} \qquad (4)$$

Note: $-\sigma = \frac{1}{2}(-a + -b)$ = arithmetic mean (midpoint) of poles $-a, -b$
$\omega_n^2 = ab$, so $\omega_n = (ab)^{1/2}$ = geometric mean of $-a, -b$

a) Overdamped cont'd

Step reponse: $y = T_2(D)\,\frac{1}{D}$

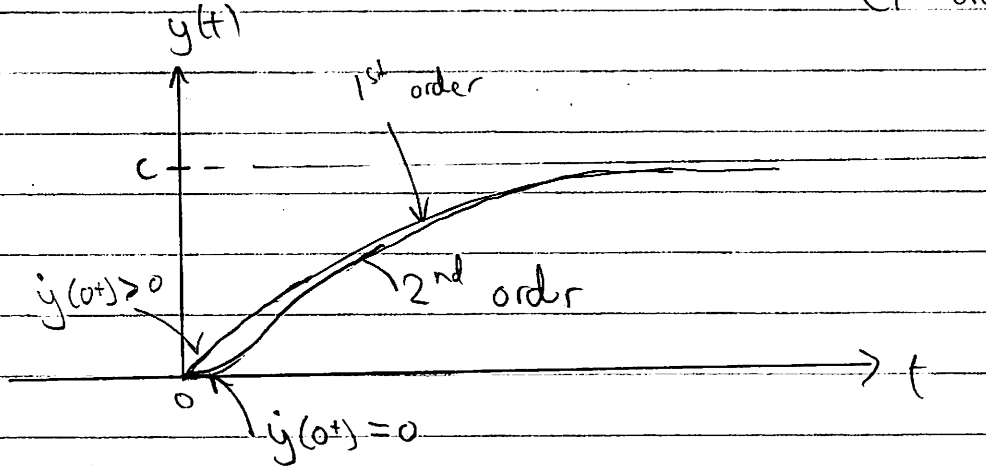
$$y = \frac{cab}{D(D+a)(D+b)} = c\left(\frac{1}{D} - \frac{1}{b-a}\left(\frac{b}{D+a} - \frac{a}{D+b}\right)\right)$$

$$= c\left[1 - \frac{1}{b-a}\left(be^{-at} - ae^{-bt}\right)\right] \qquad (5)$$

Since $a<b$, $e^{-bt} < e^{-at}$ for $t>0$, and the e^{-at} term dominates

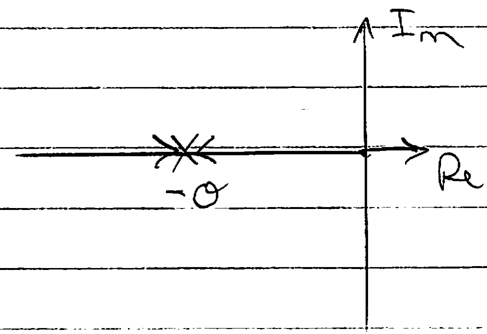
If $a \ll b$, then $y \simeq c\left[1 - e^{-at}\right]$ which is (1) (1st order response)

a) Overdamped Step Response



b) Critically-damped $\sigma = \omega_n$

Then $\lambda_1 = \lambda_2 = -a = -b = -\sigma$



$$T_2(D) = \frac{c\sigma^2}{(D+\sigma)^2}$$

Exercise: find step response y!

Qualitatively looks like a) above

b) cont'd

Step response: $y = \frac{1}{D} T_2(D) = \frac{c\sigma^2}{D(D+\sigma)^2}$

$$= c\left(\frac{1}{D} - \left(\frac{\sigma}{(D+\sigma)^2} + \frac{1}{D+\sigma}\right)\right)$$

$$= c\left[1 - e^{-\sigma t}(\sigma t + 1)\right] \qquad (6)$$

Check: $y(0) = 0$, $y_{ss} = y(\infty) = c$

Also, $\dot{y} = Dy = \frac{c\sigma^2}{(D+\sigma)^2} = c\sigma^2\left[t e^{-\sigma t}\right]$

So $\dot{y}(0) = 0$

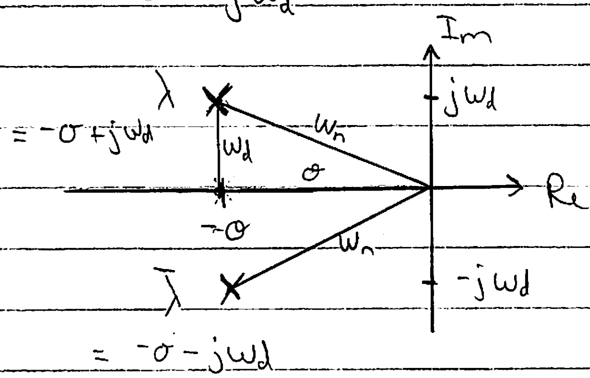
And for all $t > 0$, $\dot{y}(t) > 0$, so $y(t)$ never reaches a maximum, i.e. never peaks.

$\therefore$ Step response looks like overdamped response.

c) Underdamped : $\sigma < \omega_n$

$$T_2(D) = \frac{c\omega_n^2}{D^2 + 2\sigma D + \omega_n^2} = \frac{c\omega_n^2}{(D+\sigma)^2 + \omega_d^2} \quad \text{where } \omega_d^2 = \omega_n^2 - \sigma^2$$

Poles: $\lambda, \bar{\lambda} = -\sigma \pm j\omega_d$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$
$$= \lambda\bar{\lambda}$$

$-\sigma = \mathrm{Re}(\lambda) = \mathrm{Re}(\bar{\lambda})$ (real part of λ)

$\omega_d = \mathrm{Im}(\lambda) = -\mathrm{Im}(\bar{\lambda})$ (imag " " λ)

$\omega_n = |\lambda| = |\bar{\lambda}| = \sqrt{\lambda\bar{\lambda}}$ (mag " λ)

Step response: $y = \dfrac{c\,\omega_n^2}{D\left((D+\sigma)^2 + \omega_d^2\right)}$

$$= c\left(\frac{1}{D} - \frac{D + 2\sigma}{(D+\sigma)^2 + \omega_d^2}\right)$$

$$= \frac{c}{D} - c\left(\frac{D+\sigma}{(D+\sigma)^2 + \omega_d^2} + \frac{\sigma}{\omega_d}\,\frac{\omega_d}{(D+\sigma)^2 + \omega_d^2}\right)$$

$$= c\left[1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)\right] \qquad (7)$$

Check: $y(0) = 0$, $y(\infty) = c$

ω_d is called the damped natural frequency.

c) cont'd

Derivation of step response (7) is just the impulse response:

$$\dot{y} = yD = T_2(D) = \frac{c\,\omega_n^2}{(D+\sigma)^2 + \omega_d^2}$$

$$= c\frac{\omega_n^2}{\omega_d} \cdot \frac{\omega_d}{(D+\sigma)^2 + \omega_d^2} = c\frac{\omega_n^2}{\omega_d}\left[e^{-\sigma t}\sin\omega_d t\right] \quad (8)$$

$\therefore \quad \dot{y}(0) = 0$ ✓

Also, $\dot{y}(t) = 0$ whenever $t = \frac{n\pi}{\omega_d}$, $n = 0, 1, 2, \ldots$

First peak, $t = t_p = \frac{\pi}{\omega_d}$

Let $y_p = y(t_p)$. Then (7) gives

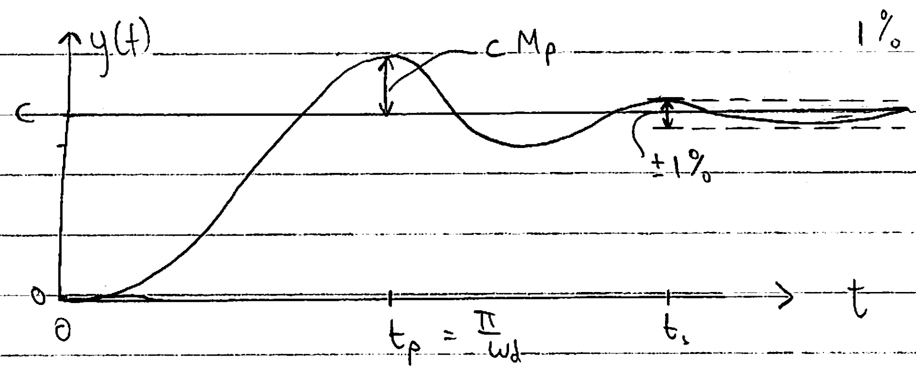
$$y_p = c\left(1 - e^{-\frac{\sigma\pi}{\omega_d}}\left(\cos\pi + \frac{\sigma}{\omega_d}\sin\pi\right)\right)$$

$$= c\left(1 + e^{-\frac{\sigma\pi}{\omega_d}}\right)$$

Overshoot (ratio):

$$M_p = \frac{y_p - y_{ss}}{y_{ss}} = \frac{y_p - c}{c} = \underline{\underline{e^{-\frac{\sigma\pi}{\omega_d}}}}, \quad 0 \le M_p \le 1$$

Step response



1% Settling time:

$$e^{-\sigma t_s} = 0.01$$

$$\sigma t_s = -\ln(0.01)$$

$$= 4.6$$

$$\therefore \boxed{t_s = \frac{4.6}{\sigma}}$$