Properties of Signals and Systems

1 Stability

Def'n: A signal $f \in \mathbb{P}$ is stable if $\lim_{t\to\infty} f(t) = 0$

Def'n: A system (transfer function) $G(D)$ is stable if its impulse response $[g(t)] = G(D)$ is stable.

$G(D)$ expands into a sum of partial fractions of the form $\frac{c}{(D-a)^n}$ where $c \in \mathbb{C}$ and $a = a_r + j a_i$ is a pole of multiplicity n.

$$(2.9) \Rightarrow \frac{c}{(D-a)^n} = \frac{1}{(n-1)!} t^{n-1} e^{at} = \frac{1}{(n-1)!} t^{n-1} e^{a_r t} e^{j a_i t} \qquad (1.1)$$

which is stable if and only if (iff) $a_r < 0$.
$\therefore$ $\mathrm{Re}\, \frac{c}{(D-a)^n}$ is stable iff $a_r < 0$. This gives

Lemma 1: $G(D)$ is stable iff the real parts of all of its poles are negative.

e.g. 1) $\frac{1}{D+1} = [e^{-t}]$ is stable since its pole is -1 and hence $\lim_{t\to\infty} e^{-t} = 0$

e.g. 2) $\frac{-3}{(D+1)(D-2)} = [e^{-t} + 3e^{2t}]$ is unstable since it has a pole at $D = 2 \geq 0$ and hence $\lim_{t\to\infty}(e^{-t} + 3e^{2t}) = \infty$.

e.g. 3) $\frac{1}{D(D+1)} = [1 - e^{-t}]$ is unstable since it has a pole at $D=0$ and hence $\lim_{t\to\infty} (1-e^{-t}) = 1 \neq 0$.

This example is 'marginally unstable' since it does not blow up.

$$G(D) = \frac{a_m D^m + \dots + a_0}{b_n D^n + \dots + b_0}, \quad m \leq n$$

Def'n: A system $G(D)$ is bounded-input bounded-output (BIBO) stable if for every causal bounded input $u(t)$ (i.e. $|u(t)| < B$ for all t), the output $u\,G(D)$ is bounded.

Lemma 2: $G(D)$ is BIBO stable iff it is stable in the sense of $\lim_{t\to\infty} g(t) = 0$ where

$[g(t)] = G(D)$ is its impulse response.

The proof consists of 2 steps:

a) Prove that $\lim_{t\to\infty} g(t) = 0 \iff$ existence of $\|g\|_1 \equiv \int_0^\infty |g(\tau)|\,d\tau$.

b) Prove that if $u(t)$ is bounded, then $\|g\|_1 < \infty$ iff $y(t) \equiv \int_0^t u(t-\tau)\, g(\tau)\, d\tau$ is bounded.

(a) & (b) are left as exercises.

BIBO Stability Examples

e.g.1) The system $\frac{y}{u} = \frac{1}{D+1}$ is BIBO stable since its impulse response $\frac{1}{D+1} = [e^{-t}]$ is stable. If the input u is causal, then the output y is

$$y(t) = [e^{-t}]u = \int_0^t u(t-\tau)\, e^{-\tau}\, d\tau$$

If $|u(t)|$ is bounded by B (i.e. $|u(t)| \leq B$), then

for $t \geq 0$, $$|y(t)| \leq \int_0^t |u(t-\tau)| |e^{-\tau}| d\tau \leq B \int_0^t |e^{-\tau}| d\tau = B(1-e^{-t}) \leq B$$

and hence $|y(t)| \leq b$ is also bounded.
This confirms Lemma 2 : $\frac{1}{D+1}$ is BIBO stable.
It also illustrates the proof of the lemma.
Note that the 'worst-case' bounded input ($|u(t)| < B$) for this system is $u = Bh$, since this gives $y(t) = B(1-e^{-t})$, which achieves the bound B in the limit : $\lim_{t \to \infty} y(t) = B$.

e.g.2) $\frac{1}{(D+1)(D-2)}$ is BIBO <u>unstable</u> since its response to almost any input will include an unstable $[ce^{2t}]$ term.

e.g.3) $\frac{1}{D(D+1)}$ is BIBO <u>unstable</u> since its response to a step input is

$$\frac{1}{D^2(D+1)} = [t - 1 + e^{-t}],$$ which grows unbounded as $t \to \infty$.

Final Value Theorem (F.V.T.)

If $f_{ss} = \lim_{t \to \infty} f(t)$ exists, then $f_{ss} = \lim_{s \to 0^+} s F(s)$ 5

F.V.T. examples

e.g. 1) If $G(D) = \frac{1}{D+1}$, find g_∞

Sol'n: $g_\infty = \lim_{D \to 0} \frac{D}{D+1} = 0$

Check: $g_\infty = \lim_{t \to \infty} [e^{-t}] = 0$ ✓

e.g. 2) If $G(D) = \frac{2}{D(D+1)}$, find g_∞

Sol'n: $g_\infty = \lim_{D \to 0} \frac{2\cancel{D}}{\cancel{D}(\cancel{D}+1)} = 2$

Check: $g_\infty = \lim_{t \to \infty} 2[1-e^{-t}] = 2$ ✓

e.g. 3) If $\frac{y}{u} = \frac{2}{D+1}$, find the steady-state response y_∞ to a unit step $u=h$.

Sol'n: $y = \frac{2}{D+1} h = \frac{2}{(D+1)D}$, so $y_\infty = 2$ by e.g.(2).

e.g. 4) If $G(D) = \frac{-3}{(D+1)(D-2)}$, find g_∞.

Sol'n: g_∞ is <u>undefined</u> since $g = G(D)$ is unstable (and not just marginally unstable).

i.e. $g = [e^{-t} + 3e^{2t}]$, so $\lim_{t \to \infty} g(t) = 3e^{\infty t} = \infty$