Newton's Method

Navier Stokes

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Outline

Navier Stokes Equation

Cauchy Momentum Equations

$$\frac{du}{dt} + u * \nabla u = f + u\nabla^2 u - \nabla p$$

Mass Continutity Equation

$$\nabla u = 0$$

Linear Terms Proof

$$\Delta u => \Delta(u + w)$$

$$\Delta(\alpha u + \beta w)? = \alpha\Delta u + \beta\Delta w$$

$$\nabla\nabla(u + w) = \nabla\nabla u + \nabla\nabla w$$

$$\nabla \begin{bmatrix} (u_1 + w_1)_x & (u_1 + w_1)_y & (u_1 + w_1)_z \\ (u_2 + w_2)_x & (u_2 + w_2)_y & (u_2 + w_2)_z \\ (u_3 + w_3)_x & (u_3 + w_3)_y & (u_3 + w_3)_z \end{bmatrix} = \nabla \begin{bmatrix} u_x^1 & u_y^1 & u_z^1 \\ u_x^3 & u_y^3 & u_z^3 \\ u_x^3 & u_y^3 & u_z^3 \end{bmatrix} + \nabla \begin{bmatrix} w_x^1 & w_y^1 & w_z^1 \\ w_x^3 & w_y^3 & w_z^3 \\ w_x^3 & w_y^3 & w_z^3 \end{bmatrix}$$

$$\begin{bmatrix} u_{xx} + w_{xx} & u_{yx} + w_{yx} & u_{zx} + w_{zx} \\ u_{xy} + w_{xy} & u_{yy} + w_{yy} & u_{zy} + w_{zy} \\ u_{xz} + w_{xz} & u_{yz} + w_{yz} & u_{zz} + w_{zz} \end{bmatrix} = \begin{bmatrix} u_{xx} & u_{yx} & u_{zx} \\ u_{xy} & u_{yy} & u_{zy} \\ u_{xz} & u_{yz} & u_{zz} \end{bmatrix} + \begin{bmatrix} w_{xx} & w_{yx} & w_{zx} \\ w_{xy} & w_{yy} & w_{zy} \\ w_{xz} & w_{yz} & w_{zz} \end{bmatrix}$$

$$\begin{bmatrix} u_{xx} + w_{xx} & u_{yx} + w_{yx} & u_{zx} + w_{zx} \\ u_{xy} + w_{xy} & u_{yy} + w_{yy} & u_{zy} + w_{zy} \\ u_{xz} + w_{xz} & u_{yz} + w_{yz} & u_{zz} + w_{zz} \end{bmatrix} = \begin{bmatrix} u_{xx} + w_{xx} & u_{yx} + w_{yx} & u_{zx} + w_{zx} \\ u_{xy} + w_{xy} & u_{yy} + w_{yy} & u_{zy} + w_{zy} \\ u_{xz} + w_{xz} & u_{yz} + w_{yz} & u_{zz} + w_{zz} \end{bmatrix}$$

Nonlinear Term Proof

$$u * \nabla u => (v + w)\nabla(v + w)$$

$$\begin{aligned}
(v + w)\nabla(v + w) \quad &?= \quad v\nabla v + w\nabla w \\
(v + w) * (\nabla v + \nabla w) \quad &?= \quad v\nabla v + w\nabla w \\
v * (\nabla v + \nabla w) + w * (\nabla v + \nabla w) \quad &?= \quad v\nabla v + w\nabla w \\
v * \nabla v + v\nabla w + w\nabla v + w\nabla w \quad &\neq \quad v\nabla v + w\nabla w
\end{aligned}$$

Newton's Method Equation

$$H(w) = 0$$

Now we come up with a linear operator Hp which estimated the value of the function for a given small displacemet dw. If w(x) is a set of state functions, we would like the estimate the change in value of the Navier Stokes equation by the set of functions (w + dw)(x).

$$H(w + dw) \approx H(w) + Hp(w) * dw$$

If that linear operator is invertible, then Newton's method follows:

$$w^{k+1} = w^k - \frac{H(w^k)}{Hp(w)}$$

Newton's Method Equation cont.

$$H(x) = H(X_0) + H'(X_0)(X - X_0) + (||x - x_0||^2)$$

$$H(X) \approx H(X_0) + H'(X_0)(X - X_0)$$

$$h(X) = H(X_0) + H'(X_0)(X - X_0)$$

$$x^* = X_0 - (H'(X_0))^{-1} H(X_0)$$

$$X^{k+1} = X^k - \frac{H(X^k)}{H'(X^k)}$$

Note that we can also write this equation in the equivalent implicit form:

$$H'(X^k)(X^{k+1} - X^k) = -H(X^k)$$

$$\delta X^k \equiv X^{k+1} - X^k$$

$$H'(X^k)\delta X^k = -H(X^k)$$

Intro to Variables

Equation we are solving for: $H'(X^k)(\delta X^k) = -H(X^k)$

$$H'_G(X_1)(\delta(X)) = \lim_{\epsilon \to} \frac{H(X_0 + \epsilon\delta X) - H(X_0)}{\epsilon}$$

$$H^{\rightarrow}(x^{\rightarrow}) = -\Delta u^{\rightarrow} +^{\rightarrow} \nabla u^{\rightarrow} + \nabla p - f^{\rightarrow} = 0$$

$$X = \begin{pmatrix} u \\ v \\ p \end{pmatrix}$$

$$X^0 + \epsilon\delta X = \begin{pmatrix} u^0 \\ v^0 \\ p^0 \end{pmatrix} + \epsilon \begin{pmatrix} \delta u \\ \delta v \\ \delta p \end{pmatrix}$$

Solving the Limit

$$\lim_{\epsilon \to} \frac{H(X_0 + \epsilon \delta X) - H(X_0)}{\epsilon}$$

$$H(X^0 + \epsilon\delta X) = -\Delta(u^0 + \epsilon\delta u) + (u^0 + \epsilon\delta u) * \nabla(u^0 + \epsilon\delta u) + \nabla(p^0 + \epsilon\delta p) - f(x, y, z)$$

$$H(X^0) = -\Delta u^0 + u^0 \nabla u^0 + \nabla p - f$$

Taking the difference of these terms yields:

$$-\Delta\epsilon\delta u + [(u^0 + \epsilon\delta u) * \nabla(u^0 + \epsilon\delta u) - u^0\nabla u^0] + \nabla\epsilon\delta p$$

Dividing by ϵ and taking the limit as ϵ goes to zero yields:

$$-\Delta\delta u + u^0\nabla\delta u + \delta u\nabla u^0 + \nabla\delta p = H'_G(X_1)(\delta(X))$$

Note: u is the vector $< u, v >$

Oseen Equations

Writing in matrix form:

$$-\Delta \begin{bmatrix} u^1 - u^0 \\ v^1 - v^0 \end{bmatrix} + (\begin{bmatrix} u^0(u^1 - u^0)_x + v^0(u^1 - u^0)_y \\ u^0(v^1 - v^0)_x + v^0(v^1 - v^0)_y \end{bmatrix} +$$

$$\begin{bmatrix} u^0_x(u^1 - u^0) + u^0_y(v^1 - v^0) \\ v^0_x(u^1 - u^0) + v^0_y(v^1 - v^0) \end{bmatrix}) + (p^1 - p^0)_x = 0$$

Here we see the previous matrix written out into linear equations.

$$-(\frac{d^2\Delta u}{dx^2} + \frac{d^2\Delta u}{dy^2}) + R(\Delta u\frac{du_0}{dx} + u_0\frac{d\Delta u}{dx} + \Delta v\frac{du_0}{dy} + v_0\frac{d\Delta u}{dy}) + \frac{d\Delta p}{dx} = 0$$

$$-(\frac{d^2\Delta v}{dx^2} + \frac{d^2\Delta v}{dy^2}) + R(\Delta u\frac{du_0}{dx} + u_0\frac{d\Delta v}{dx} + \Delta v\frac{dv_0}{dy} + v_0\frac{d\Delta v}{dy}) + \frac{d\Delta p}{dy} = 0$$

$$\frac{d\Delta u}{dx} + \frac{d\Delta v}{dy} = 0$$

Linking with the Code

Recall $H'_G(X)(\delta X) = -H(X^k)$

$$-\Delta\delta u + u^0\nabla\delta u + \delta u\nabla u^0 + \nabla\delta p = \Delta u^0 - u^0\nabla u^0 - \nabla p^0 + f$$

Plug in for δ:

$$-\Delta(u^1 - u^0) + u^0\nabla(u^1 - u^0) + (u^1 - u^0)\nabla u^0 + \nabla(p^1 - p^0) = \Delta u^0 - u^0\nabla u^0 - \nabla p^0 + f$$

Reduce:

$$-\Delta u^1 + u^0\nabla u^1 + u^1\nabla u^0 - u^0\nabla u^0 + \nabla p^1 = f$$

Now 1=k+1 and 0=k

$$-\Delta u^{k+1} + u^k\nabla u^{k+1} + u^{k+1}\nabla u^k - u^k\nabla u^k + \nabla p^{k+1} = f$$

Move known values to right-hand-side:

$$-\Delta u^{k+1} + u^k\nabla u^{k+1} + u^{k+1}\nabla u^k + \nabla p^{k+1} = f + u^k\nabla u^k$$

Completed Newton's Linearization Equation

$$-\Delta u^{k+1} + u^k \nabla u^{k+1} + u^{k+1} \nabla u^k + \nabla p^{k+1} = f + u^k \nabla u^k$$
$$\nabla * u^{k+1} = 0$$

Next we must obtain the discretized weak form by integrating over the region, multiplying by a function (like v), and doing integration by parts.

$$\int\int -\Delta u^{k+1} + u^k \nabla u^{k+1} + u^{k+1} \nabla u^k + \nabla p^{k+1} dxdy = \int\int f + u^k \nabla u^k dxdy$$

Time-Dependent

Equation we are solving for: $H'(X^k)(\delta X^k) = -H(X^k)$

$$H'_G(X_1)(\delta(X)) = \lim_{\epsilon \to} \frac{H(X_0 + \epsilon \delta X) - H(X_0)}{\epsilon}$$

$$(u_t^k + \epsilon \delta u_t) - v\Delta(u^k + \epsilon \delta u_k) + (u^k + \epsilon \delta u_k) * \nabla(u^k + \epsilon \delta u_k) + \nabla(p_k + \epsilon \ deltap) - f$$

$$-$$

$$u_t^k - v\Delta u^k + u^k * \nabla u^k + \nabla p^k - f$$

$$=$$

$$\delta u_t - v\Delta \delta u + u^k \nabla \delta u + \delta u \nabla u^k \nabla \delta p = H'(X^K)(\Delta X) = -H(X^k)$$

Expansion

$$u_t^{k+1} - u_t^k - v\Delta u^{k+1} + v\Delta u^k + u^k \nabla u^{k+1} - u^k \nabla u^k + u^{k+1} \nabla u^k - u^k \nabla u^k + \nabla p^{k+1} - \nabla p^k$$

$$= -u_t^k + v\Delta u^k - u^k \nabla u^k - \nabla p^k + f$$

Linearized Equation

$$u_t^{k+1} - v\Delta u^{k+1} + u^k \nabla u^{k+1} + u^{k+1} \nabla u^k - u^k \nabla u^k + \nabla p^{k+1} = f$$