Derivation of the Navier-Stokes Equations

Using Beamer

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Conservation of Mass

Given a controlled element with a volume $\Delta x \Delta y \Delta z$, the mass accumulation is given by the mass inflow minus the mass outflow, according to the law of conservation of mass. In three dimensions, the mass accumulation in the volume is given by the following equation:

$$\frac{\partial \rho(\Delta x \Delta y \Delta z)}{\partial t} = (\rho U)|_x \Delta y \Delta z - (\rho U)|_{x+\Delta x} \Delta y \Delta z$$

$$+(\rho V)|_y \Delta x \Delta z - (\rho V)|_{y+\Delta y} \Delta x \Delta z$$

$$+(\rho W)|_Z \Delta x \Delta y - (\rho W)|_{z+\Delta z} \Delta x \Delta y.$$

Divide both sides of the equation by $\Delta x \Delta y \Delta z$ to get

$$\frac{\partial \rho}{\partial t} = \frac{(\rho U)|_x - (\rho U)|_{x+\Delta x}}{\Delta x} + \frac{(\rho U)|_y - (\rho U)|_{y+\Delta y}}{\Delta y} + \frac{(\rho U)|_z - (\rho U)|_{z+\Delta z}}{\Delta z}$$

The definition of a limit states that

$$f'(x) = \lim_{x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Applying this to the above equation and rearranging it, we get the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} + \frac{\partial(\rho W)}{\partial z} = 0$$

In steady-state problems, the density does not change over time, so $\frac{\partial \rho}{\partial t}$ is zero, which gives us

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} + \frac{\partial(\rho W)}{\partial z} = 0$$

Further, incompressible fluids have relatively constant densities. Thus, the equation can be simplified:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

Conservation of Momentum

The momentum of a body is equal to its mass times its velocity ($m\vec{v}$). The accumulation of momentum in the volume is equal to the momentum flowing in minus the momentum flowing out plus the surface and forces acting on the surfaces and the body force acting inside the volume.

Calculating the x component of the momentum flux will allow us to then alter it slightly to match the y and z components.

First, we account for the momentum fluxes in and out in the x direction:

$$\Delta x \Delta y \Delta z [\frac{\partial(\rho U)}{\partial t}] =$$

$$\Delta y \Delta z (\rho UU)|_x - \Delta y \Delta z (\rho UU)|_{x+\Delta x}$$

$$+\Delta x \Delta z (\rho VU)|_y - \Delta x \Delta z (\rho VU)|_{y+\Delta y}$$

$$+\Delta x \Delta y (\rho WU)|_z - \Delta x \Delta y (\rho WU)|_{z+\Delta z}$$

We can now add the forces to get the total momentum accumulation in the x direction

$$\Delta x \Delta y \Delta z[\frac{\partial(\rho U)}{\partial t}] =$$

$$\Delta y \Delta z(\rho UU)|_x - \Delta y \Delta z(\rho UU)|_{x+\Delta x}$$

$$+\Delta x \Delta z(\rho VU)|_y - \Delta x \Delta z(\rho VU)|_{y+\Delta y}$$

$$+\Delta x \Delta y(\rho WU)|_z - \Delta x \Delta y(\rho WU)|_{z+\Delta z}$$

$$+[(P + \tau_{xx})|_x - (P + \tau_{xx})|_{x+\Delta x}]\Delta y \Delta z$$

$$+[(\tau_{xy})|_y - (\tau_{xy})|_{y+\Delta y}]\Delta x \Delta z$$

$$+[(\tau_{xz})|_z - (\tau_{xz})|_{z+\Delta z}]\Delta x \Delta y$$

Dividing all terms by $\Delta x \Delta y \Delta z$ gives us

$$\frac{\partial(\rho U)}{\partial t} =$$

$$\frac{(\rho UU)|_x - (\rho UU)|_{x+\Delta x}}{\Delta x}$$

$$+ \frac{(\rho VU)|_y - (\rho VU)|_{y+\Delta y}}{\Delta y}$$

$$+ \frac{(\rho WU)|_z - (\rho WU)|_{z+\Delta z}}{\Delta z}$$

$$+ \frac{(P + \tau_{xx})|_x - (P + \tau_{xx})|_{x+\Delta x}}{\Delta x}$$

$$+ \frac{(\tau_{xy})|_y - (\tau_{xy})|_{y+\Delta y}}{\Delta y}$$

$$+ \frac{(\tau_{xz})|_z - (\tau_{xz})|_{z+\Delta z}}{\Delta z} + \rho F_g$$

We can now apply the definition of limits to this equation by substituting in ϕ, similar to how we simplified the mass accumulation.

$$\frac{(\rho UU)|_x - (\rho UU)|_{x+\Delta x}}{\Delta x} \to -\frac{\partial(\rho UU)}{\partial x}$$
$$\frac{(\rho VU)|_y - (\rho VU)|_{y+\Delta y}}{\Delta y} \to -\frac{\partial(\rho VU)}{\partial y}$$
$$\frac{(\rho WU)|_z - (\rho WU)|_{z+\Delta z}}{\Delta z} \to -\frac{\partial(\rho WU)}{\partial z}$$
$$\frac{(P+\tau_{xx})|_x - (P+\tau_{xx})|_{x+\Delta x}}{\Delta x} \to -\frac{\partial P}{\partial x} - \frac{\partial(\tau_{xx})}{\partial x}$$
$$\frac{(P+\tau_{xy})|_y - (P+\tau_{xy})|_{y+\Delta y}}{\Delta y} \to -\frac{\partial(\tau_{xy})}{\partial y}$$
$$\frac{(P+\tau_{xz})|_z - (P+\tau_{xz})|_{z+\Delta z}}{\Delta z} \to -\frac{\partial(\tau_{xz})}{\partial z}$$

Next we can bring in the terms that were already simplified and rearrange to get the standard momentum equation for the x-component (U):

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho U^2)}{\partial x} + \frac{(\rho VU)}{\partial y} + \frac{(\rho WU)}{\partial z} = -\frac{\partial P}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{xz}}{\partial z} + \rho F_g$$

The momentum equation for the y-component (V):

$$\frac{\partial(\rho V)}{\partial t} + \frac{\partial(\rho UV)}{\partial x} + \frac{(\rho V^2)}{\partial y} + \frac{(\rho WV)}{\partial z} = -\frac{\partial P}{\partial y} - \frac{\partial \tau_{yx}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} - \frac{\partial \tau_{yz}}{\partial z} + \rho F_g$$

The momentum equation for the z-component (W):

$$\frac{\partial(\rho W)}{\partial t} + \frac{\partial(\rho UW)}{\partial x} + \frac{(\rho VW)}{\partial y} + \frac{(\rho W^2)}{\partial z}$$
$$= -\frac{\partial P}{\partial z} - \frac{\partial \tau_{zx}}{\partial x} - \frac{\partial \tau_{zy}}{\partial y} - \frac{\partial \tau_{zz}}{\partial z} + \rho F_g$$

If we take $\tau = -\alpha \frac{\partial U}{\partial y}$ to be a true statement, the momentum equations can be further simplified: X-component:

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho U^2)}{\partial x} + \frac{(\rho VU)}{\partial y} + \frac{(\rho WU)}{\partial z}$$

$$= -\frac{\partial P}{\partial x} + 2\frac{\partial}{\partial x}\left(\alpha\left(\frac{\partial U}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\alpha\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\alpha\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)\right)$$
$$+\rho F_g$$

Y-component:

$$\frac{\partial(\rho V)}{\partial t} + \frac{\partial(\rho UV)}{\partial x} + \frac{(\rho V^2)}{\partial y} + \frac{(\rho WV)}{\partial z}$$

$$= -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\alpha\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)\right) + 2\frac{\partial}{\partial y}\left(\alpha\left(\frac{\partial V}{\partial y}\right)\right) + \frac{\partial}{\partial z}\left(\alpha\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right)\right)$$

$$+\rho F_g$$

Z-component:

$$\frac{\partial(\rho W)}{\partial t} + \frac{\partial(\rho UW)}{\partial x} + \frac{(\rho VW)}{\partial y} + \frac{(\rho W^2)}{\partial z}$$

$$= -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\alpha\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\alpha\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right)\right) + 2\frac{\partial}{\partial z}\left(\alpha\left(\frac{\partial W}{\partial z}\right)\right)$$

$$+\rho F_g$$

The left-hand side of each equation can be broken apart using the product rule of derivatives and each component of velocity can be replaced with ϕ :

$$\rho\frac{\partial U}{\partial t} + \rho U\frac{\partial \phi}{\partial x} + \rho V\frac{\partial \phi}{\partial x} + \rho W\frac{\partial \phi}{\partial x}$$