Navier-Stokes Steady

Ja'Nya Breeden

Department of Mathematics
Francis Marion University

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Introduction

Lukas Bystricky uses steady Stokes' flow to examine fluid dynamics.
After linearizing the Navier-Stokes equations,
$-\Delta u^{k+1} + u^k \nabla u^{k+1} + u^{k+1} \nabla u^k + p^{k+1} = f + u^k \nabla u^k$ (1)
and the incompressibility constraint,
$\nabla u^{k+1} = 0$ (2)
we obtain the weak form of the equations.

Weak Form

First, the inner product is taken with a test function v$\in H_0^1$ on the function.

$-(\Delta u^{k+1}) \cdot v + (u^k \cdot \nabla u^{k+1}) \cdot v + (u^{k+1} \cdot \nabla u^k) \cdot v + (p^{k+1} \cdot v) = (f \cdot v) + (u^k \cdot \nabla u^k) \cdot v$

and then integration by parts is performed.

$\int_\Omega (\nabla u^{k+1}) : \nabla v d\Omega + \int_\Omega (u^k \cdot \nabla u^{k+1}) \cdot v d\Omega + \int_\Omega (u^{k+1} \cdot \nabla u^k) \cdot v d\Omega - \int_\Omega (p^{k+1} \cdot v) d\Omega = \int_\Omega (f \cdot v) d\Omega + \int_\Omega (u^k \cdot \nabla u^k) \cdot v d\Omega$ (3)

Weak Form

The weak form of $\nabla u^{k+1} = 0$ becomes
$-\int_\Omega (\nabla u^{k+1} \cdot q) d\Omega$ with the test function q$\in L^2$.

Linearization

Then Bystricky uses the weak form to linearize the function,
$\int_\Omega (\nabla u^{k+1}) : \nabla v d\Omega + \int_\Omega (u^k \cdot \nabla u^{k+1}) \cdot v d\Omega + \int_\Omega (u^{k+1} \cdot \nabla u^k) \cdot v d\Omega - \int_\Omega (p^{k+1} \cdot v) d\Omega = \int_\Omega (f \cdot v) d\Omega + \int_\Omega (u^k \cdot \nabla u^k) \cdot v d\Omega$
$\forall v \in H_0^1$
$-\int_\Omega (\nabla u^{k+1} \cdot q) d\Omega$
$\forall q \in L^2$

Stokes Code

Steady Stokes Flow is not time-dependent
It solves Stokes equations on the unit box with Dirichlet boundary conditions on the velocity

```
private:
void setup geometry(int cycle);
void assemble system();
void solve();
void refine grid();
void output results(int cycle);
void calculate error(int cycle);
void print errors();
```

Code can be found at this link:
https://github.com/lukasbystricky/FiniteElements/blob/master/deal.ii/StokesSteady/stokes_solver.cc

Set Up Geometry

Input is read from GMesh file to create grid
After vector is formed the unit box is created along with degrees of freedom.
Then, he sets boundary indicators based on values at the center
Dirichlet boundary conditions are set and pressure is initialized at 0.

Calculate Error

Bystricky uses the function integrateDifference() to calculate the errors of the pressure and velocity.
He then adds the values to their respective convergence tables.