# SPECTRAL ANALYSIS

Fourier transform and frequency domain

# Sound in different domains

An audio signal can be described from different perspectives, depending on which aspect of sound is being analyzed.

- Time-based descriptions reveal changes over time
- Frequency-based descriptions reveal spectral content

→ *Both views describe the same signal, but reveal different information.*

# Time domain vs. frequency domain

Audio signals are commonly represented in two domains:

- Time domain:
  - Amplitude as a function of time
- Frequency domain:
  - Distribution of energy across frequencies (magnitude and phase)

(Analog signals are continuous; digital signals are discrete.)

# Time domain

The time domain shows how a signal's amplitude changes over time.

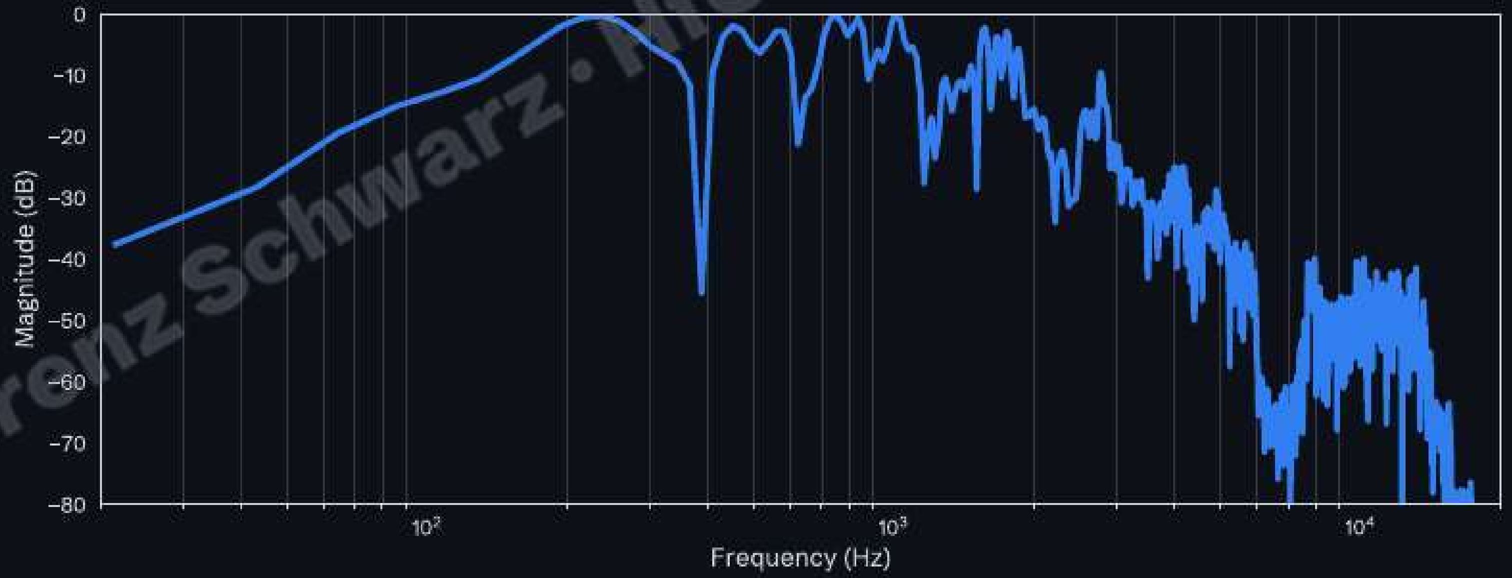
- Analog signals: continuous in time and amplitude
- Digital signals: discrete samples in time and amplitude

→ *The time domain reveals transients, timing, and amplitude changes.*

# Frequency domain

The frequency domain describes a signal in terms of its frequency components rather than time.

- shows how much energy each frequency contributes



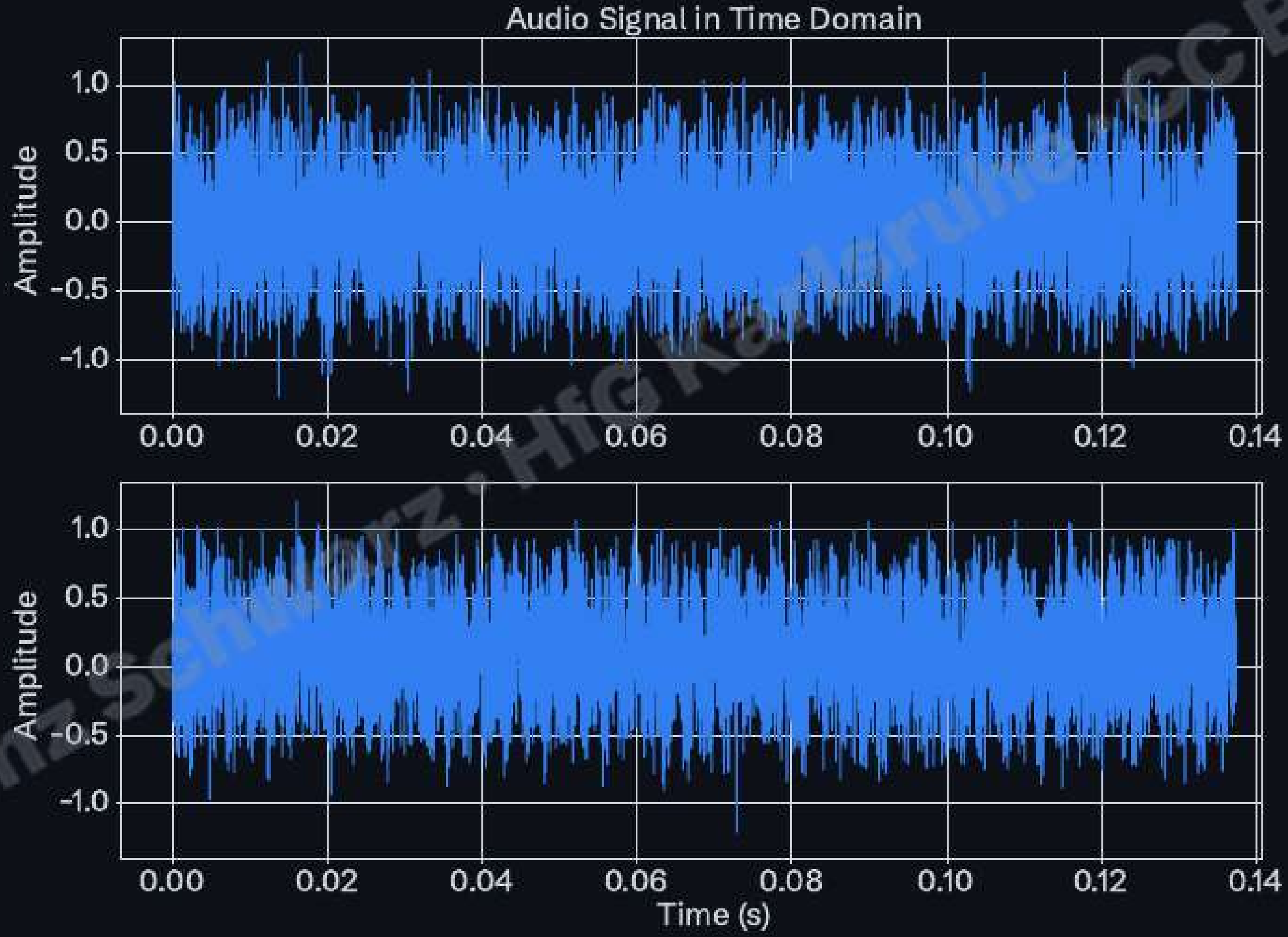
# Spectrum

The spectrum shows a moment of a signal's frequency content.

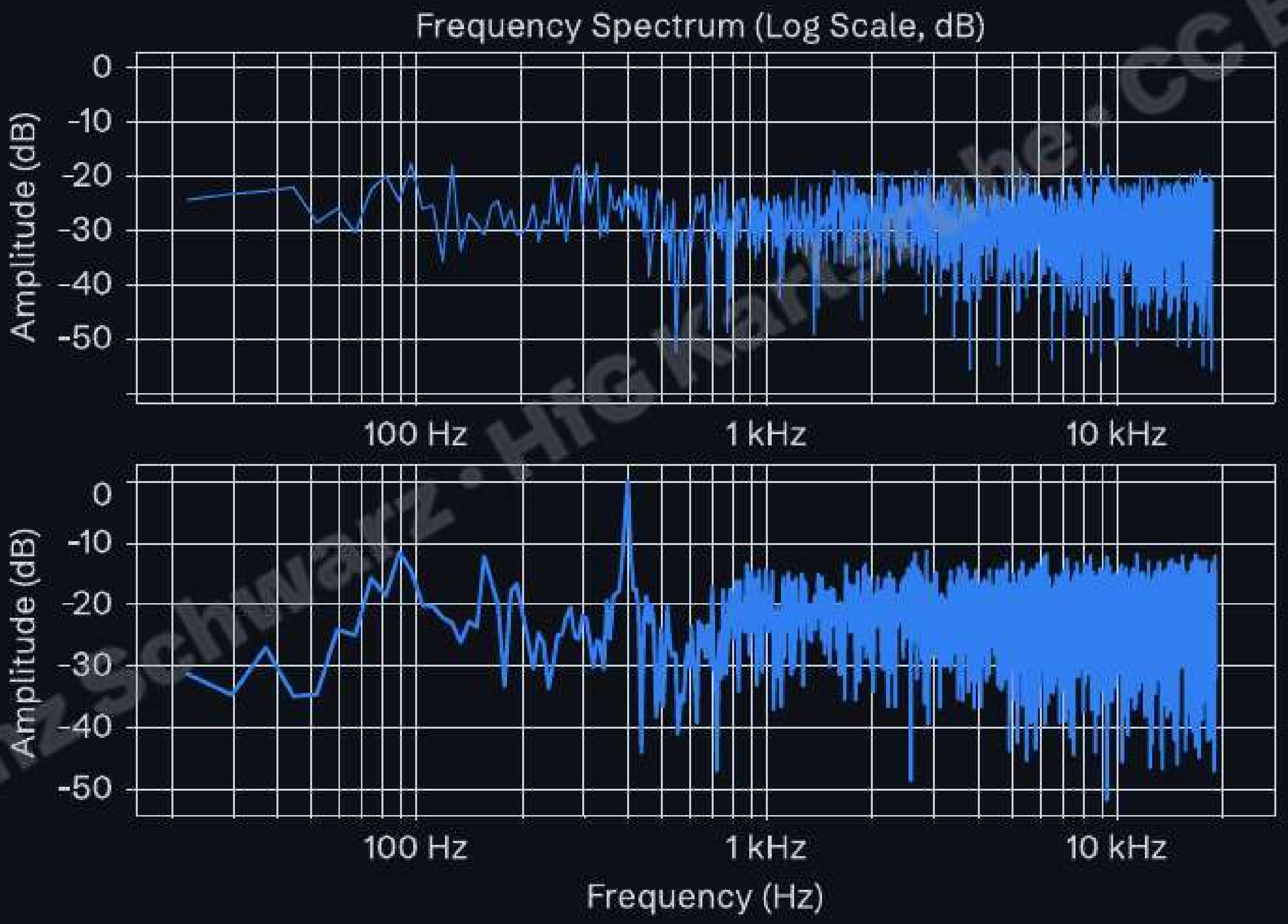
- Amplitude (or magnitude) as a function of frequency
- Optionally includes phase information

→ *The spectrum is the primary tool for analyzing timbre and harmonic structure.*

# Which signal contains a 400 Hz sine?



# The second spectrum shows a spike at 400Hz



# Timbre and spectrum

**Timbre (perceptual):**
The sonic quality that distinguishes instruments playing the same pitch

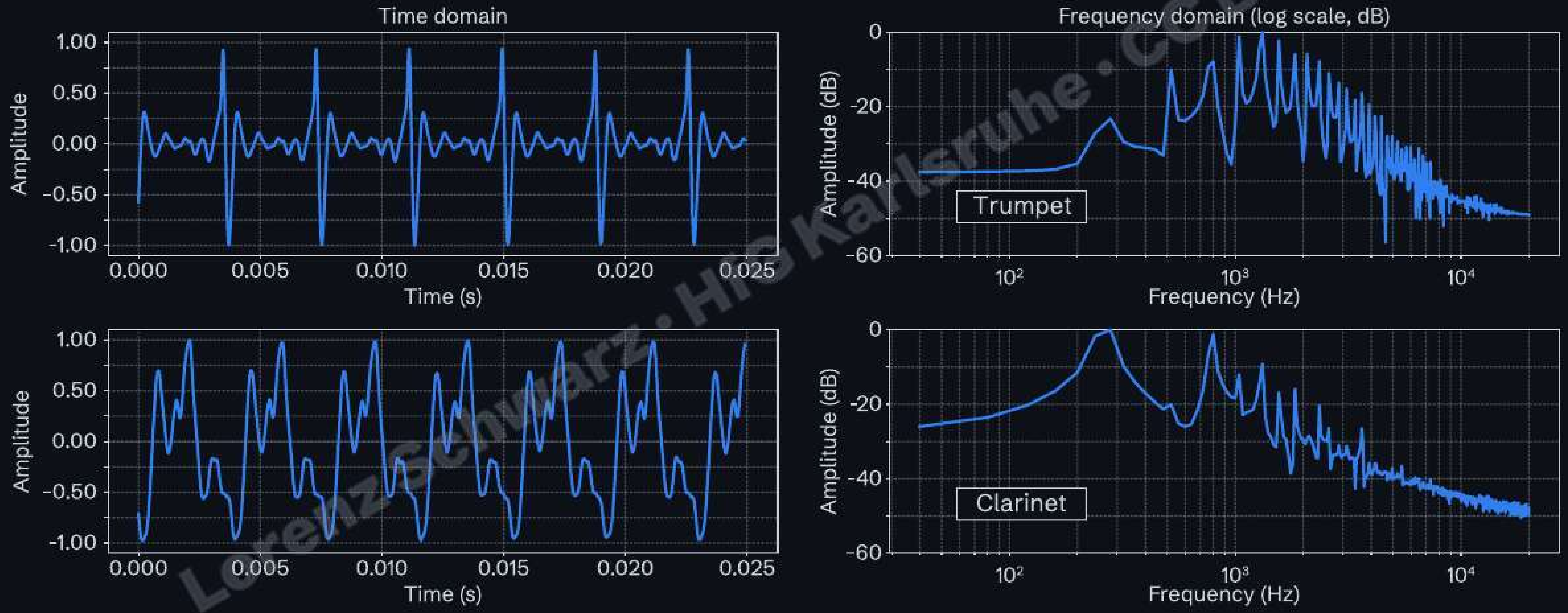
**Spectrum (technical):**
The distribution of frequency components and their amplitudes

**Listen to the same pitch (C4 ≈ 261.6 Hz):**

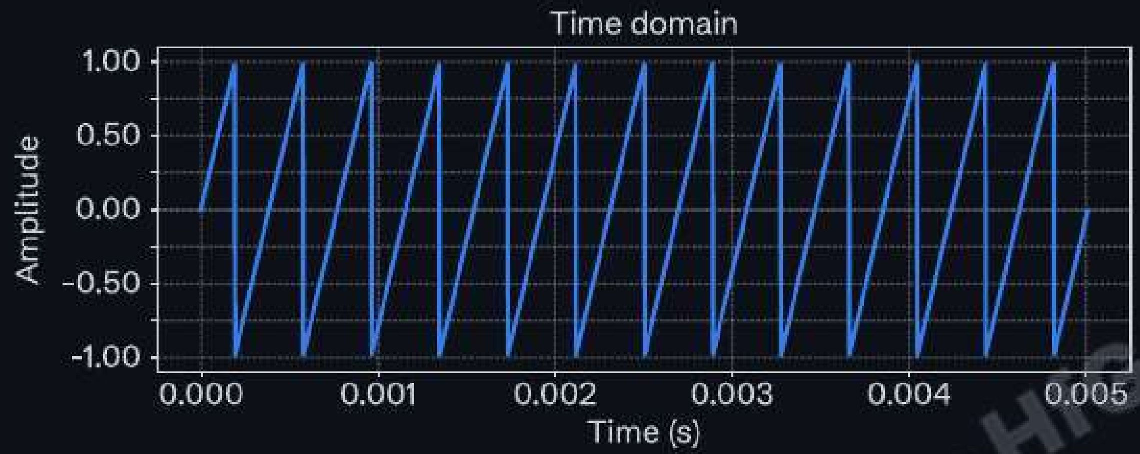
**Trumpet:** ▶ Play **Clarinet:** ▶ Play

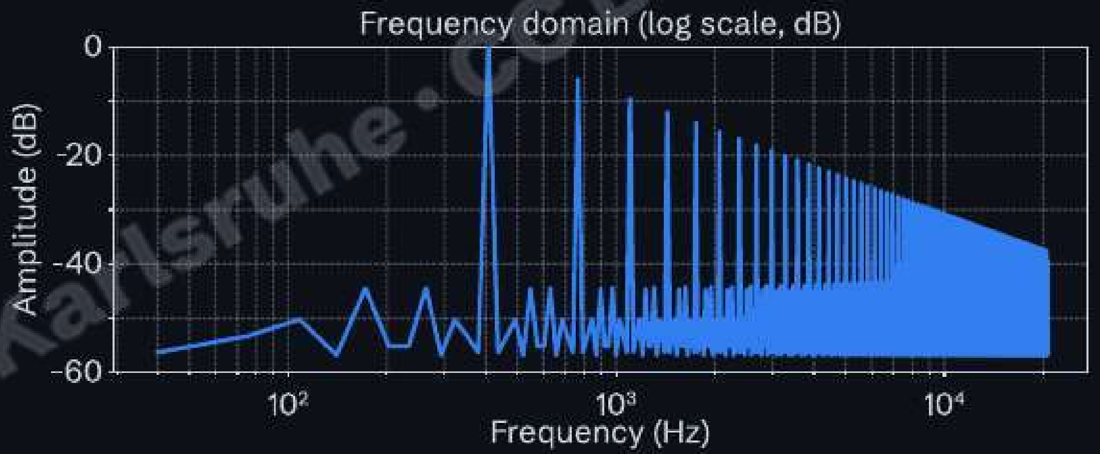
→ *Same pitch, different timbre*

# Comparing clarinet and trumpet at 260 Hz



# Understanding spectra with a sawtooth wave



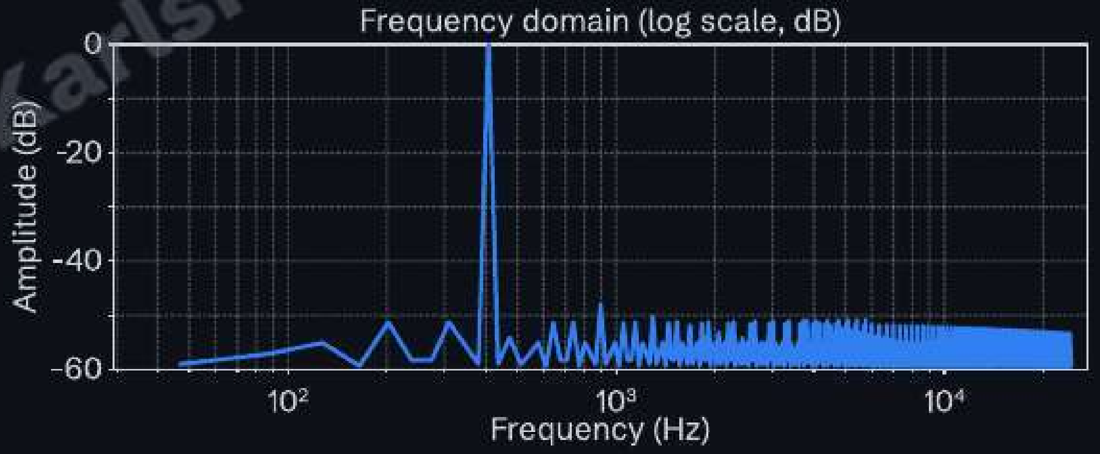




- Each peak in the spectrum represents one sine wave (partial or harmonics)
- Harmonic series 260, 520, 780, 1040… Hz

▶ Play sawtooth wave C4 ≈ 260 Hz

# Pure tone (sine wave)

A sine wave is a single frequency component, the fundamental building block of all sounds

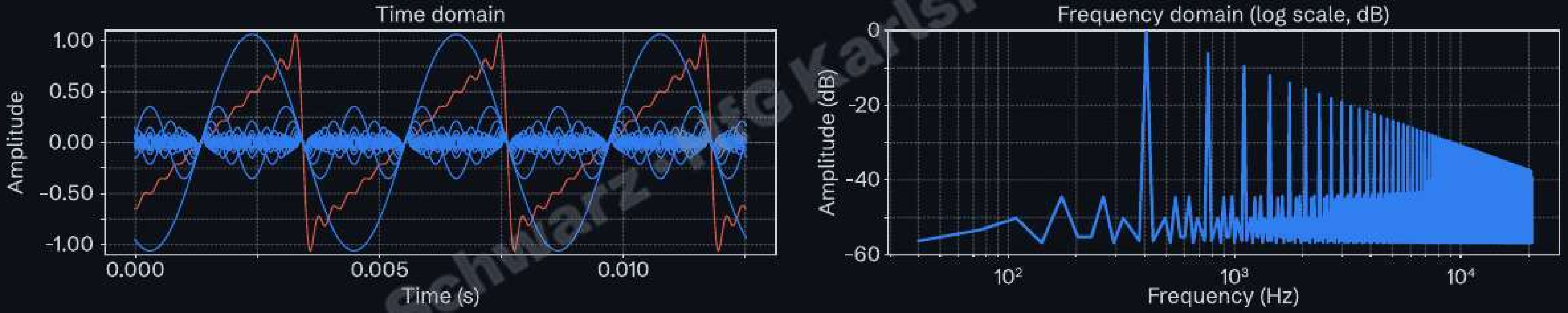




$$x(t) = A\sin(2\pi f_0 t + \varphi)$$

View sine wave on Desmos

# Complex tones (example: sawtooth)

Musical instrument sounds and basic waveforms (except sine) contain many sine waves (click for graphing calculator)




$$x(t) = \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2\pi n f_0 t)$$

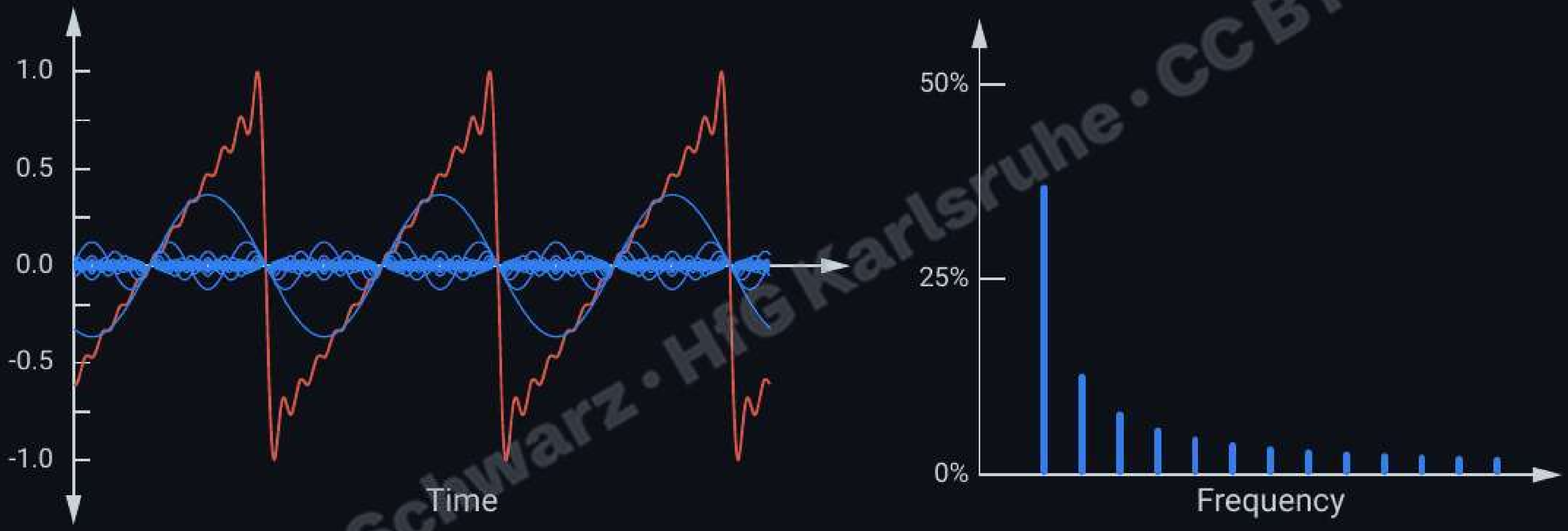
# Fourier transform

Decomposes a signal from the time domain (waveform) into the frequency domain (spectrum):

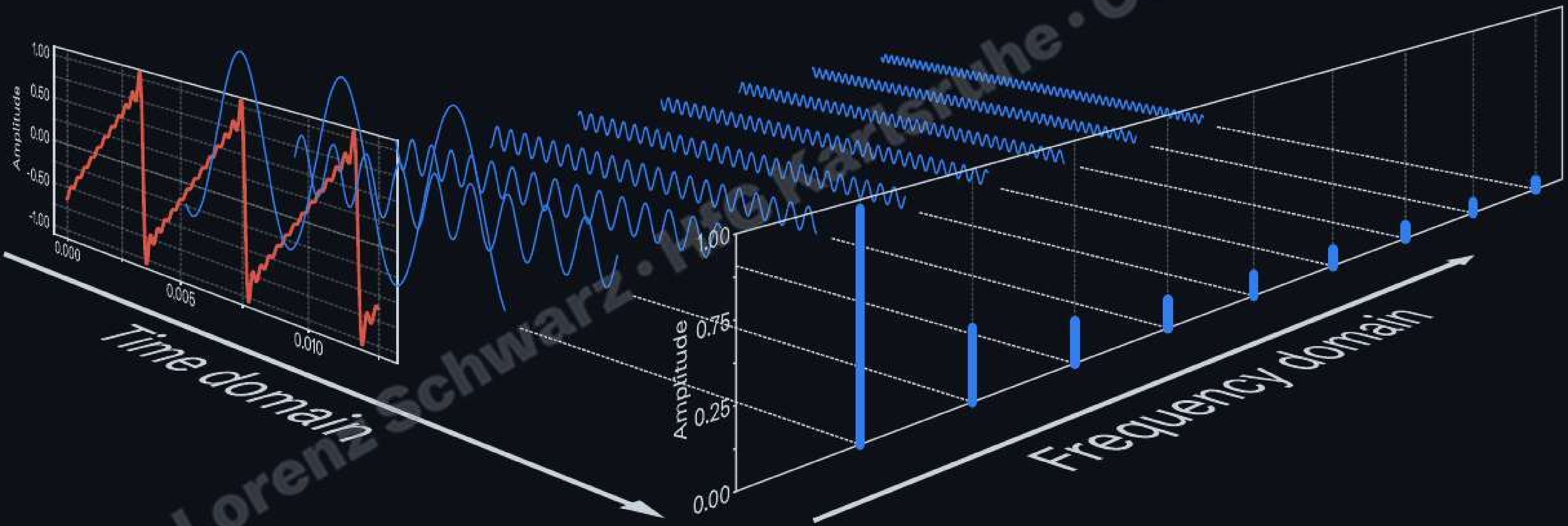
- Reveals the individual sine wave components and their amplitudes (and phases)
- Shows which frequencies are present and how strong they are

→ *Any complex sound can be represented as a sum of sine waves.*

# Fourier series of a saw tooth wave (approximation)



Fourier transform of a sawtooth wave



# Time-domain signals and spectral analysis

A time-domain signal $x(t)$ represents amplitude values over time, either continuous or discrete $x(n)$. The spectrum $X(\omega)$ is a weighting function that describes how harmonic components are combined to reconstruct the time-domain signal as a sum.

- Input: time-domain signal $x(t)$
- Output: frequency-domain spectrum $X(\omega)$

→ *The spectrum represents the amplitude and phase of each frequency component.*

# Fourier Transform

Fourier transform (analysis formula):

- Break the signal into its frequency components

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \mathrm{d}t, \quad \omega \in (-\infty, \infty)$$

Inverse transform (synthesis formula):

- Rebuild the signal from its frequency components.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \mathrm{d}\omega$$

# Euler's formula and complex numbers

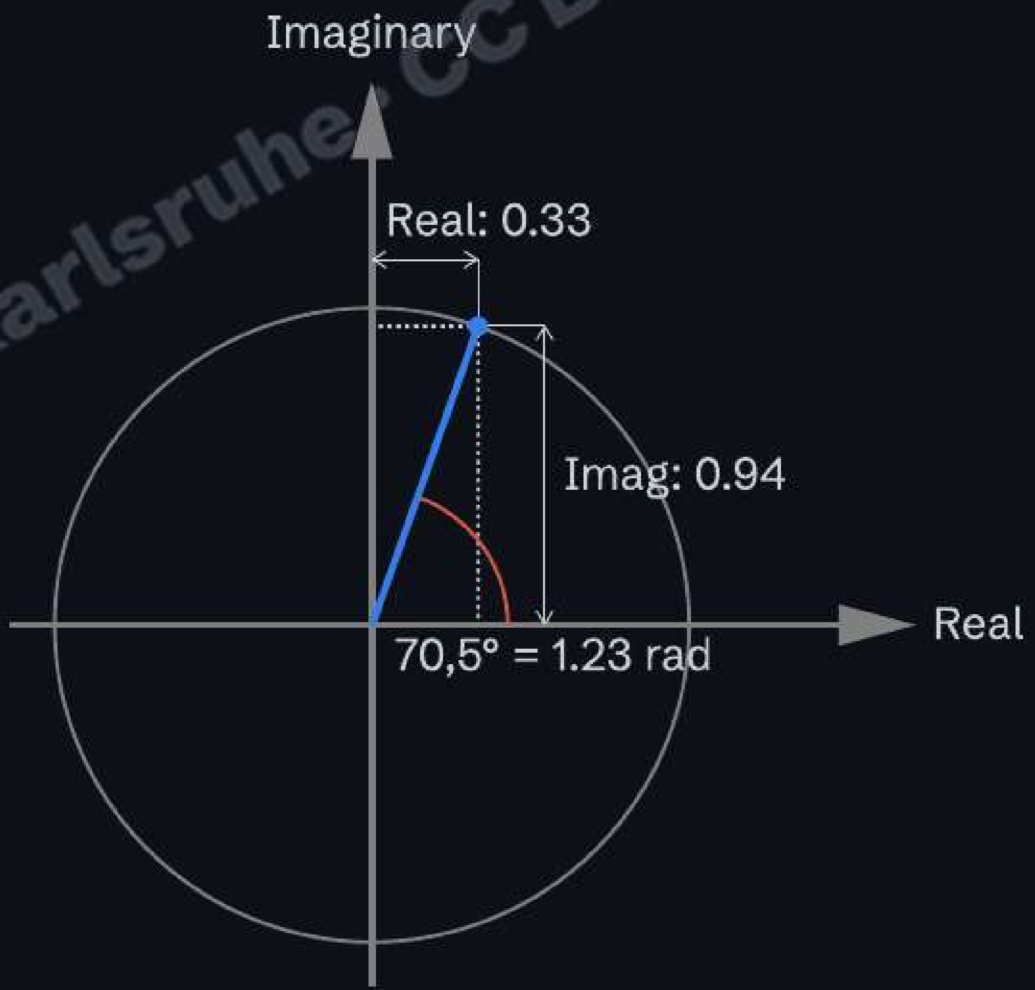
A complex exponential combines cosine and sine into a single expression representing sinusoidal components:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

where $j = \sqrt{-1}$ is the imaginary unit.

- **Real part:** $\cos(\omega t)$ — cosine component
- **Imaginary part:** $j\sin(\omega t)$ — sine component

→ *This representation is fundamental to the Fourier transform, allowing efficient encoding of both amplitude and phase.*



# Discrete Fourier transform (DFT)

For digital audio, the DFT is used, which operates on discrete, finite-duration signals by replacing the continuous integral with a finite sum. The Fast Fourier Transform (FFT) is an efficient algorithm for computing the DFT.

$$X(\omega_k) = \sum_{n=0}^{N-1} x(t_n) e^{-j\omega_k t_n}, \quad k = 0, 1, 2, \ldots, N-1$$

inverse DFT:

$$x(t_n) = \frac{1}{N} \sum_{k=0}^{N-1} X(\omega_k) e^{j\omega_k t_n}, \qquad n = 0, 1, 2, \ldots, N-1$$

# Discrete Fourier transform (DFT)

- No calculus needed — uses finite sums, avoids infinities.
- Assumes finite, sampled signals: Digital processing uses sampled signals.

→ *DFT is simpler and more computationally relevant than FT.*

# Quantities of the DFT formula:

$$
\begin{aligned}
\sum_{n=0}^{N-1} &= f(0) + f(1) + \ldots + f(N-1) \\
x(t_n) &= \text{input signal amplitude at time } t_n \text{ (sec)} \\
t_n &= nT = nth \text{ sampling instant (sec), n an integer} \geq 0 \\
T &= \text{sampling interval (sec)} \\
X(\omega_k) &= \text{spectrum of x at frequency } \omega_k \\
\omega_k &= k\Omega = kth \text{ frequency sample (rad/s)} \\
\Omega &= \frac{2\pi}{NT} = \text{radian-frequency sampling interval (rad/s)} \\
f_s &= 1/T = \text{sampling rate (samples/second, or Hertz (Hz))} \\
N &= \text{number of time samples} = \text{number of frequency samples (integer)}
\end{aligned}
$$

# Fast Fourier Transform (FFT)

The FFT is an efficient algorithm for computing the DFT, reducing computational complexity from $N^2$ to $N \log N$ operations.

This allows efficient computation for:

- Real-time spectrum analysis
- Frequency-domain filtering and equalization
- Convolution-based processing (reverberation, time-stretching)

→ *The FFT is an algorithmic optimization of the DFT computation.*

# Frequency bins

The FFT produces discrete frequency values called bins, each representing a specific frequency component. Each bin contains amplitude and phase information for its frequency component.

**Frequency of bin $k$:**

$$f_k = \frac{k}{N} f_s$$

where $k = 0, 1, 2, \ldots, N-1$ is the bin index, $f_s$ is the sampling rate, and $N$ is the FFT size.

For real signals: Number of bins = $\frac{N}{2} + 1$ (due to symmetry)

# Frequency resolution

Frequency resolution (bin spacing) determines how finely the spectrum is divided:

$$\Delta f = \frac{f_s}{N}$$

where $f_s$ is the sampling rate and $N$ is the FFT size.

- Larger FFT size → smaller $\Delta f$ → better frequency resolution
- Frequencies separated by less than $\Delta f$ cannot be distinguished

# Example

Sampling rate: 44.1 kHz
FFT 1024-sample

$$\Delta f = \frac{44100}{1024} \approx 43 \text{ Hz per bin}$$

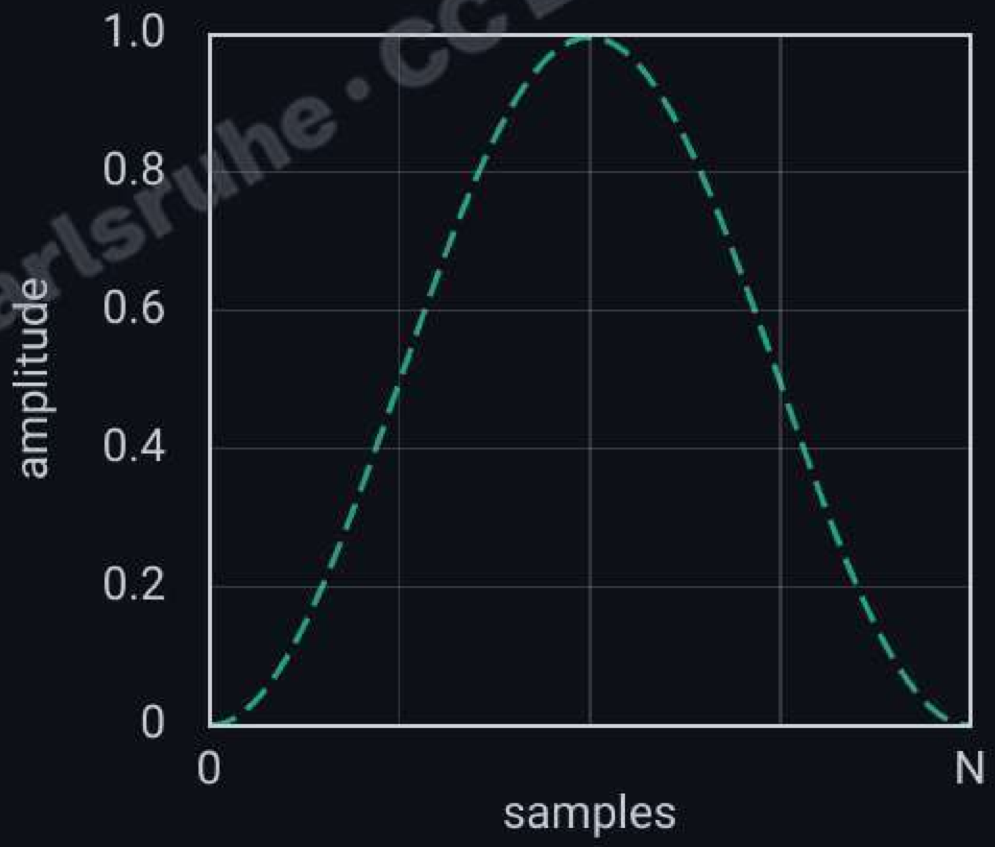
This means frequencies within a frequency band 43 Hz fall in the same bin and cannot be distinguished.

# Window function

Tapering function that smoothly reduces signal amplitude to zero at analysis window boundaries, minimizing discontinuities and spectral leakage.

- Applied when signals contain non-integer periods within the FFT window
- Typically symmetrical, bell-shaped functions
- Common types: Hann, Hamming, Blackman-Harris

→ *Trade-off: Reduced leakage vs. reduced frequency resolution.*




Hann window

# Spectral leakage and windowing

Spectral leakage occurs when the analysis window doesn't contain an exact integer number of wave cycles.

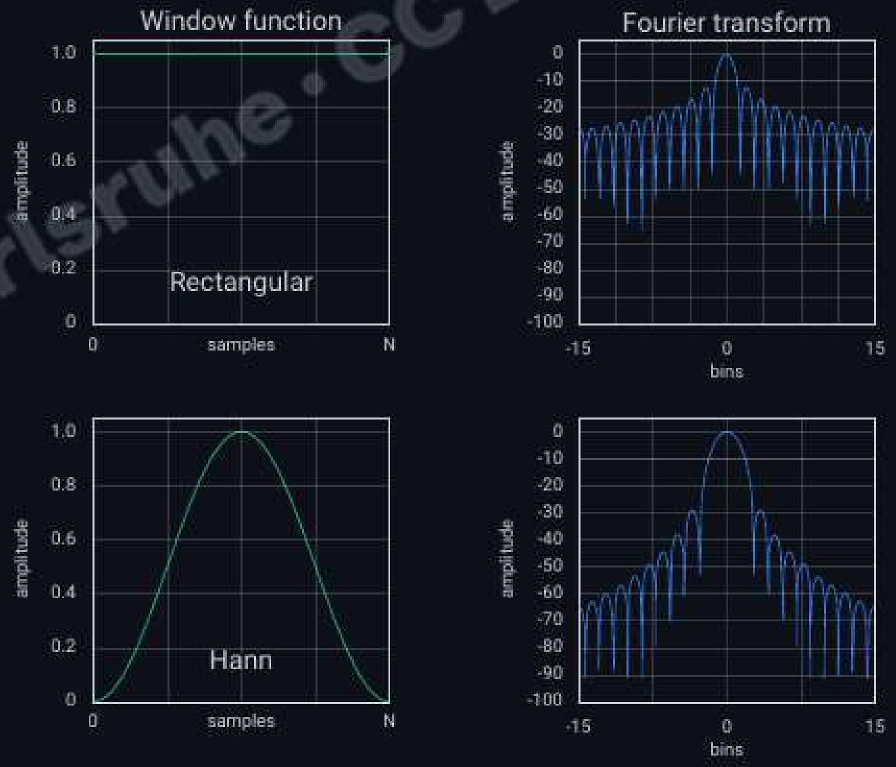
- The signal appears discontinuous at window edges
- This discontinuity creates artificial frequency components
- Energy 'leaks' from the true frequency into neighboring bins

→ *A window function tapers the signal smoothly to zero at the edges.*

# Window characteristics in frequency domain

Each window function has a characteristic frequency response with a main lobe and side lobes:

- **Main Lobe:**
  - Central peak determining frequency resolution. Width measured between first zeros (null points).
- **Side Lobes:**
  - Secondary peaks flanking the main lobe. Height (in dB) indicates leakage suppression quality.
- **Trade-off:**
  - Lower side lobes require wider main lobes, reducing frequency resolution.




Rectangular window: narrow main lobe, high side lobes (-13 dB)

# Selecting a window function

| Window | Main lobe width | Side lobe level | Use case |
|---|---|---|---|
| Rectangular (no window) | Narrowest (2 bins) | Highest (-13 dB) | Maximum frequency resolution, integer number of periods |
| Hann | Medium (4 bins) | -31 dB | General purpose, good balance of resolution and leakage |
| Hamming | Medium (4 bins) | -42 dB | Better side lobe suppression, 8-bit systems, telephony |
| Blackman-Harris | Widest (6 bins) | -92 dB (4-term) | High dynamic range, very low leakage critical applications |

Trade-off: Better side lobe suppression = wider main lobe = reduced frequency resolution.

# FFT size (window size)

Number of samples per FFT computation. Determines the time-frequency resolution trade-off:

- Larger size: Better frequency resolution, worse time resolution
- Smaller size: Better time resolution, worse frequency resolution

Common sizes: 256, 512, 1024, 2048, 4096 (powers of 2)

| FFT Size | Frequency Resolution | Time Resolution |
|---|---|---|
| Small (256) | Poor (coarse bins) | Good (fast response) |
| Large (4096) | Good (fine bins) | Poor (slow response) |

# Applications of the Fourier transform



Theoretical approaches:

- Organs: Additive synthesis for sound creation.
- Tone Wheels: Used in the Telharmonium by Thaddeus Cahill (1898).

Spectral audio signal processing:

- Additive synthesis
- Digital filter design
- Vocoder: Manipulation of speech and audio signals.

# Applications of spectral analysis

- Analysis: spectrum analyzers in DAWs
- Processing: convolution reverb, spectral effects, noise reduction, time stretching
- Synthesis: additive synthesis, spectral resynthesis

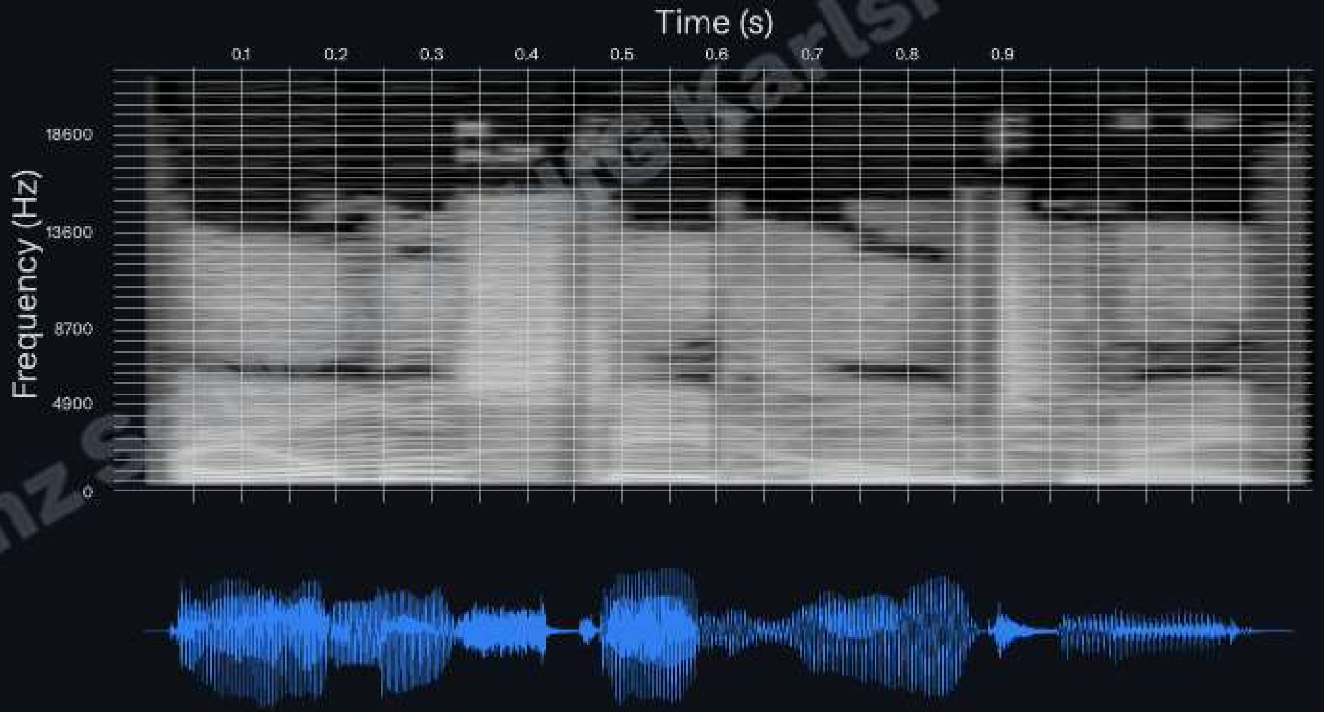
# Spectrogram, sonogram

A time-varying visual representation of a signal's frequency content:

- **X-axis:** Time
- **Y-axis:** Frequency
- **Color/brightness:** Amplitude

# Spectrogram

Spectrograms reveal temporal evolution of spectral content. Helpful for analyzing speech, music, and environmental sounds.



# Copyright and Licensing



**Contact:** lschwarz@hfg-karlsruhe.de

← Chapters · Download PDF ↓