SPECTRAL ANALYSIS

Fourier transform and frequency domain

Sound in different domains

An audio signal can be described from different perspectives, depending on which aspect of sound is being analyzed.

- Time-based descriptions reveal changes over time
- Frequency-based descriptions reveal spectral content

→ *Both views describe the same signal, but reveal different information.*

Time domain vs. frequency domain

Audio signals are commonly represented in two domains:

- Time domain:
  - Amplitude as a function of time
- Frequency domain:
  - Distribution of energy across frequencies (magnitude and phase)

(Analog signals are continuous; digital signals are discrete.)

Time domain

The time domain shows how a signal's amplitude changes over time.

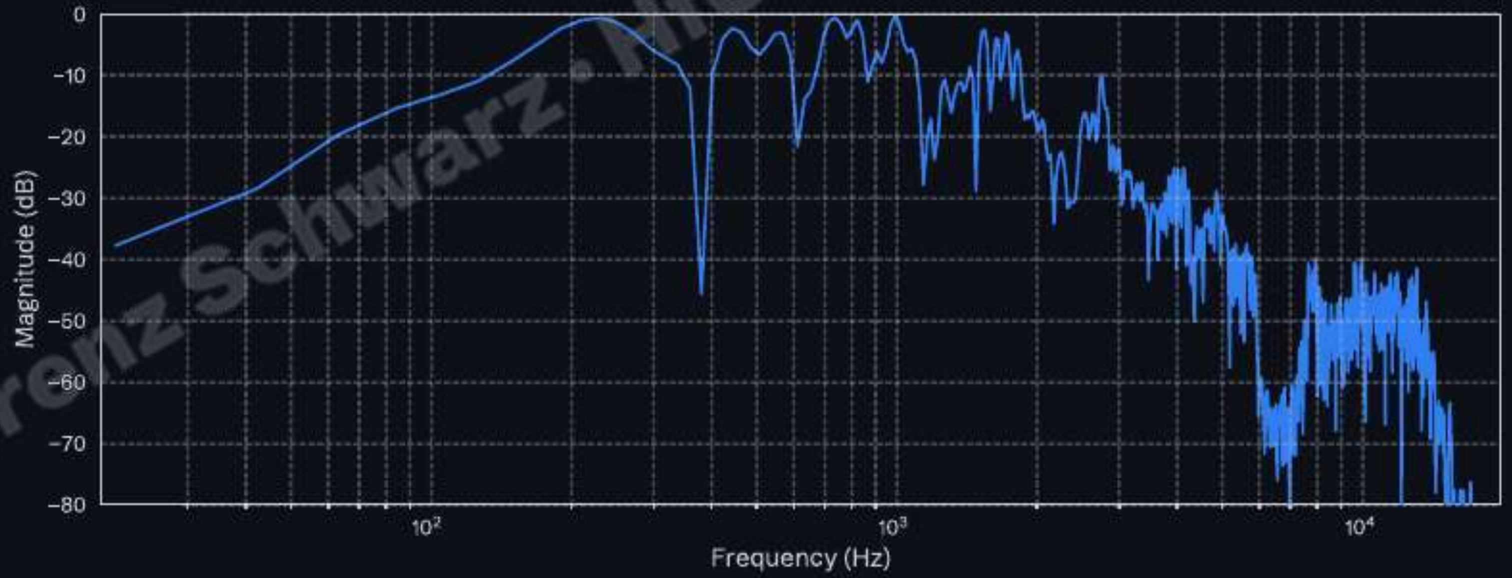
- Analog signals: continuous in time and amplitude
- Digital signals: discrete samples in time and amplitude

→ *The time domain reveals transients, timing, and amplitude changes.*

Frequency domain

The frequency domain describes a signal in terms of its frequency components rather than time.

- shows how much energy each frequency contributes



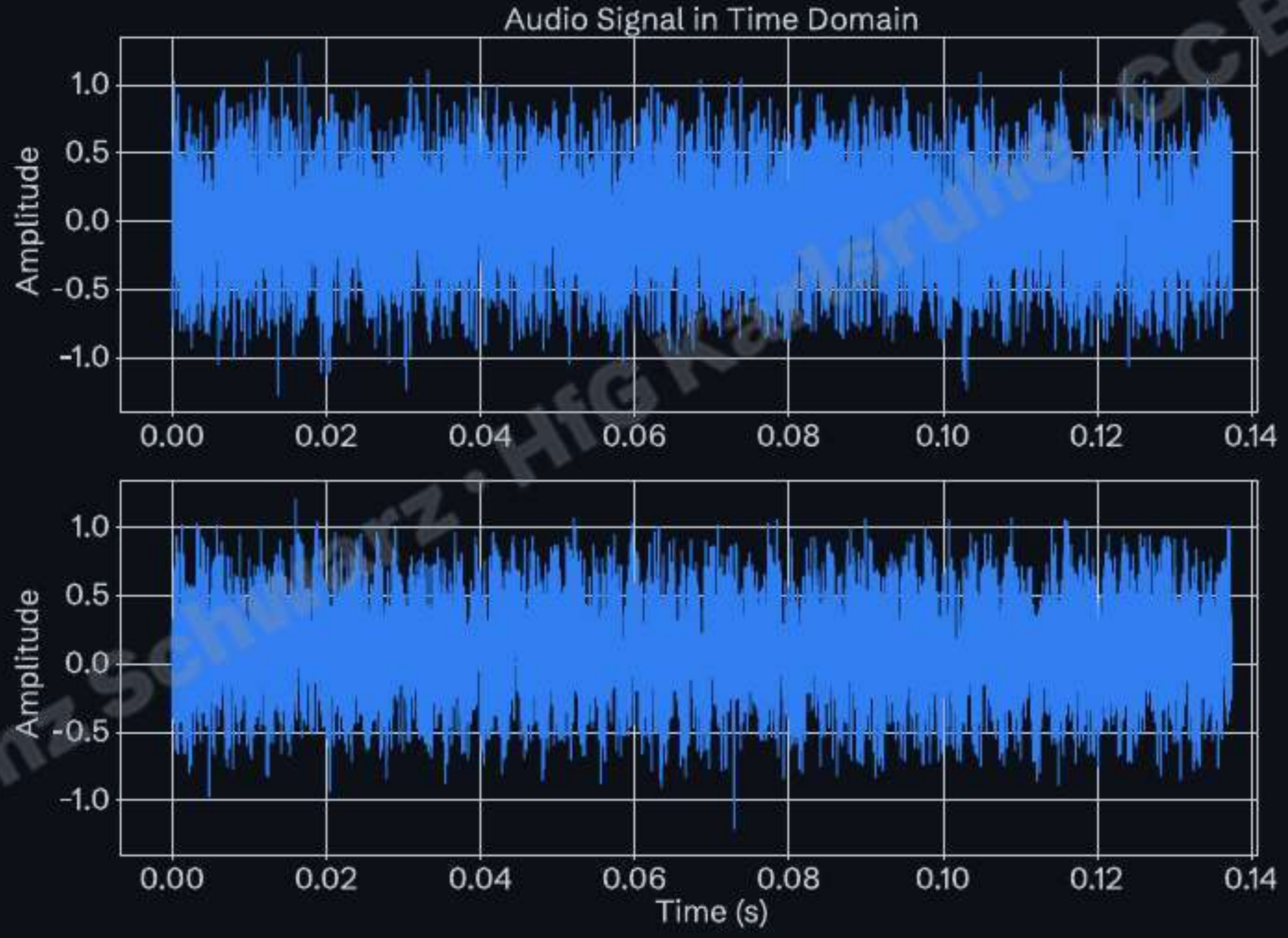
Spectrum

The spectrum shows a moment of a signal’s frequency content.

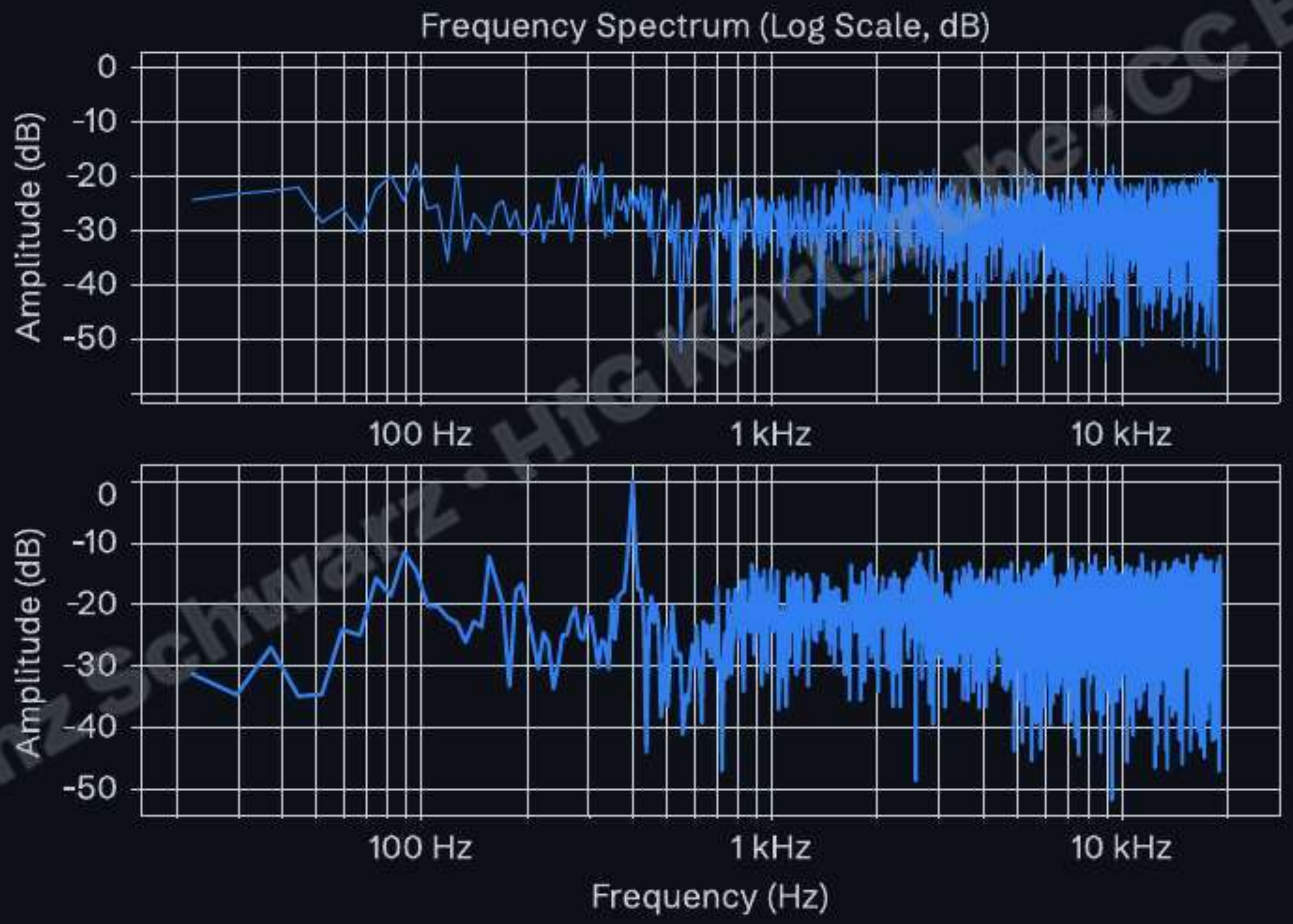
- Amplitude (or magnitude) as a function of frequency
- Optionally includes phase information

→ *The spectrum is the primary tool for analyzing timbre and harmonic structure.*

Which signal contains a 400 Hz sine?



The second spectrum shows a spike at 400Hz



Timbre and spectrum

Timbre (perceptual):
The sonic quality that distinguishes instruments playing the same pitch

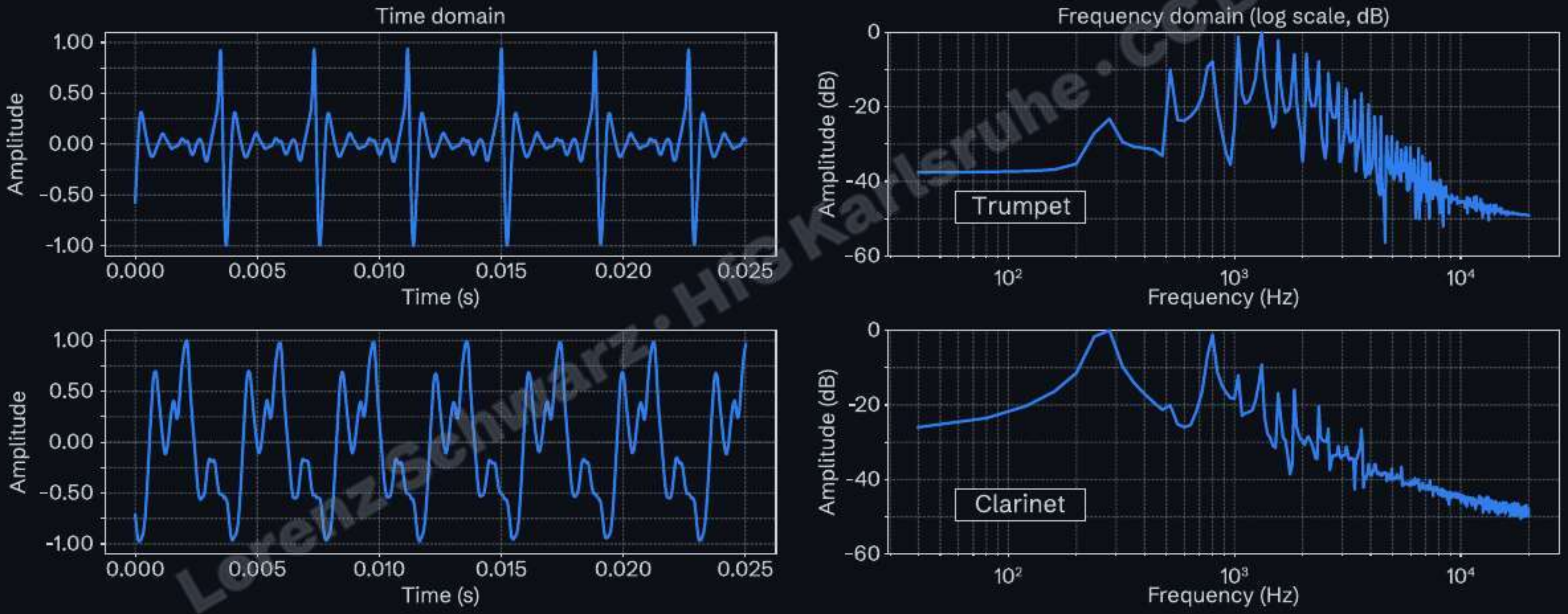
Spectrum (technical):
The distribution of frequency components and their amplitudes

Listen to the same pitch (C4 ≈ 261.6 Hz):

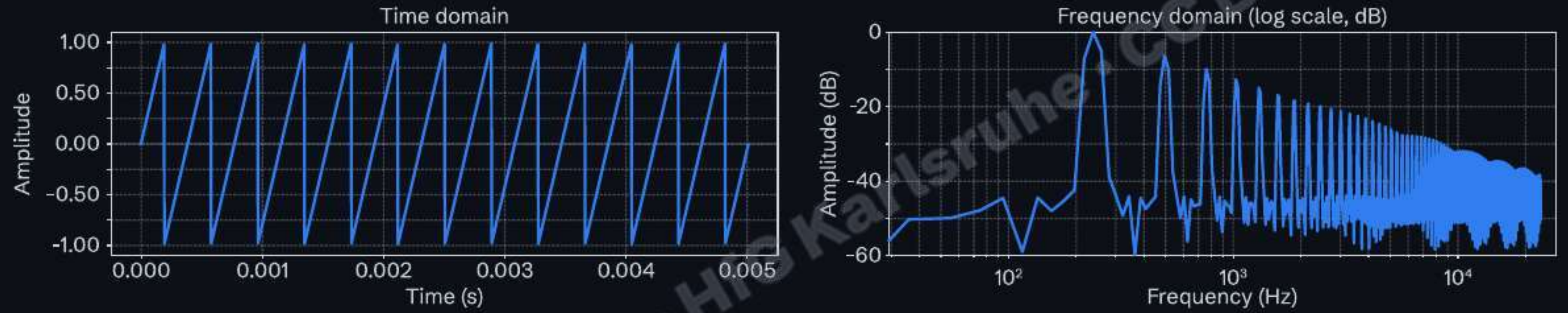
Trumpet: ▶ Play **Clarinet:** ▶ Play

→ *Same pitch, different timbre*

Comparing clarinet and trumpet at 260 Hz



Understanding spectra with a sawtooth wave

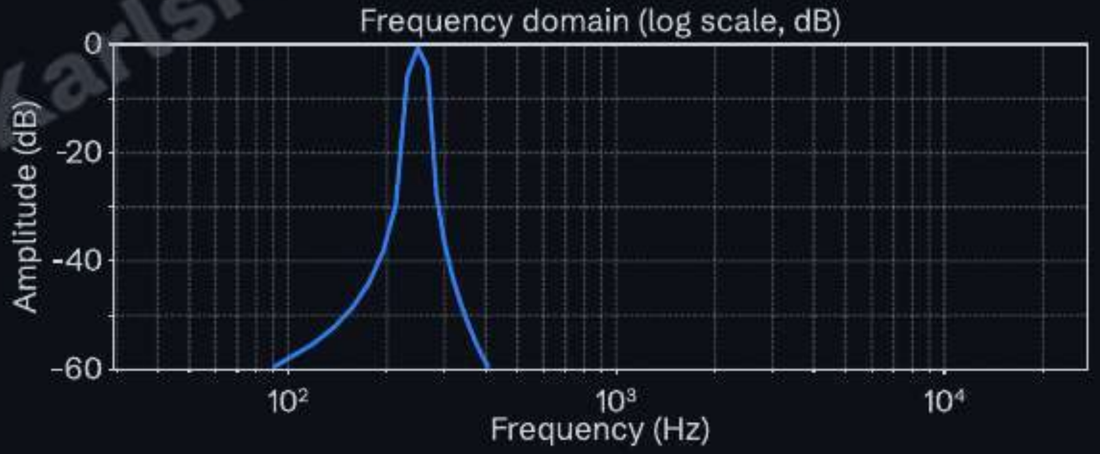


- Each peak in the spectrum represents one sine wave (partial or harmonics)
- Harmonic series 260, 520, 780, 1040… Hz

▶ Play sawtooth wave C4 ≈ 260 Hz

Pure tone (sine wave)

A sine wave is a single frequency component, the fundamental building block of all sounds

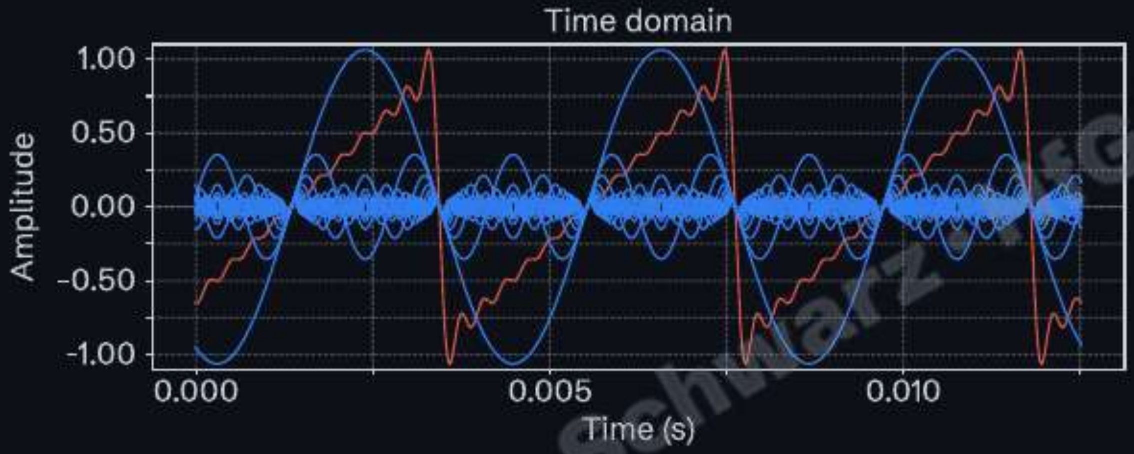


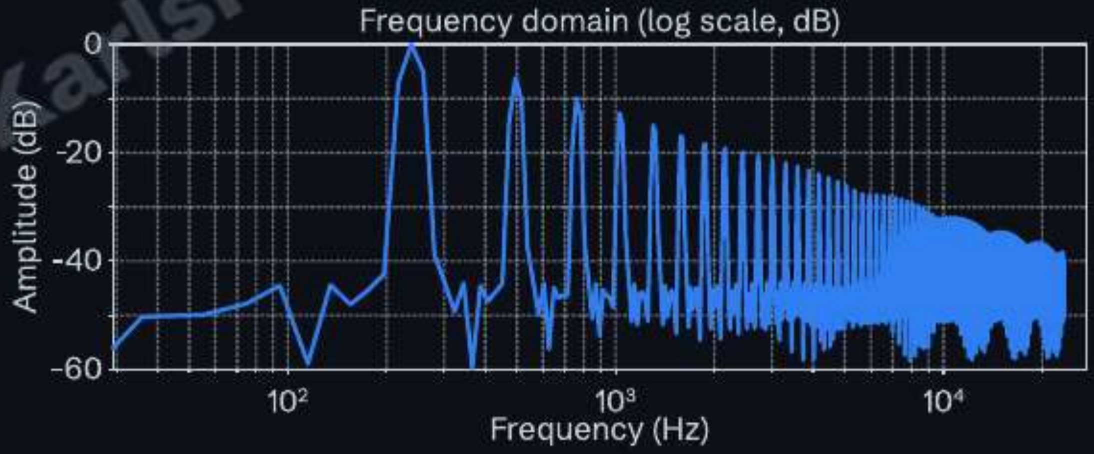
$$x(t) = A\sin(2\pi f_0 t + \varphi)$$

View sine wave on Desmos

Complex tones (example: sawtooth)

Musical instrument sounds and basic waveforms (except sine) contain many sine waves (click for graphing calculator)





$$x(t) = \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2\pi n f_0 t)$$

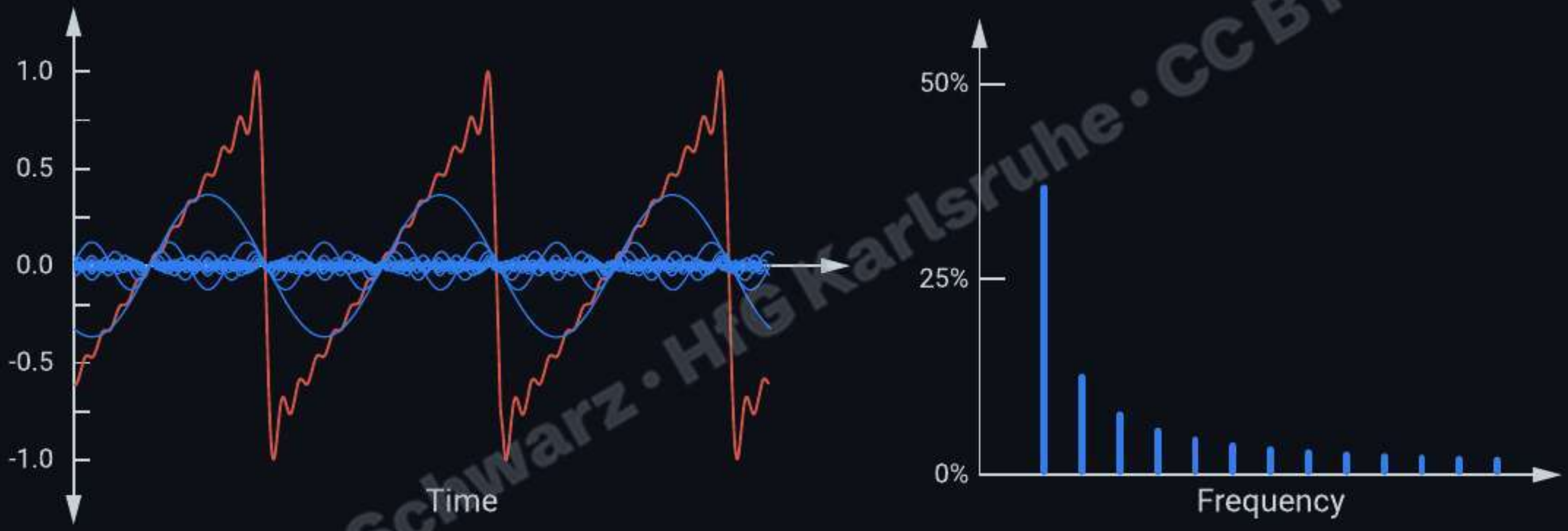
Fourier transform

Decomposes a signal from the time domain (waveform) into the frequency domain (spectrum):

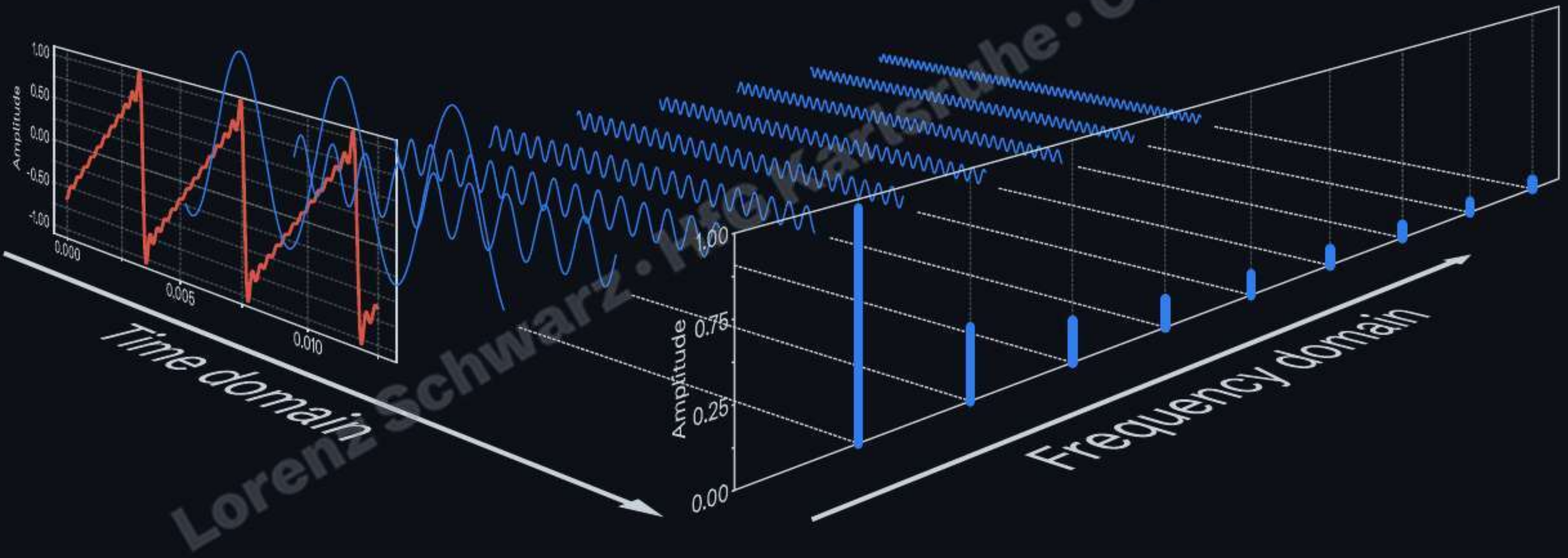
- Reveals the individual sine wave components and their amplitudes (and phases)
- Shows which frequencies are present and how strong they are

→ *Any complex sound can be represented as a sum of sine waves.*

Fourier series of a saw tooth wave (approximation)



Fourier transform of a sawtooth wave



Time-domain signals and spectral analysis

A time-domain signal $x(t)$ represents amplitude values over time, either continuous or discrete $x(n)$. The spectrum $X(\omega)$ is a weighting function that describes how harmonic components are combined to reconstruct the time-domain signal as a sum.

- Input: time-domain signal $x(t)$
- Output: frequency-domain spectrum $X(\omega)$

→ *The spectrum represents the amplitude and phase of each frequency component.*

Fourier Transform

Fourier transform (analysis formula):

- Break the signal into its frequency components

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t, \quad \omega \in (-\infty, \infty)$$

Inverse transform (synthesis formula):

- Rebuild the signal from its frequency components.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\mathrm{d}\omega$$

Euler's formula and complex numbers

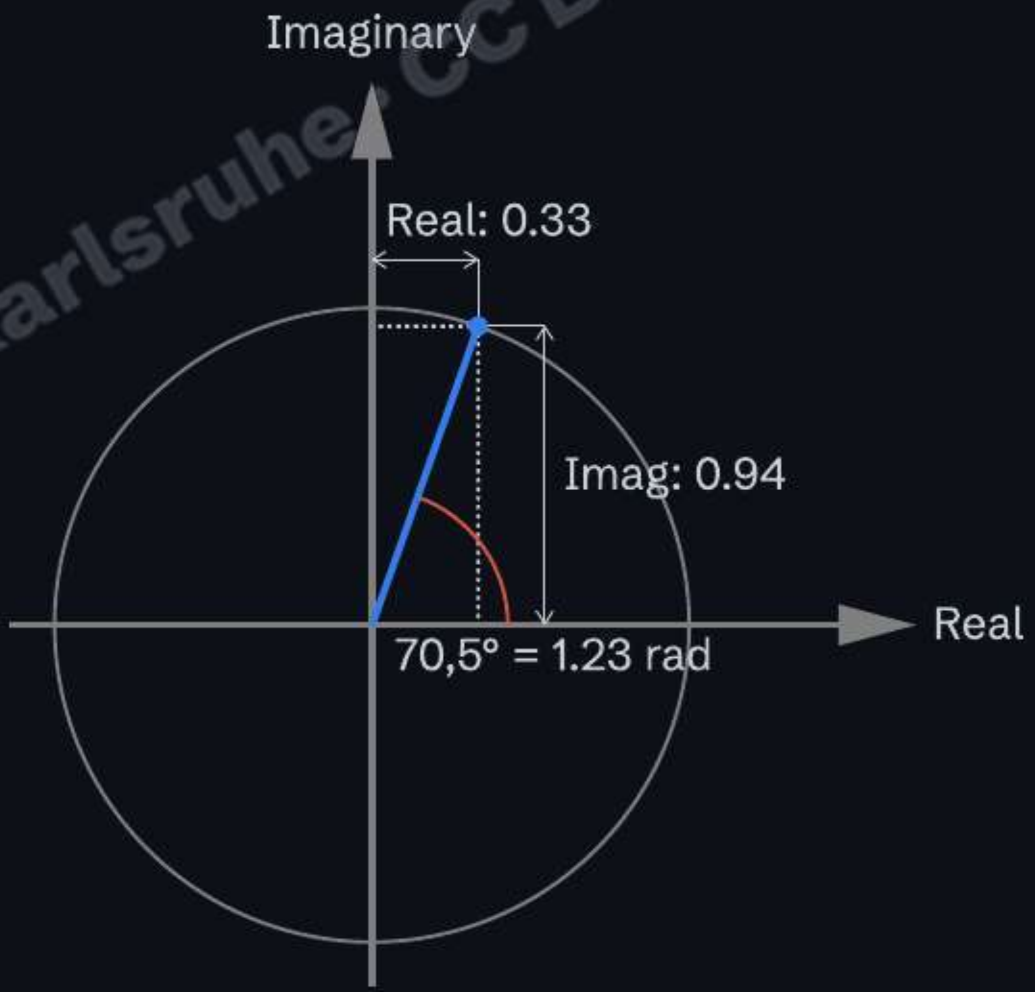
A complex exponential combines cosine and sine into a single expression representing sinusoidal components:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

where $j = \sqrt{-1}$ is the imaginary unit.

- **Real part:** $\cos(\omega t)$ — cosine component
- **Imaginary part:** $j\sin(\omega t)$ — sine component

→ *This representation is fundamental to the Fourier transform, allowing efficient encoding of both amplitude and phase.*



Discrete Fourier transform (DFT)

For digital audio, the DFT is used, which operates on discrete, finite-duration signals by replacing the continuous integral with a finite sum. The Fast Fourier Transform (FFT) is an efficient algorithm for computing the DFT.

$$X(\omega_k) = \sum_{n=0}^{N-1} x(t_n) e^{-j\omega_k t_n}, \quad k = 0, 1, 2, \ldots, N-1$$

inverse DFT:

$$x(t_n) = \frac{1}{N} \sum_{k=0}^{N-1} X(\omega_k) e^{j\omega_k t_n}, \qquad n = 0, 1, 2, \ldots, N-1$$

Discrete Fourier transform (DFT)

- No calculus needed — uses finite sums, avoids infinities.
- Assumes finite, sampled signals: Digital processing uses sampled signals.

→ *DFT is simpler and more computationally relevant than FT.*

Quantities of the DFT formula:

$$
\begin{aligned}
\sum_{n=0}^{N-1} &= f(0) + f(1) + \ldots + f(N-1) \\
x(t_n) &= \text{input signal amplitude at time } t_n \text{ (sec)} \\
t_n &= nT = nth \text{ sampling instant (sec), n an integer} \geq 0 \\
T &= \text{sampling interval (sec)} \\
X(\omega_k) &= \text{spectrum of x at frequency } \omega_k \\
\omega_k &= k\Omega = kth \text{ frequency sample (rad/s)} \\
\Omega &= \frac{2\pi}{NT} = \text{radian-frequency sampling interval (rad/s)} \\
f_s &= 1/T = \text{sampling rate (samples/second, or Hertz (Hz))} \\
N &= \text{number of time samples} = \text{number of frequency samples (integer)}
\end{aligned}
$$

Fast Fourier Transform (FFT)

The FFT is an efficient algorithm for computing the DFT, reducing computational complexity from N^2 to $N \log N$ operations.

This allows efficient computation for:

- Real-time spectrum analysis
- Frequency-domain filtering and equalization
- Convolution-based processing (reverberation, time-stretching)

→ *The FFT is an algorithmic optimization of the DFT computation.*

Frequency bins

The FFT produces discrete frequency values called bins, each representing a specific frequency component. Each bin contains amplitude and phase information for its frequency component.

Frequency of bin k:

$$f_k = \frac{k}{N} f_s$$

where $k = 0, 1, 2, \ldots, N-1$ is the bin index, f_s is the sampling rate, and N is the FFT size.

For real signals: Number of bins = $\frac{N}{2} + 1$ (due to symmetry)

Frequency resolution

Frequency resolution (bin spacing) determines how finely the spectrum is divided:

$$\Delta f = \frac{f_s}{N}$$

where f_s is the sampling rate and N is the FFT size.

- Larger FFT size → smaller Δf → better frequency resolution
- Frequencies separated by less than Δf cannot be distinguished

Example

Sampling rate: 44.1 kHz
FFT 1024-sample

$$\Delta f = \frac{44100}{1024} \approx 43 \text{ Hz per bin}$$

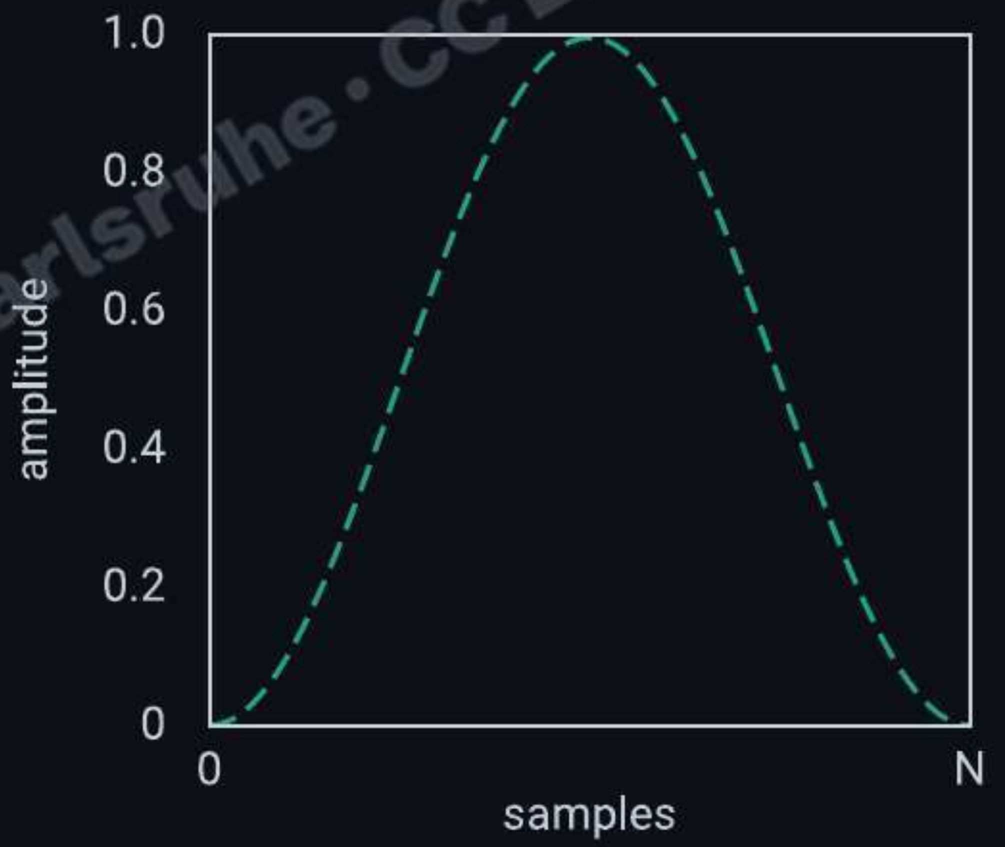
This means frequencies within a frequency band 43 Hz fall in the same bin and cannot be distinguished.

Window function

Tapering function that smoothly reduces signal amplitude to zero at analysis window boundaries, minimizing discontinuities and spectral leakage.

- Applied when signals contain non-integer periods within the FFT window
- Typically symmetrical, bell-shaped functions
- Common types: Hann, Hamming, Blackman-Harris

→ *Trade-off: Reduced leakage vs. reduced frequency resolution.*



Hann window

Spectral leakage and windowing

Spectral leakage occurs when the analysis window doesn't contain an exact integer number of wave cycles.

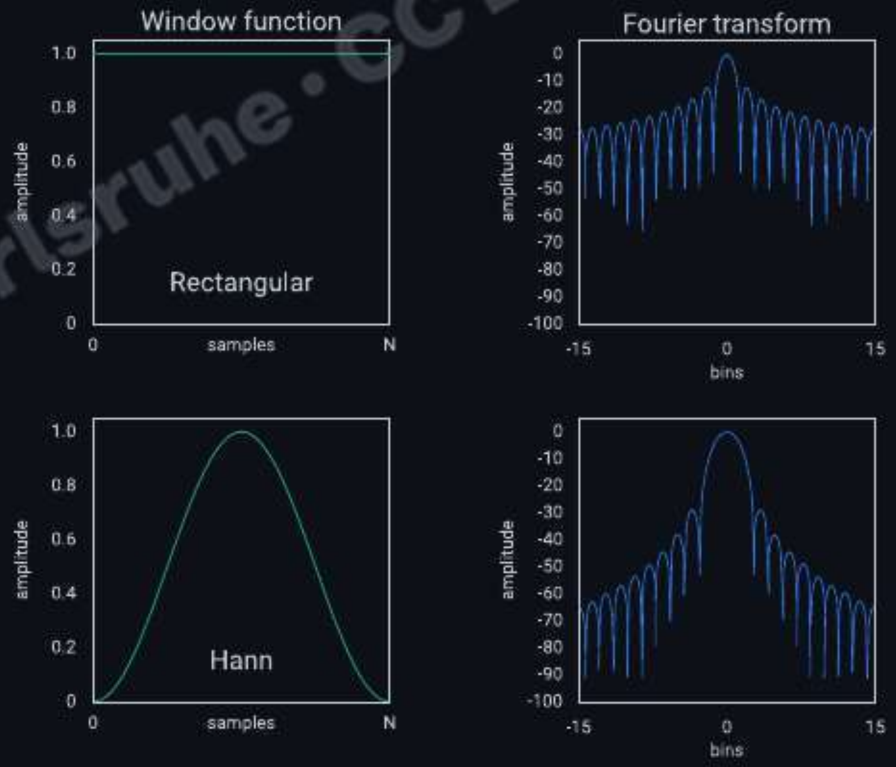
- The signal appears discontinuous at window edges
- This discontinuity creates artificial frequency components
- Energy 'leaks' from the true frequency into neighboring bins

→ *A window function tapers the signal smoothly to zero at the edges.*

Window characteristics in frequency domain

Each window function has a characteristic frequency response with a main lobe and side lobes:

- **Main Lobe:**
  - Central peak determining frequency resolution. Width measured between first zeros (null points).
- **Side Lobes:**
  - Secondary peaks flanking the main lobe. Height (in dB) indicates leakage suppression quality.
- **Trade-off:**
  - Lower side lobes require wider main lobes, reducing frequency resolution.



Rectangular window: narrow main lobe, high side lobes (-13 dB)

Selecting a window function

Window	Main lobe width	Side lobe level	Use case
Rectangular (no window)	Narrowest (2 bins)	Highest (-13 dB)	Maximum frequency resolution, integer number of periods
Hann	Medium (4 bins)	-31 dB	General purpose, good balance of resolution and leakage
Hamming	Medium (4 bins)	-42 dB	Better side lobe suppression, 8-bit systems, telephony
Blackman-Harris	Widest (6 bins)	-92 dB (4-term)	High dynamic range, very low leakage critical applications

Trade-off: Better side lobe suppression = wider main lobe = reduced frequency resolution.

FFT size (window size)

Number of samples per FFT computation. Determines the time-frequency resolution trade-off:

- Larger size: Better frequency resolution, worse time resolution
- Smaller size: Better time resolution, worse frequency resolution

Common sizes: 256, 512, 1024, 2048, 4096 (powers of 2)

FFT Size	Frequency Resolution	Time Resolution
Small (256)	Poor (coarse bins)	Good (fast response)
Large (4096)	Good (fine bins)	Poor (slow response)

Applications of the Fourier transform

Theoretical approaches:

- Organs: Additive synthesis for sound creation.
- Tone Wheels: Used in the Telharmonium by Thaddeus Cahill (1898).

Spectral audio signal processing:

- Additive synthesis
- Digital filter design
- Vocoder: Manipulation of speech and audio signals.

Applications of spectral analysis

- Analysis: spectrum analyzers in DAWs
- Processing: convolution reverb, spectral effects, noise reduction, time stretching
- Synthesis: additive synthesis, spectral resynthesis

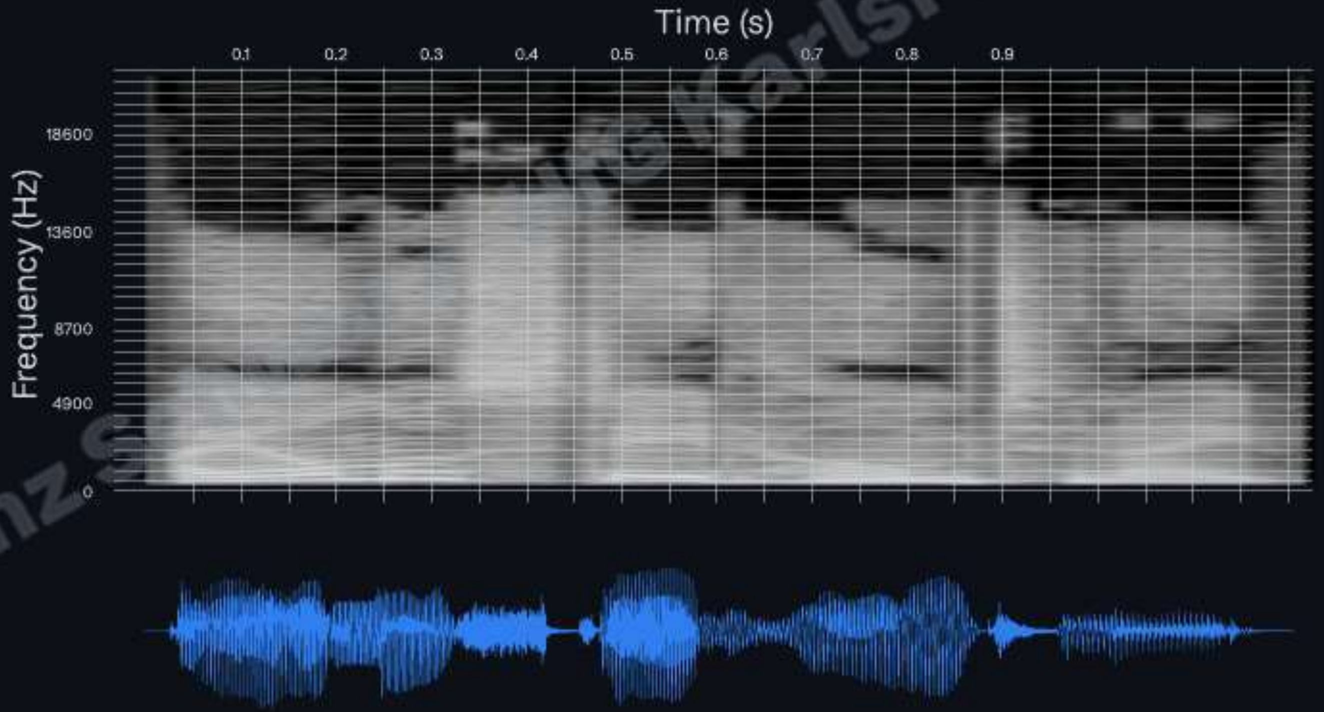
Spectrogram, sonogram

A time-varying visual representation of a signal's frequency content:

- **X-axis:** Time
- **Y-axis:** Frequency
- **Color/brightness:** Amplitude

Spectrogram

Spectrograms reveal temporal evolution of spectral content. Helpful for analyzing speech, music, and environmental sounds.



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