WAVEFORMS

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Complex vibratory systems

While a spring-mass system produces a single sinusoidal vibration, real vibrating systems (strings, air columns, membranes, plates) produce many simultaneous vibration modes.

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Modes and spectra in real sounds

Pure sine waves are rare in the real world:

- Real sound sources excite many modes, producing complex spectra.
- Each normal mode is a sinusoid.
- Superposition of modes gives the harmonic series and timbre.

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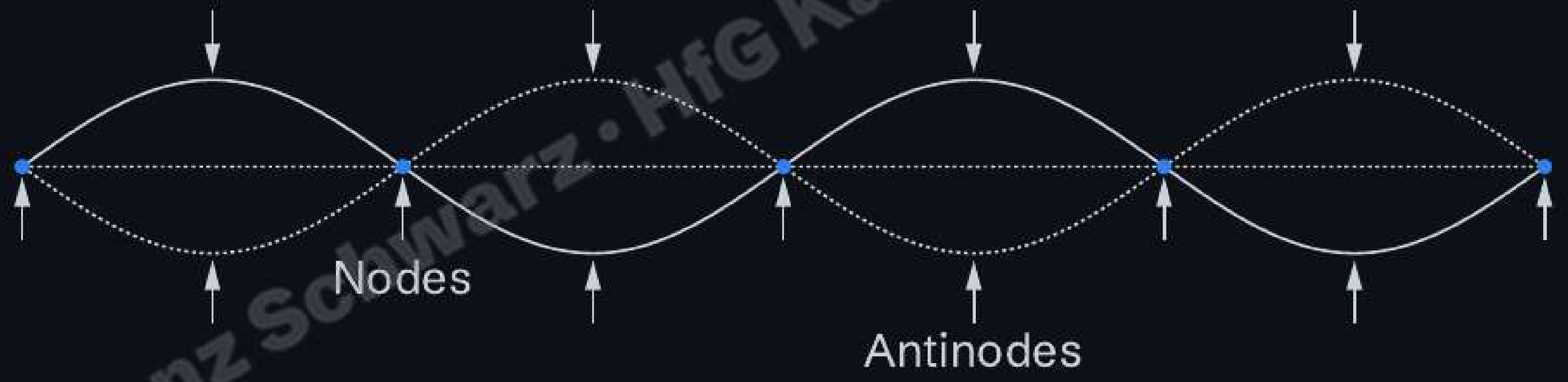
Standing waves on a vibrating string

On a vibrating string, waves travel both ways, interfere, and form standing waves with nodes (no motion) and antinodes (max motion):



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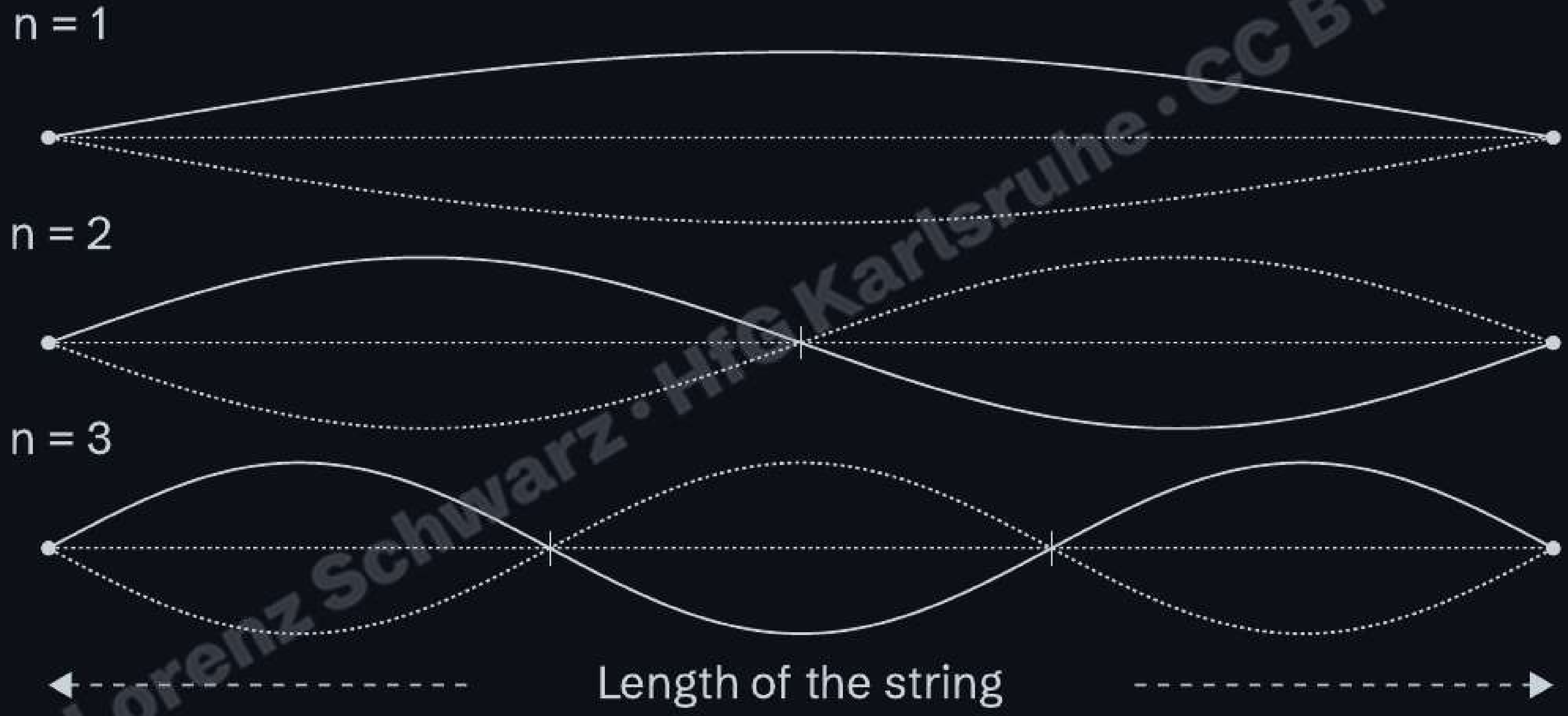
Nodes appear at rational fractions of string length.



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Normal modes on a string

Normal modes are standing-wave patterns that fit the string's boundary conditions.

→ *Their frequencies are integer multiples of the fundamental.*

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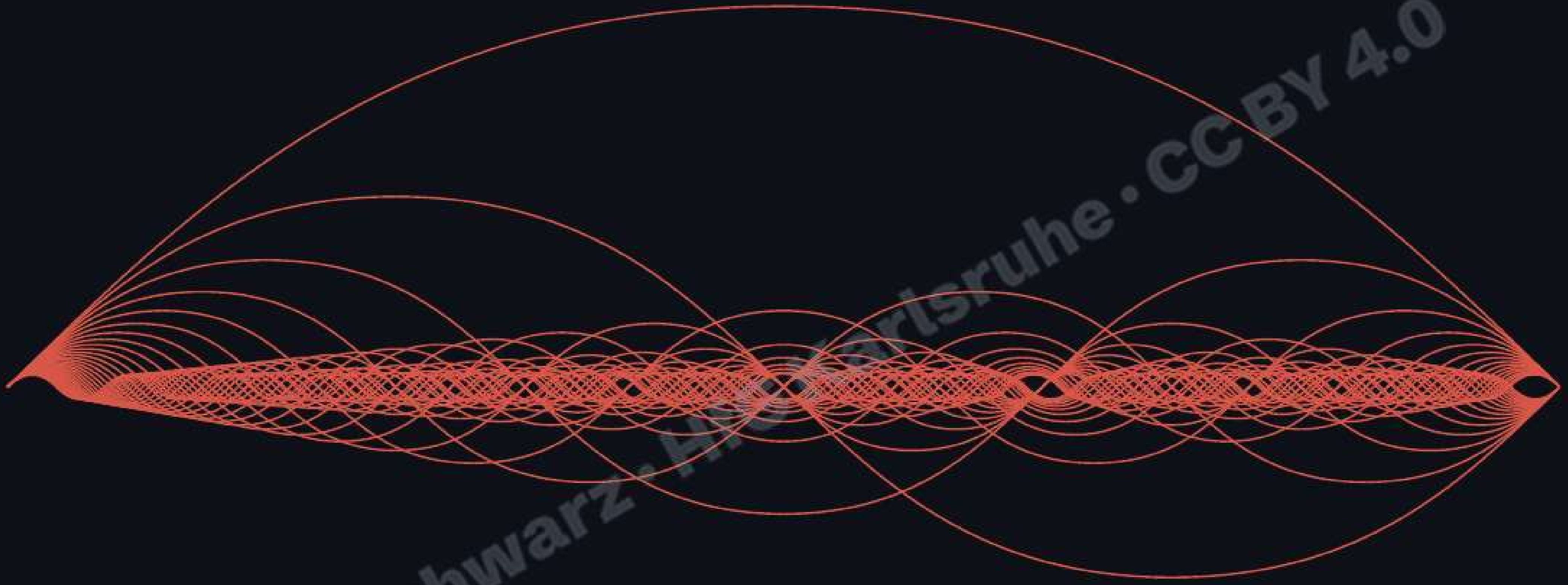

Standing waves in a string: it vibrates as a whole (fundamental) and in integer fractions of its length (harmonics).

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Mode frequencies

Allowed frequencies are integer multiples of the fundamental f_1:

$$L = n\,\frac{\lambda_n}{2} \;\Rightarrow\; \lambda_n = \frac{2L}{n}, \qquad f_n = \frac{n\,v}{2L}\;(n = 1, 2, 3, \ldots)$$

- n = mode (harmonic) number
- L = string length
- λ_n = wavelength of the n-th mode
- f_n = frequency of the n-th harmonic
- v = wave speed on the string

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Inharmonic partials in real sounds

Plates, gongs, bells, drumhead membranes and other real world sound sources have **inharmonic partials.**

→ *Inharmonic Partials that are not integer multiples of the fundamental frequency.*

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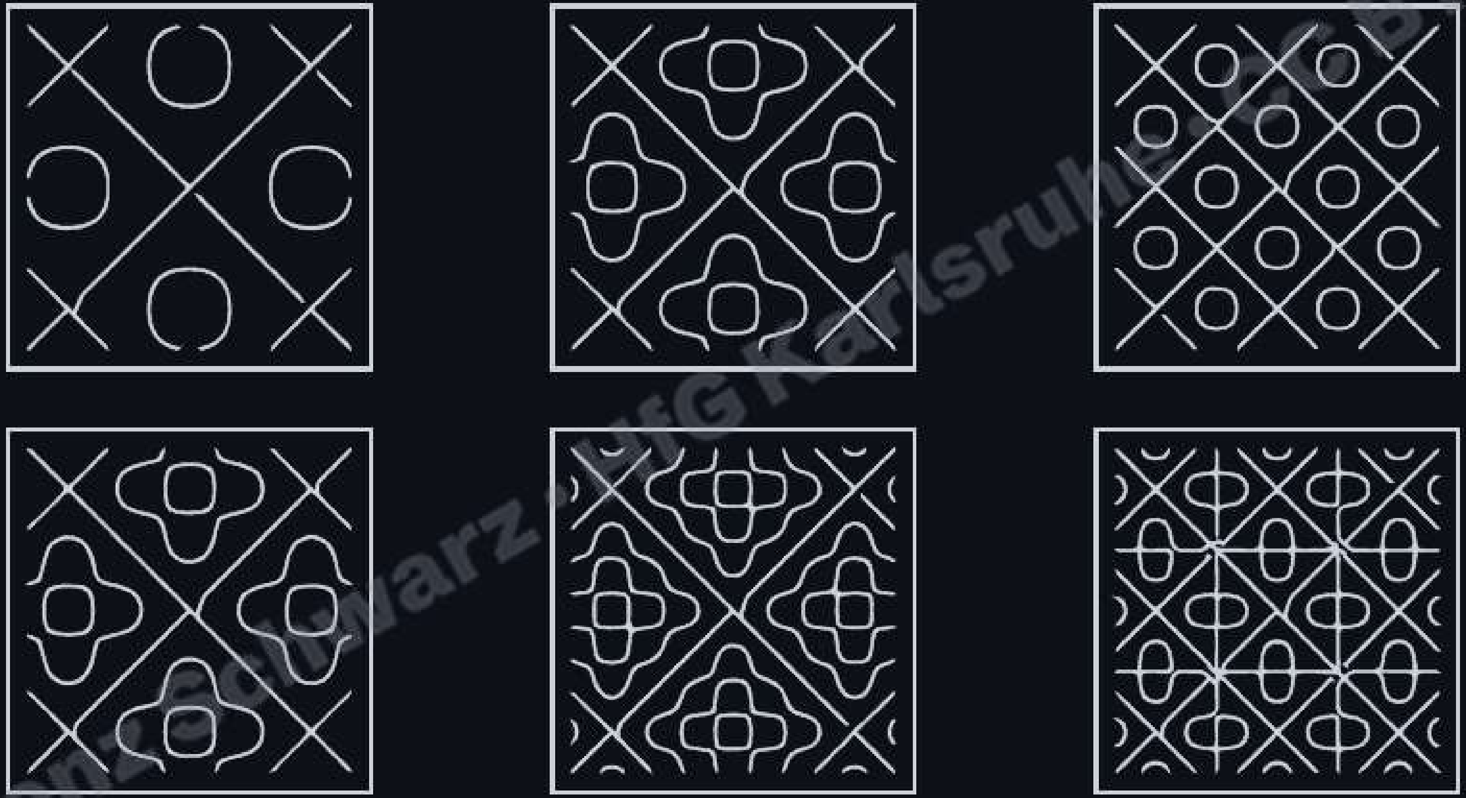
Chladni figures reveal allowed vibrational modes.



[View on Wolfram]

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Periodic functions

A periodic function in audio describes a waveform that repeats its shape at regular time intervals. Understanding these fundamental shapes and their spectral properties is essential for sound synthesis.

1. Sine wave
2. Sawtooth wave
3. Triangle wave
4. Square wave
   - Pulse wave

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Basic shapes of periodic waveforms

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Sine wave

Symmetrical and curved rise and fall with no abrupt changes:

- has no overtones (single frequency, fundamental only)
- serves as a building block of periodic signals (additive synthesis)
- rarely exists alone in nature
- resembles a pipe sound, like a flute or an organ
- used often as a test tone to assess signal integrity

▶ Sine 400 Hz

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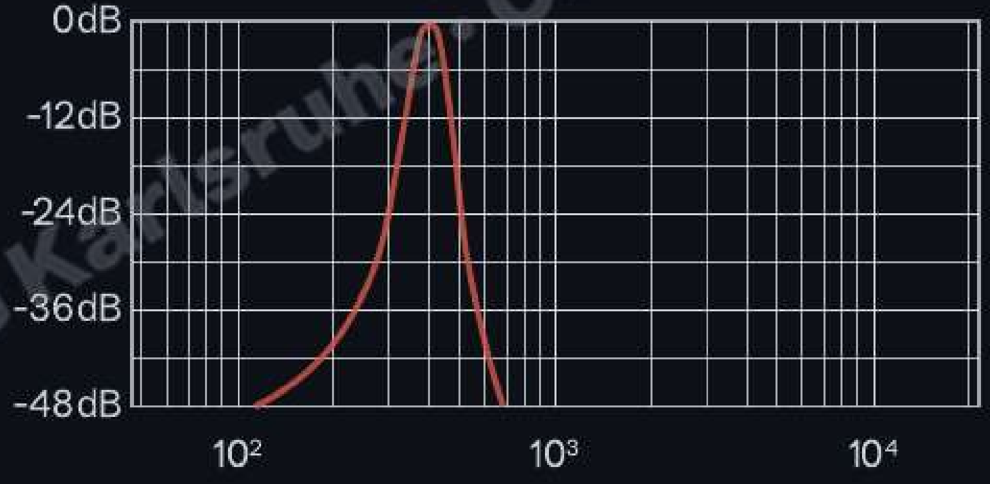
Sine wave

Time domain

Frequency domain



$$x(t) = A\sin(2\pi f_0 t + \varphi)$$

[View sine wave on Desmos]

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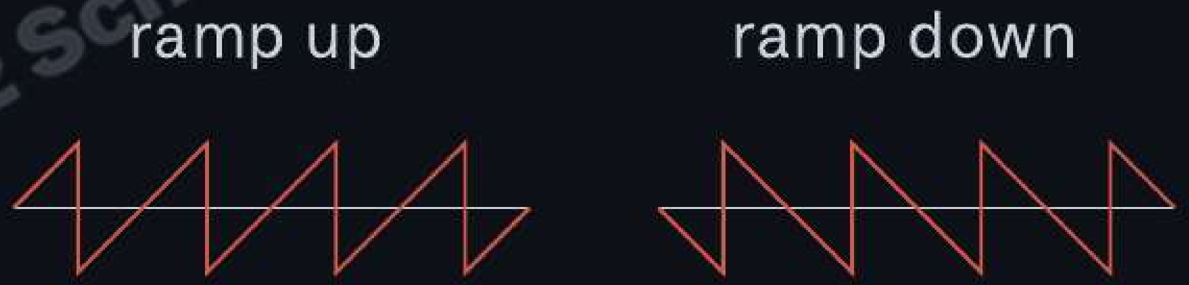
Sawtooth wave

A sawtooth is characterized by a linear rise followed by an abrupt drop:

- also called ramp ('up' or 'down')
- ramp down: shifting the phase of the even harmonics by 180°
- rich and full, great for powerful synth bass and lead sounds

▶ Sawtooth 400 Hz



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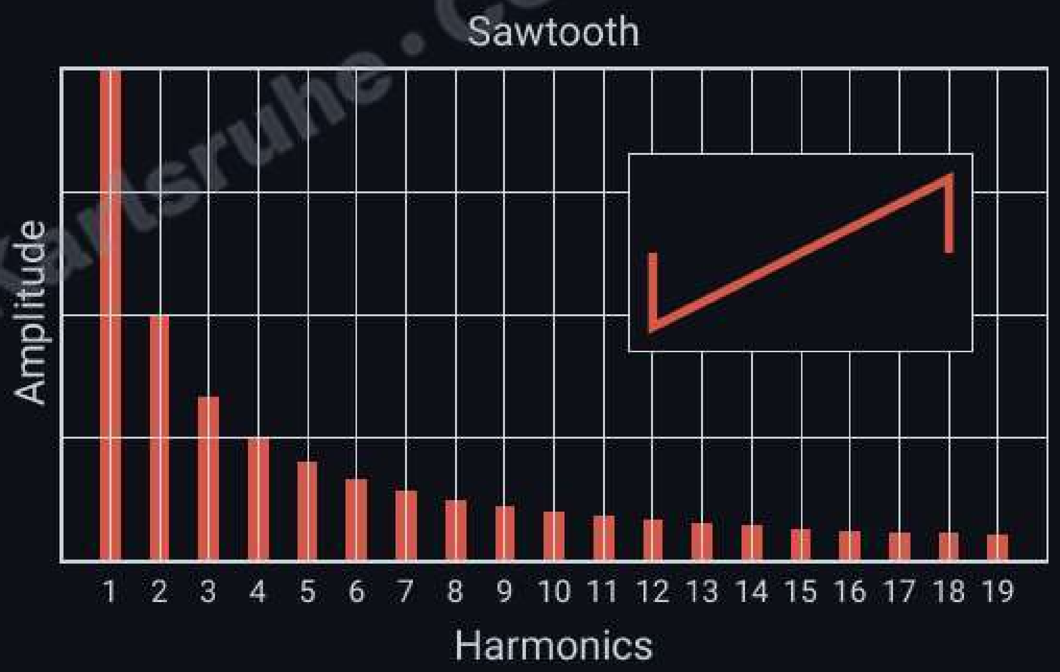
Harmonic spectrum of a sawtooth

- contains both even and odd harmonics
- relative amplitudes of harmonics are $\frac{1}{n}$



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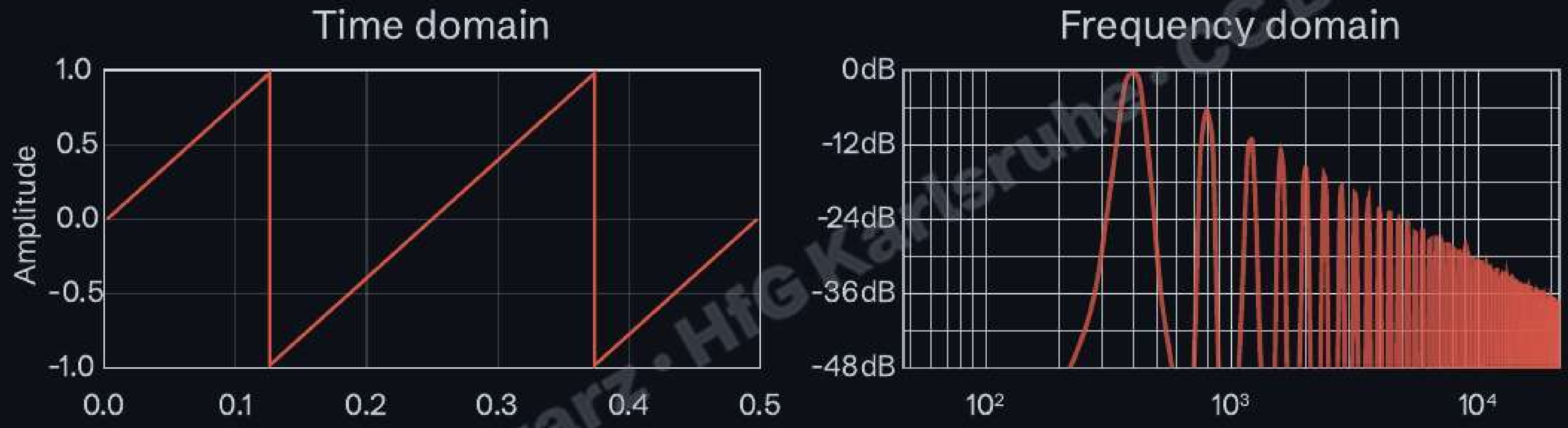
Sawtooth wave



The formula shows the waveform as a sum of sine waves (view on Desmos)

$$x(t) = \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(2\pi n f_0 t)$$

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Triangle wave

Continuous, linear rise and fall between its maximum and minimum values, forming a symmetric triangle:

- closer to a sine wave

▶ Triangle 400 Hz

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Harmonic spectrum of a triangle wave

- contains only odd harmonics (1,3,5,7...)
- relative amplitudes decay as $\frac{1}{(2n-1)^2}$
- every other harmonic is 180 degrees out of phase

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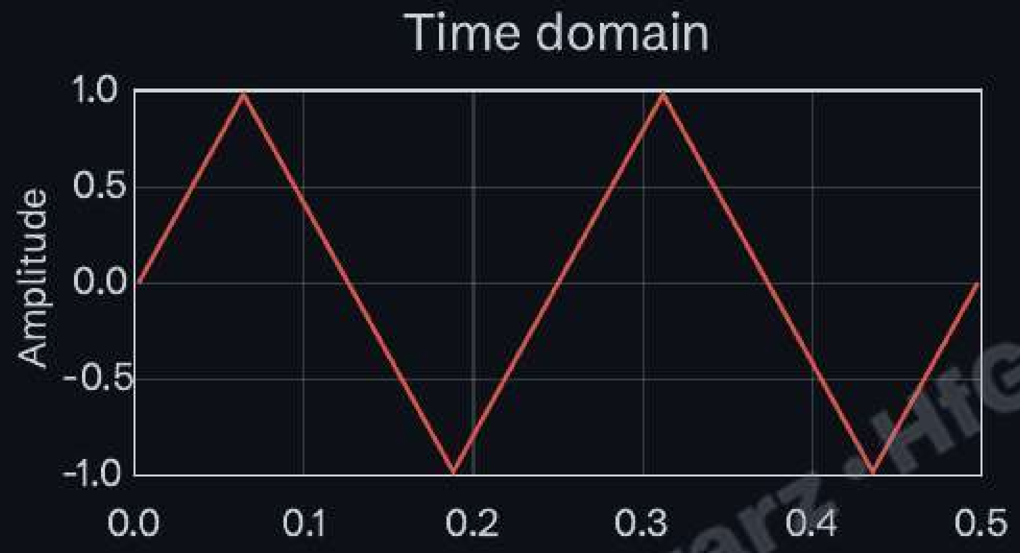
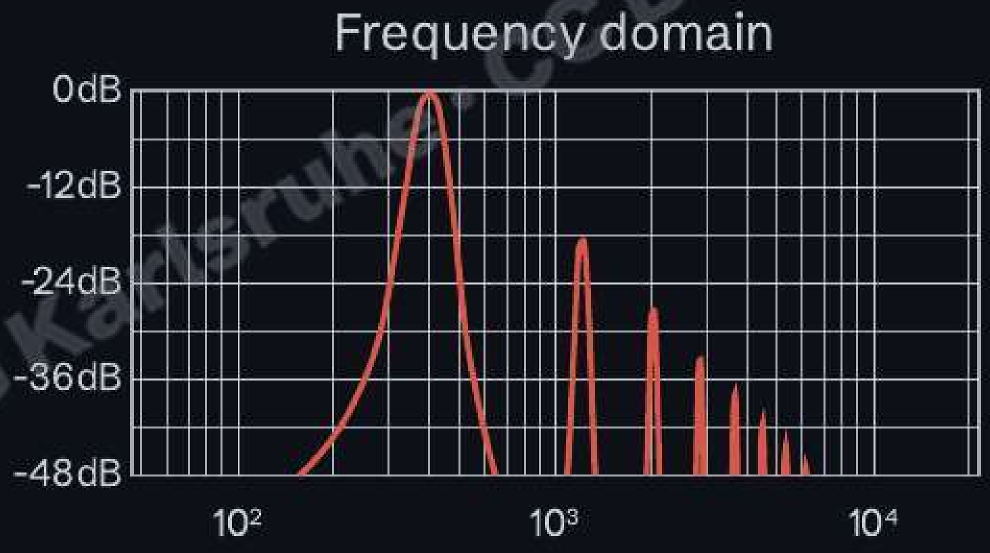
Triangle wave





The formula shows the waveform as a sum of sine waves [(view on Desmos)]

$$x(t) = \frac{8A}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin(2\pi(2n-1)\, f_0\, t)$$

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Square wave

The signal spends equal time at the maximum (high) and minimum (low) levels, making it a symmetrical waveform with a 50% duty cycle (T_{ON} = T_{OFF}):

- Square waves are often described as sounding "hollow" or "nasal". This means that they are good for creating wind instruments, like a clarinet.

▶ Square wave 400 Hz

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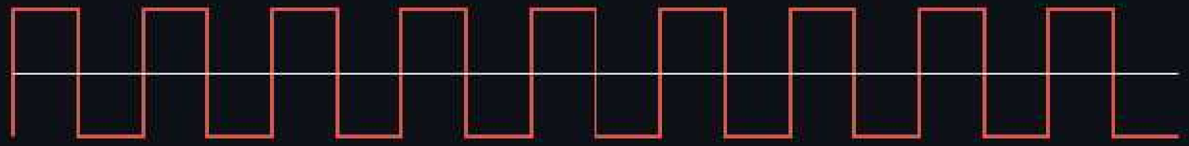
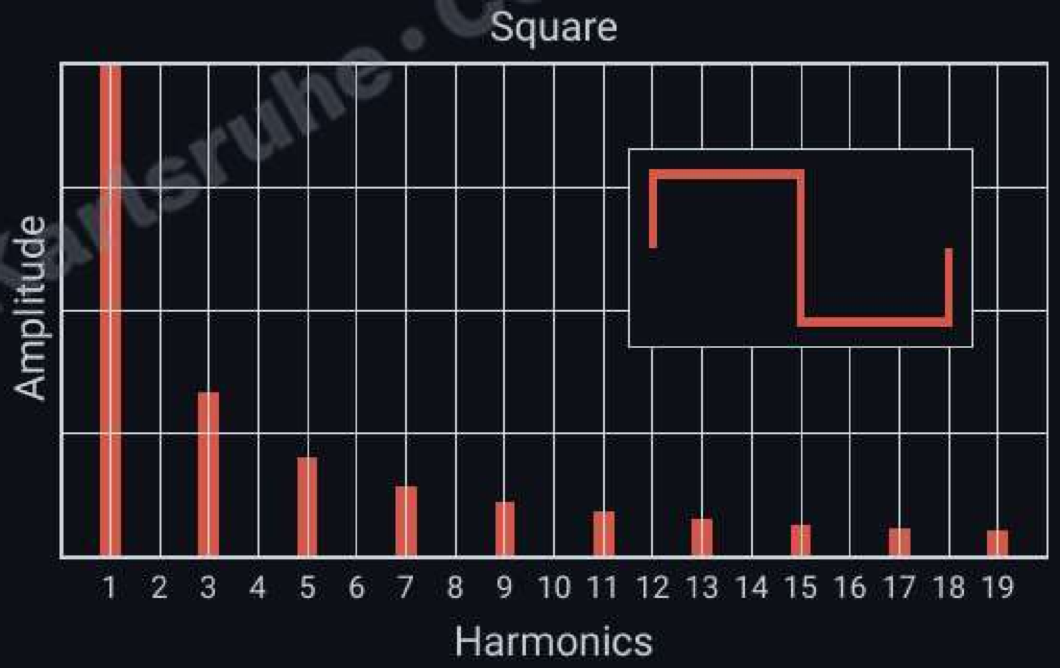


Harmonic spectrum of a square wave

- contains only odd harmonics
- relative amplitudes of harmonics are $\frac{1}{(2n-1)}$
- duty cycle of a square wave is always 50%
- all harmonics in phase



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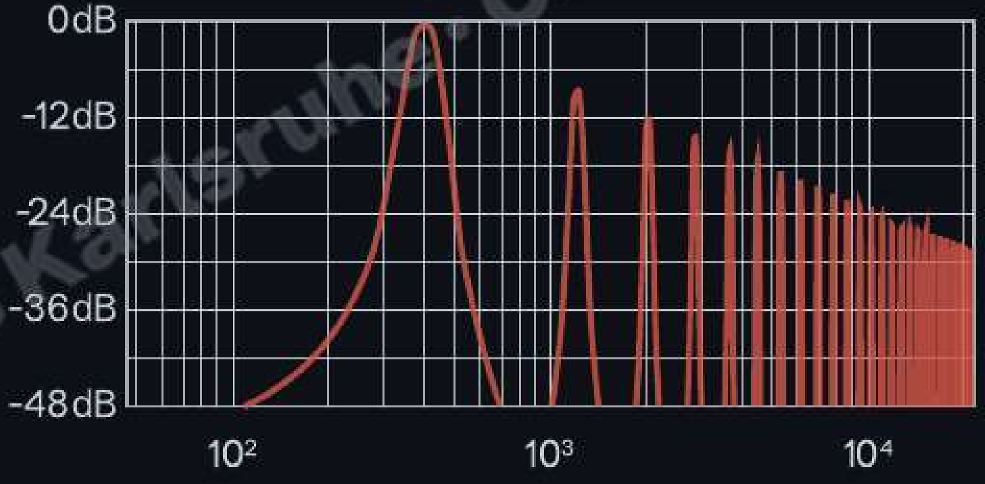
Square wave

Time domain

Frequency domain



The formula shows the waveform as a sum of sine waves [(view on Desmos)]

$$x(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2\pi(2n-1)f_0 t)$$

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Harmonic content of periodic waveforms

Waveform	Harmonics	Amplitude
Sine	Fundamental only	-
Sawtooth	Odd and even	-6 dB/octave ($\propto 1/n$)
Triangle	Odd only	-12 dB/octave ($\propto 1/n^2$)
Square	Odd only	-6 dB/octave ($\propto 1/n$)

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Pulse wave

A pulse wave is a non-sinusoidal periodic signal characterized by abrupt alternation between two amplitude levels: a maximum (T_{ON}) and a minimum (T_{OFF}):

- Durations of the high and low states differ.

→ *Asymmetrical form of a square wave.*

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Pulse width and duty cycle *D*

The duty cycle (D) is the percentage of a waveform's period (T_{ON}) during which the signal is in the "high" or "on" state (value of 1 for a square wave), calculated as the ratio of the on time to the total period (T_{ON} + T_{OFF}).

- Ratio of the pulse width to the total period

$$D = \frac{T_{ON}}{(T_{ON} + T_{OFF})} \times 100\%$$

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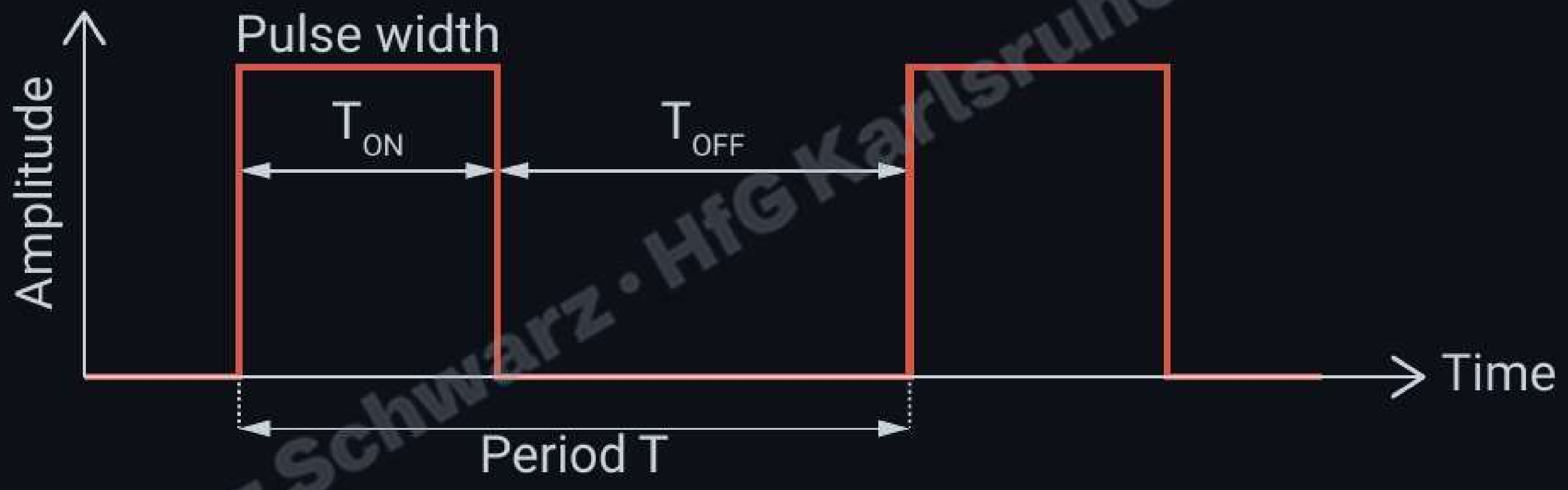
Duty cycle



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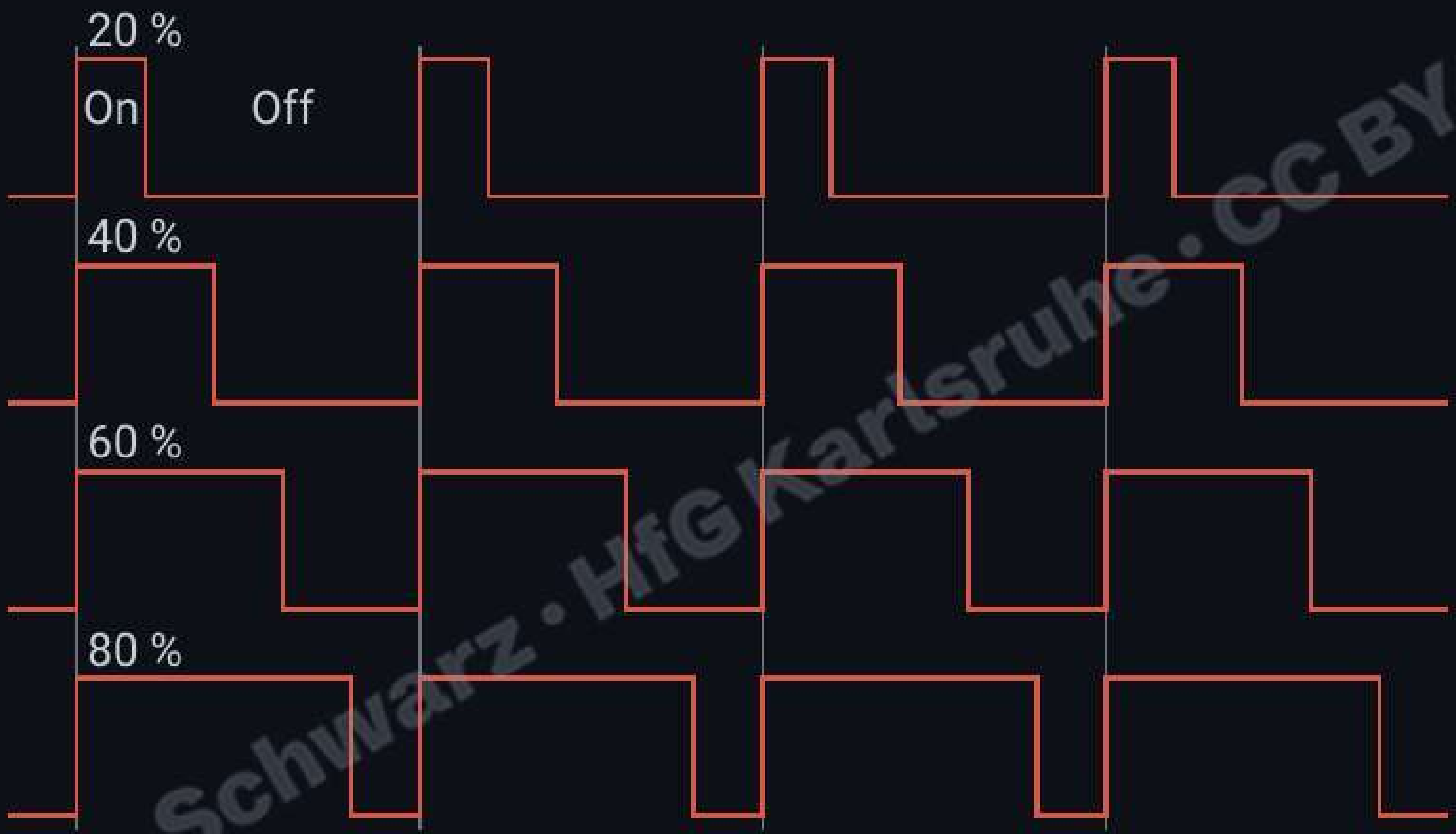

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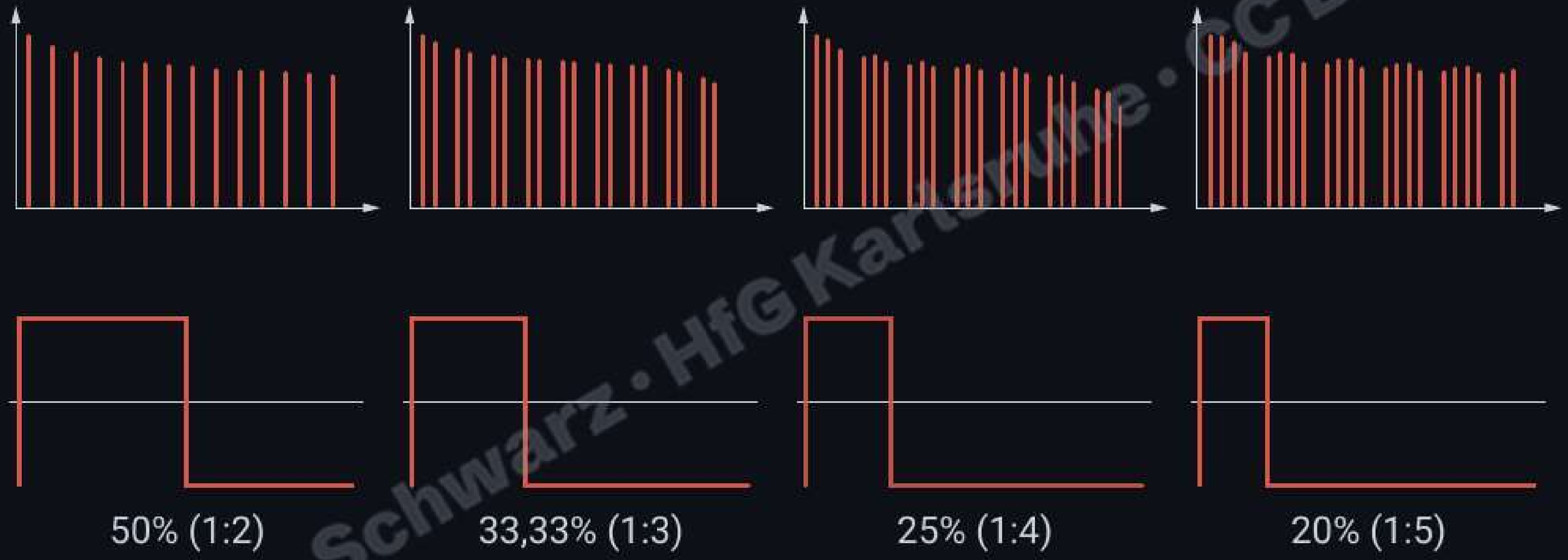
Pulse waves with different duty cycles.

Harmonic spectra of pulse waves with various duty cycles.



→ *The duty cycle determines the harmonic spectrum of the pulse wave.*

▶ Play examples

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Pulse width and timbre

Changing the duty cycle alters the harmonic structure and perceived timbre of a pulse wave.

- Narrowing the duty cycle from 50 % produces a thinner, more nasal sound
- Very narrow pulses create a characteristic reed-like quality
- Commonly used for string- and brass-like synth sounds

→ *Modulating the duty cycle over time (PWM) creates dynamic, evolving timbres*

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Pulse Width Modulation (PWM)

PWM (Pulse Width Modulation) is a type of signal modulation that converts an analog signal into a binary-coded signal by varying the duty cycle of a pulse wave in direct proportion to the amplitude of the analog signal.

Applications of PWM:

- Class-D amplifiers, light dimmers/LED brightness control
- Variable-speed control for computer fans, and servo motor
- Sound synthesis

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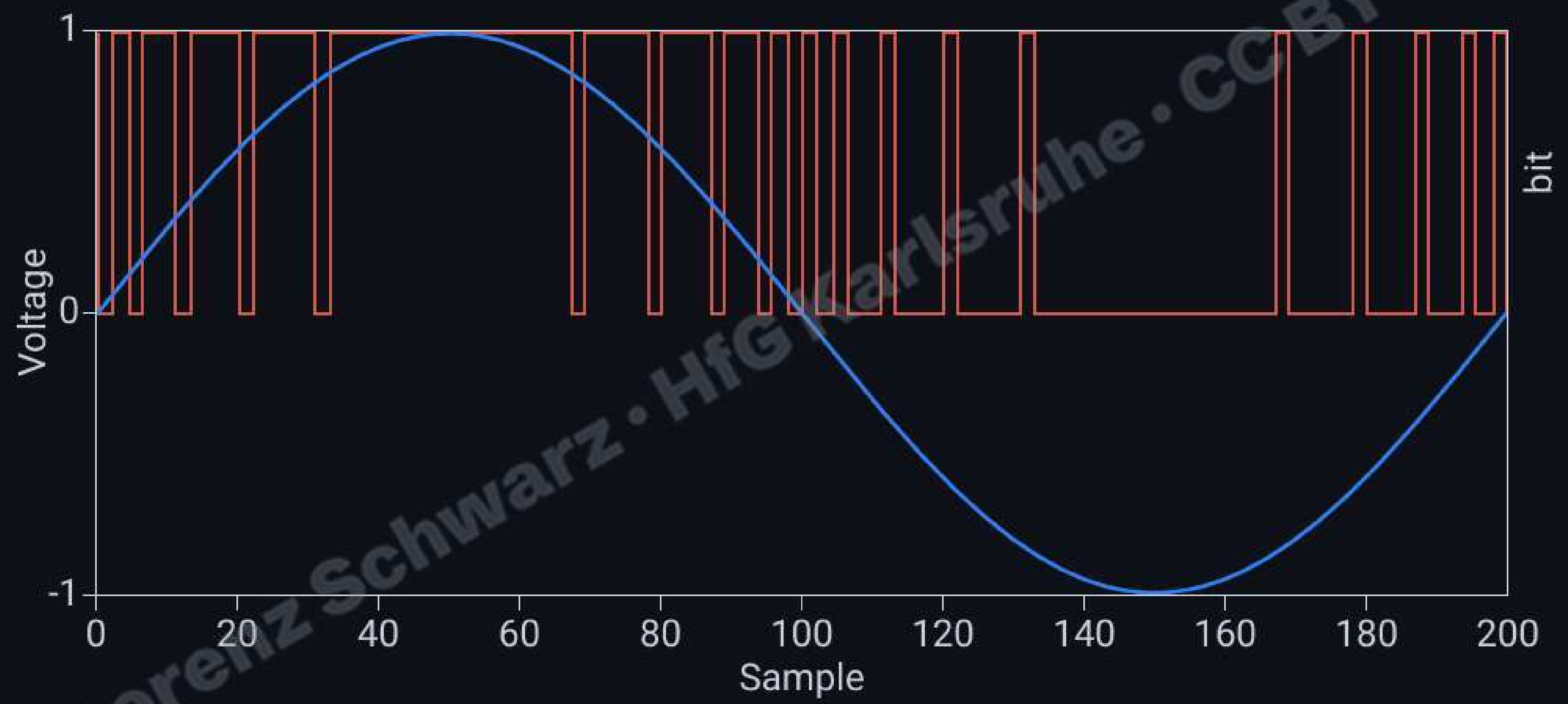
Amplitude values are encoded into pulses.



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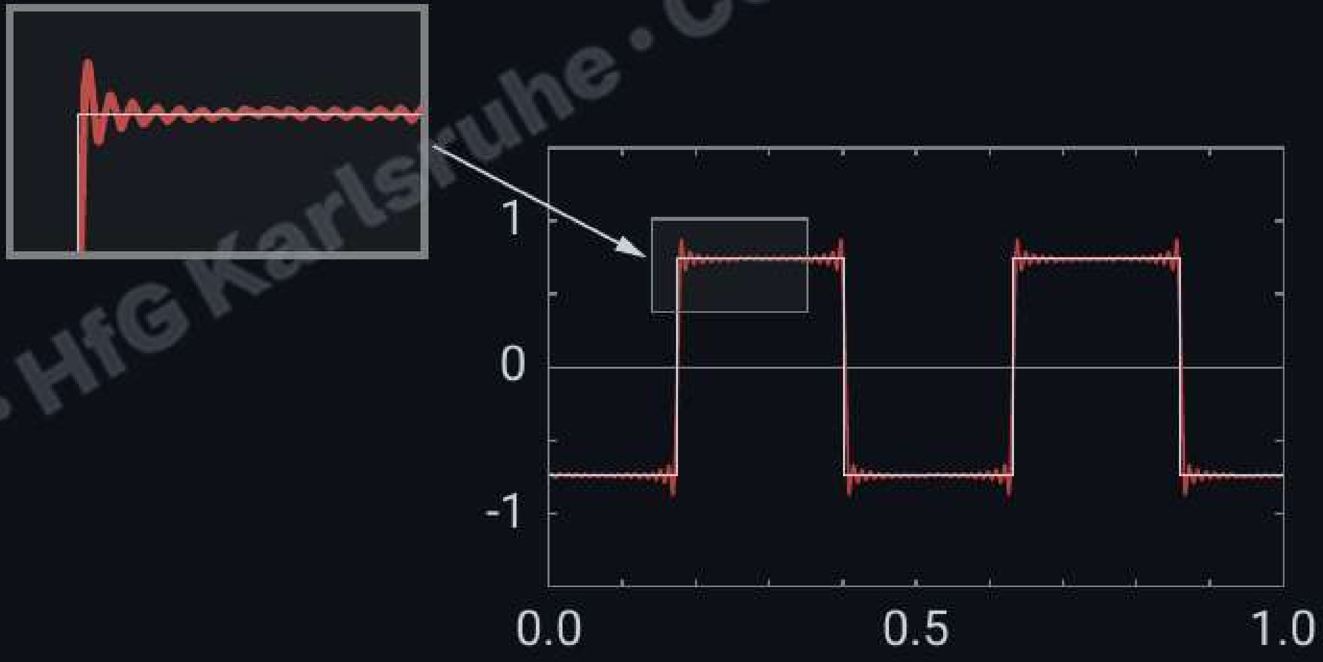
Gibbs phenomenon

Approximating a discontinuous function (such as a square wave or a sawtooth wave) by a finite sum of continuous sine waves causes:



- Oscillations at the jump discontinuities occur.
- The overshoot does not vanish, even as more terms are added.

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Digital waveforms

In digital systems, frequency components above the Nyquist frequency (half the sampling rate) are mirrored back into the audible range, creating new, non-harmonic frequencies.

This effect is called aliasing.

→ *Steep low-pass filtering before sampling minimizes aliasing (band-limited synthesis)*

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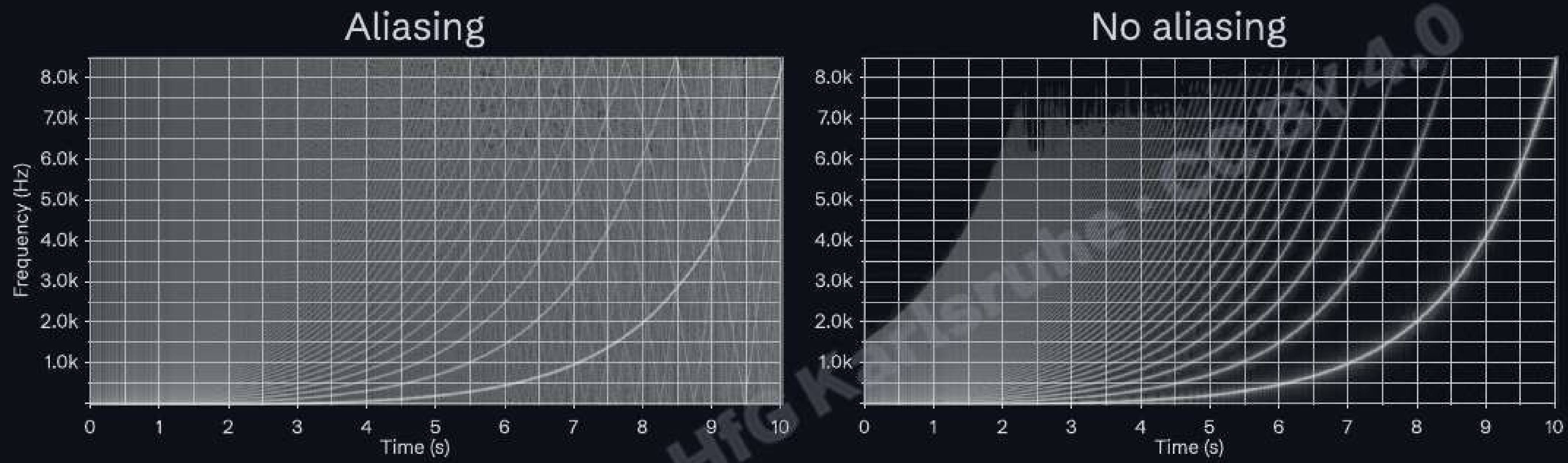

- ▶ Square wave sweep without band-limiting
- ▶ Square wave sweep with band-limiting

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Stochastic signals

In contrast to periodic signals, stochastic signals (noise) are random and non-repeating, and are described primarily by their spectral distribution rather than their waveform shape.

1. White noise
2. Pink noise
3. Brownian/Red noise
4. Blue/Azure noise
5. Violet/Purple noise

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Types of power-law colored noise

The term colored noise refers to signals whose power distribution across frequencies is roughly similar to the corresponding spectra of visible light.

→ *An imprecise analogy inspired by filtering white light.*

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White noise

White noise, analogous to white light which contains all spectral components, has equal energy distributed across all frequencies with a constant power spectral density.

Applications:

- Audio synthesis
- Testing and calibrating audio equipment

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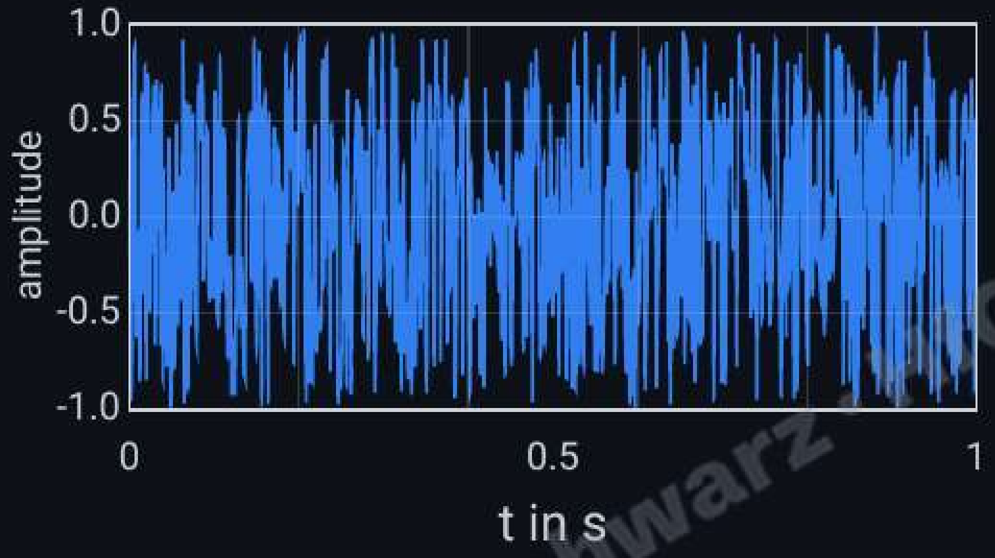
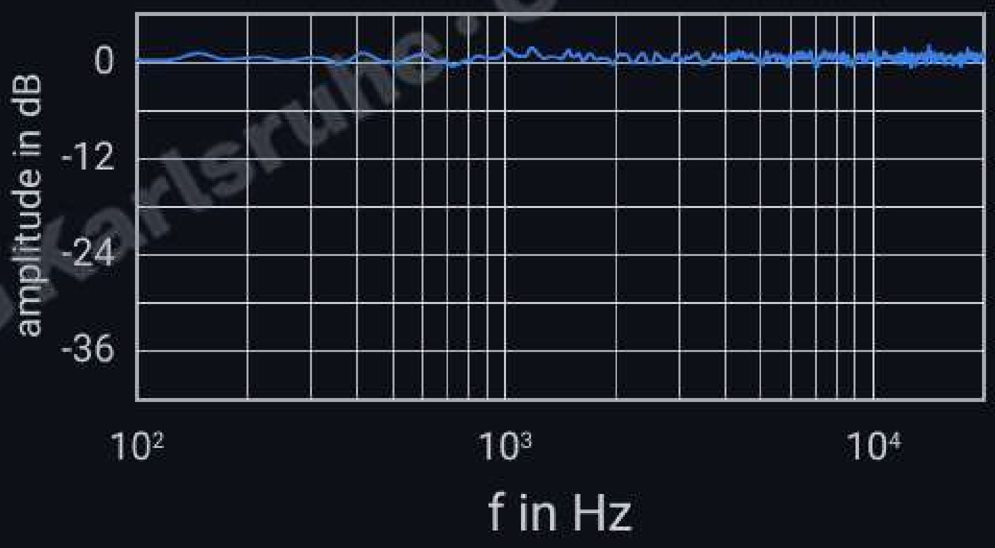
▶ White noise





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Grey noise

While white noise has physically equal energy at all frequencies, grey noise has perceptually equal loudness.

→ *Inverse of the equal-loudness curve (A-weighting) compensates for the human ear's varying sensitivity across the frequency spectrum.*

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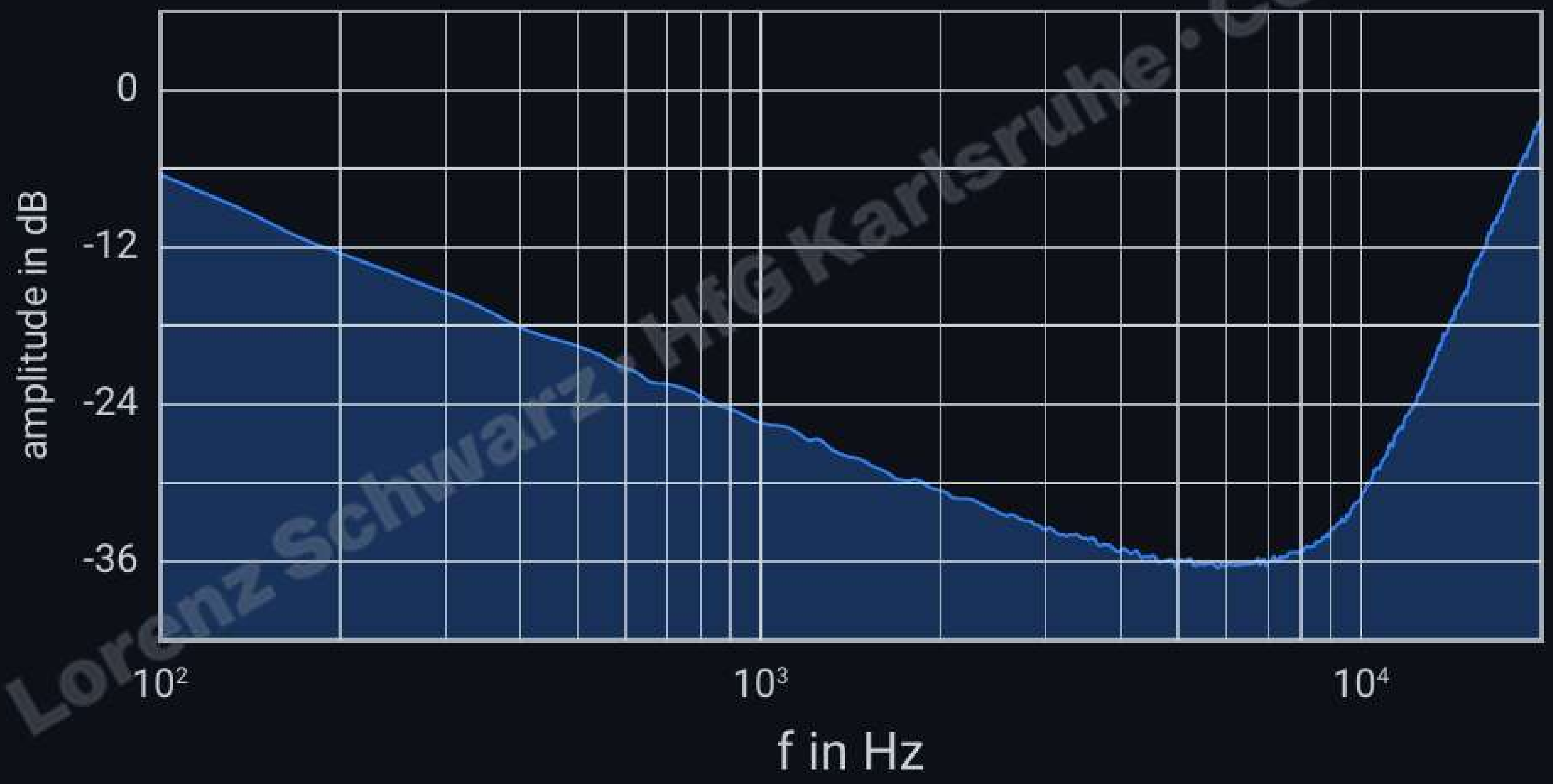
▶ Grey noise



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Pink noise

Pink noise has equal power per octave, meaning its power decreases as frequency increases.

- Power spectrum is inversely proportional to frequency ($\sim 1/f$)
- Found in natural and biological systems
- Sounds like a steady waterfall or rainfall
- Also called flicker noise in electronics

Application:

- Commonly used for acoustic measurements and sound system calibration.

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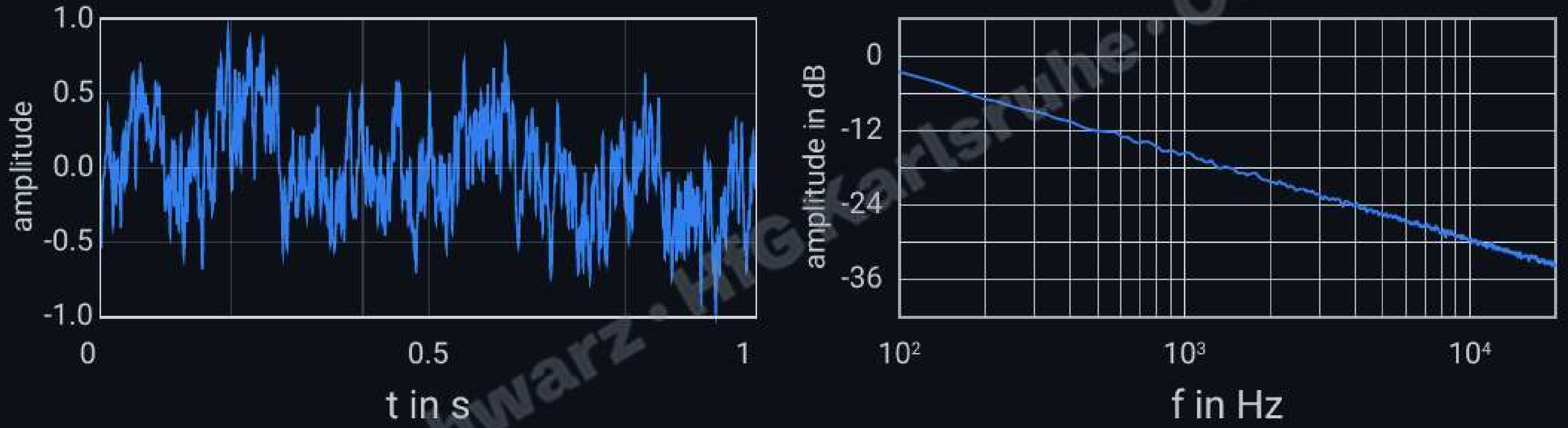
▶ Pink noise



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Brownian noise ($\sim 1/f^2$ noise)

Brownian noise, also known as Brown noise or Red noise, has a power density that decreases by 6 dB per octave (or 20 dB per decade), emphasizing lower frequencies.

- It approximates the random patterns of Brownian motion.
- Named after Robert Brown, who discovered Brownian motion in 1827.

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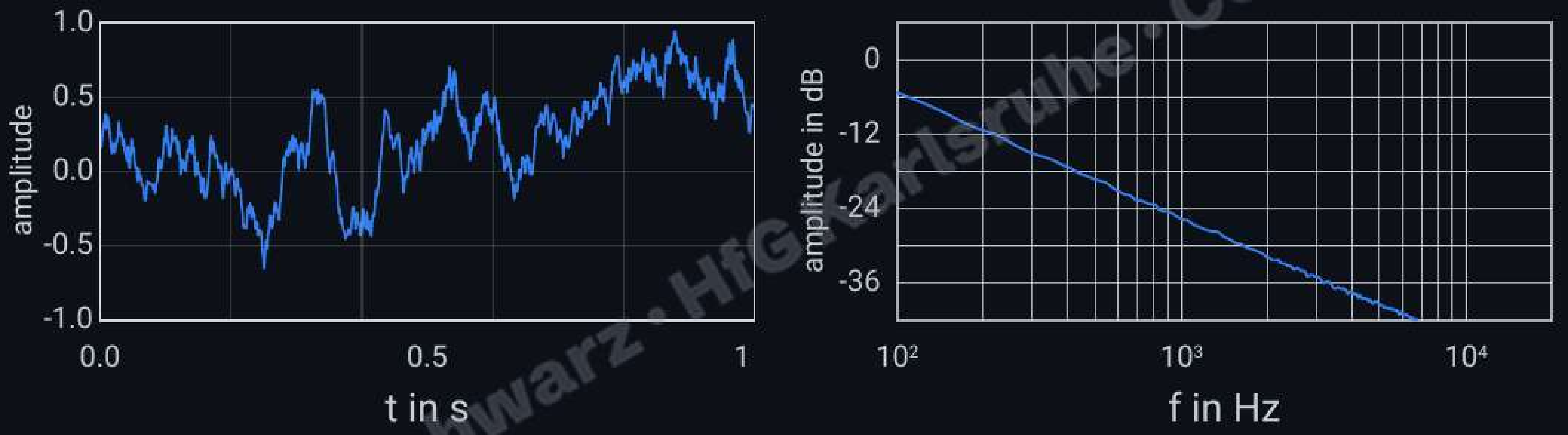
▶ Brownian noise



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Blue/Azure noise

Blue noise's power density increases by 3 dB per octave as the frequency increases.

- Inverse of pink noise.
- Proportional to frequency.

Application:

- Dithering

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▶ Blue noise

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Violet/Purple noise

Power density increases by 6 dB/octave with frequency.

- Differentiated white noise
- Opposite of brown noise

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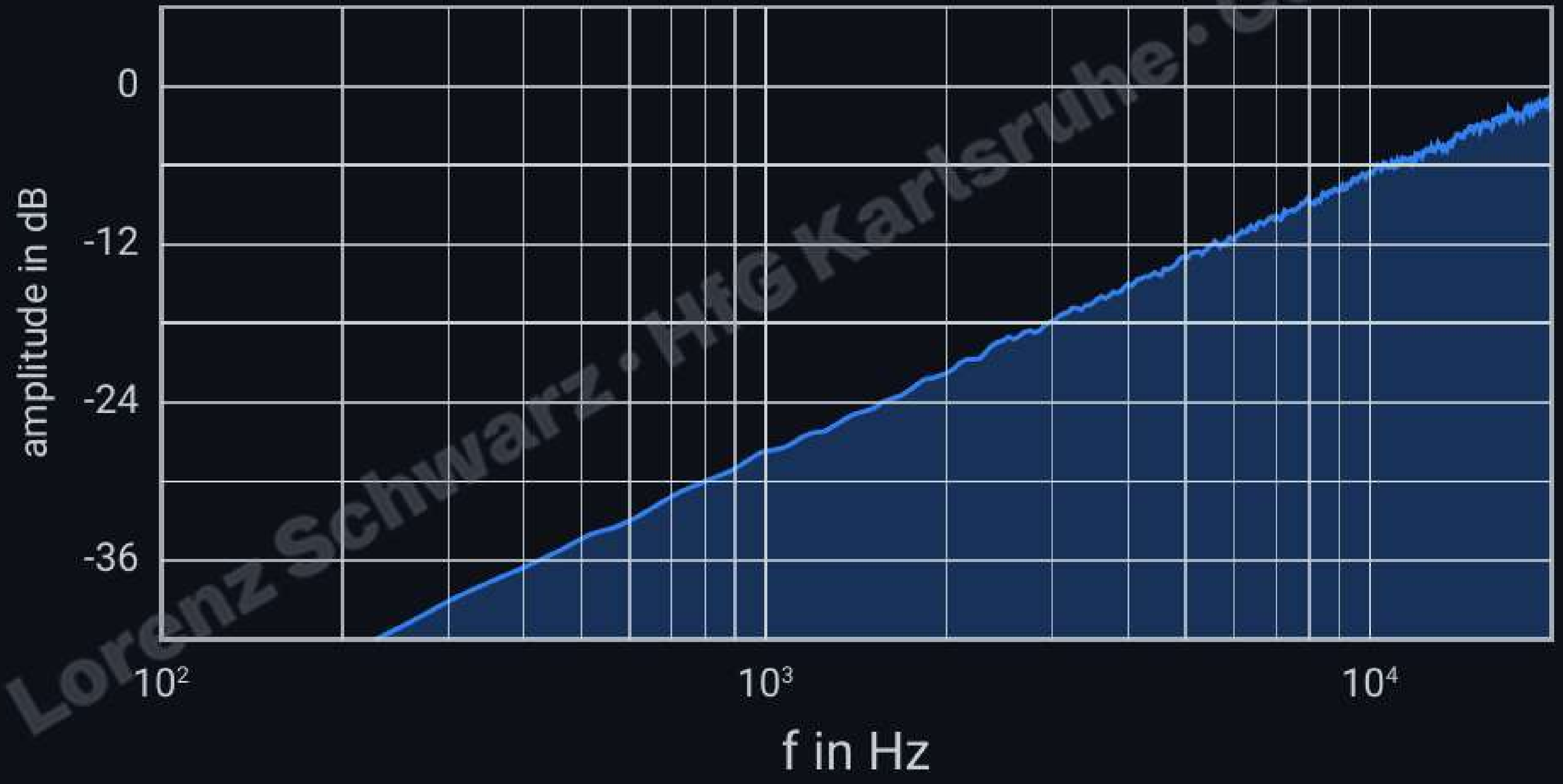
▶ Purple noise



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Spectral properties of noise types

Type of Noise	Spectral Density	Change per Octave (dB)	$1/f^\alpha$
White	$S(f) \propto 1$	0 dB	$\alpha = 0$
Pink	$S(f) \propto 1/f$	-3 dB	$\alpha = 1$
Brownian / Red	$S(f) \propto 1/f^2$	-6 dB	$\alpha = 2$
Blue / Azure	$S(f) \propto f$	+3 dB	$\alpha = -1$
Violet / Purple	$S(f) \propto f^2$	+6 dB	$\alpha = -2$

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Additive synthesis

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Build complex sounds by adding sine waves together

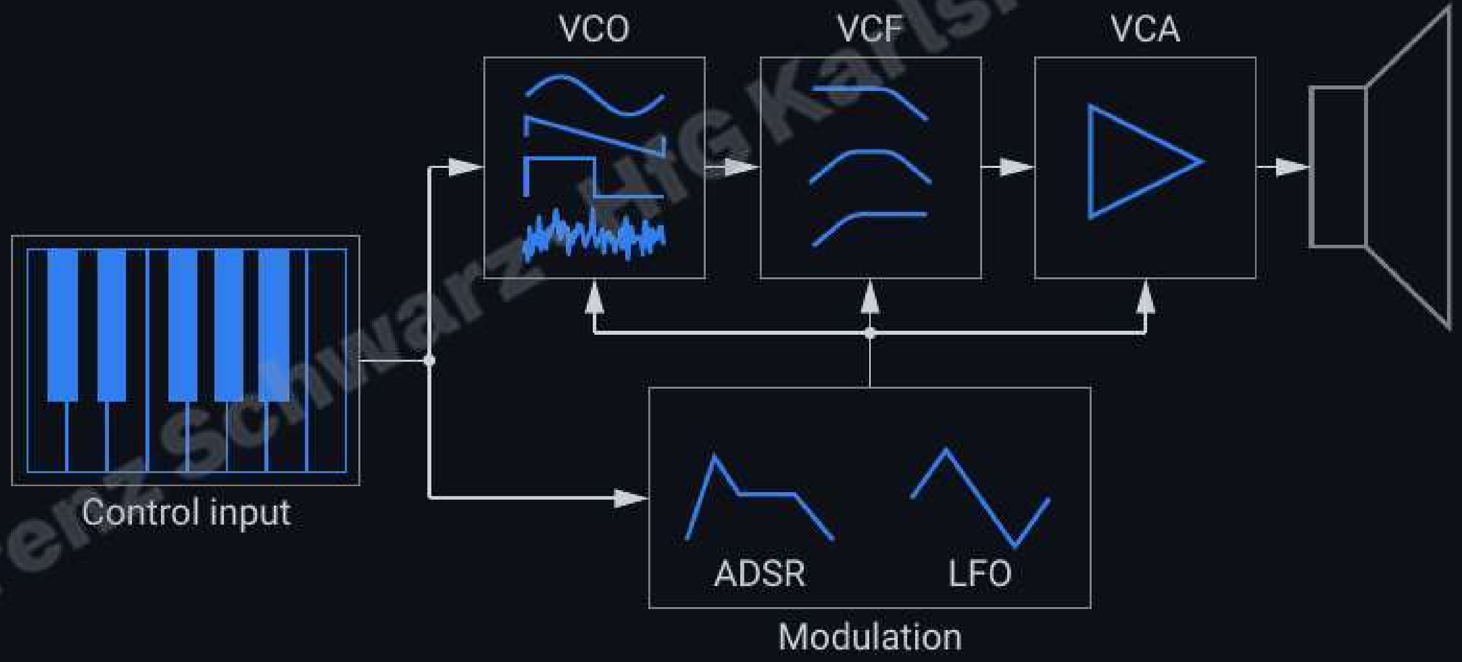
$$x(t) = \sum_{k=1}^{K} A_k(t) \sin(2\pi k f_0 t + \varphi_k)$$

- Requires many oscillators (one per partial)
- Controlling timbre changes over time is complex
- Computationally expensive

→ *Additive synthesis: conceptually simple, practically expensive*

Subtractive sound synthesis

Subtractive synthesis starts with rich, periodic waveforms (like sawtooth or square) and removes frequencies using filters.



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Signals for measurement and analysis

Beyond sound synthesis, certain signals are designed specifically for measuring and analyzing acoustic systems, such as room reverberation and loudspeaker response.

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Dirac delta function (δ distribution)

A mathematical function with infinite amplitude at a single point and infinitely small duration.

Application:

- Impulse response measurement

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Impulse and unit sample

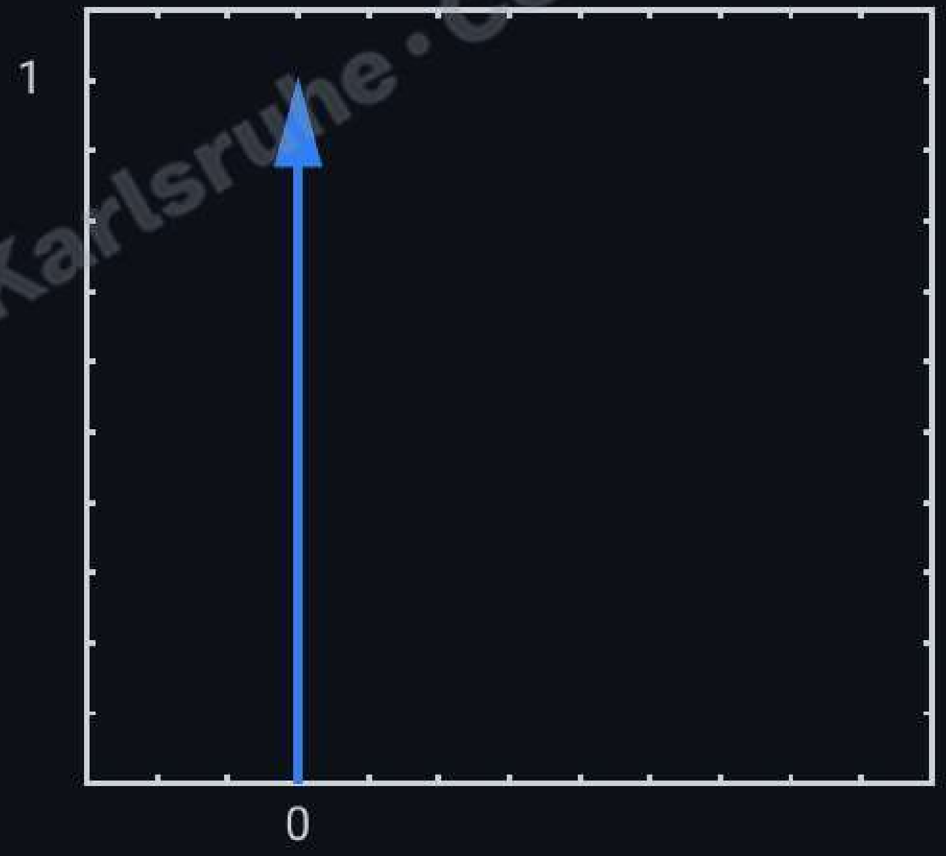
The discrete unit sample is the digital equivalent of the Dirac delta:

$$\delta(n) \mathrel{\widehat{=}} \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Real-world approximations:

- Balloon pop, gunshot
- Electromagnetic interference

▶ Dirac Impulse



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Sine sweep (chirp)

A sine sweep uses a sinusoid with an increasing frequency to excite an acoustic system, enabling the calculation of its impulse response.

Application:

- Impulse response measurement

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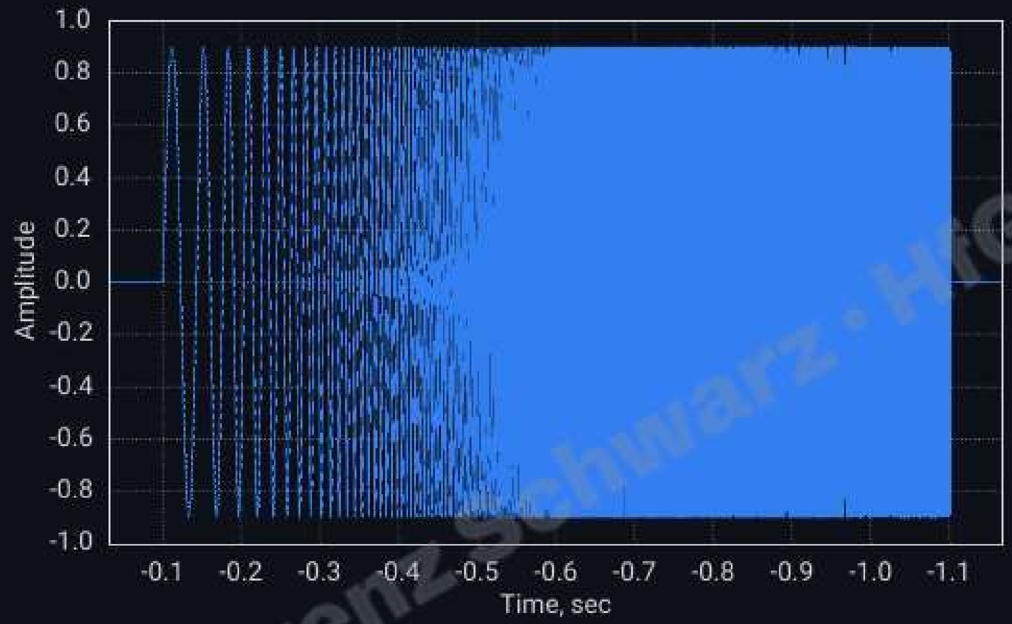
▶ Sine sweep



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