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Random ϵ-cover on compact Riemannian symmetric space

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Abstract. A randomized scheme that succeeds with probability $1 - 2\delta$ (for any $\delta > 0$) has been devised to construct (1) an equidistributed ϵ-cover, and (2) an approximate $(\lambda_r, 2)$-design — in a compact Riemannian symmetric space $\mathbb{M}$ of dimension $d_{\mathbb{M}}$ — using $n(\epsilon, \delta)$-many Haar-random isometries of $\mathbb{M}$, where

$$n(\epsilon, \delta) := \mathcal{O}_{\mathbb{M}}\left(d_{\mathbb{M}}\left(\ln\left(\frac{1}{\epsilon}\right) + \log_2\left(\frac{1}{\delta}\right)\right)\right),$$

and $\lambda_r = \mathcal{O}_{\mathbb{M}}(\epsilon^{-1-\frac{d_{\mathbb{M}}}{2}})$ is the r-th smallest eigenvalue of the Laplace-Beltrami operator on $\mathbb{M}$. The ϵ-cover so-produced can be used to compute the integral of 1-Lipschitz functions within additive $\tilde{\mathcal{O}}_{\mathbb{M}}(\epsilon)$-error, as well as in comparing persistence homology computed from data cloud to that of a hypothetical data cloud sampled from the uniform measure.
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Key Words and Phrases: symmetric space, ϵ-cover, $(\lambda, 2)$-design, equidistributed cover, random isometries, Wasserstein distance, irreducible representations, Casimir operator, Laplace-Beltrami operator, Schrier graph, expander, spectral gap, Markov chain..

1. Introduction

The purpose of this paper is to devise a randomized scheme that produces (with success probability at least $1 - 2\delta$ for any $0 < \delta \leq \frac{1}{2}$) an ϵ-cover in a $d_{\mathbb{M}}$-dimensional compact connected Riemannian symmetric space $(\mathbb{M}, \zeta_{\boldsymbol{o}})$ — where $\mathbb{M} = \mathbb{K}/\mathbb{H}$ with $\mathbb{K}$ a compact connected semisimple Lie group and $\mathbb{H} \subseteq \mathbb{K}$ a closed subgroup, and $\zeta_{\boldsymbol{o}}$ a geodesic reflecting global isometry with respect to the origin $\boldsymbol{o} := \mathbb{H}\boldsymbol{e} \in \mathbb{M}$. The underlying mechanism is to pick an alphabet consisting of $k := O(d_{\mathbb{M}} \ln(1/\epsilon) + \ln(1/\delta))$ independent Haar-random isometries from $\mathbb{K}$, take all possible concatenations of length $\ell := O(d_{\mathbb{M}} \ln(1/\epsilon))$ from this set of isometries, and make them act on a fixed (but arbitrary) point from $\mathbb{M}$. This draws a strong parallelism to the theory of geometric random walk, since the idea of the construction is essentially that of analysing certain lazy random walk on a Schrier graph on the space $\mathbb{M}$ — with edges generated by a nonempty set of isometries, and show that the walk mixes fast because the random Schrier graph essentially has an expander-like property. We show that the set of points so produced is equidistributed at a scale of ϵ. The ϵ-cover so generated, requires much smaller number of random samples than the naive Monte Carlo or other similar methods of taking a large number of i.i.d random samples from $\mathbb{M}$; for example, [26] requires $O(d_{\mathbb{M}}\epsilon^{-d_{\mathbb{M}}}\log\epsilon^{-1})$ many i.i.d random samples from $\mathbb{M}$ to generate an ϵ-cover of a $d_{\mathbb{M}}$-dimensional compact Riemannian manifold $\mathbb{M}$. The log-size of the ϵ-cover so produced differs from the log-size of a hypothetical ϵ-net — whose size is the volumetric lower-bound of ϵ-covering number of $\mathbb{M}$ — by a multiplicative $\log\log$ factor. Quantitatively, our result is most interesting when we let $\epsilon \to 0$ for symmetric spaces $\mathbb{M}$ of fixed dimension and antipodal dimension, injective radius, and sectional curvature. We are motivated in part by the possibility of using equidistributed points for the integration of Lipschitz functions over $\mathbb{M}$. The ϵ-cover we construct is equidistributed (with probability at least $1-\delta$) in the following sense: the normalized counting measure over the net is close to the Haar-induced probability measure on $\mathbb{M}$ in 1-Wasserstein distance. This implies that the integral of every 1-Lipschitz function on the $\mathbb{M}$ with respect to the normalized counting measure is "ϵ-close" to its integral with respect to Haar-induced probability measure. Moreover, such a cover is an explicit approximate $(\lambda, 2)$-design on $\mathbb{M}$. When, we have oracle access to the pairwise Riemannian-distances in $\mathbb{M}$, such an ϵ-cover immediately yields an efficient computation of the singular homologies of $\mathbb{M}$. Furthermore, the bound on the Wasserstein-1 distance ensures small Prokhorov distance between the push-forwards along the persistence homology maps of the canonical Haar-induced probability measure in $\mathbb{M}$ and the empirical measure supported on the ϵ-cover constructed in this paper, showing that the persistence homologies can be computed — to within small bottleneck distance with high probability — using point-cloud produced by such a construction.

We briefly survey earlier work relevant to the theme of this paper. It follows from theory of random walk on a Schrier graph $\chi(\mathbb{M}, \mathcal{S})$ of compact symmetric space $\mathbb{M}$ — with respect to a finite set $\mathcal{S} \subseteq \mathbb{K}$ of its isometries — that, a sufficient condition for rapid mixing of this random walk to the uniform distribution on $\mathbb{M}$ is that the averaging operator $z_\mu : \mathrm{L}^2_0(\mathbb{M}) \to \mathrm{L}^2_0(\mathbb{M})$, given by

$$z_\mu(f)(\boldsymbol{x}) = \int_{\mathbb{K}} f(\boldsymbol{x}\boldsymbol{g})\ d\mu_{\mathbb{K}}(\boldsymbol{g}) \tag{1}$$

posses a spectral gap; here $\mu_{\mathbb{K}}$ is the normalized empirical measure supported on the group of isometries underlying the Schrier graph. The existence of spectral gap for these operators looks far-fetched at this state, and we approach the problem via proving a weaker but sufficient version of the spectral gap phenomenon, as sketched below.

In [2], Alon and Roichman proved that, given any $\delta > 0$, there exists a $c(\delta) > 0$ such that for any finite group G, and a random subset $S \subset G$ of order at least $c(\delta)\log|G|$, the induced Cayley graph $\chi(G, S)$ has small normalized second largest eigenvalue (in absolute value), in that

$$\mathbb{E}\left(|\lambda_2^*(\chi(G,S))|\right) < \delta \tag{2}$$

holds. Considering random walk on expander multigraphs, it follows that every element $g \in G$ is an S-word of length at most $O_\delta(\log|G|)$. Landau and Russell in [24] deviced a short proof (with slightly better constants) of this result while rephrasing the question using representation theory. For an irreducible representation $\rho \in \hat{G}$, let d_ρ be its dimension; let R be the regular representation of G, and $D = \sum_{\rho \in \hat{G}} d_\rho$. Landau and Russell proved that eq. (2) holds for random subsets $S \subset G$ of order at least

$$\left(\tfrac{2\ln 2}{\delta} + o(1)\right)^2 \log|D|$$

This was obtained via an application of tail bounds for operator-valued random variables, as in Ahlswede and Winter [1], building upon the following observation: the normalized adjacency matrix of $\chi(G, S)$ is the operator

$$(2|S|)^{-1} \sum_{s \in S} (R(s) + R(s^{-1})), \tag{3}$$

presented in terms of the standard basis of $\mathbb{C}[G]$. In equivalent terms, the quoted result from [2] implies that random Cayley graph $\chi(G, S)$ is an expander, and also, the operator in eq. (3) has a spectral gap satisfying eq. (2). When G is a compact connected simple Lie group, and μ_G a left-invariantBorel probability measure on G, Benoist and de Saxcé in [3] showed — following earlier works [5] by Bourgain and Gamburd in $G = \mathrm{SU}(d)$ case — that spectral gap of the operator in eq. (1) is equivalent to μ being almost diophantine, a property known to be true when the support of μ is sufficiently (inexplicit) large, and consists of algebraic elements. However, note that these qualitative results are mostly not applicable in the randomized computational setting, since 1. the set of elements with algebraic entries is a measure zero subset of the Lie group, 2. the statement on the required support size is qualitative, and 3. checking almost diophantine property is not computationally feasible at this stage. In [25], quantitative version of the spectral gap question for compact Lie group G was considered. It was shown that — with high probability — the Hausdorff distance between G and the subset of fixed length words on a random finite alphabet $S \subset G$ decays exponentially fast with respect to the length of the words.

In this work, we work in the setting analogous to that of expander Cayley graphs on finite groups, as in [24]. In summary, our analysis underlying the construction of the ϵ-cover evolves around orthogonally projecting the heat kernel of $\mathbb{M}$ — induced by the Casimir operator of $\mathbb{K}$ — onto an appropriate finite-dimensional subspace of $\mathrm{L}^2(\mathbb{M})$. In [7], this idea was used to generate equidistributed random ϵ-cover on unit sphere $\mathbb{S}^d$ — a rank one compact symmetric space, and the present work contains an extension of the results in [7] to compact symmetric space of arbitrary rank, and some additional results on designs and persistent homology of compact symmetric space.

In the special case of the unitary group U_d, a similar result with a quadratic dependence on dimension for the length of the words was previously obtained by Hastings and Harrow, in [18, theorem 5]; however, in their result, the number of generators is specified in a indirect manner, whose dependence on the parameters of U_d is not explicit. This present work yields the corresponding result for $\mathbb{M} = U_d$ (and the analogous results for any compact Lie groups) as a special case, and the bounds obtained here are shown to be linear in d, both for the number of generators and the length of the words. In fact, for these parameters, the value of $(2k)^\ell$ is close to the volumetric lower bound of $(1/\epsilon)^{\Omega(d)}$ on the size of an ϵ-net of $\mathbb{M}$. Besides, the random ϵ-net — that comes out of our procedure — has a "small" description, from a computer science perspective: *viz*, it consists of the description of the random subset $\mathcal{S} \subseteq \mathbb{K}$, and of the maximum word length ℓ. And, the number of random isometry required for the construction depends only on the parameters of the symmetric space itself, and not the group of isometries $\mathbb{K}$. We show that such an ϵ-cover computes integral of 1-Lipschitz functions on $\mathbb{M}$, and the singular homology groups of $\mathbb{M}$ when given oracle access to the pairwise Riemannian-distances. Moreover, for any $n > 0$, persistent homology of an random n-samples from the ϵ-cover $\mathcal{S}$ is $\sqrt{\epsilon}$-close to the persistence homology computed from $\sigma^{\mathbb{M}}$-uniform random n-samples (corollary 4.1), showing that the data cloud so constructed can be used — in persistent homology landscape — as a proxy for hypothetical data cloud sampled uniformly from $\mathbb{M}$.

1.1. Main results

We now state our main results, as theorem 1.1, theorem 1.2 and theorem 1.3.

Throughout this subsection, $\mathbb{M} = \mathbb{K}/\mathbb{H}$ is a compact connected Riemannian symmetric space of dimension $d_{\mathbb{M}}$, with $\sigma^{\mathbb{M}}$ the probability measure on $\mathbb{M}$ corresponding to Haar probability measure on the compact connected semisimple Lie group $\mathbb{K}$.

Theorem 1.1. There is a constant $C_{\mathbb{M}} > 0$ — that depends only on (the dimension and injectivity radius of) $\mathbb{M}$ — such that for any $\delta \in (0, \frac{1}{2})$, the following statement holds with probability at least $1 - 2\delta$.

Let $\epsilon \in (0, 2^{-e})$, and assume that

$$r_{\epsilon,\mathbb{M}} = 2\epsilon\sqrt{\ln \frac{3C_{\mathbb{M}}}{\epsilon^{2d_{\mathbb{M}}}}}$$

is small enough. Let $\mathcal{S} \subset \mathbb{K}$ be a random multiset consisting of iid random points, drawn from the Haar measure on $\mathbb{K}$, such that

$$|\mathcal{S}| \geq 16\ln 2\left(\ln C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4}\log_2\left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \frac{d_{\mathbb{M}}}{2}\log_2\log_2\left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \ln\frac{1}{e\delta}\right).$$

Let $\boldsymbol{p} \in \mathbb{M}$ and define $\hat{\mathcal{S}} := \mathcal{S} \cup \mathcal{S}^{-1}$. Suppose that

$$\ell \geq d_{\mathbb{M}}\log_2\frac{1}{\epsilon} + \log_2\left(\tfrac{6C_{\mathbb{M}}}{\mathfrak{v}_{\mathbb{M}}}\right) + \frac{d_{\mathbb{M}}}{4}\log_2\left(\tfrac{1}{\pi r_{\epsilon,\mathbb{M}}^2}\right) + \frac{1}{2}\log_2\Gamma\left(\tfrac{d_{\mathbb{M}}}{2}+1\right).$$

Then

1. *(ϵ-cover on $\mathbb{M}$):* $\hat{\mathcal{S}}^\ell\boldsymbol{p} \subseteq \mathbb{M}$ is an $r_{\epsilon,\mathbb{M}}$-cover of $\mathbb{M}$ for any $\boldsymbol{p} \in \mathbb{M}$;
2. *(equidistribution of the ϵ-cover):* the empirical measure ν on $\hat{\mathcal{S}}^\ell\boldsymbol{p}$ satisfies

$$\mathrm{W}_1(\sigma^{\mathbb{M}}, \nu) \leq 2\sqrt{d_{\mathbb{M}}}\epsilon\,;$$

These are further discussed in details in sections 4.1 and 4.3 below (see theorem 4.1 and theorem 4.4).

Theorem 1.2. Let $C_{\mathbb{M}}$ be as in theorem 1.1. Let $\mathscr{S} := \hat{\mathcal{S}}^\ell \boldsymbol{o}$ — where $\mathcal{S} \subseteq \mathbb{M}$ is a random multisubset of isometries selected independently from the Haar measure on $\mathbb{K}$. Let $\delta \in (0, \frac{1}{2})$. For any $\upsilon \in (0,1)$, and any integer $r > 0$, if

$$|\mathcal{S}| = 16 \ln 2 \ln \left(\frac{C_{\mathbb{M}} \lambda_r^{\frac{d_{\mathbb{M}}}{2}}}{\delta} \right),$$

and

$$\ell \geq \log_2 \frac{1}{\upsilon} + \log_2 C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4} \log_2 \lambda_r + d_{\mathbb{M}} \log_2 \frac{1}{\epsilon},$$

where $\epsilon = \lambda_r^{-\frac{d_{\mathbb{M}}+2}{2}} C_{\mathbb{M}}^{-\frac{1}{4}} \upsilon^{\frac{1}{2}}$, then $\mathscr{S} \subseteq \mathbb{M}$ is an υ-approximate $(\lambda_r, 2)$-design with probability at least $1 - 2\delta$.

This is further discussed in details in section 4.2 below (see theorem 4.2).

Theorem 1.3. Suppose that the random subsets $\hat{\mathcal{S}} \subseteq \mathbb{K}$ as well as the integer ℓ are as ststed in theorem 1.1; then, for any $\boldsymbol{p}_0 \in \mathbb{M}$, letting $\mathscr{S}_\ell = \hat{\mathcal{S}}^\ell \boldsymbol{p}_0$, the following inequality holds for all integers $q \geq 0$, with probability at least $1 - \delta$:

$$\frac{1}{n} \cdot d_{\mathrm{Pr}} \left(\Phi^{q,n}_{\mathbb{M}, \partial_{\mathbb{M}}, \sigma^{\mathbb{M}}}, \Phi^{q,n}_{\mathscr{S}_\ell, \partial_{\mathbb{M}}, \sigma^{\mathbb{M}}_\ell} \right) \leq \sqrt[4]{2 d_{\mathbb{M}} \epsilon} \tag{4}$$

This is further discussed in details in section 4.4 below (see theorem 4.5 and corollary 4.1).

To elaborate a bit more on these results, suppose that $\mathbb{K}$ is one of the compact connected matrix groups such as $\mathrm{SU}(d)$ — the group of orthogonal matrices having determinant one. We assume given a real number model of computation, in which only standard algebraic operations are allowed on random vectors, but bits are not manipulated. Now, for $\mathcal{S}$ and ℓ as described in theorem 1.1, we consider all compositions of length ℓ of the elements in $\mathcal{S}$ and their inverses. Apply the resulting matrices to a vector $\boldsymbol{o} = \mathbb{H}\boldsymbol{e}$. Then these points form an equidistributed cover, that can be used for integrating any 1-Lipschitz function on $\mathbb{M}$ to within an additive error of ϵ. The persistence homology computed from such a point cloud well-approximates the one computed from point cloud formed out of random samples from $\mathbb{M}$. In the special case of an d-dimensional Euclidean sphere, if we assume an oracle that outputs independent d-dimensional random matrices from $\mathbb{K}$ when queried, then the whole process requires only k queries to this oracle. Note that the obvious procedure of producing an equidistributed cover would require $\epsilon^{-\Omega(d)}$ calls to the oracle. The latter method uses exponentially more randomness than our procedure using random isometries.

2. Preliminaries

Let $\mathbb{M}$ be a connected compact (real) Riemannian symmetric space of dimension $d_{\mathbb{M}}$. If $\mathbb{K}$ is the identity component of the group of isometries of $\mathbb{M}$, then $\mathbb{K}$ is a compact Lie group that acts transitively on $\mathbb{M}$, leading to the identification $\mathbb{M} = \mathbb{K}/\mathbb{H}$ for a closed subgroup $\mathbb{H} \subseteq \mathbb{K}$ — the isotopy group of a point in $\mathbb{M}$. Let θ be the Cartan involution associated to the symmetric structure of $\mathbb{M}$. Let $d_{\mathbb{K}}$ be the dimension of $\mathbb{K}$, so that $d_{\mathbb{H}} = d_{\mathbb{K}} - d_{\mathbb{M}}$. Let $\sigma^{\mathbb{M}}$ be the canonical probability measure on $\mathbb{M}$, induced by the Haar probability measure $\sigma^{\mathbb{K}}$ on $\mathbb{K}$, and let $\mu_{\mathbb{M}}$ be the canonical $\mathbb{K}$-invariant Riemannian top form on $\mathbb{M}$, induced by the Killing metric on the isometry group $\mathbb{K}$; there is a constant $\mathfrak{v}_{\mathbb{M}} > 0$ such that $\sigma^{\mathbb{M}} = \mathfrak{v}_{\mathbb{M}} \mu_{\mathbb{M}}$. We will refer to $\mathfrak{v}_{\mathbb{M}}^{-1}$ as the Riemannian volume of $\mathbb{M}$. Recall that, for any measurable $\mathcal{S} \subseteq \mathbb{K}$, one has

$$\sigma^{\mathbb{M}}(\mathcal{S}) = \int_{\mathbb{M}} 1_{\mathcal{S}}(\boldsymbol{x})\, d\sigma^{\mathbb{M}}(\boldsymbol{x}) = \int_{\mathbb{K}} 1_{\mathbb{H}\mathcal{S}}(\boldsymbol{x})\, d\sigma^{\mathbb{K}}(\boldsymbol{x})$$

The L^2-norm of $\mathbb{C}$-valued square-integrable functions on $\mathbb{M}$ will always be with respect to the probability measure $\sigma^{\mathbb{M}}$; we use the shorthand $\mathrm{L}^2(\mathbb{M})$ for $\mathrm{L}^2(\mathbb{M}, \sigma^{\mathbb{M}})$.

Example 2.1. We briefly summarise the above framework in case of some explicit examples.

- Suppose that $\mathbb{K} = \mathrm{SO}(2d)$ and $\mathbb{H} = \mathrm{SO}(2d-1)$ for sme integer $d \geq 1$. Then $\mathbb{K}$ is compact connected semisimple Lie group of dimension $d_{\mathbb{K}} = d(2d-1)$, with $\mathbb{H} \subseteq \mathbb{K}$ a closed subgroup of dimension $d_{\mathbb{H}} = (2d-1)(d-1)$, and $\mathbb{M} = \mathbb{S}^{2d-1}$ is a compact symmetric space with $d_{\mathbb{M}} = 2d-1$. Here, $\mu^{\mathbb{M}}$ is the standard Lebesgue measure on $\mathbb{M}$, and $\sigma^{\mathbb{M}}$ is the normalized Lebesgue measure, so that $\mathfrak{v}_{\mathbb{M}} = \pi^{-d+\frac{1}{2}}\Gamma(d+\frac{1}{2})$. This has been treated in details in [7], and the corresponding generalisations derived here specialise to the main results of [7].
- Let $\mathbb{K} = \mathrm{SU}(d)$ and $\mathbb{H} = \mathbb{T}^d \cap \mathrm{SU}(d)$ the maximal torus in it; here $\mathbb{T}^d$ is the standard d-dimensional torus (of d-dimensional invertible diagonal matrices) in $\mathrm{GL}(d)$. The complete flag manifold $\mathbb{M} = \mathbb{K}/\mathbb{H}$ is compact connected (simply connected) Riemannian symmetric space with (real) dimension $d_{\mathbb{M}} = d(d-1)$.

2.1. Some algebraic prelimiaries From here on, we assume — for the rest of this paper — that $\mathbb{K}$ is a semisimple compact connected separable Lie group, and $\mathbb{H}$ a closed subgroup. Note that, $\mathbb{K}$ has finite center by semisimple criterion. Consider the Cartan decomposition $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$ associated to the symmetric space $\mathbb{M}$; here, $\mathfrak{k}$ and $\mathfrak{h}$ are the Lie algebras of $\mathbb{K}$ and $\mathbb{H}$, respectively, and $\mathfrak{m}$ the tangent space to $\mathbb{M}$ at the origin $\boldsymbol{o} := \mathbb{H}\boldsymbol{e}$. Realising the Lie algebra $\mathfrak{k}$ alternatively as the algebra of Killing vector-fields on $\mathbb{M}$, one has

$$\mathfrak{h} := \{\mathfrak{p} \in \mathfrak{g} : \mathfrak{p}_{\boldsymbol{o}} = 0\}, \quad \mathfrak{m} := \{\mathfrak{p} \in \mathfrak{g} : (\nabla\mathfrak{p})_{\boldsymbol{o}} = 0\}$$

Let $\mathrm{B}_{\mathbb{K}}$ denote the Killing form on $\mathfrak{k}$; thus,

$$\mathrm{B}_{\mathbb{K}}(\mathfrak{p}_1, \mathfrak{p}_2) := \mathrm{tr}(\mathrm{ad}_{\mathfrak{p}_1} \circ \mathrm{ad}_{\mathfrak{p}_1}) \tag{5}$$

We denote the norm on $\mathfrak{k}$ induced by $\mathrm{B}_{\mathbb{K}}$ simply by $\|\cdot\|$. Note that, since $\mathrm{ad} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{k})$ is a Lie-algebra morphism, the following holds for any $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p} \in \mathfrak{k}$:

$$\mathrm{B}_{\mathbb{K}}(\mathrm{ad}_{\mathfrak{p}}(\mathfrak{p}_1), \mathfrak{p}_2) = \mathrm{B}_{\mathbb{K}}(\mathfrak{p}_1, \mathrm{ad}_{\mathfrak{p}}(\mathfrak{p}_2)).$$

By *Cartan's criterion* for semisimplicity ([22]), $\mathbb{K}$ being semi-simple implies $\mathrm{B}_{\mathbb{K}}$ is nondegenerate. The metric on $\mathbb{M}$ is induced by the Killing form $\mathrm{B}_{\mathbb{K}}$ on the Lie algebra $\mathfrak{k}$ of $\mathbb{K}$, via the decomposition $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$. Let $\nabla_{\mathbb{M}}$ denote the canonical Levi-Civita connection on $\mathbb{M}$, with respect to this metric. A curve γ in $\mathbb{M}$ is a geodesic if and only if $\nabla_{\mathbb{M}}(\dot{\gamma}, \dot{\gamma}) = 0$ holds. Recall ([29]) that the geodesics through the origin in $\mathbb{M}$ are precisely the images under the canonical projection $\pi : \mathbb{K} \to \mathbb{M}$ of the $\mathbb{H}$-transversal geodesics in $\mathbb{K}$; since the geodesics in $\mathbb{K}$ are the 1-parameter subgroups in $\mathbb{K}$, the same holds for geodesic through origin of $\mathbb{M}$.

Since the compact Lie group $\mathbb{K}$ acts on the separable ([20]) Hilbert-space $\mathrm{L}^2(\mathbb{M})$ — via the unitary 'regular' representation $(\boldsymbol{s} \cdot \phi)(\boldsymbol{x}) := \phi(\boldsymbol{x}\boldsymbol{s})$, the machinery of Peter-Weyl theorem ([6]) implies that there is Hilbert-space decomposition of $\mathrm{L}^2(\mathbb{M})$ as orthogonal direct sum of unitary irreducible $\mathbb{K}$-representations; that is,

$$\mathrm{L}^2(\mathbb{M}) \cong \bigoplus_{n=1}^{\infty} V_{\pi_n} , \tag{6}$$

where the direct sum is over a multiset whose underlying set is a subset of equivalence classes $[\pi_n]$ of irreducible $\mathbb{K}$-representations (V_{π_n}, π_n). Note that, by compactness of the Lie group $\mathbb{K}$, the separable Hilbert space $\mathrm{L}^2(\mathbb{K})$ has an orthonormal basis of matrix coefficients of all the unitary irreducible representations, which are all finite-dimensional (Peter-Weyl theorem); in particular, there are only countably many inequivalent irreducible representations of $\mathbb{K}$. Let $\mathcal{D}(\mathbb{M})$ denote the algebra of smooth functions on $\mathbb{M}$. This is a sub-representation of the unitary regular representation of $\mathbb{K}$ on $\mathrm{L}^2(\mathbb{M})$.

Proposition 2.1 (Helgason, [19])**.** Each V_{π_n} is a joint eigenspace, in $\mathcal{D}(\mathbb{M})$, of the algebra of $\mathbb{K}$-invariant differential operators on $\mathbb{M}$. Moreover, each irreducible $\mathbb{K}$-representation contained in $\mathcal{D}(\mathbb{M}) \subseteq \mathrm{L}^2(\mathbb{M})$ arises as a direct summand V_{π_n} with multiplicity one.

For the purpose of this paper, we will need to realise a somewhat more concrete version of the orthonormal decomposition of $\mathrm{L}^2(\mathbb{M})$ into finite dimensional $\mathbb{K}$-invariant subspaces. To this end, we present a brief exposition of the notion of Casimir operator from representation theory of compact Lie groups. Let $\Delta_{\mathbb{M}}$ denote the Laplace-Beltrami operator with respect to the Riemannian metric on $\mathbb{M}$. Let $\mathscr{O}(\mathbb{M})$ denote the algebra of invariant differential functions on $\mathbb{M}$. Note that, a differential operator D on $\mathbb{M}$ is in $\mathscr{O}(\mathbb{M})$ if

$$D(\Phi^{\boldsymbol{s}^{-1}}) = D(\Phi)^{\boldsymbol{s}^{-1}} \tag{7}$$

holds for all $\boldsymbol{s} \in \mathbb{K}$ and $\Phi \in \mathcal{D}(\mathbb{M})$; here $f^{\boldsymbol{s}^{-1}}(\boldsymbol{x}) := f(\boldsymbol{s}\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{M}$. The canonical projection $\pi : \mathbb{K} \to \mathbb{M}$ associated to the (normal) homogeneous structure of $\mathbb{M} = \mathbb{K}/\mathbb{H}$ implies canonical indentification of complex Hilbert spaces with $\mathbb{K}$-actions:

$$\mathrm{L}^2(\mathbb{M}) \cong \mathrm{L}^2(\mathbb{K})^{\mathbb{H}} := \{\phi \in \mathrm{L}^2(\mathbb{K}) : \phi^{\mathbb{H}} = \phi\} . \tag{8}$$

Moreover, this restricts to the space of smooth functions too:

$$\mathcal{D}(\mathbb{M}) \cong \mathcal{D}(\mathbb{M})^{\mathbb{H}} := \{\phi \in \mathcal{D}(\mathbb{M}) : \phi^{\mathbb{H}} = \phi\} . \tag{9}$$

Let $\{\mathfrak{p}_a : a \in [d_{\mathbb{K}}]\}$ be a basis of $\mathfrak{k}$; let $\{\mathfrak{p}_a^\star : a \in [d_{\mathbb{K}}]\}$ be a Killing-dual basis of $\mathfrak{k}$. The Casimir element $\Omega_{\mathbb{K}}$ of $\mathbb{K}$ is defined by

$$\Omega_{\mathbb{K}} = \sum_{a=1}^{d_{\mathbb{K}}} \mathfrak{p}_a \cdot \mathfrak{p}_a^\star .$$

A change-of-basis argument shows that the Casimir operator $\Omega_{\mathbb{K}}$ is basis-independent. Thus, we may take $\{\mathfrak{p}_a : a \in [d_{\mathbb{K}}]\}$ to be a Killing-orthonormal basis of $\mathfrak{k}$, compatible with the Cartan decomposition; then, the Casimir operator is

$$\Omega_{\mathbb{K}} = \sum_{a=1}^{d_{\mathbb{K}}} \mathfrak{p}_a^2 . \tag{10}$$

Note that, for any $a, b \in [d_{\mathbb{K}}]$, one has — following Einstein's summation convention — that $\mathrm{ad}_{\mathfrak{p}_b}(\mathfrak{p}_a) = (s_{a,b}^c)\mathfrak{p}_c$ for some scalars $s_{a,b}^c$: it follows that

$$0 = \mathrm{B}_{\mathbb{K}}(\mathrm{ad}_{\mathfrak{p}_b}(\mathfrak{p}_a), \mathfrak{p}_c) + \mathrm{B}_{\mathbb{K}}(\mathfrak{p}_a, \mathrm{ad}_{\mathfrak{p}_b}(\mathfrak{p}_c)) = s_{a,b}^c + s_{c,b}^a ,$$

which implies $0 = \mathrm{ad}_{\mathfrak{p}_b}(\Omega_{\mathbb{K}})$. Linearity yields $\mathrm{ad}_{\mathfrak{p}}(\Omega_{\mathbb{K}}) = 0$ for every $\mathfrak{p} \in \mathfrak{k}$. In effect, this shows that $\Omega_{\mathbb{K}}$ lies in $\mathrm{Z}(\mathscr{U}_{\mathfrak{k}})$ — the center of the universal enveloping algebra $\mathscr{U}_{\mathfrak{k}}$.

To each $\mathfrak{p} \in \mathfrak{k}$ is associated a linear operator $\mathrm{D}_{\mathfrak{p}} : \mathcal{D}(\mathbb{K}) \to \mathcal{D}(\mathbb{K})$, defined by

$$\mathrm{D}_{\mathfrak{p}}\, \Phi(\boldsymbol{x}) := \left.\frac{d}{dt}\right|_{t=0} \Phi\left(\exp(t\mathfrak{p}) \cdot \boldsymbol{x}\right) .$$

Definition 2.1. The Casimir operator $\mathrm{D}_{\mathbb{K},\Omega_{\mathbb{K}}} : \mathcal{D}(\mathbb{K}) \to \mathcal{D}(\mathbb{K})$ is defined by

$$\mathrm{D}_{\mathbb{K},\Omega_{\mathbb{K}}} := \sum_{a=1}^{d_{\mathbb{K}}} \mathrm{D}_{\mathfrak{p}_a}^2 . \tag{11}$$

Note that, by the identification in (9) above, $\mathrm{D}_{\mathbb{K},\Omega_{\mathbb{K}}}$ restricts to a differential operator $\mathrm{D}_{\mathbb{M},\Omega_{\mathbb{K}}}$ — the *Casimir-Laplace-Beltrami* operator — on $\mathcal{D}(\mathbb{M})$. Now, for $\Phi \in \mathcal{D}(\mathbb{K})$, it follows from the Definition of $\mathrm{D}_{\mathfrak{p}_a}$ that

$$\mathrm{D}_{\mathbb{K},\Omega_{\mathbb{K}}}(\Phi)(\boldsymbol{x}) = \sum_{a=1}^{d_{\mathbb{K}}} \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial t'}\right|_{t'=0} \Phi\left(\exp(t'\mathfrak{p}_a) \cdot \exp(t\mathfrak{p}_a) \cdot \boldsymbol{x}\right) .$$

Let $\mathrm{D}_{\mathbb{M},\Omega_{\mathbb{K}}}$ be the restriction of $\mathrm{D}_{\mathbb{K},\Omega_{\mathbb{K}}}$ to $\mathcal{D}(\mathbb{M})$. The following lemma is well-known; see [19] for a proof.

Lemma 2.1. $\Delta_{\mathbb{M}} = \mathrm{D}_{\mathbb{M},\Omega_{\mathbb{K}}}$.

By a standard argument using Rellich-Kondrachov theorem ([12]), it follows that the resolvent of the self-adjoint positive definite operator $-\mathrm{D}_{\mathbb{M},\Omega_{\mathbb{K}}} = -\Delta_{\mathbb{M}}$ — which is defined on the dense subspace $\mathcal{D}(\mathbb{M})$ in strong sense — is compact, and therefore, $-\mathrm{D}_{\mathbb{M},\Omega_{\mathbb{K}}}$ possess only a discrete spectrum. Let

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

be the eigenvalues of $-\Delta_{\mathbb{M}}$. An application of spectral theorem implies the following orthogonal decomposition of $\mathrm{L}^2(\mathbb{M})$ into the eigenspaces of $\mathrm{D}_{\mathbb{M},\Omega_{\mathbb{K}}}$:

Theorem 2.1 (Spectral Theorem)**.** Let $\mathscr{E}_{\mathbb{M}} = \{\lambda_0, \lambda_1, \cdots\}$ denote the spectrum of $-\Delta_{\mathbb{M}}$, and for each $\lambda \in \mathscr{E}_{\mathbb{M}}$, let $\mathcal{H}_\lambda(\mathbb{M}) \subseteq \mathrm{L}^2(\mathbb{M})$ denote the null-space of $\lambda\mathbb{I}+\Delta_{\mathbb{M}}$; then

$$\mathrm{L}^2(\mathbb{M}) = \bigoplus_{\lambda\in\mathscr{E}_{\mathbb{M}}} \mathcal{H}_\lambda(\mathbb{M})\,. \tag{12}$$

Note that, each eigenspace $\mathcal{H}_\lambda$ is a sub-representation of $\mathcal{D}(\mathbb{M})$. By proposition 2.1 and irreducibility of V_{π_n} — together with Peter-Weyl theorem — it follows that for each $\lambda \in \mathscr{E}_{\mathbb{M}}$ there are $n_{\lambda,1}, \cdots, n_{\lambda,\ell} \in \mathbb{Z}_+$ such that

$$\mathcal{H}_\lambda = (V^\star_{\pi_{n_{\lambda,1}}} \otimes V^{\mathbb{H}}_{\pi_{n_{\lambda,1}}}) \oplus \cdots \oplus (V^\star_{\pi_{n_{\lambda,\ell}}} \otimes V^{\mathbb{H}}_{\pi_{n_{\lambda,\ell}}})\,.$$

Let $\omega_{n_{\lambda,j}}$ be the highest dominant integral weight corresponding to the irreducible representation $(V_{\pi_{n_{\lambda,j}}}, \pi_{n_\lambda})$. An application of the Casimir-van der Waerden formula ([28]), together with lemma 2.1, implies

$$\lambda := \|\omega_{n_{\lambda,j}} + \rho\|^2 - \|\rho\|^2 \tag{13}$$

for all $j \in \{1, \cdots, \ell\}$; here, ρ denotes the half-sum of the positive weights of $\mathfrak{k}$. Note that the following inequality holds for all $\lambda \geq \|\rho\|^2$:

$$\|\omega_{n_{\lambda,j}}\|^2 \leq \lambda \leq 3\|\omega_{n_{\lambda,j}}\|^2\,. \tag{14}$$

2.2. Some estimates on the heat kernel on $\mathbb{M}$ Going forward, we will need to apply the following result that bounds the eigenvalues and eigenspace-dimensions of the Laplace-Beltrami operator on a compact Riemannian manifold.

Consider the differential operator

$$\mathcal{H}_{\mathbb{M}} := \frac{d}{dt} - \frac{1}{2}\Delta_{\mathbb{M}}\,. \tag{15}$$

For any $\boldsymbol{p} \in \mathbb{M}$, the heat kernel $\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)$ on $\mathbb{M}$ is the unique (smooth) solution to the following problem:

$$\mathcal{H}_{\mathbb{M}} u = 0, \qquad \lim_{t\to 0^+} u(\boldsymbol{x}, t) = \delta_{\boldsymbol{p}}(\boldsymbol{x})\,, \tag{16}$$

where the limit above is taken in the distribution sense. Note that the following identity is immediate from $\mathbb{K}$-invariance of the Casimir operator and lemma 2.1:

$$\forall\ \boldsymbol{s} \in \mathbb{K}\,, \qquad \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t) = \mathrm{H}_{\boldsymbol{sp}}(\boldsymbol{sx}, t)\,. \tag{17}$$

Proposition 2.2 (Donnelly, [10])**.** There is a constant $C_{\mathbb{M}} > 0$, depending only on the lower bound of the injectivity radius of $\mathbb{M}$, (upper bound on the) sectional curvature of $\mathbb{M}$, the volume of $\mathbb{M}$, and dimension $d_{\mathbb{M}}$, such that the following inequality holds:

$$d_\lambda := \dim \mathcal{H}_\lambda \leq C_{\mathbb{M}} \lambda^{\frac{d_{\mathbb{M}}-1}{4}}\,. \tag{18}$$

The Fourier decomposition ([17]) of the heat kernel on $\mathbb{M}$ is

$$\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2) = \sum_{\lambda\in\mathscr{E}_{\mathbb{M}}} e^{-\lambda\epsilon^2} \sum_{j\in[d_\lambda]} \phi_{\lambda,j}(\boldsymbol{p})\phi_{\lambda,j}(\boldsymbol{x}) \tag{19}$$

for any fixed choice of an ordered orthonormal basis $(\phi_{\lambda,1},\dots,\phi_{\lambda,\dim\mathcal{H}_\lambda})$ for each $\lambda\in\mathscr{E}_{\mathbb{M}}$. For any $\lambda_\infty>0$, let

$$\mathrm{H}_{\boldsymbol{p}}^{\lambda_\infty}(\boldsymbol{x},\epsilon^2) := \sum_{\lambda\le\lambda_\infty} e^{-\lambda\epsilon^2} \sum_{j\in[d_\lambda]} \phi_{\lambda,j}(\boldsymbol{p})\phi_{\lambda,j}(\boldsymbol{x})$$

For (integer) $\lambda_\infty>0$, let $\mathcal{E}_{\lambda_\infty}(\mathbb{M})\subseteq \mathrm{L}^2(\mathbb{M})$ denote the direct sum

$$\mathcal{E}_{\lambda_\infty}(\mathbb{M}) := \bigoplus_{\lambda\in\mathscr{E}_{\mathbb{M}}(\lambda_\infty)} \mathcal{H}_\lambda\,,$$

where $\mathscr{E}_{\mathbb{M}}(\lambda_\infty) := \{\lambda\in\mathscr{E}_{\mathbb{M}} : \lambda\le\lambda_\infty\}$. Then, $\mathrm{H}_{\boldsymbol{p}}^{(\lambda)}(\boldsymbol{x},\epsilon^2)$ is the orthogonal projection of $\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)$ on $\mathcal{E}_\lambda(\mathbb{M})$.

Sharp bounds on heat-kernel on compact symmetric space of rank one has recently been derived in [27]. See [30] for background and results on antipodal dimension of compact Riemannian symmetric space.

Proposition 2.3 (Nowak et al, [27])**.** Suppose that $\mathbb{M}$ is rank-one symmetric space. There are constants $C_{\mathbb{M}}>c_{\mathbb{M}}>0$ — depending only on the dimension $d_{\mathbb{M}}$ and antipodal dimension $\bar{d}_{\mathbb{M}}$ — such that the following holds for all $t\in(0,1)$, and all $\boldsymbol{x},\boldsymbol{p}\in\mathbb{M}$:

$$\frac{c_{\mathbb{M}}\exp\left(-\frac{\partial_{\mathbb{M}}(\boldsymbol{x},\boldsymbol{p})^2}{4t}\right)}{t_{\mathbb{M},\boldsymbol{p}}^{\frac{d_{\mathbb{M}}-\bar{d}_{\mathbb{M}}-1}{2}}\, t^{\frac{d_{\mathbb{M}}}{2}}} \le \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},t) \le \frac{C_{\mathbb{M}}\exp\left(-\frac{\partial_{\mathbb{M}}(\boldsymbol{x},\boldsymbol{p})^2}{4t}\right)}{t_{\mathbb{M},\boldsymbol{p}}^{\frac{d_{\mathbb{M}}-\bar{d}_{\mathbb{M}}-1}{2}}\, t^{\frac{d_{\mathbb{M}}}{2}}}\,.$$

Here $t_{\mathbb{M},\boldsymbol{p}} := t+\mathrm{diam}(\mathbb{M})-\partial_{\mathbb{M}}(\boldsymbol{x},\boldsymbol{p})$.

However, we need the bound to hold for compact symmetric space of arbitrary (finite) rank. To this end, we will use the following fundamental bound on the heat kernel.

Proposition 2.4 (Cheng-Li-Yau, [9])**.** Suppose that $\mathbb{M}$ is complete Riemannian manifold. There is a constant $C_{\mathbb{M}}>1$ — depending only on the dimension $d_{\mathbb{M}}$ and the curvature of $\mathbb{M}$ — such that the following holds for all $t\in(0,1)$, and all $\boldsymbol{x},\boldsymbol{p}\in\mathbb{M}$:

$$\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},t) \le \frac{C_{\mathbb{M}}\exp\left(-\frac{\partial_{\mathbb{M}}(\boldsymbol{x},\boldsymbol{p})^2}{5t}\right)}{t^{\frac{d_{\mathbb{M}}}{2}}}\,.$$

It is well-known ([17]) that $\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},t)\ge 0$ for all points $\boldsymbol{x},\boldsymbol{p}\in\mathbb{M}$ and all $t>0$. Moreover, $\mathbb{M}$ being compact Riemannian symmetric space (hence, in particular, complete and geodesically complete, as in [17]), one has

$$\int_{\mathbb{M}} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},t)\; d\sigma^{\mathbb{M}}(\boldsymbol{x}) = 1\,. \tag{20}$$

In other words, $\mathbb{M}$ is stochastically complete.

Remark 2.1. For any $\epsilon, \eta \in (0,1)$, let

$$r(\epsilon,\eta) := 2\epsilon\sqrt{\ln \frac{3C_{\mathbb{M}}}{\eta\epsilon^{2d_{\mathbb{M}}}}} \tag{21}$$

Fix $\boldsymbol{p} \in \mathbb{M}$; then, for any $\boldsymbol{x} \in \mathbb{M}$ satisfying $\partial_{\mathbb{M}}(\boldsymbol{x},\boldsymbol{p}) \geq r(\epsilon,\eta)$, it follows from proposition 2.4 that

$$0 \leq \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2) \leq \frac{\eta}{3}\,. \tag{22}$$

Notice that, when $\eta = \Omega(\mathrm{poly}\,\epsilon)$, one has $r(\epsilon,\eta) = o(\epsilon^{1-\mathfrak{t}})$ for every constant $\mathfrak{t} > 0$.

The following statement was established in course of proving *lemma 5* in [9], although the dependence of the constant on manifold parameters were not explicit; we note the statement as a lemma here, for easy reference, and to make the dependence of the constants on the manifold parameters explicit. Recall that the constant $C_{\mathbb{M}} > 0$ appearing above depends only on $d_{\mathbb{M}}$ and the intejectivity radius.

Lemma 2.2. Let $\mathbb{M}$ be a compact Riemannian symmetric space of dimension $d_{\mathbb{M}}$. For $\epsilon \in (0,1)$, the following holds:

$$\|\,\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)\|_{\mathrm{L}^2(\mathbb{M})} \leq \sqrt{C_{\mathbb{M}}2^{-d_{\mathbb{M}}}}\epsilon^{-d_{\mathbb{M}}} \tag{23}$$

Proof. By the upper-bound in proposition 2.4, it follows that

$$\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{p},2\epsilon^2) \leq C_{\mathbb{M}}(2\epsilon^2)^{-d_{\mathbb{M}}}$$

holds for $\epsilon \in (0,1)$. Now, by semigroup property of the heat kernel, one has

$$\begin{aligned}\int_{\mathbb{M}} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)^2 \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) &= \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{p},2\epsilon^2) \\ &\leq C_{\mathbb{M}}(2\epsilon^2)^{-d_{\mathbb{M}}}\,,\end{aligned}$$

which establishes the lemma. ■

Corollary 2.1. For any $\lambda_\infty > 0$, the following inequality holds for all $\epsilon \in (0,1)$:

$$\|\,\mathrm{H}_{\boldsymbol{p}}^{(\lambda_\infty)}(\boldsymbol{x},\epsilon^2)\|_{\mathrm{L}^2(\mathbb{M})} \leq \sqrt{C_{\mathbb{M}}2^{-d_{\mathbb{M}}}}\epsilon^{-d_{\mathbb{M}}}\,. \tag{24}$$

Proof. Immediate, since

$$\|\,\mathrm{H}_{\boldsymbol{p}}^{(\lambda_\infty)}(\boldsymbol{x},\epsilon^2)\|_{\mathrm{L}^2(\mathbb{M})} \leq \|\,\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)\|_{\mathrm{L}^2(\mathbb{M})}$$

by property of orthogonal projection on subspaces of Hilbert space. ■

Going forward, we make essential use of Weyl's asymptotic of the eigenvalues of $\Delta_{\mathbb{M}}$ in a Riemannian manifold $\mathbb{M}$, as formulated in [11].

Lemma 2.3 (Weyl). The following holds:

$$\lim_{\lambda_\infty\to\infty} \lambda_\infty^{-\frac{d_{\mathbb{M}}}{2}} \sum_{\lambda\in\mathscr{E}_{\mathbb{M}}(\lambda_\infty)} d_\lambda = C_{\mathbb{M}}$$

From Weyl's law, it follows that

$$\lim_{\lambda_\infty\to\infty}\left(\lambda_\infty^{-\frac{d_{\mathbb{M}}}{2}} \sum_{\lambda\in\mathscr{E}_{\mathbb{M}}(4^{\frac{1}{d_{\mathbb{M}}}}\lambda_\infty)} d_\lambda\right) - \lim_{\lambda_\infty\to\infty}\left(\lambda_\infty^{-\frac{d_{\mathbb{M}}}{2}} \sum_{\lambda\in\mathscr{E}_{\mathbb{M}}(\lambda_\infty)} d_\lambda\right) = C_{\mathbb{M}}\,,$$

and in particular,

$$\sum_{\lambda\in\mathscr{E}_{\mathbb{M}}(\lambda_\infty,4^{\frac{1}{d}}\lambda_\infty)} d_\lambda \le C_{\mathbb{M}}\lambda_\infty^{\frac{d_{\mathbb{M}}}{2}} \tag{25}$$

for all $\lambda_\infty > 0$, where $\mathscr{E}_{\mathbb{M}}(\lambda_\infty, 4^{\frac{1}{d_{\mathbb{M}}}}\lambda_\infty) := \mathscr{E}_{\mathbb{M}}(4^{\frac{1}{d_{\mathbb{M}}}}\lambda_\infty)\setminus\mathscr{E}_{\mathbb{M}}(\lambda_\infty)$.

3. Truncation of heat kernel

In this section, we show that the L^2-mass of the heat kernel can be well-approximated by the L^2-mass of its orthogonal projection onto the subspace $\mathscr{E}_{\mathbb{M}}(\lambda_\infty)\subseteq \mathrm{L}^2(\mathbb{M})$ for a suitable λ_∞ large enough. We start with an auxillery lemma.

Lemma 3.1. For any $\Delta_{\mathbb{M}}$-eigenvalue $-\lambda$, and orthonormal basis $\phi_{\lambda,1},\cdots,\phi_{\lambda,d_\lambda}$ of $\mathcal{H}_\lambda$, let $\psi_\lambda:\mathbb{M}^2\to\mathbb{C}$ be defined by

$$\psi_\lambda(\boldsymbol{x},\boldsymbol{y}) := \sum_{j=1}^{d_\lambda}\phi_{\lambda,j}(\boldsymbol{x})\overline{\phi_{\lambda,j}(\boldsymbol{y})}\,.$$

Then the following statements hold:

- ψ_λ is independent of the orthonormal basis;
- $\psi_\lambda\circ(\zeta,\zeta) = \psi_\lambda$ for any $\zeta\in\mathbb{K}$;
- $\psi_\lambda(\boldsymbol{p},\boldsymbol{p}) = d_\lambda$ for any $\boldsymbol{p}\in\mathbb{M}$.

Proof. A standard base-change argument shows that the function ψ_λ is independent of basis.

For any isometry $\zeta\in\mathbb{K}$, the elements

$$\phi_{\lambda,1}\circ\zeta,\cdots,\phi_{\lambda,d_\lambda}\circ\zeta$$

are linearly independent, since any linear relation

$$0 = a_1\phi_{\lambda,1}\circ\zeta+\cdots+a_{d_\lambda}\phi_{\lambda,d_\lambda}\circ\zeta\,,$$

together with the $\mathbb{K}$-invariance of $\sigma^{\mathbb{M}}$, implies

$$\begin{aligned} 0 &= \sum_{j_1,j_2=1}^{d_\lambda} a_{j_1}\overline{a}_{j_2} \int_{\mathbb{M}} \phi_{\lambda,j_1}(\zeta(\boldsymbol{x}))\overline{\phi_{\lambda,j_2}(\zeta(\boldsymbol{x}))}\; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\ &= \sum_{j_1,j_2=1}^{d_\lambda} a_{j_1}\overline{a}_{j_2} \int_{\mathbb{M}} \phi_{\lambda,j_1}(\boldsymbol{x})\overline{\phi_{\lambda,j_2}(\boldsymbol{x})}\; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\ &= |a_1|^2 + \cdots + |a_{d_\lambda}|^2 \,, \end{aligned}$$

which is equivalent to $a_1 = \cdots = a_{d_\lambda} = 0$. By invariance of Casimir-Laplace-Beltrami, the elments

$$\phi_{\lambda,1}\circ\zeta, \cdots, \phi_{\lambda,d_\lambda}\circ\zeta$$

are contained in $\mathcal{H}_\lambda$, and hence, form a basis of $\mathcal{H}_\lambda$; furthermore, this must be an orthonormal basis by $\mathbb{K}$-invariance of $\sigma^{\mathbb{M}}$. By the basis-independence character of ψ_λ, it follows that $\psi_\lambda \circ (\zeta,\zeta) = \psi_\lambda$.

For the third part, we notice that orthonomality of $\{\phi_{\lambda,j}\}_{j=1}^{d_\lambda}$ implies

$$\begin{aligned} d_\lambda &= \sum_{j=1}^{d_\lambda} \int_{\mathbb{M}} \phi_{\lambda,j}(\boldsymbol{x})\overline{\phi_{\lambda,j}(\boldsymbol{x})}\; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\ &= \int_{\mathbb{M}} \psi_\lambda(\boldsymbol{x},\boldsymbol{x})\; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\ &= \psi_\lambda(\boldsymbol{p},\boldsymbol{p})\,, \end{aligned}$$

where the last step follows from the identity $\psi_\lambda \circ (\zeta,\zeta) = \psi_\lambda$. This proves the lemma. ∎

In the following, we will closely follow the arguments in [25]. The following lemma proves the main approximation result for the L^2-mass of the heat kernel.

Lemma 3.2. Suppose that $\epsilon \in (0, 2^{-e})$; for any $\eta > 0$, let $\lambda_\infty \geq 4^{\frac{k_\eta}{d_{\mathbb{M}}}}$ — where

$$k_\eta := \max\left(2 + 2\log_2 \tfrac{1}{\eta},\ \tfrac{1}{2} d_{\mathbb{M}} \log_2\left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + d_{\mathbb{M}} \log_2\log_2\left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right)\right); \tag{26}$$

then the following inequality holds for any $\boldsymbol{p} \in \mathbb{M}$:

$$\|\, \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2) - \mathrm{H}_{\boldsymbol{p}}^{\lambda_\infty}(\boldsymbol{x},\epsilon^2)\|_{\mathrm{L}^2}^2 \leq C_{\mathbb{M}}\eta^2\,.$$

Proof. Write $\mathscr{E}_{\mathbb{M},\lambda_\infty} := \{\lambda \in \mathscr{E}_{\mathbb{M}} : \lambda > \lambda_\infty\}$. Note that, by eq. (19), integration with respect to the Haar-induced probability measure on $\mathbb{M}$ yields

$$\begin{aligned} \|\, \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2) - \mathrm{H}_{\boldsymbol{p}}^{\lambda_\infty}(\boldsymbol{x},\epsilon^2)\|_{\mathrm{L}^2}^2 &= \sum_{\lambda\in\mathscr{E}_{\mathbb{M},\lambda_\infty}} e^{-2\lambda\epsilon^2} \sum_{j\in[d_\lambda]} |\phi_{\lambda,j}(\boldsymbol{p})|^2 \\ \text{Lemma 3.1} \;\Rightarrow\quad &\leq \sum_{\lambda\in\mathscr{E}_{\mathbb{M},\lambda_\infty}} e^{-2\lambda\epsilon^2} d_\lambda\,. \end{aligned} \tag{27}$$

For any $k \in \mathbb{Z}_+$, write
$$\mathcal{I}_k := \left(4^{\frac{k}{d_{\mathbb{M}}}}, 4^{\frac{k+1}{d_{\mathbb{M}}}}\right].$$
One has
$$\begin{aligned}\sum_{\lambda \in \mathscr{E}_{\mathbb{M},\lambda_\infty}} e^{-2\lambda\epsilon^2} d_\lambda &\le \sum_{k \ge k_\eta} \sum_{\lambda \in \mathcal{I}_k} e^{-2\lambda\epsilon^2} d_\lambda \\ &\le \sum_{k \ge k_\eta} \left(\sup_{\lambda \in \mathcal{I}_k} e^{-2\lambda\epsilon^2}\right)\left(\sum_{\lambda \in \mathcal{I}_k} d_\lambda\right) \\ eq.\ (25) \ \Rightarrow \qquad &\le C_{\mathbb{M}} \sum_{k \ge k_\eta} 2^{1+k} e^{-(2^{\frac{2k+d_{\mathbb{M}}}{d_{\mathbb{M}}}})\epsilon^2}.\end{aligned}$$
Suppose that an integer $k \ge 0$ satisfies $2k \ge d_{\mathbb{M}} \log_2\left(\frac{k}{\epsilon^2}\right)$; then we get
$$e^{-(2^{\frac{2k+d_{\mathbb{M}}}{d_{\mathbb{M}}}})\epsilon^2} = (e^{-2^{\frac{2k}{d_{\mathbb{M}}}}})^{2\epsilon^2} \le e^{-2k}.$$
Consider the inequality
$$\frac{k}{\log_2\left(\frac{k}{\epsilon^2}\right)} \ge \frac{d_{\mathbb{M}}}{2}. \tag{28}$$
By monotone property of logarithm, the following inequality is equivalent to eq. (28) above:
$$k\left(1 - \frac{\log_2\log_2\left(\frac{k}{\epsilon^2}\right)}{\log_2\left(\frac{k}{\epsilon^2}\right)}\right) \ge \frac{d_{\mathbb{M}}}{2}\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right). \tag{29}$$
For $u \in [2^e, +\infty)$, the function
$$g(u) = 1 - \frac{\log_2\log_2 u}{\log_2 u}$$
satisfies $0 < g(u) < 1$, has global minima $g(2^e) = 1 - e^{-1}\log_2 e > 0.46$, and is increasing. Since $\epsilon \in (0, 2^{-e})$ and $d_{\mathbb{M}} \ge 1$, the condition $d_{\mathbb{M}}/2\epsilon^2 > 2^e$ is satisfied; requiring $k \ge \frac{1}{2} d_{\mathbb{M}} \log_2(2^e d_{\mathbb{M}})$, we see that the following inequality implies eq. (29):
$$k \ge \frac{d_{\mathbb{M}}}{2}\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)\left(1 - \frac{\log_2\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}{\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}\right)^{-1}. \tag{30}$$
Notice that, for $d_{\mathbb{M}} \ge 1$ and $\epsilon \in (0, 2^{-e})$, one has $\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right) \ge 2\log_2\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)$, which yields the following inequality:
$$\begin{aligned}\frac{\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right) + 2\log_2\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}{\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)} &= 1 + \frac{2\log_2\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}{\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)} \\ &\ge \left(1 - \frac{\log_2\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}{\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}\right)^{-1} \\ &= \frac{\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}{\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right) - \log_2\log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}.\end{aligned}$$

This shows that eq. (30) follows if $k_\eta \geq \frac{1}{2} d_{\mathbb{M}} \log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right) + d_{\mathbb{M}} \log_2 \log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)$, in which case we have

$$\begin{aligned}
\| \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, \epsilon^2) - \mathrm{H}_{\boldsymbol{p}}^{(\lambda_\infty)}(\boldsymbol{x}, \epsilon^2) \|_{\mathrm{L}^2}^2 &= C_{\mathbb{M}} \sum_{\lambda \geq 4^{\frac{k_\eta}{d_{\mathbb{M}}}}} e^{-2\lambda\epsilon^2} d_\lambda \\
&\leq C_{\mathbb{M}} \sum_{k \geq k_\eta} 2^{1+k} e^{-(2^{\frac{2k+d_{\mathbb{M}}}{d_{\mathbb{M}}}})\epsilon^2} \\
&\leq C_{\mathbb{M}} \sum_{k \geq k_\eta} 2^{1+k-2k} \\
&\leq 4 C_{\mathbb{M}} 2^{-k_\eta} \\
&\leq C_{\mathbb{M}} \eta^2 \,.
\end{aligned}$$

This finishes the proof. ■

Remark 3.1. Suppose that $d_{\mathbb{M}} \geq 2^e$, and that $\epsilon \in (0, d_{\mathbb{M}}^{-1})$. It is immediate that

$$\frac{1}{2} d_{\mathbb{M}} \log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right) < \frac{3}{4} d_{\mathbb{M}} \log_2 \frac{1}{\epsilon^2} \,.$$

Moreover, it follows from elementary calculus that

$$\begin{aligned}
\frac{2 \log_2 \log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right)}{\log_2 \frac{1}{\epsilon^2}} &< \frac{2 \log_2 \log_2\left(\frac{1}{2\epsilon^3}\right)}{\log_2 \frac{1}{\epsilon^2}} \qquad \because \ \epsilon < d_{\mathbb{M}}^{-1} \\
&= \frac{\log_2 \log_2\left(\frac{1}{2\epsilon^3}\right)}{\log_2 \frac{1}{\epsilon}} \\
&< \frac{3 \log_2 \log_2\left(\frac{1}{\epsilon}\right)}{\log_2 \frac{1}{\epsilon}} \\
&< \frac{3 \log_2 \log_2\left(d_{\mathbb{M}}\right)}{\log_2\left(d_{\mathbb{M}}\right)} \,.
\end{aligned}$$

This shows

$$\frac{1}{2} d_{\mathbb{M}} \log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right) + d_{\mathbb{M}} \log_2 \log_2\left(\frac{d_{\mathbb{M}}}{2\epsilon^2}\right) < \frac{3}{4} d_{\mathbb{M}} \log_2 \frac{1}{\epsilon^2} \left(1 + \frac{2 \log_2 \log_2 d_{\mathbb{M}}}{\log_2 d_{\mathbb{M}}}\right) \,.$$

The following lemma furnishes an analytically checkable criterion for a subset $\mathcal{S} \subseteq \mathbb{M}$ — in any compact Riemannian manifold $\mathbb{M}$ — to be an ϵ-cover. The underlying idea is that, if the weighted sum of the heat kernels based at points in $\mathcal{S}$ is close to the identity in L^2-norm, then the set $\mathcal{S}$ must be a cover.

Lemma 3.3. Let $\mathcal{S} \subseteq \mathbb{M}$ be a nonempty subset. For $\epsilon \in (0, 1)$, if

$$r_{\epsilon,\mathbb{M}} := r(\epsilon, 1) = 2\epsilon \sqrt{\ln \frac{3 C_{\mathbb{M}}}{\epsilon^{2 d_{\mathbb{M}}}}}$$

is sufficiently small, then the following inequality implies that $\mathcal{S} \subseteq \mathbb{M}$ is a $2r_{\epsilon,\mathbb{M}}$-net:

$$\left\| 1_{\mathbb{M}}(\boldsymbol{x}) - \frac{1}{|\mathcal{S}|} \sum_{\boldsymbol{p} \in \mathcal{S}} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, \epsilon^2) \right\|_{\mathrm{L}^2} \leq \frac{r_{\epsilon,\mathbb{M}}^{\frac{d_{\mathbb{M}}}{2}}}{3} \sqrt{\frac{\mathfrak{v}_{\mathbb{M}} \pi^{\frac{d_{\mathbb{M}}}{2}}}{\Gamma\left(\frac{d_{\mathbb{M}}}{2} + 1\right)}} \tag{31}$$

Remark 3.2. The lemma 3.3 first appeared in [25], where the statement was formulated in terms of compact Lie groups.

Proof. From remark 2.1, it follows that the inequalities

$$\frac{2}{3}\cdot 1_{\mathbb{M}}(\boldsymbol{x}) \leq 1_{\mathbb{M}}(\boldsymbol{x}) - \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2) \leq 1_{\mathbb{M}}(\boldsymbol{x}) \tag{32}$$

hold for any $\boldsymbol{x},\boldsymbol{p}\in\mathbb{M}$ satisfying $\partial_{\mathbb{M}}(\boldsymbol{x},\boldsymbol{p})\geq r_{\epsilon,\mathbb{M}}$. Write $\mathrm{B}(\boldsymbol{p},r_{\epsilon,\mathbb{M}})\subseteq\mathbb{M}$ for the open geodesic disk of radius $r_{\epsilon,\mathbb{M}}$, centered at $\boldsymbol{p}\in\mathbb{M}$; thus, $\mathrm{B}(\boldsymbol{x},r_{\epsilon,\mathbb{M}}):=\{\boldsymbol{x}\in\mathbb{M}:\partial_{\mathbb{M}}(\boldsymbol{x},\boldsymbol{p})<r_{\epsilon,\mathbb{M}}\}$. Also, let $\mathrm{B}_{d_{\mathbb{M}}}\subseteq\mathbb{R}^{d_{\mathbb{M}}}$ denote the origin-centric unit euclidean ball; one has (see [16])

$$\lim_{r\to 0}\frac{\sigma^{\mathbb{M}}(\mathrm{B}(\boldsymbol{x},r_{\epsilon,\mathbb{M}}))}{r_{\epsilon,\mathbb{M}}^{d_{\mathbb{M}}}} = \mathfrak{v}_{\mathbb{M}}\cdot\mathrm{vol}(\mathrm{B}_{d_{\mathbb{M}}}) = \frac{\mathfrak{v}_{\mathbb{M}}\pi^{\frac{d_{\mathbb{M}}}{2}}}{\Gamma\left(\frac{d_{\mathbb{M}}}{2}+1\right)} \tag{33}$$

Now assume, if possible, that $\mathcal{S}\subseteq\mathbb{M}$ is not a $2r_{\epsilon,\mathbb{M}}$-net, so that there is $\boldsymbol{p}_0\in\mathbb{M}$ with $d(\boldsymbol{p}_0,\mathcal{S})>2r_{\epsilon,\mathbb{M}}$. By triangle inequality applied to the Riemannian distance on $\mathbb{M}$, it follows that $\mathrm{B}(\boldsymbol{p}_0,r_{\epsilon,\mathbb{M}})\cap\mathrm{B}(\boldsymbol{p},r_{\epsilon,\mathbb{M}})=\emptyset$ for every $\boldsymbol{p}\in\mathcal{S}$. Writing

$$\alpha_{d_{\mathbb{M}}} := \sqrt{\frac{\mathfrak{v}_{\mathbb{M}}\pi^{\frac{d_{\mathbb{M}}}{2}}}{\Gamma\left(\frac{d_{\mathbb{M}}}{2}+1\right)}}\,,$$

we derive from eq. (31) and eq. (32) the following:

$$\begin{aligned}\frac{r_{\epsilon,\mathbb{M}}^{\frac{d_{\mathbb{M}}}{2}}\alpha_{d_{\mathbb{M}}}}{3} &\geq \left\| 1_{\mathbb{M}}(\boldsymbol{x}) - \frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{p}\in\mathcal{S}}\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)\right\|_{\mathrm{L}^2}\\ &= \left\|\frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{p}\in\mathcal{S}}\left|1_{\mathbb{M}}(\boldsymbol{x})-\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)\right|\right\|_{\mathrm{L}^2}\\ &\geq \frac{2}{3}\left(\int_{\mathrm{B}(\boldsymbol{p}_0,r_{\epsilon,\mathbb{M}})} d\sigma^{\mathbb{M}}(\boldsymbol{x})\right)^{\frac{1}{2}}.\end{aligned}$$

This implies

$$\begin{aligned}\frac{\sigma^{\mathbb{M}}(\mathrm{B}(\boldsymbol{p}_0,r_{\epsilon,\mathbb{M}}))}{r_{\epsilon,\mathbb{M}}^{d_{\mathbb{M}}}} &= \frac{1}{r_{\epsilon,\mathbb{M}}^{d_{\mathbb{M}}}}\int_{\mathrm{B}(\boldsymbol{p}_0,r_{\epsilon,\mathbb{M}})} d\sigma^{\mathbb{M}}(\boldsymbol{x})\\ &\leq \frac{\alpha_{d_{\mathbb{M}}}^2}{4}\end{aligned}$$

Considering eq. (33), this is impossible if $r_{\epsilon,\mathbb{M}}>0$ is sufficiently small. ■

4. Equidistributed cover

In what follows, we will make use of the following tail-bound on operator-valued random variables on a finite dimensional Hilbert space. We recall that for hermitian matrices A,B of same size, we write $A\preceq B$ if $B-A$ is positive semidefinite; similarly, $B\succeq A$ if $A\preceq B$.

Lemma 4.1 (Ahlswede-Winter, [1])**.** Let V be a finite dimensional (real or complex) Hilbert space, with $\dim V = D$. Let $A_1, \cdots, A_k$ be independent identically distributed random variables taking values in the cone of positive semidefinite operators on V, such that $A_j \preceq \mathrm{I}$ for each $j \in [k]$, and there is some real $\mu \geq 0$ for which $\mathbb{E}[A_j] = A \succeq \mu \mathrm{I}$ for each $j \in [k]$. Then, for all $\upsilon \in [0, 0.5]$, the following holds:

$$\mathbb{P}\left(\frac{1}{k}\sum_{j=1}^{k} A_j \notin [(1-\upsilon)A, (1+\upsilon)A]\right) \leq 2D\exp\left(\frac{-\upsilon^2\mu k}{2\ln 2}\right) \tag{34}$$

■

Now let $\mathcal{S} \subset \mathbb{K}$ be a non-empty subset. Recall that, for any $\lambda_\infty > 0$, we write $\mathcal{E}^\star_{\lambda_\infty}(\mathbb{M}) \subseteq \mathrm{L}^2_0(\mathbb{M})$ for the direct sum

$$\mathcal{E}^\star_{\lambda_\infty}(\mathbb{M}) = \bigoplus_{\lambda \in \mathscr{E}_{\mathbb{M}}(0,\lambda_\infty)} \mathcal{H}_\lambda\,,$$

where $\mathscr{E}_{\mathbb{M}}(0, \lambda_\infty) = \{\lambda \in \mathscr{E}_{\mathbb{M}} : 0 < \lambda \leq \lambda_\infty\}$; we will often simply write $\mathcal{E}^\star_{\lambda_\infty}$ et cetera, for ease of notation. Note that

$$\dim \mathcal{E}^\star_{\lambda_\infty}(\mathbb{M}) < \dim \mathcal{E}_{\lambda_\infty}(\mathbb{M}) \leq C_{\mathbb{M}} \lambda_\infty^{\frac{d_{\mathbb{M}}}{2}} \tag{35}$$

by eq. (25). Because $\Delta_{\mathbb{M}}$ is $\mathbb{K}$-invariant, the subspace $\mathcal{E}^\star_{\lambda_\infty}(\mathbb{M})$ is invariant under the operators $A_{\boldsymbol{s}}$ for all $\boldsymbol{s} \in \mathbb{K}$, where $A_{\boldsymbol{s}}$ is defined via

$$A_{\boldsymbol{s}}(\phi)(\boldsymbol{x}) := \frac{1}{2}\phi(\boldsymbol{x}) + \frac{1}{4}(\phi(\boldsymbol{s}\boldsymbol{x}) + \phi(\boldsymbol{s}^{-1}\boldsymbol{x}))\,. \tag{36}$$

Due to $\mathbb{K}$-invariance of the measure $\sigma^{\mathbb{M}}$, the operators $A_{\boldsymbol{s}} : \mathcal{E}^\star_{\lambda_\infty}(\mathbb{M}) \to \mathcal{E}^\star_{\lambda_\infty}(\mathbb{M})$ turns out to be self-adjoint. Positive semidefiniteness of $A_{\boldsymbol{s}}$ follows from the identity

$$\langle A_{\boldsymbol{s}}\phi, \phi\rangle = \frac{1}{4}\int_{\mathbb{M}} |\phi(\boldsymbol{x}) + \phi(\boldsymbol{s}\boldsymbol{x})|^2\, d\sigma^{\mathbb{M}}(\boldsymbol{x})\,.$$

Moreover, writing $\sigma^{\mathbb{K}}$ for the unique invariant Haar (probability) measure on $\mathbb{K}$, one has

$$\left(\mathbb{E}_{\boldsymbol{s}\sim\sigma^{\mathbb{M}}}[A_{\boldsymbol{s}}]\right)\phi(\boldsymbol{x}) = \frac{1}{2}\phi(\boldsymbol{x}) + \frac{1}{4}\int_{\mathbb{K}} \phi(\boldsymbol{s}\boldsymbol{x})\; d\sigma^{\mathbb{K}}(\boldsymbol{s}) + \frac{1}{4}\int_{\mathbb{K}} \phi(\boldsymbol{s}^{-1}\boldsymbol{x})\; d\sigma^{\mathbb{K}}(\boldsymbol{s})\,.$$

Writing $\tau : \mathbb{K} \to \mathbb{M}$ for the map $\boldsymbol{s} \mapsto \boldsymbol{s}\boldsymbol{o}$, one has

$$\begin{aligned}\int_{\mathbb{K}} \phi(\boldsymbol{s}\boldsymbol{o})\; d\sigma^{\mathbb{K}}(\boldsymbol{s}) &= \int_{\mathbb{K}} (\phi \circ \tau)(\boldsymbol{s})\; d\sigma^{\mathbb{K}}(\boldsymbol{s}) \\ &= \int_{\mathbb{M}} \phi(\boldsymbol{x})\; d(\tau_*\sigma^{\mathbb{K}})\,,\end{aligned}$$

where $\tau_*\sigma^{\mathbb{K}}$ is the push-forward measure, given by $\tau_*\sigma^{\mathbb{K}}(E) = \sigma^{\mathbb{K}}(\tau^{-1}(E))$ for Borel subsets $E \subseteq \mathbb{M}$. Since $\tau_*\sigma^{\mathbb{K}}$ is $\mathbb{K}$-invariant Borel probability measure on $\mathbb{M}$, one has $\sigma^{\mathbb{M}} = \tau_*\sigma^{\mathbb{K}}$. Because $\phi \in \mathcal{E}^\star_{\lambda_\infty}(\mathbb{M}) \subset \mathrm{L}^2(\mathbb{M})$, one has

$$\int_{\mathbb{K}} \phi(\boldsymbol{x}\boldsymbol{s})\; d\sigma^{\mathbb{K}}(\boldsymbol{s}) = \int_{\mathbb{M}} \phi(\boldsymbol{x})\; d\sigma^{\mathbb{M}} = 0\,. \tag{37}$$

Therefore, $(\mathbb{E}_{\boldsymbol{s}\sim\sigma^{\mathbb{K}}}[A_{\boldsymbol{s}}])\,\phi(\boldsymbol{x}) = \frac{1}{2}\phi(\boldsymbol{x})$, which makes

$$\mathbb{E}_{\boldsymbol{s}\sim\sigma^{\mathbb{K}}}[A_{\boldsymbol{s}}] = \frac{1}{2}\,\mathrm{I}, \qquad \boldsymbol{s}\in\mathbb{K}\,.$$

Furthermore, the operators $\mathrm{I} - A_{\boldsymbol{s}}$ are positive semidefinite for all $\boldsymbol{s}\in\mathbb{K}$, because

$$\langle \phi - A_{\boldsymbol{s}}\phi, \phi\rangle = \frac{1}{4}\int_{\mathbb{M}} |\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}\boldsymbol{s})|^2\, d\sigma^{\mathbb{M}}(\boldsymbol{x})\,.$$

4.1. Random cover In the following lemma, we derive a high probability upper-bound on the L^2-norm of the non-constant tail of the heat kernel on $\mathbb{M}$.

Lemma 4.2. Let $\mathcal{S}\subset\mathbb{K}$ be a nonempty finite multisubset whose elements are selected independently at random from the Haar measure on $\mathbb{K}$ and let $\hat{\mathcal{S}} := \mathcal{S}\sqcup\mathcal{S}^{-1}$ be the (multi)set of all elements in $\mathcal{S}$ and their inverses. Suppose that $\eta > 0$ satisfies

$$2 + 2\log_2\frac{1}{\eta} \geq \frac{1}{2} d_{\mathbb{M}} \log_2\left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + d_{\mathbb{M}}\log_2\log_2\left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right)\,. \tag{38}$$

Let $\epsilon\in(0, 2^{-e})$; if

$$\delta := \frac{C_{\mathbb{M}}}{\eta}\exp\left(\tfrac{-|\mathcal{S}|}{16\ln 2}\right)\,,$$

then for any integer $\ell > 0$, and any $\boldsymbol{p}\in\mathbb{M}$, the following inequality holds:

$$\mathbb{P}_{\mathcal{S}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^{\ell}}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^{\ell}} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{s}\boldsymbol{x}, \epsilon^2) - 1_{\mathbb{M}}(\boldsymbol{x})\right\|_{\mathrm{L}^2} \leq \sqrt{C_{\mathbb{M}}}\left(2^{-\ell}\epsilon^{-d_{\mathbb{M}}} + \eta\right)\right) \geq 1 - 2\delta\,. \tag{39}$$

Proof. Setting $\upsilon = \mu = 0.5$ and $\lambda_\infty = \eta^{-\frac{2}{d}}$ in eq. (34) yields

$$\mathbb{P}_{\mathcal{S}}\left(\frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{s}\in\mathcal{S}} A_{\boldsymbol{s}} \notin \left[\frac{1}{4}\,\mathrm{I}, \frac{3}{4}\,\mathrm{I}\right]\right) \leq \frac{2C_{\mathbb{M}}}{\eta}\exp\left(\frac{-|\mathcal{S}|}{16\ln 2}\right)\,.$$

Note that the combination of eq. (38) and eq. (35) — together with the choice of $\lambda_\infty = \eta^{-\frac{2}{d}}$ — ensures that lemma 3.2 can be applied if required. Now, disentangling the inequality $\frac{1}{4}\,\mathrm{I} \preccurlyeq |\mathcal{S}|^{-1}\sum_{\boldsymbol{s}\in\mathcal{S}} A_{\boldsymbol{s}} \preccurlyeq \frac{3}{4}\,\mathrm{I}$ in the cone of positive-definite operators on $\mathcal{E}^{\star}_{\lambda_\infty}(\mathbb{M})$, we obtain

$$-\frac{1}{4}\,\mathrm{I} \preccurlyeq \left(\frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{s}\in\mathcal{S}} A_{\boldsymbol{s}} - \frac{1}{2}\,\mathrm{I}\right) \preccurlyeq \frac{1}{4}\,\mathrm{I}\,.$$

Thus, the eigenvalues of the hermitian operator $|\mathcal{S}|^{-1}\sum_{\boldsymbol{s}\in\mathcal{S}} A_{\boldsymbol{s}} - \frac{1}{2}\,\mathrm{I}$ are all in $[-\frac{1}{4}, \frac{1}{4}]$, which shows that — for all $\phi\in\mathcal{E}^{\star}_{\lambda_\infty}(\mathbb{M})$ — the following holds:

$$\left\|\frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{s}\in\mathcal{S}}\left(\frac{1}{4}\phi_{\boldsymbol{s}} + \frac{1}{4}\phi_{\boldsymbol{s}^{-1}}\right)\right\|_{\mathrm{L}^2}^2 \leq \frac{1}{4}\|\phi\|_{\mathrm{L}^2}^2\,.$$

In particular, one has the concentration inequality

$$\mathop{\mathbb{P}}_{\mathcal{S}}\left(\left\|\frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}}\frac{1}{4}\phi_{\boldsymbol{s}}\right\|_{\mathrm{L}^2}\leq\frac{1}{4}\|\phi\|_{\mathrm{L}^2}\ \forall\ \phi\in\mathcal{E}^\star_{\lambda_\infty}(\mathbb{M})\right)\geq 1-\frac{2C_{\mathbb{M}}}{\eta}\exp\left(\frac{-|\mathcal{S}|}{16\ln 2}\right),$$

where we denote $\phi_{\boldsymbol{s}}(\boldsymbol{x})=\phi(\boldsymbol{s}\boldsymbol{x})$ for all $\boldsymbol{x}\in\mathbb{M}$. Now, writing $\mathrm{H}^{\star,\lambda_\infty}_{\boldsymbol{p}}$ for the function $\boldsymbol{x}\mapsto\mathrm{H}^{\lambda_\infty}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)-1_{\mathbb{M}}(\boldsymbol{x})$, one has

$$\mathop{\mathbb{P}}_{\mathcal{S}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}}\mathrm{H}^{\star,\lambda_\infty}_{\boldsymbol{sp}}\right\|_{\mathrm{L}^2}\leq\frac{1}{2}\|\mathrm{H}^{\star,\lambda_\infty}_{\boldsymbol{p}}\|_{\mathrm{L}^2}\right)\geq 1-\frac{2C_{\mathbb{M}}}{\eta}\exp\left(\frac{-|\mathcal{S}|}{16\ln 2}\right).$$

Iterating this inequality $\ell>0$ times fetches

$$\mathop{\mathbb{P}}_{\mathcal{S}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\mathrm{H}^{\star,\lambda_\infty}_{\boldsymbol{sp}}\right\|_{\mathrm{L}^2}\leq\frac{1}{2^\ell}\left\|\mathrm{H}^{\star,\lambda_\infty}_{\boldsymbol{p}}\right\|_{\mathrm{L}^2}\right)\geq 1-\frac{2C_{\mathbb{M}}}{\eta}\exp\left(\frac{-|\mathcal{S}|}{16\ln 2}\right). \tag{40}$$

Let $\mathrm{H}^\star$ denote the function $\boldsymbol{x}\mapsto\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x},\epsilon^2)-1_{\mathbb{M}}(\boldsymbol{x})$; then we have

$$\begin{aligned}\mathrm{H}^\star&=(\mathrm{H}^\star-\mathrm{H}^{\star,\lambda_\infty})+\mathrm{H}^{\star,\lambda_\infty}\\&=(\mathrm{H}-\mathrm{H}^{\lambda_\infty})+\mathrm{H}^{\star,\lambda_\infty},\end{aligned}$$

where $\mathrm{H}^{\lambda_\infty}=1_{\mathbb{M}}(\boldsymbol{x})+\mathrm{H}^{\star,\lambda_\infty}_{\boldsymbol{p}}$ and $\mathrm{H}=1_{\mathbb{M}}(\boldsymbol{x})+\mathrm{H}^\star$. Then, using triangle inequality — along with lemma 3.2 and corollary 2.1 — in inequality eq. (40), one derives

$$\begin{aligned}&\mathop{\mathbb{P}}_{\mathcal{S}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\mathrm{H}^\star_{\boldsymbol{sp}}\right\|_{\mathrm{L}^2}\leq\sqrt{C_{\mathbb{M}}}\left(\eta+2^{-\ell}\epsilon^{-d_{\mathbb{M}}}\right)\right)\\
\geq\ &\mathop{\mathbb{P}}_{\mathcal{S}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\mathrm{H}^\star_{\boldsymbol{sp}}\right\|_{\mathrm{L}^2}\leq 2^{-\ell}\|\mathrm{H}^{\star,\lambda_\infty}\|_{\mathrm{L}^2}+\|\mathrm{H}^{\lambda_\infty}-\mathrm{H}\|_{\mathrm{L}^2}\right)\\
\geq\ &\mathop{\mathbb{P}}_{\mathcal{S}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\mathrm{H}^{\star,\lambda_\infty}_{\boldsymbol{sp}}\right\|_{\mathrm{L}^2}\leq 2^{-\ell}\|\mathrm{H}^{\star,\lambda_\infty}\|_{\mathrm{L}^2}\right)\\
\geq\ &1-\frac{2C_{\mathbb{M}}}{\eta}\exp\left(\frac{-|\mathcal{S}|}{16\ln 2}\right),\end{aligned}$$

which proves the lemma. ∎

Remark 4.1. We note, for use in the next section, that the condition eq. (38) is not used in the proof, except at the very end. In particular, eq. (40) has been derived independently, without using eq. (38).

We now prove that, for any $\boldsymbol{p}\in\mathbb{M}$, and a random subset $\mathcal{S}\subseteq\mathbb{M}$, the orbit $\hat{\mathcal{S}}^\ell\boldsymbol{p}\subseteq\mathbb{M}$ is an $r_{\epsilon,\mathbb{M}}$-cover, provided that the cardinality of the random subset and the integer ℓ are appropriately large.

Theorem 4.1. Let $\mathcal{S} \subset \mathbb{K}$ be a nonempty finite multisubset whose elements are selected independently at random from the Haar measure on $\mathbb{K}$. Let $\delta \in (0, \frac{1}{2})$, and assume that $\epsilon \in (0, 2^{-e})$ is sufficiently small. Suppose that the cardinality of $\mathcal{S}$ satisfies

$$|\mathcal{S}| = 16 \ln 2 \left(\ln C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \frac{d_{\mathbb{M}}}{2} \log_2 \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \ln \frac{1}{e\delta} \right) . \tag{41}$$

Let $\ell > 0$ be an integer satisfying

$$\ell \geq d_{\mathbb{M}} \log_2 \frac{1}{\epsilon} + \log_2 \left(\tfrac{6C_{\mathbb{M}}}{\mathfrak{v}_{\mathbb{M}}}\right) + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{1}{\pi r_{\epsilon,\mathbb{M}}^2}\right) + \frac{1}{2} \log_2 \Gamma \left(\tfrac{d_{\mathbb{M}}}{2} + 1\right) . \tag{42}$$

Then, for any $\boldsymbol{p}_0 \in \mathbb{M}$, the random multisubset $\mathscr{S} := \hat{\mathcal{S}}^{\ell} \boldsymbol{p}_0 \subseteq \mathbb{M}$ is an $r_{\epsilon,\mathbb{M}}$-cover of $\mathbb{M}$ with with probability at least $1 - 2\delta$; here,

$$r_{\epsilon,\mathbb{M}} = 2\epsilon \sqrt{\ln \frac{3C_{\mathbb{M}}}{\epsilon^{2d_{\mathbb{M}}}}} .$$

Proof. Fix $\boldsymbol{p}_0 \in \mathbb{M}$; in order to prove the theorem, it suffices — by lemma 3.3 — to show that

$$\mathop{\mathbb{P}}_{\mathcal{S}} \left(\left\| 1_{\mathbb{M}}(\boldsymbol{x}) - \frac{1}{|\hat{\mathcal{S}}|^{\ell}} \sum_{\boldsymbol{p} \in \hat{\mathcal{S}}^{\ell} \boldsymbol{p}_0} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, \epsilon^2) \right\|_{\mathrm{L}^2} \leq \frac{\alpha_{d_{\mathbb{M}}} r_{\epsilon,\mathbb{M}}^{\frac{d_{\mathbb{M}}}{2}}}{3} \right) \geq 1 - 2\delta ,$$

where $\alpha_{d_{\mathbb{M}}}$ satisfies $\alpha_{d_{\mathbb{M}}}^2 \Gamma\left(\frac{d_{\mathbb{M}}}{2} + 1\right) = \mathfrak{v}_{\mathbb{M}} \pi^{\frac{d_{\mathbb{M}}}{2}}$.

Suppose that the real number $\eta > 0$ and the integer $\ell > 0$ satisfy the following condition:

$$3C_{\mathbb{M}}(\eta + 2^{-\ell} \epsilon^{-d_{\mathbb{M}}}) \leq \alpha_{d_{\mathbb{M}}} r_{\epsilon,\mathbb{M}}^{\frac{d_{\mathbb{M}}}{2}} . \tag{43}$$

Applying the condition in eq. (43) to lemma 4.2, we derive

$$\begin{aligned} &\mathop{\mathbb{P}}_{\mathcal{S}} \left(\left\| \frac{1}{|\hat{\mathcal{S}}|^{\ell}} \sum_{\boldsymbol{s} \in \hat{\mathcal{S}}^{\ell}} \mathrm{H}_{\boldsymbol{p}_0}(\boldsymbol{s}\boldsymbol{x}, \epsilon^2) - 1_{\mathbb{M}}(\boldsymbol{x}) \right\|_{\mathrm{L}^2} \leq \frac{\alpha_{d_{\mathbb{M}}} r_{\epsilon,\mathbb{M}}^{\frac{d_{\mathbb{M}}}{2}}}{3} \right) \\ &\geq 1 - \frac{2C_{\mathbb{M}}}{\eta} \exp\left(\frac{-|\mathcal{S}|}{16 \ln 2} \right) , \end{aligned} \tag{44}$$

provided that

$$2 + 2 \log_2 \frac{1}{\eta} \geq \frac{1}{2} d_{\mathbb{M}} \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + d_{\mathbb{M}} \log_2 \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) . \tag{45}$$

We force an equality in eq. (45), and demand that the inequality $2^{-\ell} \epsilon^{-d_{\mathbb{M}}+} \leq \eta$ be true, so that it suffices to require — instead of eq. (43) — the condition

$$2^{-\ell} \epsilon^{-d_{\mathbb{M}}} \leq \eta \leq \frac{\alpha_{d_{\mathbb{M}}} r_{\epsilon,\mathbb{M}}^{\frac{d_{\mathbb{M}}}{2}}}{6C_{\mathbb{M}}} . \tag{46}$$

This is guaranteed if we set

$$\ell \geq d_{\mathbb{M}} \log_2 \frac{1}{\epsilon} + \log_2 \left(\tfrac{6C_{\mathbb{M}}}{\mathfrak{v}_{\mathbb{M}}}\right) + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{1}{\pi r_{\epsilon,\mathbb{M}}^2}\right) + \frac{1}{2} \log_2 \Gamma \left(\tfrac{d_{\mathbb{M}}}{2} + 1\right) . \tag{47}$$

In fact, since $\epsilon \in (0, 2^{-e})$ is sufficiently small, we have $\epsilon \ll r_{\epsilon,\mathbb{M}}$, and thus, an equality in eq. (45) ensures that

$$\eta \leq \frac{\alpha_{d_{\mathbb{M}}} r_{\epsilon,\mathbb{M}}^{\frac{d_{\mathbb{M}}}{2}}}{6C_{\mathbb{M}}} .$$

Furthermore, the other inequality, $2^{-\ell}\epsilon^{-d_{\mathbb{M}}} \leq \eta$, holds because the reverse inequality, $2^{-\ell}\epsilon^{-d_{\mathbb{M}}} > \eta$, contradicts eq. (47). Finally, set

$$\delta = \frac{C_{\mathbb{M}}}{\eta} \exp \left(\tfrac{-|\mathcal{S}|}{16 \ln 2}\right) , \tag{48}$$

and recall that eq. (45) is an equality, which translates to the condition

$$|\mathcal{S}| = 16 \ln 2 \left(\ln C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \frac{d_{\mathbb{M}}}{2} \log_2 \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \ln \frac{1}{e\delta} \right) . \tag{49}$$

This completes the proof. ■

4.2. Approximate $(\lambda, 2)$-design

Definition 4.1. For any eigenvalue λ of the Laplace-Beltrami $-\Delta_{\mathbb{M}}$ on $\mathbb{M}$, a finite subset $\mathscr{S} \subset \mathbb{M}$ is a λ-design if for every $\phi \in \mathcal{E}_\lambda(\mathbb{M})$, the following equality holds:

$$\int_{\mathbb{M}} \phi(\boldsymbol{x}) \ \sigma^{\mathbb{M}}(d\boldsymbol{x}) = \frac{1}{|\mathscr{S}|} \sum_{\boldsymbol{x} \in \mathscr{S}} \phi(\boldsymbol{x}) \tag{50}$$

Recently, [15] established the existence of λ-design of "small" size on compact Riemannian manifolds. However, their result is not constructive; more generally, efficient algorithmic construction of such designs of optimal size are still far fetched.

Definition 4.2. Let λ be an eigenvalue of the Laplace-Beltrami $-\Delta_{\mathbb{M}}$. An υ-approximate $(\lambda, 2)$-design on $\mathbb{M}$ is a finite subset $\mathscr{S} \subseteq \mathbb{M}$ such that for every element $\phi \in \mathcal{E}_\lambda(\mathbb{M})$ with unit L^2-norm, the following inequality holds:

$$\left| \frac{1}{|\mathscr{S}|} \sum_{\boldsymbol{x} \in \mathscr{S}} \phi(\boldsymbol{x}) - \int \phi \ d\sigma^{\mathbb{M}} \right| \leq \upsilon \tag{51}$$

The main result in this section is a random construction that gives an approximate $(\lambda, 2)$-design with arbitrarily high probability.

Theorem 4.2. Let $\mathscr{S} := \hat{\mathcal{S}}^\ell \boldsymbol{o}$ — where $\mathcal{S} \subseteq \mathbb{M}$ is a random multisubset of isometries selected independently from the Haar measure on $\mathbb{K}$. Let $\delta \in (0, \frac{1}{2})$. For any $\upsilon \in (0, 1)$, and any integer $r > 0$, if

$$|\mathcal{S}| = 16 \ln 2 \ln \left(\tfrac{C_{\mathbb{M}} \lambda_r^{\frac{d_{\mathbb{M}}}{2}}}{\delta} \right) ,$$

and

$$\ell \geq \log_2 \frac{1}{\upsilon} + \log_2 C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4} \log_2 \lambda_r + d_{\mathbb{M}} \log_2 \frac{1}{\epsilon} \,,$$

where

$$\epsilon = \lambda_r^{-\frac{d_{\mathbb{M}}+2}{2}} C_{\mathbb{M}}^{-\frac{1}{4}} \upsilon^{\frac{1}{2}} \,, \tag{52}$$

then $\mathscr{S} \subseteq \mathbb{M}$ is an υ-approximate $(\lambda_r, 2)$-design with probability at least $1 - 2\delta$.

Proof. Let us fix an orthonormal basis for each of the eigenspaces $\mathcal{H}_\lambda$ on $\mathbb{M}$, say $(\phi_{\lambda,j})_{j=1}^{d_\lambda}$. Let $\mathcal{S} \subseteq \mathbb{K}$ be a finite nonempty multi-subset, consisting of independent Haar-distributed elements, and write $\hat{\mathcal{S}} = \mathcal{S} \sqcup \mathcal{S}^{-1}$. We first take ϕ to be an element in this orthogonal basis, corresponding to eigenvalue λ_ϕ of the Casimir-Laplace-Beltrami $-\Delta_{\mathbb{M}}$, and aim to show that — for the subset $\hat{\mathcal{S}}^\ell \boldsymbol{o} \subseteq \mathbb{M}$ — the left hand side of eq. (51) is small. Since

$$\mathrm{H}_{\boldsymbol{o}}(\boldsymbol{x}, \epsilon^2) = \sum_{\lambda \in \mathscr{E}_{\mathbb{M}}} e^{-\lambda \epsilon^2} \sum_{j \in [d_\lambda]} \phi_{\lambda,j}(\boldsymbol{o}) \phi_{\lambda,j}(\boldsymbol{x}) \,,$$

one has

$$\int_{\mathbb{M}} \phi(\boldsymbol{x}) \, \mathrm{H}_{\boldsymbol{o}}(\boldsymbol{x}, \epsilon^2) \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) = e^{-\lambda_\phi \epsilon^2} \phi(\boldsymbol{o}) \int_{\mathbb{M}} |\phi(\boldsymbol{x})|^2 d\sigma^{\mathbb{M}}(\boldsymbol{x}) = e^{-\lambda_\phi \epsilon^2} \phi(\boldsymbol{o}) \,, \tag{53}$$

from which it follows that

$$\begin{aligned}\left| \frac{1}{|\hat{\mathcal{S}}|^\ell} \sum_{\boldsymbol{s} \in \hat{\mathcal{S}}^\ell} \phi(\boldsymbol{s}\boldsymbol{o}) - \int \phi \; d\sigma^{\mathbb{M}} \right| &= \left| e^{\lambda_\phi \epsilon^2} \frac{1}{|\hat{\mathcal{S}}|^\ell} \sum_{\boldsymbol{s} \in \hat{\mathcal{S}}^\ell} \langle \phi, \mathrm{H}_{\boldsymbol{s}\boldsymbol{o}} \rangle - \int \phi \; d\sigma^{\mathbb{M}} \right| \\ &\leq \left\| \sum_{\boldsymbol{s} \in \hat{\mathcal{S}}^\ell} \frac{1}{|\hat{\mathcal{S}}|^\ell} e^{\lambda_\phi \epsilon^2} \, \mathrm{H}_{\boldsymbol{s}\boldsymbol{o}} - 1_{\mathbb{M}} \right\|_{\mathrm{L}^2} ,\end{aligned}$$

where the last step is due to Hölder inequality. We aim to have

$$\left\| |\hat{\mathcal{S}}|^{-\ell} \sum_{\boldsymbol{s} \in \hat{\mathcal{S}}^\ell} e^{\lambda_\phi \epsilon^2} \, \mathrm{H}_{\boldsymbol{s}\boldsymbol{o}} - 1_{\mathbb{M}} \right\|_{\mathrm{L}^2} \leq e^{\lambda_\phi \epsilon^2} \left\| |\hat{\mathcal{S}}|^{-\ell} \sum_{\boldsymbol{s} \in \hat{\mathcal{S}}^\ell} \mathrm{H}_{\boldsymbol{s}\boldsymbol{o}} - 1_{\mathbb{M}} \right\|_{\mathrm{L}^2} + e^{\lambda_\phi \epsilon^2} - 1$$

sufficiently small. Since

$$\lambda_\phi \epsilon^2 \leq e^{\lambda_\phi \epsilon^2} - 1 \leq 2 \lambda_\phi \epsilon^2 \tag{54}$$

if $0 \leq \lambda_\phi \epsilon^2 \leq \ln 2$, it suffices to have $e^{\lambda_\phi \epsilon^2} \leq 2$ and

$$\max \left\{ 2 \left\| |\hat{\mathcal{S}}|^{-\ell} \sum_{\boldsymbol{s} \in \hat{\mathcal{S}}^\ell} \mathrm{H}_{\boldsymbol{s}\boldsymbol{o}} - 1_{\mathbb{M}} \right\|_{\mathrm{L}^2} , \; 2 \lambda_\phi \epsilon^2 \right\}$$

sufficiently small. In the general case, for an element $\phi \in \mathcal{E}_{\lambda_r}$ with unit L^2-norm, say

$$\phi = c_1 \phi_{\lambda_{k_1}, j_1} + \cdots + c_q \phi_{\lambda_{k_q}, j_q} \,,$$

the above derivation takes the form

$$\begin{aligned}
&\left|\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\phi(\boldsymbol{so})-\int\phi\ d\sigma^{\mathbb{M}}\right| \\
&=\left|\sum_{a=1}^{q}c_a\left(e^{\lambda_{k_a}\epsilon^2}\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\langle\phi_{\lambda_{k_a},j_a},\mathrm{H}_{\boldsymbol{so}}\rangle-\int_{\mathbb{M}}\phi_{\lambda_{k_a},j_a}d\sigma^{\mathbb{M}}\right)\right| \\
&\le\sum_{a=1}^{q}|c_a|\cdot\left\|\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\frac{1}{|\hat{\mathcal{S}}|^\ell}e^{\lambda_{k_a}\epsilon^2}\,\mathrm{H}_{\boldsymbol{so}}-1_{\mathbb{M}}\right\|_{\mathrm{L}^2}
\end{aligned}$$

Recall that H^{λ_r} denotes the orthogonal projection of the heat kernel on $\mathcal{E}_{\lambda_r}$ for integer $r\ge 0$. For any $r_0\ge r$, we can replace $\mathrm{H}_{\boldsymbol{p}}$ by the truncated heat kernel $\mathrm{H}_{\boldsymbol{p}}^{\lambda_{r_0}}$ in the above computations, and this gets us the following:

$$\begin{aligned}
&\left|\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\phi(\boldsymbol{so})-\int\phi\ d\sigma^{\mathbb{M}}\right| \\
&\le\sum_{a=1}^{q}|c_a|\cdot\left\|\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\frac{1}{|\hat{\mathcal{S}}|^\ell}e^{\lambda_{k_a}\epsilon^2}\,\mathrm{H}_{\boldsymbol{so}}^{\lambda_{r_0}}-1_{\mathbb{M}}\right\|_{\mathrm{L}^2} \\
&\le\left\|\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\frac{1}{|\hat{\mathcal{S}}|^\ell}e^{\lambda_r\epsilon^2}\,\mathrm{H}_{\boldsymbol{so}}^{\lambda_{r_0}}-1_{\mathbb{M}}\right\|_{\mathrm{L}^2}\cdot\sum_{a=1}^{q}|c_a| \\
&\le\sqrt{\dim\mathcal{E}_{\lambda_r}}\left(e^{\lambda_r\epsilon^2}\cdot\left\|\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\frac{1}{|\hat{\mathcal{S}}|^\ell}\,\mathrm{H}_{\boldsymbol{so}}^{\lambda_{r_0}}-1_{\mathbb{M}}\right\|_{\mathrm{L}^2}+2\lambda_r\epsilon^2\right), \qquad (55)
\end{aligned}$$

where the last step is due to an application of Cauchy-Schwartz inequality and the inequality eq. (54). By eq. (35), we have

$$s\le\dim\mathcal{E}_{\lambda_r}=C_{\mathbb{M}}\lambda_r^{\frac{d_{\mathbb{M}}}{2}}$$

Given $\upsilon\in(0,1)$, suppose that we chose $\epsilon\in(0,2^{-e})$ small and $\ell>0$ large, and also $r_0\ge r$ appropriately, so that

$$\max\left\{\lambda_r\epsilon^2,\left\|\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\frac{1}{|\hat{\mathcal{S}}|^\ell}\,\mathrm{H}_{\boldsymbol{so}}^{\lambda_{r_0}}-1_{\mathbb{M}}\right\|_{\mathrm{L}^2}\right\}\le\frac{\upsilon}{\lambda_r^{\frac{d_{\mathbb{M}}}{4}}C_{\mathbb{M}}^{\frac{1}{2}}}. \qquad (56)$$

From eq. (55), it then follows that $\hat{\mathcal{S}}^\ell\boldsymbol{o}\subseteq\mathbb{M}$ is a 3υ-approximate design. Now, by lemma 2.2, for any $\epsilon\in(0,1)$ and any integer $r_0\ge 0$, we have

$$\left\|\mathrm{H}_{\boldsymbol{so}}^{\star,\lambda_{r_0}}\right\|_{\mathrm{L}^2}\le\sqrt{C_{\mathbb{M}}2^{-d_{\mathbb{M}}}}\epsilon^{-d_{\mathbb{M}}}.$$

And, by inequality eq. (40), we have

$$\mathop{\mathbb{P}}_{\mathcal{S}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^\ell}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^\ell}\mathrm{H}_{\boldsymbol{so}}^{\lambda_{r_0}}-1_{\mathbb{M}}\right\|_{\mathrm{L}^2}\le\frac{1}{2^\ell}\left\|\mathrm{H}_{\boldsymbol{o}}^{\star,\lambda_{r_0}}\right\|_{\mathrm{L}^2}\right)\ge 1-\frac{2C_{\mathbb{M}}}{\eta}\exp\left(\frac{-|\mathcal{S}|}{16\ln 2}\right),$$

with $\eta = \lambda_{r_0}^{-\frac{d_{\mathbb{M}}}{2}}$. From these, it follows that

$$\begin{aligned}
&1 - \frac{2C_{\mathbb{M}}}{\eta}\exp\left(\frac{-|\mathcal{S}|}{16\ln 2}\right) \\
&\leq \underset{\mathcal{S}}{\mathbb{P}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^{\ell}}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^{\ell}} \mathrm{H}^{\lambda_{r_0}}_{\boldsymbol{so}} - 1_{\mathbb{M}}\right\|_{\mathrm{L}^2} \leq \frac{1}{2^{\ell}}\left\|\mathrm{H}^{\star,\lambda_{r_0}}_{\boldsymbol{o}}\right\|_{\mathrm{L}^2}\right) \\
&\leq \underset{\mathcal{S}}{\mathbb{P}}\left(\left\|\frac{1}{|\hat{\mathcal{S}}|^{\ell}}\sum_{\boldsymbol{s}\in\hat{\mathcal{S}}^{\ell}} \mathrm{H}^{\lambda_{r_0}}_{\boldsymbol{so}} - 1_{\mathbb{M}}\right\|_{\mathrm{L}^2} \leq \frac{1}{2^{\ell}}\sqrt{C_{\mathbb{M}}2^{-d_{\mathbb{M}}}}\epsilon^{-d_{\mathbb{M}}}\right)
\end{aligned} \tag{57}$$

From eq. (56), it is imperative that we select $\epsilon \in (0, 2^{-e})$ such that the following holds:

$$\epsilon = \lambda_r^{-\frac{d_{\mathbb{M}}+2}{2}} C_{\mathbb{M}}^{-\frac{1}{4}} \upsilon^{\frac{1}{2}} . \tag{58}$$

Suppose, moreover, that we select integer $\ell > 0$ that satisfies

$$\ell \geq \log_2 \frac{1}{\upsilon} + \log_2 C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4}\log_2 \lambda_r + d_{\mathbb{M}}\log_2 \frac{1}{\epsilon} \tag{59}$$

Then, eq. (56) is guaranteed. Finally, we set $r_0 = r$, and for any $\delta \in (0, \frac{1}{2})$, we set

$$|\mathcal{S}| = 16\ln 2 \ln\left(\frac{C_{\mathbb{M}}\lambda_r^{\frac{d_{\mathbb{M}}}{2}}}{\delta}\right) , \tag{60}$$

and the discussion above shows that — with probability at least $1 - 2\delta$ — the inequality eq. (51) is satisfied if we set $\mathscr{S} := \hat{\mathcal{S}}^{\ell}\boldsymbol{o}$ — where $\mathcal{S} \subseteq \mathbb{M}$ is a random multisubset of isometries, whose elements are choosen independently from Haar measure on $\mathbb{K}$ — and demand that eq. (59) and eq. (60) hold, as desired. ∎

4.3. Bound on Wasserstein distance In this subsection, we assume that we have oracle access to function values at queried points. we start with briefly recalling the following notion of quadratic exposedness of submanifolds of $\mathbb{R}^n$, first appeared in [14].

Definition 4.3. A compact submanifold $M \subseteq \mathbb{R}^n$ is said to be quadratically R_m-exposed at a point $m \in M$ for some $R_m > 0$ if there is a $\boldsymbol{x}_m \in \mathbb{R}^n$ such that $M \subseteq B_{R_m}(\boldsymbol{x}_m, \mathbb{R}^n)$ and $m \in \partial B_{R_m}(\boldsymbol{x}_m, \mathbb{R}^n) \cap M$. If M is quadratically R_m-exposed at each $m \in M$, then we say that M is uniformly quadratically exposed.

Remark 4.2. Any submanifold of euclidean sphere is uniformly quadratically exposed. Torus $(\mathbb{S}^1)^d$ is not uniformly quadratically exposed for $d > 1$.

Remark 4.3. Let $\partial_{\mathbb{M}}(\cdot,\cdot)$ denote the Riemannian distance on $\mathbb{M}$. By Nash' embedding theorem, there is an isometric embedding

$$\chi : (\mathbb{M},\ \partial_{\mathbb{M}}) \to (\mathbb{R}^{d_{\mathbb{M}}^2 + d_{\mathbb{M}}}, \|\ \|_2) .$$

Let $R > 0$ be such that a closed ball $\mathbb{B}_R(\boldsymbol{x}_0)$ of radius $R > 0$ and center $\boldsymbol{x}_0 \in \mathbb{R}^{d_{\mathbb{M}}^2+d_{\mathbb{M}}}$ satisfies $\chi(\mathbb{M}) \subseteq \mathbb{B}_R(\mathbf{0})$. By decreasing R continuously, we can find an R_0 such that $\chi(\mathbb{M}) \subseteq \mathbb{B}_{R_0}(\boldsymbol{x}_0)$ and $\chi(\mathbb{M}) \cap \mathbb{S}_{R_0}(\boldsymbol{x}_0) \neq \emptyset$, where $\mathbb{S}_{R_0}(\boldsymbol{x}_0)$ denotes the sphere of radius R_0 and center at $\boldsymbol{x}_0 \in \mathbb{R}^{d_{\mathbb{M}}^2+d_{\mathbb{M}}}$. Fix a point $\boldsymbol{p} = \chi(\boldsymbol{m}_0) \in \chi(\mathbb{M}) \cap \partial\mathbb{B}_{R_0}(\mathbf{0})$. For any $\boldsymbol{m} \in \mathbb{M}$, let $\sigma^{\mathbb{M}} : [-1, 1] \to \mathbb{M}$ be a geodesic such that $\sigma^{\mathbb{M}}(-1) = \boldsymbol{m}$ and $\sigma^{\mathbb{M}}(1) = \boldsymbol{m}_0$. Let $\zeta : \mathbb{M} \to \mathbb{M}$ be an isometry that satisfies $\zeta(\sigma^{\mathbb{M}}(t)) = \sigma^{\mathbb{M}}(-t)$; then $(\chi \circ \zeta)(\mathbb{M}) = \chi(\mathbb{M}) \subseteq \mathbb{B}_{R_0}(\boldsymbol{x}_0)$, and $\boldsymbol{p} = (\chi \circ \zeta)(\boldsymbol{m})$. Thus, for any point $\boldsymbol{m} \in \mathbb{M}$, there is an isometric embedding

$$\chi_{\boldsymbol{m}} : (\mathbb{M},\ \partial_{\mathbb{M}}) \to (\mathbb{R}^{d_{\mathbb{M}}^2+d_{\mathbb{M}}}, \|\ \|_2)\,,$$

such that $\chi_{\boldsymbol{m}}(\mathbb{M})$ is quadratically exposed at $\chi_{\boldsymbol{m}}(\boldsymbol{m})$.

Lemma 4.3. Let $\sigma^{\mathbb{M}}$ the probability measure on $\mathbb{M}$ induced by Haar measure on $\mathbb{K}$. For any fixed $\boldsymbol{x}_0 \in \mathbb{M}$, and any $t > 0$ be a random variable on $\mathbb{M}$ with density $\mathrm{H}_{\boldsymbol{x}_0}(\boldsymbol{x}, t)$, and let $\boldsymbol{x}_t := \partial_{\mathbb{M}}(\boldsymbol{x}_0, \boldsymbol{X}_t)$. Then

$$\mathbb{E}[\boldsymbol{x}_t^2] \leq d_{\mathbb{M}} t \tag{61}$$

Proof of lemma 4.3. For notational simplicity, we write $\mathrm{H}_t(\boldsymbol{x}) := \mathrm{H}_{\boldsymbol{x}_0}(\boldsymbol{x}, t)$ for the time-t heat kernel on $\mathbb{M}$, and let ν_t^* be the Borel measure corresponding to the top form

$$d\nu_t^* = \mathrm{H}_t(\boldsymbol{x})\ d\sigma^{\mathbb{M}}$$

Let $\{\boldsymbol{x}_u \mid u \in [0, t]\}$ be the Brownian motion in $\mathbb{M}$, starting at $\boldsymbol{x}_0 \in \mathbb{M}$, with infinitesimal generator

$$\mathcal{H}_{\mathbb{M}} := \frac{d}{dt} - \frac{1}{2}\Delta_{\mathbb{M}}\,.$$

For each positive integer $m > 0$, consider the equipartition

$$0 = t_0 < t_1 < \cdots < t_m = t$$

where $t_{j+1} - t_j = m^{-1}t$ for $j \in \{0, 1, \cdots, m-1\}$. Define $\{\boldsymbol{x}_j^{(m)}\}_{j=0}^m$ such that $\boldsymbol{x}_j^{(m)} = \boldsymbol{x}_{\frac{jt}{m}}$. As stated in remark remark 4.3, we fix a closed isometric embedding

$$\chi_{\boldsymbol{x}_{j-1}^{(m)}} : (\mathbb{M},\ \partial_{\mathbb{M}}) \to (\mathbb{R}^{d_{\mathbb{M}}^2+d_{\mathbb{M}}}, \|\ \|_2)\,,$$

such that $\chi_{\boldsymbol{x}_{j-1}^{(m)}}(\mathbb{M}) \subseteq \mathbb{S}_{R_0}(\boldsymbol{a})$, and

$$\boldsymbol{x}_{j-1}^{(m)} \in \chi_{\boldsymbol{x}_{j-1}^{(m)}}(\mathbb{M}) \cap \mathbb{S}_{R_0}(\boldsymbol{a})\,,$$

ensuring quadratic exposedness at $\chi_{\boldsymbol{x}_{j-1}^{(m)}}(\boldsymbol{x}_{j-1}^{(m)})$. Moreover, by applying an affine transformation of $\mathbb{R}^{d_{\mathbb{M}}^2+d_{\mathbb{M}}}$, we may assume that $\boldsymbol{x}_0 = \mathbf{0}$.

We now identify $\mathbb{M}$ with $\chi_{\boldsymbol{x}_{j-1}^{(m)}}(\mathbb{M})$. Note that

$$\begin{aligned}\partial_{\mathbb{M}}(\boldsymbol{x}_t,\boldsymbol{x}_0)^2 &= \|\boldsymbol{x}_t-\boldsymbol{x}_0\|_2^2\\ &= \sum_{j=1}^m \|\boldsymbol{x}_j^{(m)}-\boldsymbol{x}_{j-1}^{(m)}\|_2^2 + 2\sum_{j=1}^m\langle \boldsymbol{x}_j^{(m)}-\boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)}\rangle\\ &= \sum_{j=1}^m \partial_{\mathbb{M}}(\boldsymbol{x}_j^{(m)},\boldsymbol{x}_{j-1}^{(m)})^2 + 2\sum_{j=1}^m\langle \boldsymbol{x}_j^{(m)}-\boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)}\rangle\,, \end{aligned} \tag{62}$$

Suppose that $\boldsymbol{v}_{j-1}^{(m)} \in T_{\boldsymbol{x}_{j-1}^{(m)}}(\mathbb{M})$ is a unit normed tangent vector. Write $\mathbb{S}_{R_0} := \mathbb{S}_{R_0}(\boldsymbol{a})$, and let Π be the orthogonal projection onto $T_{\boldsymbol{x}_{j-1}^{(m)}}(\mathbb{S}_{R_0})$. For any $\epsilon_0 > 0$ sufficiently small, Federer's tangency condition (see [13]) implies that there is $\boldsymbol{p}_{\epsilon_0} \in \mathbb{M}$ such that $\|\boldsymbol{p}-\boldsymbol{x}_{j-1}^{(m)}\|_2 < \epsilon_0$, and

$$\left\|\boldsymbol{v}-\frac{\boldsymbol{p}-\boldsymbol{x}_{j-1}^{(m)}}{\|\boldsymbol{p}-\boldsymbol{x}_{j-1}^{(m)}\|_2}\right\|_2 < \epsilon_0\,.$$

If $\boldsymbol{q}\in\mathbb{S}_{R_0}$ is the point of intersection of the line segment joining $\boldsymbol{p}$ and $\Pi(\boldsymbol{p})$, it is easy to verify that $\|\boldsymbol{q}-\boldsymbol{x}_{j-1}^{(m)}\|_2<\epsilon_0$, and

$$\left\|\boldsymbol{v}-\frac{\boldsymbol{q}-\boldsymbol{x}_{j-1}^{(m)}}{\|\boldsymbol{q}-\boldsymbol{x}_{j-1}^{(m)}\|_2}\right\|_2 < \epsilon_0\,.$$

In particular, $T_{\boldsymbol{x}_{j-1}^{(m)}}(\mathbb{M}) \subseteq T_{\boldsymbol{x}_{j-1}^{(m)}}\mathbb{S}_{R_0}$.

For each $j\in[m]$, let $T_{j-1}^{(m)}$ denote the affine tangent space $T_{\boldsymbol{x}_{j-1}^{(m)}}(\mathbb{S}_{R_0})$ at the point $\boldsymbol{x}_{j-1}^{(m)}$. Write

$$\boldsymbol{z}_{j-1}^{(m)} := \operatorname*{argmin}_{\boldsymbol{u}\in T_{j-1}^{(m)}}\|\boldsymbol{u}\|_2,\qquad \boldsymbol{p}_{j-1}^{(m)} := \operatorname*{argmin}_{\boldsymbol{u}\in T_{j-1}^{(m)}}\|\boldsymbol{u}-\boldsymbol{x}_j^{(m)}\|_2\,. \tag{63}$$

One has the following orthogonality relations:

$$\boldsymbol{p}_{j-1}^{(m)}-\boldsymbol{x}_{j-1}^{(m)} \perp \boldsymbol{z}_{j-1}^{(m)},\qquad \text{and}\qquad \boldsymbol{x}_j^{(m)}-\boldsymbol{p}_{j-1}^{(m)} \perp \boldsymbol{x}_{j-1}^{(m)}-\boldsymbol{z}_{j-1}^{(m)}\,. \tag{64}$$

Claim 4.1.

$$\langle \boldsymbol{x}_j^{(m)}-\boldsymbol{x}_{j-1}^{(m)},\boldsymbol{x}_{j-1}^{(m)}\rangle \le 0 \tag{65}$$

Proof of claim 4.1. Let $\boldsymbol{q}_0^{(m)}\in\mathbb{S}_{R_0}$ be the point where $\mathbb{S}_{R_0}$ intersects the line segment joining $\boldsymbol{x}_0^{(m)}$ and $\boldsymbol{z}_{j-1}^{(m)}$; similarly, let $\boldsymbol{q}_j^{(m)}\in\mathbb{S}_{R_0}$ be the point where $\mathbb{S}_{R_0}$ intersects the line segment joining $\boldsymbol{x}_j^{(m)}$ and $\boldsymbol{p}_{j-1}^{(m)}$. By quadratic exposedness, one has

$$\begin{aligned}\operatorname{sgn}\langle \boldsymbol{x}_j^{(m)}-\boldsymbol{p}_{j-1}^{(m)},\ \boldsymbol{z}_{j-1}^{(m)}\rangle &= \operatorname{sgn}\langle \boldsymbol{x}_j^{(m)}-\boldsymbol{p}_{j-1}^{(m)},\ \boldsymbol{z}_{j-1}^{(m)}\rangle\\ &= \operatorname{sgn}\langle \boldsymbol{q}_j^{(m)}-\boldsymbol{p}_{j-1}^{(m)},\ \boldsymbol{z}_{j-1}^{(m)}-\boldsymbol{q}_0^{(m)}\rangle\end{aligned}$$

Let $\mathbf{n}$ be the unit normal to $T_{j-1}^{(m)}$ pointing inward of $\mathbb{S}_{R_0}$; then the orthogonality relations

$$\boldsymbol{q}_j^{(m)} - \boldsymbol{p}_{j-1}^{(m)} \| \ \boldsymbol{n}, \quad \boldsymbol{z}_{j-1}^{(m)} - \boldsymbol{q}_0^{(m)} \| - \boldsymbol{n}\,,$$

which proves the claim. ■

Claim 4.2.

$$\mathbb{E}\left[\langle \boldsymbol{x}_j^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)}\rangle\right] \leq 0 \tag{66}$$

Proof of claim 4.2. Note that

$$\begin{aligned}
\langle \boldsymbol{x}_j^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)}\rangle &= \langle \boldsymbol{p}_{j-1}^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{z}_{j-1}^{(m)}\rangle + \langle \boldsymbol{x}_j^{(m)} - \boldsymbol{p}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)} - \boldsymbol{z}_{j-1}^{(m)}\rangle \\
&\quad + \langle \boldsymbol{x}_j^{(m)} - \boldsymbol{p}_{j-1}^{(m)},\ \boldsymbol{z}_{j-1}^{(m)}\rangle + \langle \boldsymbol{p}_{j-1}^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)} - \boldsymbol{z}_{j-1}^{(m)}\rangle \\
&= \langle \boldsymbol{x}_j^{(m)} - \boldsymbol{p}_{j-1}^{(m)},\ \boldsymbol{z}_{j-1}^{(m)}\rangle + \langle \boldsymbol{p}_{j-1}^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)} - \boldsymbol{z}_{j-1}^{(m)}\rangle \\
&\leq \langle \boldsymbol{p}_{j-1}^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)} - \boldsymbol{z}_{j-1}^{(m)}\rangle\,,
\end{aligned}$$

where the last two steps follow due to eq. (64) and eq. (65). An application of dominated convergence theorem (due to compactness of $\mathbb{M}$) implies

$$\begin{aligned}
&\lim_{m\to\infty} \mathbb{E}\left[\langle \boldsymbol{p}_{j-1}^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)} - \boldsymbol{z}_{j-1}^{(m)}\rangle\right] \\
&= \mathbb{E}\left[\lim_{m\to\infty} \langle \boldsymbol{p}_{j-1}^{(m)} - \boldsymbol{x}_{j-1}^{(m)},\ \boldsymbol{x}_{j-1}^{(m)} - \boldsymbol{z}_{j-1}^{(m)}\rangle\right] \\
&= 0\,,
\end{aligned} \tag{67}$$

which establishes the claim. ■

Claim 4.3. For $j \in [m]$, let

$$s_j := \mathbb{E}^{\boldsymbol{x}_0}\left[\partial_{\mathbb{M}}\left(\boldsymbol{x}_{t_j}^{(m)}, \boldsymbol{x}_{t_{j-1}}^{(m)}\right)^2\right]\ ; \tag{68}$$

then

$$\lim_{m\to\infty} \sum_{j=1}^{m} s_j \leq \frac{1}{2} d_{\mathbb{M}} t\,. \tag{69}$$

Proof of claim 4.3. Define $u(t, \boldsymbol{x}) := \mathbb{E}^{\boldsymbol{x}}[\partial_{\mathbb{M}}(\boldsymbol{x}, \boldsymbol{x}_t)^2]$; then $u(t, \boldsymbol{x})$ solves [21, theorem 4.2.1] the initial-boundary value problem

$$\mathcal{H}_{\mathbb{M}} u = 0, \quad \lim_{t\to 0^+} u(t, \boldsymbol{x}) = 0 \tag{70}$$

In particular, one has

$$\begin{aligned}
\lim_{t\to 0^+} \frac{u(t, \boldsymbol{x})}{t} &= \frac{d}{dt} u(t, \boldsymbol{x})\bigg|_{t=0^+} \\
&= \frac{1}{2} \Delta_{\mathbb{M}} \partial_{\mathbb{M}}(\boldsymbol{x}, \boldsymbol{x}_t)^2\bigg|_{\boldsymbol{x}=\boldsymbol{x}_t}
\end{aligned}$$

An application of local Taylor expansion of the Riemannian metric on $\mathbb{M}$ (see [16]) implies

$$\Delta_{\mathbb{M}} \operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_t)^2 \Big|_{\boldsymbol{x}=\boldsymbol{x}_t} \le d_{\mathbb{M}} \,. \tag{71}$$

Since

$$\begin{aligned} s_j &= \mathbb{E}^{\boldsymbol{x}_0}\left[\partial_{\mathbb{M}}\left(\boldsymbol{x}_{t_j}^{(m)}, \boldsymbol{x}_{t_{j-1}}^{(m)}\right)^2\right] \\ &= \mathbb{E}^{\boldsymbol{x}_0}\left[\partial_{\mathbb{M}}\left(\boldsymbol{x}_{\frac{t}{m}}, \boldsymbol{x}_0\right)^2\right] \end{aligned}$$

for all $j \in [m]$, we conclude

$$\begin{aligned} \lim_{m\to\infty} \sum_{j=1}^{m} s_j &= t \lim_{m\to\infty} \frac{\mathbb{E}^{\boldsymbol{x}_0}\left[\partial_{\mathbb{M}}\left(\boldsymbol{x}_{\frac{t}{m}}, \boldsymbol{x}_0\right)^2\right]}{\frac{t}{m}} \\ &\le \frac{1}{2} d_{\mathbb{M}} t \,, \end{aligned}$$

finishing the proof of the claim. ■

Finally, to finish the proof of lemma 4.3, notice that

$$\begin{aligned} \mathbb{E}[\boldsymbol{x}_t^2] &= \int_{\mathbb{M}} \partial_{\mathbb{M}}(\boldsymbol{x}_0, \boldsymbol{x})^2 \, \mathrm{H}_t(\boldsymbol{x}) \; \sigma^{\mathbb{M}}(d\boldsymbol{x}) \\ &= \mathbb{E}^{\boldsymbol{x}_0}[\partial_{\mathbb{M}}(\boldsymbol{x}_0, \boldsymbol{x}_t)^2] \,, \end{aligned}$$

where the last expectation is over a Brownian motion $\{\boldsymbol{x}_t; t \ge 0\}$ starting at $\boldsymbol{x}_0$, with infinitesimal generator $\mathcal{H}_{\mathbb{M}}$ as in eq. (15). The lemma now follows by eq. (62), eq. (67), and eq. (69). ■

To proceed further, we will need to apply the following duality theorem by Kantorovič and Rubinšteín, relating the 1-Wasserstein distance to integration of 1-Lipschitz functions; see [23] for more details and a proof.

Theorem 4.3 (Kantorovič - Rubinšteín)**.** Let $(\mathbb{M}, \partial_{\mathbb{M}})$ be a compact connected Riemannian manifold, and $\mathscr{P}(\mathbb{M})$ the space of probability measures on $\mathbb{M}$ — equipped with the 1-Wasserstein distance $W_1(\cdot,\cdot)$. For any $\sigma^{\mathbb{M}}, \nu \in \mathscr{P}(\mathbb{M})$, the following equality holds:

$$W_1(\mu, \nu) = \sup_{\phi \in \mathrm{Lip}_1(\mathbb{M})} \left(\textstyle\int_{\mathbb{M}} \phi \; d\sigma^{\mathbb{M}} - \int_{\mathbb{M}} \phi \; d\nu\right) \tag{72}$$

We are now ready to formulate our result on equidistribution of the cover constructed above.

Theorem 4.4. Let $\mathcal{S} \subset \mathbb{K}$ be a nonempty finite multisubset whose elements are selected independently at random from the Haar measure on $\mathbb{K}$. Let $\delta \in (0, \frac{1}{2})$, and assume that $\epsilon \in (0, 2^{-e})$ is sufficiently small. Suppose that the cardinality of $\mathcal{S}$ satisfies

$$|\mathcal{S}| = 16 \ln 2 \left(\ln C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \frac{d_{\mathbb{M}}}{2} \log_2 \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \ln \frac{1}{e\delta} \right) . \tag{73}$$

Let $\ell > 0$ be an integer satisfying

$$\ell \geq d_{\mathbb{M}} \log_2 \frac{1}{\epsilon} + \log_2 \left(\tfrac{6C_{\mathbb{M}}}{\mathfrak{v}_{\mathbb{M}}}\right) + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{1}{\pi r_{\epsilon,\mathbb{M}}^2}\right) + \frac{1}{2} \log_2 \Gamma \left(\tfrac{d_{\mathbb{M}}}{2} + 1\right) . \tag{74}$$

Fix a $\boldsymbol{p}_0 \in \mathbb{M}$, and let ν be the empirical measure supported on the random multisubset $\mathscr{S} := \hat{\mathcal{S}}^\ell \boldsymbol{p}_0 \subseteq \mathbb{M}$; if ϵ is sufficiently small, then

$$W_1(\sigma^{\mathbb{M}}, \nu) \leq 2\sqrt{d_{\mathbb{M}}\epsilon} .$$

Proof. Let $\mathrm{Lip}_{1,0}(\mathbb{M})$ be the set of mean-zero Lip_1-functions on $\mathbb{M}$. By theorem theorem 4.3, it suffices to show that

$$\sup_{\phi \in \mathrm{Lip}_{1,0}(\mathbb{M})} \left(\int_{\mathbb{M}} \phi \; d\sigma^{\mathbb{M}} - \int_{\mathbb{M}} \phi \; d\nu \right) < 2\sqrt{d_{\mathbb{M}}}\epsilon \tag{75}$$

We note that, for any such function $\phi \in \mathrm{Lip}_{1,0}(\mathbb{M})$, if $\phi(\boldsymbol{x}_0) = \|\phi\|_{L^\infty}$ then

$$\begin{aligned}
0 &= \int_{\mathbb{M}} \phi(\boldsymbol{x}) \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\
&= \int_{\mathbb{M}} \phi(\boldsymbol{x}_0) \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) + \int_{\mathbb{M}} \big(\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_0)\big) \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\
&= \phi(\boldsymbol{x}_0) + \int_{\mathbb{M}} \big(\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_0)\big) \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\
\Rightarrow \qquad \phi(\boldsymbol{x}_0) &\leq \int_{\mathbb{M}} |\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_0)| \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\
&\leq \int_{\mathbb{M}} \partial_{\mathbb{M}}(\boldsymbol{x}, \boldsymbol{x}_0) \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\
&\leq \mathrm{diam}(\mathbb{M})
\end{aligned}$$

For sufficiently small $t > 0$, we let ν_t^* be the Borel probability measure on $\mathbb{M}$ whose density is

$$\frac{d\nu_t^*(\boldsymbol{x})}{d\sigma^{\mathbb{M}}} = \frac{1}{|\hat{\mathcal{S}}^\ell|} \sum_{\boldsymbol{p} \in \hat{\mathcal{S}}^\ell \boldsymbol{p}_0} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)$$

Then, for $t = \epsilon^2$, one has

$$\begin{aligned}
W_1(\sigma^{\mathbb{M}}, \nu_t^*) &= \sup_{\phi \in \mathrm{Lip}_{1,0}(\mathbb{M})} \left| \int_{\mathbb{M}} \phi(\boldsymbol{x}) \; d\sigma^{\mathbb{M}}(\boldsymbol{x}) - \int_{\mathbb{M}} \phi(\boldsymbol{x}) \; d\nu_t^*(\boldsymbol{x}) \right| \\
&\leq \sup_{\phi \in \mathrm{Lip}_{1,0}(\mathbb{M})} \int_{\mathbb{M}} |\phi(\boldsymbol{x})| \cdot \left| 1_{\mathbb{M}} - \tfrac{1}{|\hat{\mathcal{S}}|^\ell} \textstyle\sum_{\boldsymbol{p} \in \hat{\mathcal{S}}^\ell \boldsymbol{p}_0} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t) \right| d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\
&\leq \sup_{\phi \in \mathrm{Lip}_{1,0}(\mathbb{M})} \left\| \phi \right\|_{L^\infty} \cdot \int_{\mathbb{M}} \left| 1_{\mathbb{M}} - \tfrac{1}{|\hat{\mathcal{S}}|^\ell} \textstyle\sum_{\boldsymbol{p} \in \hat{\mathcal{S}}^\ell \boldsymbol{p}_0} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t) \right| d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\
&\leq \mathrm{diam}(\mathbb{M}) \epsilon^{\frac{d_{\mathbb{M}}}{2}} d_{\mathbb{M}}^{-\frac{d_{\mathbb{M}}}{4}}
\end{aligned} \tag{76}$$

with probability at least $1-\delta$; here, the last step is due to the forced equality in eq. (45).

For any function $\phi \in \mathrm{Lip}_{1,0}(\mathbb{M})$, define $\tilde{\phi}_t : \mathbb{M} \to \mathbb{R}$ to be

$$\tilde{\phi}_t(\boldsymbol{x}) = \frac{1}{|\hat{S}^\ell|} \sum_{\boldsymbol{p} \in \hat{S}^\ell \boldsymbol{p}_0} \phi(\boldsymbol{p})\, \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)$$

From stochastic completeness of $\mathbb{M}$, it follows that $\int_{\mathbb{M}} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)\ d\sigma^{\mathbb{M}}(\boldsymbol{x}) = 1$; hence, putting $t = \epsilon^2$, one has

$$\begin{aligned} \int_{\mathbb{M}} \tilde{\phi}_{\epsilon^2}(\boldsymbol{x})\ d\sigma^{\mathbb{M}}(\boldsymbol{x}) &= \frac{1}{|\hat{S}^\ell|} \sum_{\boldsymbol{p} \in \hat{S}^\ell \boldsymbol{p}_0} \phi(\boldsymbol{p}) \int_{\mathbb{M}} \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)\ d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\ &= \int_{\mathbb{M}} \phi(\boldsymbol{x})\ d\nu(\boldsymbol{x})\,, \end{aligned} \tag{77}$$

for any $\phi \in \mathrm{L}^1(\sigma^{\mathbb{M}})$. With $t = \epsilon^2$, this implies

$$\begin{aligned} W_1(\nu^*_{\epsilon^2}, \nu) &= \ \left|\int_{\mathbb{M}} \phi(\boldsymbol{x})\ d\nu^*_t(\boldsymbol{x}) - \int_{\mathbb{M}} \tilde{\phi}_t(\boldsymbol{x})\ d\sigma^{\mathbb{M}}(\boldsymbol{x})\right| \\ &= \ |\hat{\mathcal{S}}|^{-\ell} \left|\sum_{\boldsymbol{p} \in \hat{\mathcal{S}}^\ell \boldsymbol{p}_0} \int_{\mathbb{M}} \big(\phi(\boldsymbol{x}) - \phi(\boldsymbol{p})\big)\, \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)\ d\sigma^{\mathbb{M}}(\boldsymbol{x})\right| \\ &\le \ |\hat{\mathcal{S}}|^{-\ell} \sum_{\boldsymbol{p} \in \hat{\mathcal{S}}^\ell \boldsymbol{p}_0} \int_{\mathbb{M}} \big|\phi(\boldsymbol{x}) - \phi(\boldsymbol{p})\big|\, \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)\ d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\ &\le \ |\hat{\mathcal{S}}|^{-\ell} \sum_{\boldsymbol{p} \in \hat{\mathcal{S}}^\ell \boldsymbol{p}_0} \int_{\mathbb{M}} \partial_{\mathbb{M}}(\boldsymbol{p}, \boldsymbol{x})\ \ \mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)\ d\sigma^{\mathbb{M}}(\boldsymbol{x}) \\ &\le \ \sqrt{d_{\mathbb{M}}}\epsilon \end{aligned} \tag{78}$$

by lemma 4.3 and Hölder inequality applied to $\partial_{\mathbb{M}}(\boldsymbol{p}, \boldsymbol{x}) = \partial_{\mathbb{M}}(\boldsymbol{p}, \boldsymbol{x}) \cdot 1_{\mathbb{M}}$ while integrating with respect to $\mathrm{H}_{\boldsymbol{p}}(\boldsymbol{x}, t)\ d\sigma^{\mathbb{M}}(\boldsymbol{x})$. Therefore, for $t = \epsilon^2 > 0$ sufficiently small, equations eq. (76), eq. (77), and eq. (78) yield

$$\begin{aligned} W_1(\sigma^{\mathbb{M}}, \nu) &\le W_1(\sigma^{\mathbb{M}}, \nu^*_t) + W_1(\nu^*_t, \nu) \\ &\le \epsilon^{\frac{d_{\mathbb{M}}}{2}} d_{\mathbb{M}}^{-\frac{d_{\mathbb{M}}}{4}} + \sqrt{d_{\mathbb{M}}}\epsilon \\ &\le 2\sqrt{d_{\mathbb{M}}\epsilon}\,. \end{aligned}$$

This completes the proof. ■

Remark 4.4. It is immediate from the foregoing proof — in association with remark 3.1 — that the upper-bound in theorem 4.4 improves to $2\sqrt{d_{\mathbb{M}}}\epsilon$ if we ignore the low-dimensional cases $d_{\mathbb{M}} \le 6$.

4.4. Persistent Homology In this subsection, we assume that we have oracle access to pairwise geodesic distance in $\mathbb{M}$.

Definition 4.4. A family

$$\{\chi_x^y : \mathbb{S}_{\boldsymbol{x}} \to \mathbb{S}_y\}_{0 \le x \le y}$$

of morphisms of finite simplicial complexes is said to be persistent if the following holds for all $0 \le x \le y \le z$:

$$\chi_y^z \circ \chi_x^y = \chi_x^z$$

Remark 4.5. In functorial terms, let FSC be the category of finite simplicial complexes and morphisms. A persistent family is then a functor from the poset $(\mathbb{R}, \le)$ to FSC.

Definition 4.5. For a persistent family

$$\{\chi_{\boldsymbol{x}}^y : \mathbb{S}_{\boldsymbol{x}} \to \mathbb{S}_y\}_{0 \le x \le y} \,,$$

the q-th persistent p-homology of $\mathbb{S}_{\boldsymbol{x}}$, denoted $\mathrm{PH}_{p,q}(\mathbb{S}_{\boldsymbol{x}})$, is the image of the natural map

$$(\chi_{\boldsymbol{x}}^{x+q})_\star : \mathrm{H}_p(\mathbb{S}_{\boldsymbol{x}}) \to \mathrm{H}_p(\mathbb{S}_{x+q}) \,.$$

Further, when $\{\chi_{\boldsymbol{x}}^{\boldsymbol{y}} : \mathbb{S}_x \to \mathbb{S}_y\}_{0 \le x \le y}$ is the Vietoris-Rips complex, then the groups

$$\mathrm{PH}_p(\mathbb{M}) := \chi_{\boldsymbol{x}}^{x+q})_\star(\mathrm{H}_p(\mathbb{S}_{\boldsymbol{x}})$$

are called the persistent homology of the complex.

Definition 4.6. Let $\mathcal{S} \subseteq \mathbb{K}$ be a random subset consisting of independent Haar samples from $\mathbb{K}$, and let $\hat{\mathcal{S}} := \mathcal{S} \sqcup \mathcal{S}^{-1}$. For any integer $\ell > 0$, consider the subset $\mathscr{S}_\ell := \hat{\boldsymbol{\mathcal{S}}}^\ell \boldsymbol{p}_0 \subseteq \mathbb{M}$ constructed in the preceding section. Once and for all, fix an ordering of the points in each $\mathscr{S}_\ell$ that is compatible across the range of ℓ. For any $x \ge 0$, let $\mathscr{S}_{\ell,x}$ be the Vietoris-Rips complex with vertex set $\mathscr{S}_\ell$ and radius x. Thus, a t-simplex in $\mathscr{S}_{\ell,x}$ is a subset $\{\boldsymbol{m}_0, \ldots, \boldsymbol{m}_t\} \subseteq \mathscr{S}_\ell$ such that $\partial_{\mathbb{M}}(\boldsymbol{m}_{j_1}, \boldsymbol{m}_{j_2}) < x$ for every $j_1, j_2 \in \{0, 1, \ldots, t\}$. For each $0 \le x \le y$, let $\chi_\ell^{x,y} : \mathscr{S}_{\ell,x} \to \mathscr{S}_{\ell,y}$ be the natural inclusion of simplicial complexes. Thus, the morphisms

$$\chi_\ell^{x,y} : \mathscr{S}_{\ell,x} \to \mathscr{S}_{\ell,y}, \; 0 \le x \le y$$

form a persistent family of finite simplicial complexes.

For compact connected Riemannian symmetric spaces of dimension $d_{\mathbb{M}}$, the following result follows via standard arguments.

Lemma 4.4. Let $\tau > 0$ be the (infimum of the) injectivity radius of $\mathbb{M}$. Suppose that $\mathcal{S} \subset \mathbb{K}$ is a nonempty finite multisubset whose elements are selected independently at random from the Haar measure on $\mathbb{K}$. Let $\delta \in (0, \frac{1}{2})$, and assume that $\epsilon \in (0, 2^{-e})$ is sufficiently small. Suppose that the cardinality of $\mathcal{S}$ satisfies

$$|\mathcal{S}| = 16 \ln 2 \left(\ln C_{\mathbb{M}} + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \frac{d_{\mathbb{M}}}{2} \log_2 \log_2 \left(\tfrac{d_{\mathbb{M}}}{2\epsilon^2}\right) + \ln \frac{1}{e\delta} \right) . \tag{79}$$

Let $\ell > 0$ be an integer satisfying

$$\ell \geq d_{\mathbb{M}} \log_2 \frac{1}{\epsilon} + \log_2 \left(\tfrac{6C_{\mathbb{M}}}{\mathfrak{v}_{\mathbb{M}}}\right) + \frac{d_{\mathbb{M}}}{4} \log_2 \left(\tfrac{1}{\pi r_{\epsilon,\mathbb{M}}^2}\right) + \frac{1}{2} \log_2 \Gamma\left(\tfrac{d_{\mathbb{M}}}{2} + 1\right) , \tag{80}$$

where

$$r_{\epsilon,\mathbb{M}} = 2\epsilon \sqrt{\ln \frac{3C_{\mathbb{M}}}{\epsilon^{2d-1}}}$$

satisfies $r_{\epsilon,\mathbb{M}} < 4^{-1}\tau$; then, with probability at least $1-2\delta$, the geometric realization of $\mathscr{S}_{\ell,r_{\epsilon,\mathbb{M}}}$ — as in definition 4.6 — is homotopy equivalent to $\mathbb{M}$.

Proof. We assume that the random subset $\mathscr{S}_\ell \subseteq \mathbb{M}$ is an $r_{\epsilon,\mathbb{M}}$-cover of $\mathbb{M}$; by theorem 4.1, this assumption is valid with probability at least $1 - 2\delta$. The families

$$\mathscr{O}_{r_{\epsilon,\mathbb{M}}} := \{\mathrm{B}_{r_{\epsilon,\mathbb{M}}}(\boldsymbol{m}) : \boldsymbol{m} \in \mathscr{S}_\ell\}, \quad \mathscr{O}_{2r_{\epsilon,\mathbb{M}}} := \{\mathrm{B}_{2r_{\epsilon,\mathbb{M}}}(\boldsymbol{m}) : \boldsymbol{m} \in \mathscr{S}_\ell\}$$

of open geodesic balls — of radius $r_{\epsilon,\mathbb{M}} \in (0, \frac{\tau}{4})$ and $2r_{\epsilon,\mathbb{M}} \in (0, \frac{\tau}{2})$, respectively — with centers at the points in $\mathscr{S}_\ell$ forms an open cover of $\mathbb{M}$; moreover, the guarantee that $r_{\epsilon,\mathbb{M}} \in (0, \frac{\tau}{4})$ ensures that each geodesic ball in these covers is diffeomorphic — via the exponential map — to an open Euclidean ball, and finite intersections of such geodesic balls are diffeomorphic to star-shaped open subsets in Euclidean space. Therefore, by nerve lemma, the Ćech nerves $\mathscr{C}_{r_{\epsilon,\mathbb{M}}}$ and $\mathscr{C}_{2r_{\epsilon,\mathbb{M}}}$ — based on $\mathscr{O}_{r_{\epsilon,\mathbb{M}}}$ and $\mathscr{O}_{2r_{\epsilon,\mathbb{M}}}$, respectively — have the same homotopy type as $\mathbb{M}$. Since the Vietoris-Rips nerve $\mathscr{S}_{r_{\epsilon,\mathbb{M}}}$ interlaces the Ćech nerves as

$$\mathscr{C}_{r_{\epsilon,\mathbb{M}}} \subseteq \mathscr{S}_{\ell,r_{\epsilon,\mathbb{M}}} \subseteq \mathscr{C}_{2r_{\epsilon,\mathbb{M}}} ,$$

the lemma follows. ∎

Remark 4.6. Via application of the Vietoris-Rips complex and geometric realization functor, the topological invariants of the abstract Riemannian manifold $\mathbb{M}$ are identified to those of a subspace of an Euclidean space (albeit of a humongous dimension).

Recall (see [4]) that, for a compact metric measure space (X, d_X, μ_X) with metric d_X and measure μ_X, one defines

$$\Phi^{q,n}_{X,d_X,\mu_X} := (\mathrm{PH}_q)_\star(\mu_X^{\otimes n}) ,$$

which is a measure in the *Barcode* space $\mathcal{B}$. Let $\mathbb{M}$ be a compact connected Riemannian symmetric space of dimension d, Riemannian metric $\partial_{\mathbb{M}}$, and Haar-induced $\mathbb{K}$-invariant probability measure $\sigma^{\mathbb{M}}$. Let $\mathscr{S}_\ell := \hat{\mathcal{S}}^\ell \boldsymbol{p}_0$, and $\sigma_\ell^{\mathbb{M}}$ the empirical probability measure supported on $\mathscr{S}_\ell$.

Theorem 4.5. The following inequality holds for all nonempty subset $\mathcal{S} \subseteq \mathbb{K}$ of isometries of $\mathbb{M}$, all integers $\ell \geq 1$ and $q \geq 0$, with $n = |\hat{\mathcal{S}}|^\ell$:

$$\frac{1}{n} \cdot d_{\mathrm{Pr}} \left(\Phi^{q,n}_{\mathbb{M},\partial_{\mathbb{M}},\sigma^{\mathbb{M}}}, \Phi^{q,n}_{\mathscr{S}_\ell,\partial_{\mathbb{M}},\sigma_\ell^{\mathbb{M}}}\right) \leq \mathrm{W}_1^{\frac{1}{2}}(\sigma^{\mathbb{M}}, \sigma_\ell^{\mathbb{M}}) \tag{81}$$

Proof. We follow the arguments as in [4, theorem 5.2]. Since Wasserstein distance induces compact topology on $\mathscr{P}(\mathbb{M}\times\mathbb{M})$ — the space of probability measures on $\mathbb{M}\times\mathbb{M}$, there is a coupling $\lambda\in\Pi(\sigma^{\mathbb{M}},\sigma^{\mathbb{M}}_{\ell})$ such that

$$W_1(\sigma^{\mathbb{M}},\sigma^{\mathbb{M}}_{\ell}) = \mathop{\mathbb{E}}_{\lambda}\left[\partial_{\mathbb{M}}(\boldsymbol{m}_1,\boldsymbol{m}_2)\right]$$

Suppose that $W_1(\sigma^{\mathbb{M}},\sigma^{\mathbb{M}}_{\ell}) = \epsilon^2_{q,n,\ell}$ for some $\epsilon_{q,n,\ell}>0$; it follows that

$$\mathbb{P}_{\lambda}\left[\partial_{\mathbb{M}}(\boldsymbol{m}_1,\boldsymbol{m}_2)\geq\epsilon_{q,n,\ell}\right]\leq\epsilon_{q,n,\ell} \tag{82}$$

Since the n-fold product $\lambda^{\otimes n}$ induces a probability measure on $\mathbb{M}^n\times\mathscr{S}^n_{\ell}$, the push-forward $\hat{\lambda}_n := (\mathrm{PH}_q^{\otimes n},\mathrm{PH}_q^{\otimes n})_{\star}(\lambda^{\otimes n})$ is a coupling between the push-forward measures $(\mathrm{PH}_q^{\otimes n})_{\star}(\sigma^{\mathbb{M}})$ and $(\mathrm{PH}_q^{\otimes n})_{\star}(\sigma^{\mathbb{M}}_{\ell})$. Given n independent λ-samples

$$(\boldsymbol{m}_1,\boldsymbol{s}_1),\cdots,(\boldsymbol{m}_n,\boldsymbol{s}_n),$$

an application of triangle inequality implies

$$|\partial_{\mathbb{M}}(\boldsymbol{m}_j,\boldsymbol{m}_{j'})-\partial_{\mathbb{M}}(\boldsymbol{s}_j,\boldsymbol{s}_{j'})|\leq\partial_{\mathbb{M}}(\boldsymbol{m}_j,\boldsymbol{s}_j)+\partial_{\mathbb{M}}(\boldsymbol{m}_{j'},\boldsymbol{s}_{j'})$$

and this further yields

$$\begin{aligned}\mathbb{P}\left[\sup_{j,j'\in[n]}\left(\partial_{\mathbb{M}}(\boldsymbol{m}_j,\boldsymbol{s}_j)+\partial_{\mathbb{M}}(\boldsymbol{m}_{j'},\boldsymbol{s}_{j'})\right)\geq 2\epsilon_{q,n,\ell}\right] &\leq \mathbb{P}\left[\sup_{j\in[n]}\partial_{\mathbb{M}}(\boldsymbol{m}_j,\boldsymbol{s}_j)\geq\epsilon_{q,n,\ell}\right]\\ &\leq 1-(1-\epsilon_{q,n,\ell})^n\\ &\leq n\epsilon_{q,n,\ell}\end{aligned}$$

By *stability theorem* of Chazal et al (see [8]), we have

$$\mathbb{P}\left[d_{\mathcal{B}}\left(\mathrm{PH}_q(\{\boldsymbol{m}_j\}),\mathrm{PH}_q(\{\boldsymbol{s}_j\})\right)\geq n\epsilon_{q,n,\ell}\right]\leq n\epsilon_{q,n,\ell}\,,$$

which concludes the proof. ■

Together with theorem 4.4, this yields the following immediate corollary:

Corollary 4.1. Suppose that the nonempty subset $\mathcal{S}\subseteq\mathbb{K}$ is as in lemma 4.4, and that the integer $\ell\geq 0$ satisfies the inequality in lemma 4.4; then, the following inequality holds for all integers $q\geq 0$, with probability at least $1-\delta$:

$$\frac{1}{n}\cdot d_{\mathrm{Pr}}\left(\Phi^{q,n}_{\mathbb{M},\partial_{\mathbb{M}},\sigma^{\mathbb{M}}},\Phi^{q,n}_{\mathbb{S}_{k,\ell},\partial_{\mathbb{M}},\sigma^{\mathbb{M}}_{\ell}}\right)\leq\sqrt[4]{2d_{\mathbb{M}}\epsilon}\,. \tag{83}$$

Remark 4.7. It is immediate from the foregoing proof — in association with remark 4.4 — that the upper-bound in corollary 4.1 improves to $\sqrt[4]{2d_{\mathbb{M}}\epsilon^2}$ if we ignore the low-dimensional cases $d_{\mathbb{M}}\leq 6$.

5. Conclusions

On a compact Riemannian symmetric space $\mathbb{M} = \mathbb{K}/\mathbb{H}$ of dimension $d_{\mathbb{M}}$, a random Markov Chain — whose transitions correspond to isometries selected independently at random from the Haar measure of the group of isometries of $\mathbb{M}$ — has been explored and the finite time behaviour of the chain has been established. Explicitly, the first of the four main results states that for $|\hat{\mathcal{S}}| = O_{\mathbb{M}}(d_{\mathbb{M}} \ln \frac{1}{\epsilon} + \ln \frac{1}{\delta})$ random isometries and $\ell = O(d_{\mathbb{M}} \ln 1/\epsilon)$, if one takes the image $\mathscr{S}$ of any fixed (but arbitrary) point on the symmetric space under all possible composition of length ℓ in the $|\hat{\mathcal{S}}|$-length random inverse-symmetric subset $\mathcal{S} \subseteq \mathbb{K}$, one obtains an $r_{\epsilon,\mathbb{M}}$-cover with high probability. For these parameters, the value of $|\hat{\mathcal{S}}|^{\ell}$ is close to the volumetric lower bound of $(1/\epsilon)^{\Omega(d_{\mathbb{M}})}$ on the size of an hypothetical optimum ϵ-cover of $\mathbb{M}$. Secondly, we show that this ϵ-cover is equidistributed with probability at least $1-2\delta$, in the sense that the 1-Wasserstein distance of the uniform empirical measure supported on the cover is within $\sqrt{d_{\mathbb{M}}\epsilon}$ of the uniform measure on $\mathbb{M}$. Next, we show that such random subset $\mathscr{S} \subseteq \mathbb{M}$ is an approximate λ_r-design, for eigenvalue λ_r of the Laplace-Beltrami on $\mathbb{M}$, provided $|\hat{\mathcal{S}}| = O(\ln 1/\delta + d_{\mathbb{M}} \ln \lambda_{r_{\epsilon,\mathbb{M}}})$, and ℓ as above. Finally, we show that the Prokhorov distance between the persistent push-forwards of the uniform measure and the empirical measure supported on the random ϵ-cover is at most $\sqrt[4]{d_{\mathbb{M}}\epsilon}$.

These results can respectively be applied to (1) approximately minimize any given 1- Lipschitz function on the symmetric space via a locally constant function by considering the Voronoi-cells of the space induced by the ϵ-cover and evaluating the function on the ϵ-cover, (2) to approximately integrate a 1-Lipschitz function on the symmetric space, (3) to approximately integrate a smooth function in the subspace $\mathcal{E}_{\lambda_r}$, and (4) to approximate the persistence homology of a data cloud sampled from the uniform measure, with that of the constructed ϵ-cover. In the first two cases the approximation is within an additive ϵ of the true value, whereas the third case has an error of order $\sqrt{\epsilon}$.

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