seminar on Differential Geometry　　　　Date.

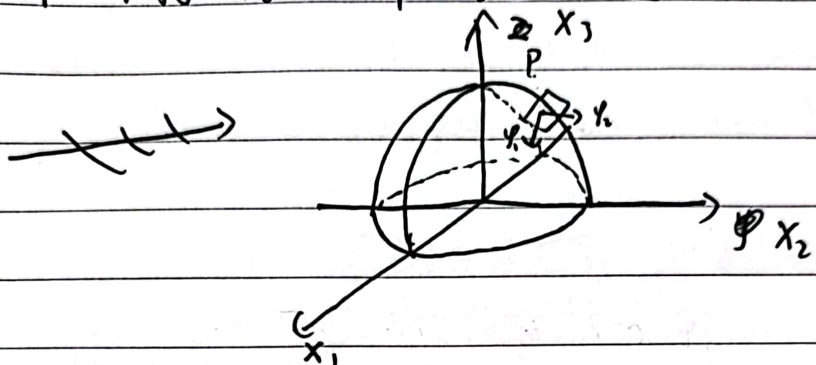
Tangant space: we have a ~~som~~ ~~smooth~~

m-dimensional smooth manifold. (we have define)

$P\in M$. consider a chart (U,φ) such $P\in U$.

我们现在尝试研究Manifold在P附近的几何特征

我们现在已知的几何是，在P点附近Manifold局部同胚于一m-dimensional 欧式空间，我们不妨将Manifold设为一些非常具体的例子 比如一个二维~~柱~~球面



φ_1 φ_2 为局部坐标(同胚)

$$S_2 : \{ x_1^2+x_2^2+x_3^2=1 \mid x_1,x_2,x_3\in\mathbb{R} \}$$

obviously S_2 is 2-dimensional smooth manifold

在数学分析中我们常研究过P点 S_2 上的一维曲线

(~~同胚~~ homeomorphic to 1-dimensional space)

the directional derivative, the ~~function~~ contin function on S_2, the differential of the function ---

一般分析老师将其称之为局域线性，这也是"连续"的一般几何特征

这启发我们是否可以在smooth manifold 构造出类似的东西

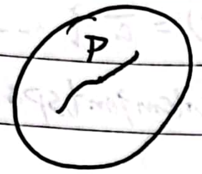
当然经典现代微分几何肯定不会局限到上面的讨论中

例如　我们令　g_{ik}　$ds^2 = g_{ik}dx^i dx^k$

当中的几何 ---

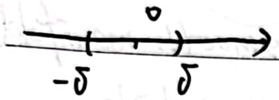
Curves: obviously, we should make curves on smooth manifold homeomorphic to 1-dimensional space.

we shall define



$\gamma : (-\delta, \delta) \to M$

$\gamma(0) = p$, and we make map γ is a C^∞ map which means



$\frac{d^n}{dt^n}(\varphi_U \circ \gamma)$ exist and cont. for every positive number n, $n \in \mathbb{N}$

and we define Γ_p is the set of all curves which go through p. Then we can get a very ~~beaut~~ beautiful picture

我们注意到在左图 这些经过P的曲线构成了我们非常熟悉的切平面

(确切的说 $\frac{d}{dt}(\varphi_U \circ \gamma)$，构成切平面 切线!)

稍加观察 我们可以发现 切平面的维数和"流形" manifold的相同，这并非是巧合。实际上大家可以将其认为 切空间的几何直观。In general, this is the most general way to introduce a tangent space. we can define a equivalence relation in Γ_p.

if ~~$\frac{d}{dt}(\varphi_U \circ \gamma)_p$~~ $\gamma_1 \sim \gamma_2$ then we have

$$\frac{d}{dt}(\varphi_U \circ \gamma_1)_p = \frac{d}{dt}(\varphi_U \circ \gamma_2)_p$$

~~det~~ denote the equivalence class $[\gamma]$. because

$\frac{d}{dt}(\varphi \circ \gamma)$ is a element of m-dimensional Euclidean space

we have the operation of vector addition. the we can define addition in $\Gamma_p / \sim = T_p$

$$[\gamma_1] + [\gamma_2] = \frac{d}{dt}(\varphi_U \circ \gamma_1) + \frac{d}{dt}(\varphi_U \circ \gamma_2) = \frac{d}{dt}(\varphi_0 (\gamma_1 + \gamma_2))$$
$$= [\gamma_1 + \gamma_2]$$

$\alpha\xi\ [c\gamma_1] = c[\gamma_1]$, Then $\Gamma_p / \sim = T_p$ is a vector space

, the we can define T_p is the tangent space of manifold at p.

Date. this is the most general way to define a tangent space

当然 大家肯定可以发现 我们现在的这个定义还有很多非常不严谨的地方, such as, given any vector $\vec{a}$ in m-dimensional vectorspace, Does there exist a curve in Γ_p such that $\frac{d}{dt}(\varphi\circ\gamma) = \vec{a}$? ...

我们现在不做验证, 因为我们今天准备先引入余切空间, (cotangent space) and tagent space as a dual space of cotangent space.

Cotangent space: suppose M is an m-dimensional smooth manifold. It's a very natural ideal that consider a function on M which $f: M \to \mathbb{R}$. and we shall let f must be "smooth enough". But how do we define "smooth enough"? we know there exist (U, φ) $p \in U$ φ smooth (C^∞), ~~we~~ Then we let $(f\circ\varphi^{-1})|_p$ is C^∞.

and denote the set of all these function C_p^∞, Naturally, the domains of two different functions in C_p^∞ may be different, but we can define $f\cdot g$, $f+g$ their domain is $U\cap V$, obviously $f+g$, $f\cdot g \in C_p^\infty$. Now consider a relation on C_p^∞ $\sim$.

suppose $f, g \in C_p^\infty$ $f \sim g$ if and only if there exist a open neighborhood ~~of~~ H of p such $f|_H = g|_H$

ob, $\sim$ is a equivalence relation in C_p^∞, denote the equivalence class of f by $[f]$, called a C^∞- germ. ~~at~~ p. Let

$$\mathcal{F}_p = C_p^\infty/\sim = \{[f] \mid f \in C_p^\infty\}$$

because we define ~~and~~ can define addition and scalar.. on C_p^∞, $[f]+[g] = [f+g]$ $a[f] = [af]$

ob, $\mathcal{F}_p$ is an infinite-dimensional real linear space

suppose γ is a curve in M through a point p

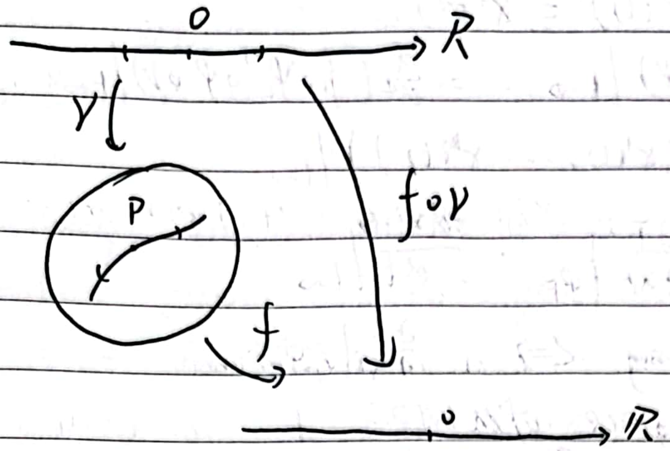
$\gamma: (-\delta, \delta) \to M$ $\gamma(0) = p$ γ, C^∞ map, for $\gamma \in \Gamma_p$ $[f] \in \mathcal{F}_p$

let

$$\langle\langle \gamma, [f] \rangle\rangle = \frac{d}{dt}(f\circ\gamma)\Big|_{t=0} \qquad -\delta < t < \delta$$

and we can draw a picture to

Date.



Obviously, for a fixed γ, the value on the right hand side is linear. for arbitrary $\gamma \in \Gamma_p$, $[f], [g] \in \mathcal{F}_p$, $a \in \mathbb{R}$, we have

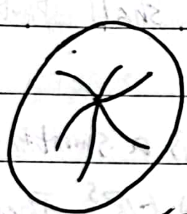
$$\langle\langle \gamma, [f]+[g] \rangle\rangle = \langle\langle \gamma, [f] \rangle\rangle + \langle\langle \gamma, [g] \rangle\rangle$$

$$\langle\langle \gamma, a[f] \rangle\rangle = a \langle\langle \gamma, [f] \rangle\rangle$$

now let the set

$$\mathcal{H}_p = \{ [f] \in \mathcal{F}_p \mid \langle\langle \gamma, [f] \rangle\rangle = 0 \text{ for } \forall \gamma \in \Gamma_p \}.$$

ob. $\mathcal{H}_p$ is a linear subspace of $\mathcal{F}_p$.



the equivalence relation we have define, ~~first~~ could not tell us the geometry (differential) manifold have. because the definition just cotain open neighborhood. so we must do some work to add some geometry.

Consider $\mathcal{H}_p$. if $[f] \notin \mathcal{H}_p$, then we have γ

$$\langle\langle \gamma, [f] \rangle\rangle = \frac{d}{dt}(f \circ \gamma)\Big|_{t=0} \neq 0,$$

means that there must exist $\gamma \in$ such $\langle\langle \gamma, [f] \rangle\rangle \neq 0$, ⇒ there must exist nonzero "directional derivative". so.

consider $\mathcal{F}_p / \mathcal{H}_p$, any $[f] \in \mathcal{F}_p / \mathcal{H}_p$ means that obviously we choose the function which have the " "

suppose $[f] \in \mathcal{F}_p$. For an admissible coordinate chart (U, φ).

Let $F(x^1 \cdots x^m) = f \circ \varphi^{-1}(x^1 \cdots x^m)$

Then. $[f] \in \mathcal{H}_p$, if and only if

$$\frac{\partial F}{\partial x^i}\Big|_{\varphi(p)} = 0 \quad \text{for} \quad 1 \leq i \leq m.$$

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Proof: suppose $\gamma \in I_p$, with coordinate

$$(\varphi_u \circ \gamma(t))^i = x^i(t)$$

$$\langle\langle \gamma, [f] \rangle\rangle = \frac{d}{dt}(f \circ \gamma)\Big|_{t=0} = \frac{d}{dt}(f \circ \varphi_u^{-1} \circ \varphi \circ \gamma)\Big|_{t=0}$$

$$= \frac{d}{dt}(F(x^1(t), \cdots x^m(t)))\Big|_{t=0}$$

$$= \sum_i \frac{\partial F}{\partial x^i}\Big|_{\varphi(p)} \cdot \frac{dx^i(t)}{dt}\Big|_{t=0}$$

since γ is arbitrary $\Leftrightarrow$ $\frac{\partial F}{\partial x^i}\Big|_{\varphi(p)} = 0$

Definition of cotangent space of M at p:

Define: the qutient space $\mathcal{F}_p / \mathcal{H}_p$ is called cotangent space of M at p, denoted by T_p^* (or $T_p^*(M)$). The $\mathcal{H}_p$-equivalence class of the function germ $[f]$ is denoted by $\widetilde{[f]}$ or $(df)_p$ and is called cotangent vector on M at p.

ob, T^* is a linear space, we have

$$\widetilde{[f]} + \widetilde{[g]} = \widetilde{[f]+[g]}$$

$$a \cdot \widetilde{[f]} = (\widetilde{a[f]})$$

Before we analyses the cotangent space T_p^*, we shall prove a lemma first:

lemma: suppose $f^1 \cdots f^s \in C_p^\infty$ and $F(y^1 \cdots y^s)$ is a smooth function in a neighborhood of $(f^1(p), f^2(p) \cdots f^s(p)) \in \mathbb{R}^s$, Then let $f = F(f^1 \cdots f^s) \in C_p^\infty$ and

$$(df)_p = \sum_{k=1}^{s}\left[\left(\frac{\partial F}{\partial f^k}(f^1(p) \cdots f^s(p))\right)(df^k)_p\right]$$

proof: since F is a smooth function, $f \in C_p^\infty$ let

$$a_k = \frac{\partial F}{\partial f^k}(f^1(p) \cdots f^s(p))$$

Then for any $\gamma \in I_p$

$$\langle\langle \gamma, [f] \rangle\rangle = \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

$$= \frac{d}{dt}(F(f^1 \circ \gamma(t), f^2 \circ \gamma(t) \cdots f^s \circ \gamma(t)))\Big|_{t=0}$$

$$= \sum_{k=1}^{n} a_k \frac{d}{dt}(f^k \circ \gamma)\Big|_{t=0} = \langle\langle \gamma, \sum_{k=1}^{s} a_k [f^k] \rangle\rangle$$

Thus $[f]-\sum_{1}^{k} a_k [f^k] \in \mathcal{H}_p$. So we have

$$(df)_p = \sum_{1}^{s} a_k (df^k)_p. \qquad \square$$

let us consider the

~~obv~~

obviously, the lemma is a bridge between admissible cooridnate chart and cotangent. the lemma will be useful when proof $\dim T_p^* = m$.

now consider any $f, g \in C_p^\infty$, $a \in \mathbb{R}$, we have $d(f+g)_p = (df)_p + (dg)_p$

$$d(af)_p = a(df)_p$$

$$d(fg)_p = f(p)\cdot (dg)_p + g(p)\cdot (df)_p \quad \text{(lemma)}$$

now we try to prove $\dim T_p^* = m$

choose an admissible coordinate chart (U, φ), and define

$u^i \qquad u^i(q) = \varphi^i(q) = x^i \circ \varphi(q) \qquad q \in U$

obviously, x^i ~~is give~~ is a given coordinate system.

ok, $u^i \in C_p^\infty$ $(du^i)_p \in T_p^*$. now we try to prove $(du^i)_p$ is a basis for T_p^*.

suppose $(df)_p \in T_p^*$, $f\circ \varphi_U^{-1} = F$, Thus

$$F(u^1 \cdots u^m) = f$$

By ~~Theorem~~ the lemma we have ~~be~~ proved.

$$(df)_p = \sum \frac{\partial F}{\partial u^i}(u^1(p) \cdots u^m(p))\,(du^i)_p$$

Thus $(df)_p$ is a linear combination of the $(du^i)_p$, now we try to prove du^i are independent. If there real numbers a_i such that

$$\sum_{1}^{m} a_i (du^i)_p = 0$$

$$\Rightarrow \sum_{1}^{m} a_i [u^i] \in \mathcal{H}_p$$

Then for any $\gamma \in \Gamma_p$, we have

$$\langle\langle \gamma, \sum_{1}^{m} a_i [u^i] \rangle\rangle = \sum a_i \frac{d}{dt}(u^i \circ \gamma)\Big|_{t=0} = 0$$

choose $\lambda_k \in \Gamma_p$ $1 \le k \le m$ such that

$$u^i \circ \lambda_k(t) = u^i(p) + \delta^i_k t$$

Date. Then: ~~γ_k~~ $\frac{d}{dt}(u^i \circ \lambda_k(t))\Big|_{t=0} = \delta^i_k$

$\Rightarrow a_k = 0 \quad 1 \le k \le m \Rightarrow \{(du^i)_p\}$ is linearly independent.

Therefore it forms a basis for T_p^*; called natural basis of T_p^*.

so $\dim T_p^* = m$

By definition $[f]-[g] \in \mathcal{H}_p$ if and only if $\langle\langle \gamma, [f] \rangle\rangle = \langle\langle \gamma, [g] \rangle\rangle$.

for any $\gamma \in \Gamma_p$

now we define a relation $\sim$ in Γ_p as follows.

Suppose $\gamma, \gamma' \in \Gamma_p$. Then $\gamma \sim \gamma'$ if and only if for any $(df)_p$

$$\langle\langle \gamma, (df)_p \rangle\rangle = \langle\langle \gamma', (df)_p \rangle\rangle$$

we will prove that the $[\gamma]$, $\gamma \in \Gamma_p$, form the dual space of T_p^*. For this purpose, we will use local coordinate

Under u^i $\quad \gamma \in \Gamma_p \Rightarrow u^i(t)$

$$\langle [\gamma], (df)_p \rangle = \sum^m a_i \xi^i$$

$$a_i = \left(\frac{\partial (f \circ \varphi^{-1})}{\partial u^i} \right)_{\varphi(p)}$$

$$\xi^i = \frac{du^i}{dt}\Big|_{t=0}$$

obviously, we can find that the coefficients a_i are exactly the components of the cotangent vector $(df)_p$ under the natural basis $(du^i)_p$, so it's a function on T_p^* (linear)

obviously is determined by the component ξ^i

so choose curves such that

$$u^i(t) = u^i(p) + \xi^i t$$

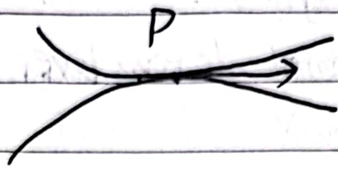
Thus $\langle [\gamma], (df)_p \rangle$ $\gamma \in \Gamma_p$ is a functionals on T_p^*

and form its dual space, T_p called the tangent space of M at p. Elements in the T_p called tangent vectors.

The geometric meaning of ~~the~~ tangent vectors is quite simple if $\gamma' \in \Gamma_p$

$$[\gamma] = [\gamma']$$

$\Rightarrow \quad \frac{du^i}{dt}\Big|_{t=0} = \frac{du'^i}{dt}\Big|_{t=0}$



Hence the equivalence of γ and γ' means these curves have the same tangent vectors at P,

denote $X=[\gamma]\in T_p$, $(df)_p \in T_p^*$

$\langle X,(df)_p\rangle$ is a bilinear function.

suppose some curves λ_k $1\le k\le m$, are given in

$$\langle[\lambda_k],(du^i)_p\rangle=\delta^i_k$$

~~There fore~~ Therefore $\{[\lambda_k]\}$ is the dual basis of $\{(du^i)_p\}$

now consider $\langle[\lambda_k],(df)_p\rangle = \langle[\lambda_k],\sum\left(\frac{\partial f}{\partial u^i}\right)_p\cdot(du^i)_p\rangle$

$= \left(\frac{\partial f}{\partial u^k}\right)_p$

Thus the $[\lambda_k]$ are the partial differential operators $(\partial/\partial u^k)$ on the function germs $[f]$, then we have

$\langle \frac{\partial}{\partial u^k}\Big|_p,(du^i)_p\rangle=\delta^i_k$ ~~beau~~ beautiful

and obviously we have

$$[\gamma]=\sum_1^m \xi^i \frac{\partial}{\partial u^i}\Big|_p$$

If $[\gamma']\in T_p$ has the components ξ'^i $\Rightarrow [\gamma]+[\gamma'] \Rightarrow \xi^i+\xi'^i$

说了这么多我们似乎还没有解释切空间向量的含义.

Date.
Tangent map: suppose $X \in T_p$, $f, g \in C_p^\infty$, $\alpha, \beta \in \mathbb{R}$, denote

$$Xf = \langle X, df|_p \rangle$$

Xf is called the directional derivative of the function f along the vector X

(1) $X(\alpha f + \beta g) = \alpha \cdot Xf + \beta \cdot Xg$

(2) $X(fg) = f(p) \cdot Xg + g(p) \cdot Xf$

Remark: suppose $X = [\gamma] \in T_p$ $\quad a = df \in T_p^*$

then under the natural bases:

$$X = \sum_{i}^{m} \xi^i \frac{\partial}{\partial u^i} \qquad a = a_i du^i$$

where $\quad \xi^i = \frac{d(u^i \circ \gamma)}{dt} \qquad a_i = \frac{\partial f}{\partial u^i}$

under another local coordinate u'^i, then we have

$$X = \sum \xi'^i \frac{\partial}{\partial u'^i} \qquad a = \sum_{i}^{m} a_i' \, du'^i$$

$$\xi'^j = \frac{d}{dt}(u'^j) = \frac{d}{dt}\left((\varphi' \circ \varphi^{-1})^j u^i \right)$$

$$= \sum_{i=1}^{m} \frac{\partial (\varphi' \circ \varphi^{-1})^j}{\partial u^i} \xi^i$$

$$\Rightarrow \quad \xi'^j = \sum_{i=1}^{m} \frac{\partial (\varphi' \circ \varphi^{-1})^j}{\partial u^i} \xi^i = \sum_{i=1}^{m} \xi^i \frac{\partial u'^j}{\partial u^i}$$

at the same time, we can get

$$a_i = \sum_{j=1}^{m} a_j' \frac{\partial u'^j}{\partial u^i}$$

where $\quad \frac{\partial u'^j}{\partial u^i} = \frac{\partial (\varphi' \circ \varphi^{-1})^j}{\partial u^i}$

~~is the~~ In classical tensor analysis, the vector satisfy ~~(…)~~ ξ^i: are called contravariant. And those satisfying a_i are called covariant vectors.

In general, smooth maps between smooth manifolds induce linear maps between tangent spaces and ~~co~~between cotangent space

Suppose. $F: M \to N$ is a smooth map, $p \in M$ $q = F(p)$

Date.

now define the map F^* $T_q^* \to T_p^*$

F^* $\quad F^*(df) = d(f\circ F) \quad df \in T_q^*$

obviously, this is a linear map, called differential of the map F

consider F_* $T_p \to T_q$ definded as (~~ajd~~ adjoint)

$$\langle F_* X, a\rangle = \langle X, F^* a\rangle \qquad \langle Av, u\rangle = \langle v, A^* u\rangle$$

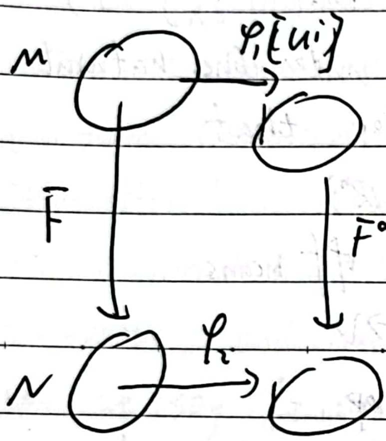
F_* is called the tangent map induced by F^*.

suppose u^i and v^α are local coordinates near p and q.

obviously, the map F can be expressed near p by the functions.

$$v^\alpha = F^\alpha(u^1 \cdots u^m)$$

we may draw a picture to help us understant



Thus the action of F^* on the natural basis $\{dv^\alpha \ 1\le\alpha\le n\}$

$$F^*(dv^\alpha) = d(v^\alpha\circ F) = \sum_1^n \left(\frac{\partial F^\alpha}{\partial u^i}\right)_p du^i$$

Similarly, the action of F_* on the natural basis $\{\partial/\partial u^i\}$

$$\langle F_* \frac{\partial}{\partial u^i}, dv^\alpha\rangle = \langle \frac{\partial}{\partial u^i}, F^*(dv^\alpha)\rangle$$

$$= \sum_{j=1}^n \langle \frac{\partial}{\partial u^i}, du^j\rangle \left(\frac{\partial F^\alpha}{\partial u^j}\right)_p$$

$$= \langle \sum_{\beta=1}^n \left(\frac{\partial F^\beta}{\partial u^i}\right)_p \frac{\partial}{\partial v^\beta}, dv^\alpha\rangle$$

then

$$F_*\left(\frac{\partial}{\partial u^i}\right) = \sum_1^n \left(\frac{\partial F^\beta}{\partial u^i}\right)_p \frac{\partial}{\partial v^\beta}$$

~~fw y. group~~

Date: Submanifolds:

suppose W is an open subset of $\mathbb{R}^n$ and $f: W \to \mathbb{R}^n$ is a smooth map, If at a point $x_0 \in M$ the determinant of the Jacobian matrix is nonzero

$$\det\left(\frac{\partial f^i}{\partial x^j}\right) \neq 0$$

then there exist a neighborhood $U \subset W$ of x_0 in $\mathbb{R}^n$ such that $V = f(U)$ is a neighborhood of the point $f(x_0)$ in $\mathbb{R}^n$ such

$$g = f^{-1} : V \to U$$

~~by~~ and why we ~~remember~~ mention the inverse function theorem in calculus. Because Dose everyone notice that $\frac{\partial f^i}{\partial x^j}$ is precisely the matrix representation of f_* under the natural basis. ~~Therefore~~ Therefore $\left(\frac{\partial f^i}{\partial x^j}\right)\Big|_{x_0} \neq 0$ implies that

$$f_* : T_{x_0}(W) (\cong \mathbb{R}^n) \to T_{f(x_0)}(\mathbb{R}^n) (\cong \mathbb{R}^n)$$

is an isomorphism. That g is the inverse of f means

$$g \circ f = id \quad U \to U \qquad f \circ g = id \quad V \to V$$

Since f and g are both smooth map, so the restriction of f to U provides a diffeomorphism from U to V. The inverse function theorem says that if the tangent map f_* of f is an isomorphism at a point the f is a diffeomorphism from a neighborhood of that point to an open set in $\mathbb{R}^n$

Suppose M and N are both n-dimensional smooth manifolds. and $f: M \to N$ is a smooth map. If at a point $p \in M$, the tangent map $f_*: T_p(M) \to T_{f(p)}(N)$ is an isomorphism, then there exists a neighborhood U of p in M such that $V = f(U)$ is a neighborhood of $f(p)$ in N. and $f|_U : U \to V$ is a diffeomorphism.

Proof: Since $M \to N$ is a smooth map, we can chose local coordinates (U_0, φ) at $p \in M$ and (V_0, ψ) at $q = f(p) \in N$ such $f(U_0) \subset V_0$ and $\tilde{f} = \psi \circ f \circ \varphi^{-1}$: $\varphi(U_0) \to \psi(V_0) \subset \mathbb{R}^n$

obviously is a smooth map. Obviously the determinant of the Jacobian of $\tilde{f}$ at point $\varphi(p)$ is nonzero, so there exist neighborhood $\tilde{U} \subset \varphi(U_0)$ and $\tilde{V} \subset \psi(V_0)$ of $\varphi(p)$ and $\psi(q)$ in $\mathbb{R}^n$ such that

$\tilde{f}|_{\tilde{U}} : \tilde{U} \to \tilde{V}$ is a diffeomorphim

let $U = \varphi^{-1}(\tilde{U})$ $V = \psi^{-1}(\tilde{V})$ Then U and V are neighborhoods of p and q in M and N, and

$$f = \psi^{-1} \circ \tilde{f} \circ \varphi : U \to V$$

is a diffeomorphism $\square$

Remark: Since the manifolds M and N in the theorem have the same dimension. the condition "the ~~th~~ tangent map f_* is an isomorphism" is equivalent to "f_* is injective". If M is an m-dimensional and N an n-dimensional manifold, $f: M \to N$ is smooth, and the tangent map f_* is injective at a point p. then we say f_* is nondegenerate at p. Obviously, in this case $m \le n$, and the rank of Jacobian matrix of f at p is m

Suppose M is an m-dimensional and N an n-dimensional manifold $m \le n$. If $f: M \to N$ is a smooth map and the tangent mat f_* is nondegenerate at a point $p \in M$, then there exist local coordinate systems (U, u^i) near p and (V, v^α) near $q = f(p)$ s.t. $f(U) \subset V$ and the map $f|_U$ can be expressed as by local co..

$$\begin{cases} v^i(f(x)) = u^i(x) \\ v^r(f(x)) = 0 \quad m+1 \le r \le n \end{cases}$$

Date.

Definition:

suppose M and N are smooth manifolds, If there is a smooth map $\varphi: M \to N$ such that

(1) φ is injective

and at any point $p \in M$ the f_* is nondegenerate

$\Rightarrow (\varphi, M)$ called a smooth sub manifold or imbedded submanifold of N

dual space: suppose V is a linear vector space the exist a linear function $V \xrightarrow{f} \mathbb{R}$

suppose V is a m-dimensional linear vector space then we can chose a basis $\{e_i\}_{i=1}^{n}$

& for any v in V we have

$v = \sum_{i=1}^{m} a_i e_i$ obviously we have

$f(v) = \sum_{i=1}^{m} a_i f(e_i)$

now we define $f^{*i}(e_j) = \delta^i_j$ $\delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

obviously we have $f^{*i}(v) = a_i$

$f(v) = \sum a_i f(e_i) = \sum f(e_i) f^{*i}(v)$

$\Rightarrow f = \sum_{i=1}^{n} f(e_i) f^{*i}$

define V^* is the space f, obviously for any f can be expressed by the linear com. of f^{*i} so f^{*i} is a basis of V^*

and $\dim V^* = m$ $\{f^{*i}\}$ is called the dual basis.

V^* and V are ~~dual~~ dual space of each other Define

$\langle v, v^* \rangle = v^*(v)$. $v \in V$ $v^* \in V^*$

obviously
$$\begin{cases} \langle \alpha_1 v_1 + \alpha_2 v_2 , v^* \rangle = \alpha_1 \langle v_1 , v^* \rangle + \alpha_2 \langle v_2 , v^* \rangle \\ \langle v, \alpha_1 v^{*1} + \alpha_2 v^{*2} \rangle = \alpha_1 \langle v, v^{*1} \rangle + \alpha_2 \langle v, v^{*2} \rangle \end{cases}$$

If we fix a vector $v \in V$ the we have

$\langle v, \_ \rangle$ is a linear function on V^*

suppose φ is an F-valued linear function on V^*. Simply

$v = \sum_{i=1}^{m} \varphi(a^{*i}) a_i \Rightarrow$ for any v^*

$$\langle v, v^* \rangle = \sum_{i=1}^{n} \varphi(a^{*i}) \langle a_i , v^* \rangle$$

$$= \sum v^*(a_i) \varphi(a^{*i})$$

$$= \varphi\left(\sum v^*(a_i) \cdot a^{*i}\right)$$

$$= \varphi(v^*)$$

In other words ~~V~~ V is the dual space of V^*

Tensor ~~pro~~ product:

the motivation: now consider V and W ~~and~~ and Z are linear space function $V \times W \xrightarrow{f} Z$ bilinear.

Now our problem is to transform a bilinear map on $V \times W$ into a linear map. More precisely, given two vector space V and W, we will construct a vector space Y and a bilinear h $V \times W \to Y$ such that h depend on V and W only and satisfy the following for any bilin.. f $V \times W \to Z$ there exist $g : Y \to Z$ such

$$f = g \circ h$$

we ~~c~~ can draw a picture to dis..

$$\begin{tikzcd} V \times W \arrow[r, "h\,(bi)"] \arrow[dr, "f\,(bi)"'] & Y \arrow[d, "g\,(li)"] \\ & Z \end{tikzcd}$$

generally: the space Y we constructed is called tensor product of V and W

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