basic knowledge of topology

我们先来回忆"Euclidean space"中有关开集、闭集、连续函数的定义以及内点(interior point) ---

首先我们知道在 Euclidean space 中有"距离"的概念，

即 $\vec{x}, \vec{y} \in \mathbb{R}^m$ "d"

$$d(\vec{x},\vec{y}) = d(\vec{y},\vec{x}) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots(x_m-y_m)^2} \geq 0$$

我们有"open ball"

$$B(\vec{x}, r): \{\vec{y} \mid \vec{y} \in \mathbb{R}^m \ d(\vec{y},\vec{x})<r\}$$

在多元微积分中我们这样来定义开集，we have set $A \subset \mathbb{R}^m$ for any $\vec{x} \in A$, there exist $r>0$ s.t

$B(\vec{x},\vec{r}) \subset A$, the we call A is an open set of Euclidean space。

这是多元微积分中定义的"开集"，对于"闭集"我们有 ---

开集的补，$\Rightarrow$ if B is a close set, then ~~B must~~ there exist an open set A such that

$$B = \mathbb{R}^m - A$$

在多元微积分中，我们要证明欧式空间下的开集具有如下性质

for any index set I, if A_λ, $\lambda \in I$ are open sets

$$B = \bigcup_{\lambda \in I} A_\lambda \quad B \text{ is also an open set.}$$

$$B = \bigcap_{i=1}^{n} A_i \quad n \in \mathbb{N} \Rightarrow B \text{ is also an open set.}$$

而闭集刚好相反。(这里不再证明)

以上两条性质是欧式空间中"开集"的基本性质。

而在多元微积分中又对连续函数有定义，最一般的定义是

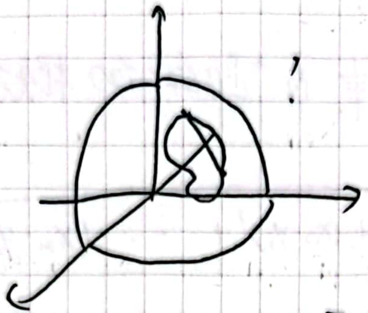
$f: D \to \mathbb{R}^m$ $D \subset \mathbb{R}^n$ if f is continuous at $\vec{p} \in \mathbb{R}^n$

for any $B_\varepsilon(\vec{p}) \subset \mathbb{R}^m$, there exist a $B_\delta(\vec{a}) \subset \mathbb{R}^n$,

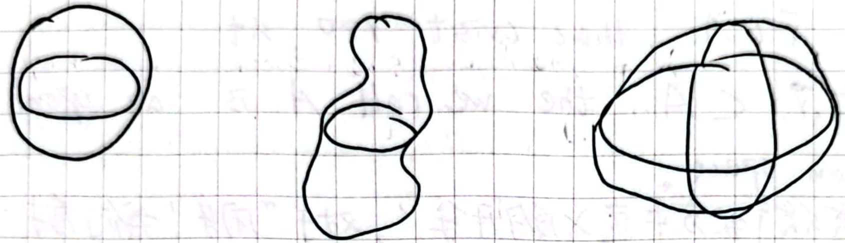
such that $f(B_\delta(\vec{a})) \subset B_\varepsilon(\vec{p})$

但我们显然的发现，有关"开闭集"以及连续函数的定义全部基于 $d(\vec{x},\vec{y}) = \sqrt{(\ )^2+(\ )^2 \cdots (\ )^2}$

但这个定义是非常不"general"，例如当我们的研究对象不在局限于平面....
例如 一个"球面"上的几何 时，d(x,y)的定义便不那么的好用.



但而且这些"几何"体都具有一些 明显不依赖于 d(x,y)的性质
例如



等都可以将区分出"内""外"
但像"莫比乌斯环"，"克莱因瓶"都是不可以的
所以我们~~需要~~一个更加"general"的定义"开""闭""连续"的方式. 现在我们可以进入拓扑学。(topology)

Topology: 我们现在有一个 set A.
now we define a topology τ on A is a collection of subsets of A, such that

(1) $A, \phi \in \tau$

(2) for any ~~$A_i, A_j \in \tau$~~ indexset I
$A_\lambda,\ \lambda \in I \quad A_\lambda \in \tau$
$\bigcup_\lambda A_\lambda \in \tau$

(3) for any $n \in \mathbb{N} \quad A_i \in \tau$
$\bigcap_{i=1}^{n} A_i \in \tau$

then we call τ is a topology on A
and if $A_\lambda \in \tau$, the we define A_λ is an open set of A

某种意义上我们可以理解为 τ 是所有开集的集合 (but.)

我们发现在这种定义下，开集只保留了无限并是开和有限交是开这两条性质 (第一条只是为保证定义的完备性)

而这样定义"开"集是非常简洁的 (定义涉及 $\cap$ $\cup$ ..)

闭集仍定义为开集的补集

if B is a close set of A, then there exist an open set A_λ of A

such that $B = A - A_\lambda$

同时我们对于一般的集合我们有"内点" $A \subset X$ $X \to$ top space.

$a \in A$ a is an interior point of A, if there exist an open neighborhood B of A such $a \in B$ ~~B~~. then we call a an interior point of A.

obviously if a subset A of X, such that for any $a \in A$ a is interior of A, then A must be an open set of X.

现在同理我们引入定义 "limit point"

the set of limit point of A, we A'

$\bar{A} = A \cup A'$ if $A = \bar{A} \Rightarrow$ A is close set.

我们现在引入 "basic" 概念， ~~now give a top space X~~ a set X,

a collation $\mathcal{B}$ of subsets of X, $\mathcal{B}$ called a basis for top space X

if:

(1) if given any $x \in X$, there exist $B \in \mathcal{B}$ such that $x \in B$

(2) given any $V, W \in \mathcal{B}$ $x \in V \cap W$, there exist a T such that $T \in \mathcal{B}$ $T \subset V \cap W$ $x \in T$

given a basis $\mathcal{B}$, the topology generated by the basis $\mathcal{B}$

Def A subset U of X is said to be open in X

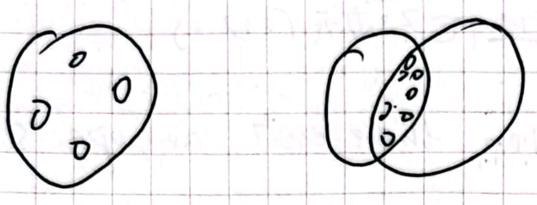
for each $x \in U$, there exist $B \in \mathcal{B}$ such that $x \in B$

$B \subset U$ $\to$ $U = \bigcup_{\lambda} B_\lambda$

how we can define the Euclidean space.

$\mathcal{B} = \{ B(\vec{x}, r) \mid r > 0 \ \vec{x} \in \mathbb{R}^n \}$

the $\mathcal{B}$ as above generate the Euclidean space.



$B(\vec{x}, \vec{r}) = \{ \vec{y} \mid d(\vec{x}, \vec{y}) < r \}$

$d(\vec{x}, \vec{y}) =$ ~ ~

(metric topology)

我们现在讲明连续映射的概念便可以进入 smooth Manifolds.

given to top space X, Y (τ_X, τ_Y)

suppose exist a map $f: X \to Y$

~~if~~ $x \in X$, if given any open neighborhood V of $f(x) \in Y$

there exist an open neighborhood U of X such that

$f(U) \subset V$, then we call f is continuous at X

if for every $x \in X$ f is conti at X

then we say f is continuous

then following are equiv.

① f is continuous

② if V any open set in Y, $\Rightarrow f^{-1}(V)$ is an open set in X

③ if V ~~a~~ any ~~open~~ close set in $Y \Rightarrow f^{-1}(V)$ is a close set in X

(这里由于时间关系不进行证明)

def: if there exist $f: X \to Y$, f is continuous, if f is a bijective function, the we call f is a homomorphism

if f^{-1} also ~~a~~ continuous function

the we call X is homeomorphic to Y (同胚)

example

(1) $X_1 = E^2 \setminus \{0\}$

(2) $X_2 = \{ (x, y, z) \in E^3 \mid x^2 + y^2 = 1 \}$

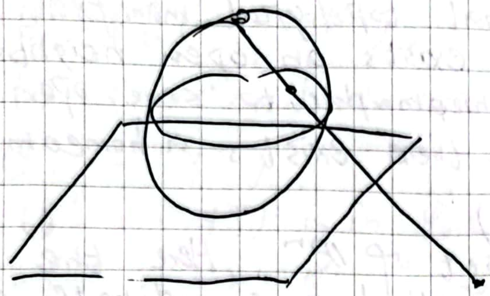
(3) $X_3 = \{ (x, y, z) \in E^3 \mid x^2 + y^2 - z^2 = 1 \}$

we can prove these top space are homeomorphic to each other; now we can prove (1) (2) ...

$$f: X_1 \to X_2$$

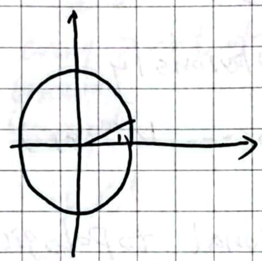
$$(x,y) \to \left\{ \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \ln(x^2+y^2) \right\}$$

E2.



3.

$[0,1) \longrightarrow$



?

Continuous

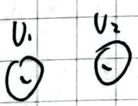
but not homeomorphic

$$\lambda \longrightarrow (\cos 2\pi\lambda, \sin 2\pi\lambda)$$

Smooth Manifolds:

Hausdoff topological space:

A topological space X is called Hausdoff space if for each pair x_1, x_2 of distinct points of X, there exist open neighborhood U_1 and U_2 of x_1 and x_2, respectively, that are disjoint



$$U_1 \cap U_2 = \phi$$

Every finite point set in a Hausdoff space X is closed.

we can easily prove this theorem

Definition:

Let M be a Hausdoff space. Let m be a positive intege. If for every ~~x~~ x in M, there exists an open neighborhood U of x such that U is homeomorphic to some open subset of Euclidean space $\mathbb{R}^m$ then we say that M is an m-dimensional topological manifold.

Let M be an m-dimensional topological manifold. Let x be an element of M. So there exists an open neighborhood U of x such that U is homeomorphic to some open set of Euclidean space $\mathbb{R}^m$. Hence, there exists a homeomorphism

$$\varphi_U : U \to \varphi_U(U)$$

s.t $\varphi_U(U)$ is an open set of $\mathbb{R}^m$. Here the ordered pair (U, φ_U) is called a coordinate chart of M. (U, φ_U) is also simply denote by (U, φ)

for every x in U, obviously

$$\varphi_U(x) = (u^1(x), u^2(x) \cdots u^m(x))$$

Let M be an m-dimensional topological manifold. (U, φ_U) (V, φ_V) be coordinate ~~charts~~ charts of M such that $U \cap V \neq \phi$, Then

1. $\mathrm{dom}(\varphi_V \circ \varphi_U^{-1}) = \varphi_U(U \cap V)$
2. $\mathrm{ran}(\varphi_V \circ (\varphi_U)^{-1}) = \varphi_V(U \cap V)$
3. $\varphi_V \circ \varphi_U^{-1}$ function
4. 1-1
5. $\varphi_U(U \cap V)$ $\varphi_V(U \cap V)$ are open sets of $\mathbb{R}^m$
6. $(\quad)^{-1} = \varphi_U \circ \varphi_V^{-1}$
7. homemorphisms.

we can easily prove all of these theorem

then we can define ~~smooth~~ the concepts of smooth M.

Let M be an m-dimensional topological manifold.
Let $(U, \varphi_U), (V, \varphi_V)$ be coordinate charts of M

$U \cap V \neq \phi$

$\varphi_V \circ \varphi_U^{-1} : \varphi_U(U\cap V) \to \varphi_V(U \cap V)$

is a homeomorphism from ...

For ever $(x^1 \cdots x^m)$ in $\varphi_U(U\cap V)$

put $\varphi_V \circ \varphi_U^{-1}(x^1 \cdots x^m)$

$= (f^1(x^1 \cdots x^m), f^2(x^1 \cdots x^m) \cdots f^m(x^1 \cdots x^m))$

$= (y^1, \cdots y^m)$

where $(y^1, \cdots y^m)$ is in $\varphi_V(U \cap V)$ similarly,

$\varphi_U \circ \varphi_V^{-1} \to g^i \to y \to x$

For fixed i and for every $j = 1, \cdots m$

$D_j f^i$ exists and continuous at every point of $\varphi_U(U\cap V)$

then we say that f^i is C^1

Similarly by g^i is C^1

then

By (U, φ_U) and (V, φ_V) are C^1-compatible
we mean that either $U \cap V = \phi$ or (f^i, g^i is C^1 for every i)

if $D_K(D_j f^i) \equiv D_{Kj} f^i$ exists and continuous at every point of $\varphi_U(U\cap V)$, we say f^i is C^2 similarly by g^i is C^2.

By (U, φ_U) (V, φ_V) are C^2-compatible, we mean that ---

similarly, we can define C^n-compatible. r-

Def: Let M be an m-dimensional topological manifold.
Let r be a positive integer let

$$A = \{(U, \varphi_U), (V, \varphi_V), (W, \varphi_W) \cdots\}$$

1. $\{U, V, W \cdots\}$ is a cover of M. $\cup\{U : (U, \varphi_U) \text{ is in } A\} = M$
2. all pairs of members in A are C^r compatible
3. A is maximal (in the sense that if $(\tilde{U}, \varphi_{\tilde{U}})$ is a coordinat chart of M but not a member of A, then there exist (U, φ_U) in A, such that $(\tilde{U}, \varphi_{\tilde{U}})$ (U, φ_U) are not C^r-compatible)

then we say that A is a C^r-differentiable structure on M and the order pair (M, A) is called a C^r-differentiable manifold.
Here, members of A are called admissible coordinate charts of M

if $r = \infty$ (in the sense that every pair of members in $\mathcal{A}$ is C^r-compatible for every positive integer r)
then we say that $\mathcal{A}$ is a C^∞-differentiable structure on M, and the pair $(M, \mathcal{A})$ C^∞-differentiable manifold.
C^∞-differentiable structure is also called smooth structure. C^∞-differentiable manifold is also called smooth manifold.
and we can proof that $\mathcal{A}$ is Unique.
if $\mathcal{B}_1, \mathcal{B}_2$ are two C^∞-differentiable structures on M

when there exists a coordinate chart (U, φ_U) of M
s.t $(U, \varphi_U) \in \mathcal{B}_1$ but $\notin \mathcal{B}_2$

let $\mathcal{B}$ be the collection of all coordinate chats (U, φ_U) of M such that (U, φ_U) is C^∞-compatible with every member of A.
$\mathcal{B}$ cotains A, obvimsly
now we try to prove $\mathcal{B}$ is a C^∞-differentiable structure on M

1. $\cup \mathcal{B} = M$ ob

2. all pairs of members of $\mathcal{B}$ are C^∞-compatible

3. $\mathcal{B}$ is maximal and unique

2.: take any (U, φ_U) in $\mathcal{B}$ and (V, φ_V)

Examples of smooth Manifolds.

1. For $m = 1, 2, 3 \cdots$ by the m-dimensional unit sphere S^m, we mean the set

$$\left\{ (x^1, \cdots, x^{m+1}) : \sqrt{x^{1^2} + x^{2^2} \cdots + x^{m+1^2}} = 1 \right\}$$

Since $\mathbb{R}^{m+1}$ is a Hausdorff second contable top space, S^m is a subset of $\mathbb{R}^{m+1}$ $\Rightarrow$ S^m with induced topology is a Hausdorff se...

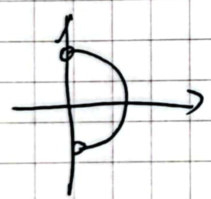
$$S^1 = \left\{ (x^1, x^2), \sqrt{x^{1^2} + x^{2^2}} = 1 \right\}$$

Now we try to prove S^1 is a 1-dimensional top space.

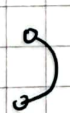
take any (a^1, a^2) in S^1 we have to find an open neighborhood G of (a^1, a^2) and $\varphi\ G \to \varphi(G)$ s.t. $\varphi(G)$ is open in $\mathbb{R}^1$

1: $a^1 \neq 0$ $a^1 > 0$

let us take $\{(x^1, x^2), (x^1, x^2) \in S^1 \text{ and } x^1 > 0\}$ for G_1 $(a^1, a^2) \in G_1$ ok G_1 is open

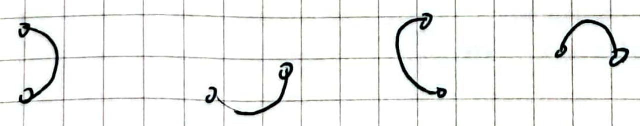


defind $P_1 : G_1 \to \{x^2; (x_1, x_2) \in G_1\}$

 $\to$ $(-1, 1)$

ok, bijective conti... for ~~base~~ both 对于 P_1, P_1^{-1}

Then



$\{ (G_1, P_1)\ (G_2, P_2)\ (G_3, P_3)\ (G_4, P_4) \} = A$.

1. $\cup A = S^1$ ok
2. all pairs of members in A are C^∞-compatible

now how he pr

(G_1, P_1) (G_2, P_2)

$P_1 \circ P_2^{-1}$: $(0,1) \to (-1,0)$

$\Rightarrow$ $t \to -\sqrt{1-t^2}$

ob. is C^∞- compatible

Hence, by the Theorem, there exists a unique C^∞- differentiable structure $\mathcal{B}$ on S^1 which cotain A.

Thus, the ordered pair $(S^1, \mathcal{B})$ is an example of a smooth manifold of dimension 1

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