线性代数同步练习册参考解答

苏州大学数学科学学院　　陈凤娟　朱广俊　潘洪亮

2017年10月8日

[1]本文档利用科学工作平台Scientific WorkPlce V5.5配置的PDFLaTeX进行排版，并且利用MuPAD V3.1进行有关的运算。

1.1-1.2 线性方程组的初等变换与高斯消元法

2.1 矩阵的定义　　2.2 矩阵的计算(1)

一、用消元法解下列线性方程：

1. $\begin{cases} 2x-y=0 \\ -x+2y=3 \end{cases}$

解：$\begin{cases} 2x-y=0 \\ -x+2y=3 \end{cases}$, Solution is: $[x=1, y=2]$.

$$\begin{cases} 2x-y=0 \\ -x+2y=3 \end{cases} \xrightarrow{r_1 \leftrightarrow r_2} \begin{cases} -x+2y=3 \\ 2x-y=0 \end{cases} \xrightarrow{-r_1} \begin{cases} x-2y=-3 \\ 2x-y=0 \end{cases}$$

$$\xrightarrow{-2r_1+r_2} \begin{cases} x-2y=-3 \\ 3y=6 \end{cases} \xrightarrow{\frac{1}{3}r_2} \begin{cases} x-2y=-3 \\ y=2 \end{cases} \xrightarrow{2r_2+r_1} \begin{cases} x=1 \\ y=2 \end{cases}$$

2. $\begin{cases} x_1+2x_2-x_3=0 \\ x_2-2x_3=-1 \\ x_3=1 \end{cases}$

解：$\begin{cases} x_1+2x_2-x_3=0 \\ x_2-2x_3=-1 \\ x_3=1 \end{cases}$, Solution is: $[x_1=-1, x_2=1, x_3=1]$.

$$\begin{cases} x_1+2x_2-x_3=0 \\ x_2-2x_3=-1 \\ x_3=1 \end{cases} \xrightarrow[r_3+r_1]{2r_3+r_2,} \begin{cases} x_1+2x_2=1 \\ x_2=1 \\ x_3=1 \end{cases} \xrightarrow{-2r_2+r_1} \begin{cases} x_1=-1 \\ x_2=1 \\ x_3=1 \end{cases}$$

3. $\begin{cases} x_1+2x_2-x_3=0 \\ x_2-2x_3=-1 \end{cases}$

解：$\begin{cases} x_1+2x_2-x_3=0 \\ x_2-2x_3=-1 \end{cases}$, Solution is: $[x_1=2-3x_3, x_2=2x_3-1]$.

$$\begin{cases} x_1+2x_2-x_3=0 \\ x_2-2x_3=-1 \end{cases} \xrightarrow{-2r_2+r_1} \begin{cases} x_1+3x_3=2 \\ x_2-2x_3=-1 \end{cases} \to \begin{cases} x_1=-3x_3+2 \\ x_2=2x_3-1 \\ x_3=x_3 \end{cases},$$

（x_3是任意实数）.

4. $\begin{cases} x_1-2x_2+x_3=1 \\ -2x_1+x_2+x_3=1 \\ x_1+x_2-2x_3=-2 \end{cases}$

解：$\begin{cases} x_1-2x_2+x_3=1 \\ -2x_1+x_2+x_3=1 \\ x_1+x_2-2x_3=-2 \end{cases}$, Solution is: $[x_1=x_3-1, x_2=x_3-1]$.

$$\begin{cases} x_1-2x_2+x_3=1 \\ -2x_1+x_2+x_3=1 \\ x_1+x_2-2x_3=-2 \end{cases} \xrightarrow[-r_1+r_2]{2r_1+r_2,} \begin{cases} x_1-2x_2+x_3=1 \\ -3x_2+3x_3=3 \\ 3x_2-3x_3=-3 \end{cases} \xrightarrow[-\frac{1}{3}r_2]{r_2+r_3,}$$
$$\begin{cases} x_1-2x_2+x_3=1 \\ x_2-x_3=-1 \\ 0=0 \end{cases} \xrightarrow{2r_2+r_1} \begin{cases} x_1-x_3=-1 \\ x_2-x_3=-1 \\ 0=0 \end{cases} \rightarrow \begin{cases} x_1=x_3-1 \\ x_2=x_3-1 \\ x_3=x_3 \end{cases}$$
，（x_3是任意实数）.

5. $\begin{cases} x_2-x_3=2 \\ x_1-x_2+2x_3=-2 \\ 3x_1+2x_2+2x_3=1 \end{cases}$

解：$\begin{cases} x_2-x_3=2 \\ x_1-x_2+2x_3=-2 \\ 3x_1+2x_2+2x_3=1 \end{cases}$, Solution is: $[x_1=3, x_2=-1, x_3=-3]$.

$$\begin{cases} x_2-x_3=2 \\ x_1-x_2+2x_3=-2 \\ 3x_1+2x_2+2x_3=1 \end{cases} \xrightarrow{r_1\leftrightarrow r_2} \begin{cases} x_1-x_2+2x_3=-2 \\ x_2-x_3=2 \\ 3x_1+2x_2+2x_3=1 \end{cases} \xrightarrow{-3r_1+r_3}$$
$$\begin{cases} x_1-x_2+2x_3=-2 \\ x_2-x_3=2 \\ 5x_2-4x_3=7 \end{cases} \xrightarrow{-5r_2+r_3} \begin{cases} x_1-x_2+2x_3=-2 \\ x_2-x_3=2 \\ x_3=-3 \end{cases} \xrightarrow[-2r_3+r_1]{r_3+r_2,}$$
$$\begin{cases} x_1-x_2=4 \\ x_2=-1 \\ x_3=-3 \end{cases} \xrightarrow{r_2+r_1} \begin{cases} x_1=3 \\ x_2=-1 \\ x_3=-3 \end{cases}$$

6. $\begin{cases} 2x_1-x_2-x_3=1 \\ x_1+x_2-2x_3=1 \\ 2x_1-3x_2+x_3=-1 \end{cases}$

解：$\begin{cases} 2x_1-x_2-x_3=1 \\ x_1+x_2-2x_3=1 \\ 2x_1-3x_2+x_3=-1 \end{cases}$, No solution found.

$$\begin{cases} 2x_1 - x_2 - x_3 = 1 \\ x_1 + x_2 - 2x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \xrightarrow{r_1 \leftrightarrow r_2} \begin{cases} x_1 + x_2 - 2x_3 = 1 \\ 2x_1 - x_2 - x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \xrightarrow[-2r_1+r_3]{-2r_1+r_2,}$$

$$\begin{cases} x_1 + x_2 - 2x_3 = 1 \\ -3x_2 + 3x_3 = -1 \\ -5x_2 + 5x_3 = -3 \end{cases} \xrightarrow{-\frac{1}{3}r_2} \begin{cases} x_1 + x_2 - 2x_3 = 1 \\ x_2 - x_3 = \frac{1}{3} \\ -5x_2 + 5x_3 = -3 \end{cases} \xrightarrow{5r_2+r_3} \begin{cases} x_1 + x_2 - 2x_3 = 1 \\ x_2 - x_3 = \frac{1}{3} \\ 0 = -\frac{4}{3} \end{cases}$$,原方程无解.

7. $\begin{cases} x_1 + x_2 - 3x_3 - x_5 = 0 \\ -2x_2 + 2x_3 + 2x_4 + x_5 = 0 \\ 3x_4 - x_5 = 0 \end{cases}$

解：$\begin{cases} x_1 + x_2 - 3x_3 - x_5 = 0 \\ -2x_2 + 2x_3 + 2x_4 + x_5 = 0 \\ 3x_4 - x_5 = 0 \end{cases}$, Solution is: $\left[x_1 = 2x_3 + \frac{1}{6}x_5, x_2 = x_3 + \frac{5}{6}x_5, x_4 = \frac{1}{3}x_5\right]$

$$\begin{cases} x_1 + x_2 - 3x_3 - x_5 = 0 \\ -2x_2 + 2x_3 + 2x_4 + x_5 = 0 \\ 3x_4 - x_5 = 0 \end{cases} \xrightarrow{\frac{1}{3}r_3} \begin{cases} x_1 + x_2 - 3x_3 - x_5 = 0 \\ -2x_2 + 2x_3 + 2x_4 + x_5 = 0 \\ x_4 - \frac{1}{3}x_5 = 0 \end{cases} \xrightarrow{-2r_3+r_2,}$$

$$\begin{cases} x_1 + x_2 - 3x_3 - x_5 = 0 \\ -2x_2 + 2x_3 + \frac{5}{3}x_5 = 0 \\ x_4 - \frac{1}{3}x_5 = 0 \end{cases} \xrightarrow{-\frac{1}{2}r_2} \begin{cases} x_1 + x_2 - 3x_3 - x_5 = 0 \\ x_2 - x_3 - \frac{5}{6}x_5 = 0 \\ x_4 - \frac{1}{3}x_5 = 0 \end{cases} \xrightarrow{-r_2+r_1}$$

$$\begin{cases} x_1 + 2x_3 - \frac{1}{6}x_5 = 0 \\ x_2 - x_3 - \frac{5}{6}x_5 = 0 \\ x_4 - \frac{1}{3}x_5 = 0 \end{cases} \to \begin{cases} x_1 = 2x_3 + \frac{1}{6}x_5 \\ x_2 = x_3 + \frac{5}{6}x_5 \\ x_3 = x_3 \\ x_4 = \frac{1}{3}x_5 \\ x_5 = x_5 \end{cases}$$ ，（x_3, x_5是任意实数）.

8. $\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 = 0 \\ x_1 - x_2 - 4x_3 - 3x_4 = 0 \end{cases}$

解：$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 = 0 \\ x_1 - x_2 - 4x_3 - 3x_4 = 0 \end{cases}$, Solution is: $\left[x_1 = 2x_3 + \frac{5}{3}x_4, x_2 = -2x_3 - \frac{4}{3}x_4\right]$.

$$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 = 0 \\ x_1 - x_2 - 4x_3 - 3x_4 = 0 \end{cases} \xrightarrow[-r_1+r_3]{-2r_1+r_2,} \begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ -3x_2 - 6x_3 - 4x_4 = 0 \\ -3x_2 - 6x_3 - 4x_4 = 0 \end{cases} \xrightarrow{-\frac{1}{3}r_2}$$

$$\begin{cases} x_1+2x_2+2x_3+x_4=0 \\ x_2+2x_3+\frac{4}{3}x_4=0 \\ -3x_2-6x_3-4x_4=0 \end{cases} \xrightarrow{3r_2+r_3} \begin{cases} x_1+2x_2+2x_3+x_4=0 \\ x_2+2x_3+\frac{4}{3}x_4=0 \\ 0=0 \end{cases} \xrightarrow{-2r_2+r_1}$$

$$\begin{cases} x_1-2x_3-\frac{5}{3}x_4=0 \\ x_2+2x_3+\frac{4}{3}x_4=0 \\ 0=0 \end{cases} \rightarrow \begin{cases} x_1=2x_3+\frac{5}{3}x_4 \\ x_2=-2x_3-\frac{4}{3}x_4 \\ x_3=x_3 \\ x_4=x_4 \end{cases}$$ ，（x_3,x_4是任意实数）

.

二、讨论参数k为何值时，线性方程组$\begin{cases} x_1+x_2-x_3=2+k \\ -x_1-2x_2+x_3=1+3k \\ x_1-x_2-x_3=3-k \end{cases}$,无解，有解.若有解，求出其一般解.

解：$\begin{cases} x_1+x_2-x_3=2+k \\ -x_1-2x_2+x_3=1+3k \\ x_1-x_2-x_3=3-k \end{cases}$, Solution is: $\begin{cases} \left\{\left[x_1=x_3+\frac{5}{2}, x_2=-1\right]\right\} & \text{if} \quad k=-\frac{1}{2} \\ \emptyset & \text{if} \quad k\neq-\frac{1}{2} \end{cases}$

$$\begin{cases} x_1+x_2-x_3=2+k \\ -x_1-2x_2+x_3=1+3k \\ x_1-x_2-x_3=3-k \end{cases} \xrightarrow[-r_1+r_3]{r_1+r_2,} \begin{cases} x_1+x_2-x_3=2+k \\ -x_2=3+4k \\ -2x_2=1-2k \end{cases} \xrightarrow[2r_2+r_3]{-r_2,}$$

$\begin{cases} x_1+x_2-x_3=2+k \\ x_2=-3-4k \\ 0=-10k-5 \end{cases}$.当$-10k-5\neq0$即$k\neq-\frac{1}{2}$时，方程组无解; 当$k=-\frac{1}{2}$时，方程组有无穷多个解.

$$\begin{cases} x_1+x_2-x_3=\frac{3}{2} \\ x_2=-1 \\ 0=0 \end{cases} \xrightarrow{-r_2+r_1} \begin{cases} x_1-x_3=\frac{5}{2} \\ x_2=-1 \\ 0=0 \end{cases} \rightarrow \begin{cases} x_1=x_3+\frac{5}{2} \\ x_2=-1 \\ x_3=x_3 \end{cases}$$ ，（x_3是任意实数）.

三、设矩阵$A=\begin{pmatrix} 2 & 1 & 4 & 0 \\ 1 & -1 & 3 & 4 \end{pmatrix}, B=\begin{pmatrix} 1 & 0 & 3 & -1 \\ 2 & 1 & 0 & 2 \end{pmatrix}, C=\begin{pmatrix} 4 & 3 & 2 & 1 \\ -2 & 1 & -2 & 1 \end{pmatrix}$.

(1) 矩阵X满足$A+X=C$,求X;

(2) 矩阵X,Y满足$3X+Y=A, X-Y=B$,求X,Y.

解：（1）已知$A=\begin{pmatrix} 2 & 1 & 4 & 0 \\ 1 & -1 & 3 & 4 \end{pmatrix}, B=\begin{pmatrix} 1 & 0 & 3 & -1 \\ 2 & 1 & 0 & 2 \end{pmatrix}, C=\begin{pmatrix} 4 & 3 & 2 & 1 \\ -2 & 1 & -2 & 1 \end{pmatrix}$,又$A+X=C$,则

$$X=C-A=\begin{pmatrix} 2 & 2 & -2 & 1 \\ -3 & 2 & -5 & -3 \end{pmatrix};$$

（2）$\begin{cases} 3X+Y=A \\ X-Y=B \end{cases} \xrightarrow{r_1 \leftrightarrow r_2} \begin{cases} X-Y=B \\ 3X+Y=A \end{cases} \xrightarrow{-3r_1+r_2} \begin{cases} X-Y=B \\ 4Y=A-3B \end{cases} \xrightarrow[r_2+r_1]{\frac{1}{4}r_2,}$
$\begin{cases} X=\frac{1}{4}A+\frac{1}{4}B \\ Y=\frac{1}{4}A-\frac{3}{4}B \end{cases}$ $, X=\frac{1}{4}A+\frac{1}{4}B=\begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{7}{4} & -\frac{1}{4} \\ \frac{3}{4} & 0 & \frac{3}{4} & \frac{3}{2} \end{pmatrix}, Y=\frac{1}{4}A-\frac{3}{4}B=$
$\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{5}{4} & \frac{3}{4} \\ -\frac{5}{4} & -1 & \frac{3}{4} & -\frac{1}{2} \end{pmatrix}.$

2.2 矩阵的计算(2)　　2.3 逆矩阵

一、选择填空题：

1. $(AB)^T=$_______, $(kA)^T=$__________（k是常数）.

2. 已知为A,B可逆矩阵，常数$k\neq 0$，则
   $\left(A^T\right)^{-1}=$_______, $(kA)^{-1}=$_________, $(AB)^{-1}=$_______.

3. 设A是n阶矩阵，若_______，则称A是对称矩阵; 若$A^T=-A$，则称A是_______.

4. 设α为3维列向量，α^T是α的转置，若$\alpha\alpha^T=\begin{pmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{pmatrix}$，则$\alpha^T\alpha=$ _______.

5. 设A是n阶对称矩阵，B是n阶反对称矩阵，则下列矩阵为反对称矩阵的是_______.

   A. $AB-BA$　　　B. $(AB)^2$　　　C. $AB+BA$　　　D. BAB

提示：1. B^TA^T, kA^T;　　2. $\left(A^{-1}\right)^T, k^{-1}A^{-1}; B^{-1}A^{-1}$.　　3. $A^T=A$,反对称矩阵.　　4. $\alpha=\begin{pmatrix}x\\y\\z\end{pmatrix}, \alpha\alpha^T=\begin{pmatrix}x^2&xy&xz\\xy&y^2&yz\\xz&yz&z^2\end{pmatrix}=\begin{pmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{pmatrix}, x^2=1, y^2=1, z^2=1, \alpha^T\alpha=x^2+y^2+z^2=3$.　　5. C.

二、解答题

1. 设$A=\begin{pmatrix}2&0\\4&3\end{pmatrix}, B=\begin{pmatrix}2&-3\\1&0\end{pmatrix}$,求$AB, BA$.

   解：因为$A=\begin{pmatrix}2&0\\4&3\end{pmatrix}, B=\begin{pmatrix}2&-3\\1&0\end{pmatrix}$,则$AB=\begin{pmatrix}4&-6\\11&-12\end{pmatrix}, BA=\begin{pmatrix}-8&-9\\2&0\end{pmatrix}$.

2. 设$A=\begin{pmatrix}1&4&2\\2&5&1\\3&6&0\end{pmatrix}, B=\begin{pmatrix}2&-2&6\\-1&1&-3\\1&-1&3\end{pmatrix}$,求$AB$.

   解：因为$A=\begin{pmatrix}1&4&2\\2&5&1\\3&6&0\end{pmatrix}, B=\begin{pmatrix}2&-2&6\\-1&1&-3\\1&-1&3\end{pmatrix}$,则$AB=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$.

3. 若$AB=BA$，则称A,B是可交换的.求出所有与$A=\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}$可交换的矩阵.

解：设$B=\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}$,如果$AB=BA$,则$\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}=\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}$.因为$\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}=\begin{pmatrix}x_1+x_2 & 2x_1-x_2\\ x_3+x_4 & 2x_3-x_4\end{pmatrix}$, $\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}=\begin{pmatrix}x_1+2x_3 & x_2+2x_4\\ x_1-x_3 & x_2-x_4\end{pmatrix}$,故$\begin{pmatrix}x_1+x_2 & 2x_1-x_2\\ x_3+x_4 & 2x_3-x_4\end{pmatrix}=\begin{pmatrix}x_1+2x_3 & x_2+2x_4\\ x_1-x_3 & x_2-x_4\end{pmatrix}$,即$\begin{cases}x_1+x_2=x_1+2x_3\\ 2x_1-x_2=x_2+2x_4\\ x_3+x_4=x_1-x_3\\ 2x_3-x_4=x_2-x_4\end{cases}$，$\begin{cases}x_2-2x_3=0\\ 2x_1-2x_2-2x_4=0\\ x_1-2x_3-x_4=0\\ x_2-2x_3=0\end{cases}\xrightarrow{r_1\leftrightarrow r_3}$ $\begin{cases}x_1-2x_3-x_4=0\\ 2x_1-2x_2-2x_4=0\\ x_2-2x_3=0\\ x_2-2x_3=0\end{cases}\xrightarrow[-r_3+r_4]{-2r_1+r_2,}\begin{cases}x_1-2x_3-x_4=0\\ -2x_2+4x_3=0\\ x_2-2x_3=0\\ 0=0\end{cases}\xrightarrow[2r_2+r_3]{r_2\leftrightarrow r_3,}$ $\begin{cases}x_1-2x_3-x_4=0\\ x_2-2x_3=0\\ 0=0\\ 0=0\end{cases}\to\begin{cases}x_1=2x_3+x_4\\ x_2=2x_3\\ x_3=x_3\\ x_4=x_4\end{cases}$，（$x_3,x_4$是任意实数）.所求矩阵为$\begin{pmatrix}2a+b & 2a\\ a & b\end{pmatrix}$,（$a,b$是任意实数）.

4. 已知矩阵$A=\begin{pmatrix}a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{pmatrix}$, $a_i\neq 0(i=1,2,3)$,求：

(1) A^{-1}和A^n;

(2) 所有与A交换的矩阵.

解：（1）设$B=\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}$,则$AB=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.因为$AB=\begin{pmatrix}a_1x_1 & a_1x_2 & a_1x_3\\ a_2x_4 & a_2x_5 & a_2x_6\\ a_3x_7 & a_3x_8 & a_3x_9\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $a_i\neq 0(i=1,2,3)$,故$x_1=$

$a_1^{-1}, x_5 = a_2^{-1}, x_9 = a_3^{-1}, x_2 = x_3 = x_4 = x_6 = x_7 = x_8 = 0$.即$A^{-1} =$ $B = \begin{pmatrix} a_1^{-1} & 0 & 0 \\ 0 & a_2^{-1} & 0 \\ 0 & 0 & a_3^{-1} \end{pmatrix}$;

$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, A^2 = AA = \begin{pmatrix} a_1^2 & 0 & 0 \\ 0 & a_2^2 & 0 \\ 0 & 0 & a_3^2 \end{pmatrix}, \cdots, A^n = \begin{pmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{pmatrix}$.

（可用数学归纳法证明）

（2）设$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$,则$XA = AX$.因为$XA = \begin{pmatrix} a_1x_1 & a_2x_2 & a_3x_3 \\ a_1x_4 & a_2x_5 & a_3x_6 \\ a_1x_7 & a_2x_8 & a_3x_9 \end{pmatrix}$
,

$AX = \begin{pmatrix} a_1x_1 & a_1x_2 & a_1x_3 \\ a_2x_4 & a_2x_5 & a_2x_6 \\ a_3x_7 & a_3x_8 & a_3x_9 \end{pmatrix}$,故$\begin{cases} a_2x_2 = a_1x_2 \\ a_3x_3 = a_1x_3 \\ a_1x_4 = a_2x_4 \\ a_3x_6 = a_2x_6 \\ a_1x_7 = a_3x_7 \\ a_2x_8 = a_3x_8 \end{cases}$,即$\begin{cases} (a_1 - a_2)\, x_2 = 0 \\ (a_1 - a_3)\, x_3 = 0 \\ (a_1 - a_2)\, x_4 = 0 \\ (a_2 - a_3)\, x_6 = 0 \\ (a_1 - a_3)\, x_7 = 0 \\ (a_2 - a_3)\, x_8 = 0 \end{cases}$.

因为a_1, a_2, a_3互不相等，故$x_2 = x_3 = x_4 = x_6 = x_7 = x_8 = 0, x_1, x_5, x_9$为任意实数，从而$X = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_5 & 0 \\ 0 & 0 & x_5 \end{pmatrix}, x_1, x_5, x_9$是任意实数.

5. 已知多项式$f(x) = x^2 + 3x - 5, n$阶方阵A满足$f(A) = 0, E$为n阶单位矩阵，求$(A + 5E)^{-1}$.

   解：因为$f(x) = x^2 + 3x - 5\,(x+5)\,(x-2) + 5, f(A) = 0$故$A^2 + 3A - 5E = 0$,即$(A + 5E)(A - 2E) + 5E = 0$,从而$(A + 5E)\left[-\frac{1}{5}(A - 2E)\right] = E$,由此$(A + 5E)^{-1} = -\frac{1}{5}(A - 2E)$.

6. 已知$A = \begin{pmatrix} 2 & 1 & -1 \\ 6 & 3 & -3 \\ -4 & -2 & 2 \end{pmatrix}$,求$A^n$.

   解：因为$A = \begin{pmatrix} 2 & 1 & -1 \\ 6 & 3 & -3 \\ -4 & -2 & 2 \end{pmatrix}$,故$A^2 = \begin{pmatrix} 14 & 7 & -7 \\ 42 & 21 & -21 \\ -28 & -14 & 14 \end{pmatrix} =$ $7A, \cdots, A^n = 7^{(n-1)}A = \begin{pmatrix} 2 \times 7^{n-1} & 7^{n-1} & -7^{n-1} \\ 6 \times 7^{n-1} & 3 \times 7^{n-1} & -3 \times 7^{n-1} \\ -4 \times 7^{n-1} & -2 \times 7^{n-1} & 2 \times 7^{n-1} \end{pmatrix}$.

7. 设$A=\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$,整数$n\geqslant 2$,求$A^n-2A^{n-1}$.

解：因为$A=\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$, $A^2=\begin{pmatrix}2&0&2\\0&4&0\\2&0&2\end{pmatrix}=2A$,故$n\geqslant 2$时，$A^n-2A^{n-1}=A^{n-2}(A^2-2A)=0$.

三、证明题：

1. 设A是n阶实对称矩阵，如果$A^2=0$,证明：$A=0$.

   证明：因为$A=(a_{ij})$是n阶实对称矩阵，故$a_{ij}=a_{ji}\,(i,j=1,2,\cdots,n)$.由已知$A^2$的第$i$行第$j$列元素$c_{ij}=\sum\limits_{k=1}^{n}a_{ik}a_{kj}=0$,从而$c_{ii}=\sum\limits_{k=1}^{n}a_{ik}a_{ki}=\sum\limits_{k=1}^{n}a_{ik}^2=0$,即$a_{ik}=0\,(i,k=1,2,\cdots,n)$,从而$A=0$.

2. 设矩阵$X=(x_1,x_2,\cdots,x_n)^T$满足$X^TX=1$, E是n阶单位矩阵，$H=E-2XX^T$.证明：H是对称矩阵，且$HH^T=E$.

   证明：因为$H=E-2XX^T$,故$H^T=E^T+\left(-2XX^T\right)^T=E-2\left(XX^T\right)^T=E-2(X^T)^TX^T=E-2XX^T=H$,即$H$是对称矩阵.又因为矩阵$X=(x_1,x_2,\cdots,x_n)^T$满足$X^TX=1$,故$HH^T=H^2=E^2-4XX^T+4\left(XX^T\right)^2=E-4XX^T+4X\left(X^TX\right)X^T=E-4XX^T+4XX^T=E$.

3. 设A,B均为n阶对称矩阵，证明：AB是对称矩阵的充要条件为$AB=BA$.

   证明：设A,B均为n阶对称矩阵，则$A^T=A, B^T=B$.如果$AB=BA$,则$(AB)^T=B^TA^T=BA=AB$,即AB是对称矩阵; 反之，如果AB是对称矩阵，则$(AB)^T=AB$,从而$AB=(AB)^T=B^TA^T=BA$.

2.4 线性方程组的矩阵解法

一、对增广矩阵作初等变换解下列方程组：

1. $\begin{cases} 2x-y=0 \\ -x+2y-z=-1 \\ -2y+4z=4 \end{cases}$

解：$\begin{cases} 2x-y=0 \\ -x+2y-z=-1 \\ -2y+4z=4 \end{cases}$, Solution is: $[x=0, y=0, z=1]$.

$\begin{cases} 2x-y=0 \\ -x+2y-z=-1 \\ -2y+4z=4 \end{cases}$, Corresponding matrix: $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -2 & 4 & 4 \end{pmatrix}$.

$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -2 & 4 & 4 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, Corresponding equations: $\{x=0, y=0, z=1\}$.

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -2 & 4 & 4 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} -1 & 2 & -1 & -1 \\ 2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 4 \end{pmatrix} \xrightarrow[-2r_1+r_2]{-r_1,}$$
$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 3 & -2 & -2 \\ 0 & -2 & 4 & 4 \end{pmatrix} \xrightarrow[-\frac{1}{2}r_2, -3r_2+r_3]{r_2 \leftrightarrow r_3,} \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 4 & 4 \end{pmatrix} \xrightarrow{\frac{1}{4}r_3}$$
$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[-r_3+r_1]{2r_3+r_2,} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{2r_2+r_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
.原方程的解为$\begin{cases} x=0 \\ y=0 \\ z=1 \end{cases}$.

2. $\begin{cases} x_1+2x_2+x_3=1 \\ x_1+3x_2+2x_3=3 \\ -2x_1-x_2+x_3=0 \end{cases}$

解：$\begin{cases} x_1+2x_2+x_3=1 \\ x_1+3x_2+2x_3=3 \\ -2x_1-x_2+x_3=0 \end{cases}$, No solution found.

$\begin{pmatrix} x_1+2x_2+x_3=1 \\ x_1+3x_2+2x_3=3 \\ -2x_1-x_2+x_3=0 \end{pmatrix}$, Corresponding matrix: $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 3 \\ -2 & -1 & 1 & 0 \end{pmatrix}$.

$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 3 \\ -2 & -1 & 1 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 3 \\ -2 & -1 & 1 & 0 \end{pmatrix} \xrightarrow[2r_1+r_3]{-r_1+r_2,} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 2 \end{pmatrix} \xrightarrow{-3r_2+r_3} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$,原方程无解.

3. $\begin{cases} x_1+x_2+x_3+x_4+x_5=7 \\ 3x_1+2x_2+x_3+x_4-3x_5=-2 \\ x_2+2x_3+2x_4+6x_5=23 \\ 5x_1+4x_2+3x_3+3x_4-x_5=12 \end{cases}$

解：$\begin{cases} x_1+x_2+x_3+x_4+x_5=7 \\ 3x_1+2x_2+x_3+x_4-3x_5=-2 \\ x_2+2x_3+2x_4+6x_5=23 \\ 5x_1+4x_2+3x_3+3x_4-x_5=12 \end{cases}$, Solution is:
$[x_1=x_3+x_4+5x_5-16, x_2=23-2x_4-6x_5-2x_3]$.

$\begin{cases} x_1+x_2+x_3+x_4+x_5=7 \\ 3x_1+2x_2+x_3+x_4-3x_5=-2 \\ x_2+2x_3+2x_4+6x_5=23 \\ 5x_1+4x_2+3x_3+3x_4-x_5=12 \end{cases}$, Corresponding matrix: $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 7 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 5 & 4 & 3 & 3 & -1 & 12 \end{pmatrix}$.

$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 7 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 5 & 4 & 3 & 3 & -1 & 12 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -1 & -1 & -5 & -16 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,
Corresponding equations: $\{0=0, x_1-x_3-x_4-5x_5=-16, x_2+2x_3+2x_4+6x_5=23\}$
.

$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 7 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 5 & 4 & 3 & 3 & -1 & 12 \end{pmatrix} \xrightarrow[-5r_1+r_4]{-3r_1+r_2,} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & -2 & -2 & -6 & -23 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & -1 & -2 & -2 & -6 & -23 \end{pmatrix} \xrightarrow[r_2+r_4]{-r_2,-r_2+r_3,}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-r_2+r_1} \begin{pmatrix} 1 & 0 & -1 & -1 & -5 & -16 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

原方程的解为$\begin{cases} x_1 = x_3 + x_4 + 5x_5 - 16 \\ x_2 = -2x_3 - 2x_4 - 6x_5 + 23 \\ x_3 = x_3 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases}$，（$x_3, x_4, x_5$是任意实数）

.

二、设线性方程组：(I)$\begin{cases} x_1 + x_2 = 0, \\ x_2 - x_4 = 0; \end{cases}$ (II)$\begin{cases} x_1 - x_2 + x_3 = 0, \\ x_2 - x_3 + x_4 = 0. \end{cases}$ 求方程组(I)(II)的公共解.

解：$\begin{cases} x_1 + x_2 = 0 \\ x_2 - x_4 = 0 \end{cases}$, Solution is: $[x_1 = -x_4, x_2 = x_4]$, $\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases}$, Solution is: $[x_1 = -x_4, x_2 = x_3 - x_4]$.

由$x_4 = x_3 - x_4$得$x_4 = \frac{1}{2}x_3$,从而方程组(I)(II)的公共解为$\begin{cases} x_1 = -\frac{1}{2}x_3 \\ x_2 = \frac{1}{2}x_3 \\ x_3 = x_3 \\ x_4 = \frac{1}{2}x_3 \end{cases}$,

（x_3是任意实数）.

另解：方程组(I)(II)的公共解，则只要求$\begin{cases} x_1 + x_2 = 0 \\ x_2 - x_4 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases}$的解.$\begin{cases} x_1 + x_2 = 0 \\ x_2 - x_4 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases}$,

Solution is: $[x_1 = -x_4, x_2 = x_4, x_3 = 2x_4]$,（$x_4$是任意实数）.

$\begin{cases} x_1 + x_2 = 0 \\ x_2 - x_4 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases}$, Corresponding matrix: $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{-r_1+r_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow[-r_2+r_4]{2r_2+r_3.}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{pmatrix} \xrightarrow[-r_2+r_1]{r_3+r_4,} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$, 对应的方程为

$$\begin{cases} x_1 + x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 - 2x_4 = 0 \\ 0 = 0 \end{cases}$$,原方程的解为$$\begin{cases} x_1 = -x_4 \\ x_2 = x_4 \\ x_3 = 2x_4 \\ x_4 = x_4 \end{cases}$$，（x_4是任意实数）.

三、已知线性方程组$$\begin{cases} x_1+x_2+x_3+x_4+x_5 = a \\ x_1+2x_3+3x_4+2x_5 = 3 \\ 4x_1+5x_2+3x_3+2x_4+3x_5 = 2 \\ x_1+x_4+2x_5 = 1 \end{cases}$$有解，试确定a的值，并求方程组的解.

解：$$\begin{cases} x_1+x_2+x_3+x_4+x_5 = a \\ x_1+2x_3+3x_4+2x_5 = 3 \\ 4x_1+5x_2+3x_3+2x_4+3x_5 = 2 \\ x_1+x_4+2x_5 = 1 \end{cases}$$, Solution is:

$$\begin{cases} \{[x_1 = -x_4 - 2x_5 + 1, x_2 = x_4 + x_5 - 1, x_3 = -x_4 + 1]\} & \text{if} \quad a = 1 \\ \emptyset & \text{if} \quad a \neq 1 \end{cases}$$

$$\begin{cases} x_1+x_2+x_3+x_4+x_5 = a \\ x_1+2x_3+3x_4+2x_5 = 3 \\ 4x_1+5x_2+3x_3+2x_4+3x_5 = 2 \\ x_1+x_4+2x_5 = 1 \end{cases}$$, Corresponding matrix: $$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & a \\ 1 & 0 & 2 & 3 & 2 & 3 \\ 4 & 5 & 3 & 2 & 3 & 2 \\ 1 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & a \\ 1 & 0 & 2 & 3 & 2 & 3 \\ 4 & 5 & 3 & 2 & 3 & 2 \\ 1 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$, row echelon form: $$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ $(a \neq 1)$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & a \\ 1 & 0 & 2 & 3 & 2 & 3 \\ 4 & 5 & 3 & 2 & 3 & 2 \\ 1 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow[-4r_1+r_3, -r_1+r_4]{-r_1+r_2,} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & a \\ 0 & -1 & 1 & 2 & 1 & 3-a \\ 0 & 1 & -1 & -2 & -1 & 2-4a \\ 0 & -1 & -1 & 0 & 1 & 1-a \end{pmatrix}$$

$$\xrightarrow[r_2+r_3, r_2+r_4]{r_2 \leftrightarrow r_3,} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & a \\ 0 & 1 & -1 & -2 & -1 & 2-4a \\ 0 & 0 & 0 & 0 & 0 & 5-5a \\ 0 & 0 & -2 & -2 & 0 & 3-5a \end{pmatrix}.$$

$a\neq 1$时，方程组无解; $a=1$时，方程组$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & -2 \end{pmatrix} \xrightarrow[-\frac{1}{2}r_3]{r_3\leftrightarrow r_4,}$

$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[-r_3+r_1]{r_3+r_2,} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-r_2+r_1}$

$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,原方程组的解为$\begin{cases} x_1=-x_4-2x_5+1 \\ x_2=x_4+x_5-1 \\ x_3=-x_4+1 \\ x_4=x_4 \\ x_5=x_5 \end{cases}$.

四、证明：线性方程组$\begin{cases} x_1-x_2=a_1 \\ x_2-x_3=a_2 \\ x_3-x_4=a_3 \\ x_4-x_5=a_4 \\ x_5-x_1=a_5 \end{cases}$有解的充分必要条件为$a_1+a_2+a_3+a_4+a_5=0$,在解的情况下，求它的一般解.

证明：$\begin{cases} x_1-x_2=a_1 \\ x_2-x_3=a_2 \\ x_3-x_4=a_3 \\ x_4-x_5=a_4 \\ x_5-x_1=a_5 \end{cases}$, Corresponding matrix: $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ -1 & 0 & 0 & 0 & 1 & a_5 \end{pmatrix}$.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ -1 & 0 & 0 & 0 & 1 & a_5 \end{pmatrix} \xrightarrow{r_1+r_5} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & -1 & 0 & 0 & 1 & a_1+a_5 \end{pmatrix} \xrightarrow{r_2+r_5}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & -1 & 0 & 1 & a_1+a_2+a_5 \end{pmatrix} \xrightarrow{r_3+r_5} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & -1 & 1 & a_1+a_2+a_3+a_5 \end{pmatrix}$$

$$\xrightarrow{r_4+r_5}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_1+a_2+a_3+a_4+a_5 \end{pmatrix}\xrightarrow{r_4+r_3}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & -1 & a_3+a_4 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_1+a_2+a_3+a_4+a_5 \end{pmatrix}\xrightarrow{r_3+r_2}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & -1 & a_2+a_3+a_4 \\ 0 & 0 & 1 & 0 & -1 & a_3+a_4 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_1+a_2+a_3+a_4+a_5 \end{pmatrix}\xrightarrow{r_2+r_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & a_1+a_2+a_3+a_4 \\ 0 & 1 & 0 & 0 & -1 & a_2+a_3+a_4 \\ 0 & 0 & 1 & 0 & -1 & a_3+a_4 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_1+a_2+a_3+a_4+a_5 \end{pmatrix},$$

当且仅当$a_1+a_2+a_3+a_4+a_5=0$时，原方程组有解$\begin{cases} x_1=x_5+a_1+a_2+a_3+a_4 \\ x_2=x_5+a_2+a_3+a_4 \\ x_3=x_5+a_3+a_4 \\ x_4=x_5+a_4 \\ x_5=x_5 \end{cases}$，

（x_5是任意实数）.

3.1-3.2 线性方程组的行列式解法　行列式的定义及性质

一、填空题：

1. $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} =$ ______.

2. $\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$ ______.

3. $\begin{vmatrix} a & b & c & d \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix} =$ ______.

4. $\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & -3 \\ 1 & 3 & 0 \end{vmatrix} =$ ______.

5. 已知A为3×3的方阵，$|A| = 2$,则$|3A| =$ ______.

6. 若$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = m$,则$\begin{vmatrix} a_1 & 2c_1 - 5b_1 & 3b_1 \\ a_2 & 2c_2 - 5b_2 & 3b_2 \\ a_3 & 2c_3 - 5b_3 & 3b_3 \end{vmatrix} =$ ______.

7. 设$\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} = 4$, $\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} = 1$,则$\begin{vmatrix} a_1 + b_1 & 2d_1 & 3c_1 \\ a_2 + b_2 & 2d_2 & 3c_2 \\ a_3 + b_3 & 2d_3 & 3c_3 \end{vmatrix} =$ ______.

8. $\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \end{vmatrix} =$ ______.

提示：1. 上三角行列式$a_{11}a_{22}\cdots a_{nn}$.　2. 下三角行列式$a_{11}a_{22}\cdots a_{nn}$.　3. $\begin{vmatrix} a & b & c & d \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ b & c & a & 1 \\ b+c & a+c & a+b & 2 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix} = 0$.　4. $\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & -3 \\ 1 & 3 & 0 \end{vmatrix} = 0$.　5. $|3A| = 3^3\,|A| = 54$.　6. $\begin{vmatrix} a_1 & 2c_1-5b_1 & 3b_1 \\ a_2 & 2c_2-5b_2 & 3b_2 \\ a_3 & 2c_3-5b_3 & 3b_3 \end{vmatrix} = \begin{vmatrix} a_1 & 2c_1 & 3b_1 \\ a_2 & 2c_2 & 3b_2 \\ a_3 & 2c_3 & 3b_3 \end{vmatrix} = 6\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = -6\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -6m$.　7. $\begin{vmatrix} a_1+b_1 & 2d_1 & 3c_1 \\ a_2+b_2 & 2d_2 & 3c_2 \\ a_3+b_3 & 2d_3 & 3c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 2d_1 & 3c_1 \\ a_2 & 2d_2 & 3c_2 \\ a_3 & 2d_3 & 3c_3 \end{vmatrix} + \begin{vmatrix} b_1 & 2d_1 & 3c_1 \\ b_2 & 2d_2 & 3c_2 \\ b_3 & 2d_3 & 3c_3 \end{vmatrix} = 6\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} + 6\begin{vmatrix} b_1 & d_1 & c_1 \\ b_2 & d_2 & c_2 \\ b_3 & d_3 & c_3 \end{vmatrix} = -6\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} - 6\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} = -6\times 4 - 6\times 1 = -30$.　8. 各行提取$-1$,故$|A| = -\left|A^T\right| = -|A|$,故$|A| = 0$.（奇数阶反对称矩阵的行列式为零）

二、计算下列行列式：

1. $\begin{vmatrix} 103 & 100 & 204 \\ 199 & 200 & 395 \\ 301 & 300 & 600 \end{vmatrix}$.

解：原式$= 100\begin{vmatrix} 103 & 1 & 204 \\ 199 & 2 & 395 \\ 301 & 3 & 600 \end{vmatrix} = 100\begin{vmatrix} 3 & 1 & 4 \\ -1 & 2 & -5 \\ 1 & 3 & 0 \end{vmatrix} = 100\begin{vmatrix} 3 & 1 & 4 \\ -1 & 2 & -5 \\ 0 & 5 & -5 \end{vmatrix} = 100\begin{vmatrix} 3 & 5 & 4 \\ -1 & -3 & -5 \\ 0 & 0 & -5 \end{vmatrix} = -500\begin{vmatrix} 3 & 5 \\ -1 & -3 \end{vmatrix} = 2000$.

2. $\begin{vmatrix} -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \\ \frac{1}{2} & 2 & 1 & -\frac{5}{2} \\ 0 & 1 & -2 & 4 \end{vmatrix} = \frac{39}{2}$

解：原式$=\begin{vmatrix} 0 & 11 & 4 & -11 \\ 0 & -6 & -1 & 5 \\ \frac{1}{2} & 2 & 1 & -\frac{5}{2} \\ 0 & 1 & -2 & 4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 11 & 4 & -11 \\ -6 & -1 & 5 \\ 1 & -2 & 4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 26 & -55 \\ 0 & -13 & 29 \\ 1 & -2 & 4 \end{vmatrix} =$
$\frac{1}{2}\begin{vmatrix} 26 & -55 \\ -13 & 29 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 3 \\ -13 & 29 \end{vmatrix} = \frac{39}{2}.$

3. $\begin{vmatrix} 2 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \\ 4 & 4 & 2 & 4 \\ 4 & 4 & 4 & 2 \end{vmatrix}.$

解：原式$=(2+3\times 4)(2-4)^3=-112.$

或原式$=\begin{vmatrix} 14 & 4 & 4 & 4 \\ 14 & 2 & 4 & 4 \\ 14 & 4 & 2 & 4 \\ 14 & 4 & 4 & 2 \end{vmatrix} = 14\begin{vmatrix} 1 & 4 & 4 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 4 & 2 & 4 \\ 1 & 4 & 4 & 2 \end{vmatrix} = 14\begin{vmatrix} 1 & 4 & 4 & 4 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} =$
$-112.$

4. $\begin{vmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{vmatrix}.$

解：原式$=\begin{vmatrix} 3+\lambda & 1 & 1 \\ 3+\lambda & 1+\lambda & 1 \\ 3+\lambda & 1 & 1+\lambda \end{vmatrix} = (3+\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{vmatrix} =$
$(3+\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda & 0 \\ 1 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda+3).$

5. $D_n = \begin{vmatrix} -a_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & & -a_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{n-1} & a_{n-1} \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}.$

解：按最后一列展开，$D_n = \begin{vmatrix} -a_1 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & -a_2 & a_2 & 0 & \cdots & 0 \\ 0 & & -a_3 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{n-1} \end{vmatrix} -$

$a_{n-1} \begin{vmatrix} -a_1 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & -a_2 & a_2 & 0 & \cdots & 0 \\ 0 & & -a_3 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{n-2} \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$

$= (-1)^{n-1} a_1 a_2 \cdots a_{n-1} - a_{n-1} D_{n-1}$（第二个行列式按最后一列展开），从而$D_n = (-1)^{n-1} n a_1 a_2 \cdots a_{n-1}$.（可用数学归纳法证明）.

或第j列加到第j+1列$(j = 1, 2, \cdots, n-1)$,得$D_n = \begin{vmatrix} -a_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & & -a_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{n-1} & 0 \\ 1 & 2 & 3 & 4 & \cdots & n-1 & n \end{vmatrix} =$

$(-1)^{n-1} n a_1 a_2 \cdots a_{n-1}$.

三、用行列式的性质证明下列等式：

1. $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

证明：左边$= \begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = \begin{vmatrix} 2a+2b+2c & c+a & a+b \\ 2a+2b+2c & b+c & c+a \\ 2a+2b+2c & a+b & b+c \end{vmatrix}$

$= 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 1 & b+c & c+a \\ 1 & a+b & b+c \end{vmatrix} = 2(a+b+c) \left(\begin{vmatrix} 1 & c & a+b \\ 1 & b & c+a \\ 1 & a & b+c \end{vmatrix} + \begin{vmatrix} 1 & a & a+b \\ 1 & c & c+a \\ 1 & b & b+c \end{vmatrix} \right)$

$= 2(a+b+c) \left(\begin{vmatrix} 1 & c & a+b \\ 1 & b & c+a \\ 1 & a & b+c \end{vmatrix} + \begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} \right)$

$$=2\left(a+b+c\right)\left(\begin{vmatrix}1&c&a+b+c\\1&b&c+a+b\\1&a&b+c+a\end{vmatrix}+\begin{vmatrix}1&a&b\\1&c&a\\1&b&c\end{vmatrix}\right)=2\left(a+b+c\right)\begin{vmatrix}1&a&b\\1&c&a\\1&b&c\end{vmatrix},$$

右边$=2\begin{vmatrix}a&b&c\\c&a&b\\b&c&a\end{vmatrix}=2\begin{vmatrix}a+b+c&b&c\\c+a+b&a&b\\b+c+a&c&a\end{vmatrix}=2\left(a+b+c\right)\begin{vmatrix}1&b&c\\1&a&b\\1&c&a\end{vmatrix}=$

$-2\left(a+b+c\right)\begin{vmatrix}1&a&b\\1&b&c\\1&c&a\end{vmatrix}=2\left(a+b+c\right)\begin{vmatrix}1&a&b\\1&c&a\\1&b&c\end{vmatrix}$,故左边=右边.

2. 若$abcd=1$,则$\begin{vmatrix}a^2+\frac{1}{a^2}&a&\frac{1}{a}&1\\b^2+\frac{1}{b^2}&b&\frac{1}{b}&1\\c^2+\frac{1}{c^2}&c&\frac{1}{c}&1\\d^2+\frac{1}{d^2}&d&\frac{1}{d}&1\end{vmatrix}=0.$

证明：$\begin{vmatrix}a^2+\frac{1}{a^2}&a&\frac{1}{a}&1\\b^2+\frac{1}{b^2}&b&\frac{1}{b}&1\\c^2+\frac{1}{c^2}&c&\frac{1}{c}&1\\d^2+\frac{1}{d^2}&d&\frac{1}{d}&1\end{vmatrix}=\begin{vmatrix}a^2&a&\frac{1}{a}&1\\b^2&b&\frac{1}{b}&1\\c^2&c&\frac{1}{c}&1\\d^2&d&\frac{1}{d}&1\end{vmatrix}+\begin{vmatrix}\frac{1}{a^2}&a&\frac{1}{a}&1\\\frac{1}{b^2}&b&\frac{1}{b}&1\\\frac{1}{c^2}&c&\frac{1}{c}&1\\\frac{1}{d^2}&d&\frac{1}{d}&1\end{vmatrix}=$

$\frac{1}{abcd}\begin{vmatrix}a^3&a^2&1&a\\b^3&b^2&1&b\\c^3&c^2&1&c\\d^3&d^2&1&b\end{vmatrix}+\left(\frac{1}{abcd}\right)^2\begin{vmatrix}1&a^3&a&a^2\\1&b^3&b&b^2\\1&c^3&c&c^2\\1&d^3&d&d^3\end{vmatrix}=\begin{vmatrix}a^3&a^2&1&a\\b^3&b^2&1&b\\c^3&c^2&1&c\\d^3&d^2&1&b\end{vmatrix}+$

$\begin{vmatrix}1&a^3&a&a^2\\1&b^3&b&b^2\\1&c^3&c&c^2\\1&d^3&d&d^3\end{vmatrix}=\begin{vmatrix}a^3&a^2&1&a\\b^3&b^2&1&b\\c^3&c^2&1&c\\d^3&d^2&1&b\end{vmatrix}+\begin{vmatrix}a^3&1&a^2&a\\b^3&1&b^2&b\\c^3&1&c^2&c\\d^3&1&d^3&b\end{vmatrix}=0$

3.4 行列式的展开 3.5 行列式的计算(1)

一、填空题

1. $\begin{vmatrix} 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & 0 & a_4 & 0 \end{vmatrix} =$ __________.

2. $\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix} =$ __________.

3. $\begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & \cdots & a_{2(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} =$ __________.

4. 设a,b是实数，则当$a =$ _____, $b =$ _____时，行列式$\begin{vmatrix} a & b & 0 \\ -b & a & 0 \\ -1 & 0 & -1 \end{vmatrix} = 0$.

5. 四阶行列式中所有含有因子$a_{11}a_{22}$的项为__________.

6. 已知$a_{23}a_{i1}a_{42}a_{56}a_{j4}a_{65}$为6阶行列式$|a_{ij}|$中带正号的项，则$i =$ __, $j =$ ___.

7. 已知A,B均为4阶矩阵，且$|A| = 2, |B| = 3$,则$|AB^T| =$ _______, $||A|B| =$ ______.

8. 已知为5阶矩阵，且$|A| = 2$,则$\left|(\frac{1}{2}A)^{-1}\right| =$ __________.

9. 设$f(x) = \begin{vmatrix} 2x & x & 2 & 3 \\ 1 & x & 2 & 4 \\ 3 & 2 & 3x & -1 \\ 2 & 5 & 3 & x \end{vmatrix}$,则$f(x)$中$x^4$的系数为______，$x^3$的系数为______，常数项为______.

10. 设$D = \begin{vmatrix} a & b & c & d \\ b & a & c & d \\ b & d & c & a \\ d & a & c & b \end{vmatrix}$,则$A_{11} + A_{21} + A_{31} + A_{41} =$ __________.

11. $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 2 & 5 & 2 & -1 \\ 6 & -3 & 1 & 0 \end{vmatrix} =$ ________.

12. 已知n阶方阵A的行列式$|A|=a$,且A的每行的元素之和为$b(b\neq 0)$,则$|A|$的第一列元素的代数余子式之和为______.

提示：1. $\begin{vmatrix} 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & 0 & a_4 & 0 \end{vmatrix} = (-1)^5 a_1 \begin{vmatrix} a_2 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_4 \end{vmatrix} = -a_1a_2a_3a_4.$ 2. $D_n = \begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix} = (-1)^{n+1} a_{n1} \begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & & \vdots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)(n-1)} \end{vmatrix} = (-1)^{n+1} a_{n1} D_{n-1} \, (n \geqslant 2) \, , D_1 = a_{11}$.故$D_n == (-1)^{\frac{(n-1)n}{2}} a_{n1}a_{(n-1)2}\cdots a_{1n}$. 3. $D_n = \begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & \cdots & a_{2(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n+1} a_{n1} \begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & \cdots & a_{2(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ a_{(n-1)1} & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n+1} a_{n1} D_{n-1} \, (n \geqslant 2) \, , D_1 = a_{11}$.故$D_n == (-1)^{\frac{(n-1)n}{2}} a_{n1}a_{(n-1)2}\cdots a_{1n}$.

4. $\begin{vmatrix} a & b & 0 \\ -b & a & 0 \\ -1 & 0 & -1 \end{vmatrix} = -a^2 - b^2 = 0, a = b = 0.$ 5. $a_{11}a_{22}a_{33}a_{44}$或$-a_{11}a_{22}a_{34}a_{43}$. 6. 由行列式的定义知$i=3, j=1$或$i=1, j=3$.但$\tau\,(312645)=4$为偶数，故$\tau\,(2i45j6)$为偶数，当$i=1, j=3$时，$\tau\,(2i45j6)=3$,舍去，从而$i=3, j=1$. 7. $|AB^T| = |A||B^T| = |A||B| = 6, ||A|B| = |A|^4|B| = 48.$ 8. $\left|(\frac{1}{2}A)^{-1}\right| = |2A^{-1}| = 2^5\,|A^{-1}| = 2^5|A|^{-1} = 2^4 = 16.$ 9. $f(x) = \begin{vmatrix} 2x & x & 2 & 3 \\ 1 & x & 2 & 4 \\ 3 & 2 & 3x & -1 \\ 2 & 5 & 3 & x \end{vmatrix} = 6x^4 - 3x^3 - 116x^2 + 65x + 14, x^4$的系数为6，$x^3$的系数为$-3$,数项为14. 10. $A_{11}+A_{21}+A_{31}+A_{41} = \begin{vmatrix} a & c & d \\ d & c & a \\ a & c & b \end{vmatrix} - \begin{vmatrix} b & c & d \\ d & c & a \\ a & c & b \end{vmatrix} +$

$\begin{vmatrix} b & c & d \\ a & c & d \\ a & c & b \end{vmatrix} - \begin{vmatrix} b & c & d \\ a & c & d \\ d & c & a \end{vmatrix} = \begin{vmatrix} a-b & 0 & 0 \\ d & c & a \\ a & c & b \end{vmatrix} + \begin{vmatrix} b & c & d \\ a & c & d \\ a-d & 0 & b-a \end{vmatrix} = \begin{vmatrix} a-b & 0 & 0 \\ d & c & a \\ a & c & b \end{vmatrix} +$
$\begin{vmatrix} b-a & 0 & 0 \\ a & c & d \\ a-d & 0 & b-a \end{vmatrix} = 0$.或：$A_{11}+A_{21}+A_{31}+A_{41} = \begin{vmatrix} 1 & b & c & d \\ 1 & a & c & d \\ 1 & d & c & a \\ 1 & a & c & b \end{vmatrix} =$
0.　　11. $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 2 & 5 & 2 & -1 \\ 6 & -3 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 6 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$.或原式$=$
$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = -2$.　　12. 设$|A|$的第一列元素的代数余子式之和为$x$,且$A$的每行的元素之和为$b(b \neq 0)$,则$a = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$
$\begin{vmatrix} b & a_{12} & \cdots & a_{1n} \\ b & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ b & a_{n2} & \cdots & a_{nn} \end{vmatrix} = bx, x = \frac{a}{b}(b \neq 0)$.

二、设$D = \begin{vmatrix} 2 & -5 & 2 & 1 \\ -3 & 1 & 0 & -5 \\ 1 & 3 & 1 & 3 \\ -2 & -4 & -1 & -3 \end{vmatrix}$, D的(i,j)元的余子式和代数余子式依次记作M_{ij}和A_{ij}，计算：

（1）$2A_{41}-5A_{42}+2A_{43}+A_{44}$；（2）$A_{11}-A_{21}+A_{31}-A_{41}$；（3）$M_{11}-M_{12}+M_{13}-M_{14}$.

解：（1）D的第一行元素$2,-5,2,1$与D的第四行对应元素的代数余子式$A_{41},A_{42},A_{43},A_{44}$,故$2A_{41}-5A_{42}+2A_{43}+A_{44}=0$;

（2）$A_{11}-A_{21}+A_{31}-A_{41} = \begin{vmatrix} 1 & -5 & 2 & 1 \\ -1 & 1 & 0 & -5 \\ 1 & 3 & 1 & 3 \\ -1 & -4 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 2 & 1 \\ 0 & -4 & 2 & -4 \\ 0 & 8 & -1 & 2 \\ 0 & -9 & 1 & -2 \end{vmatrix} =$

$$\begin{vmatrix} -4 & 2 & -4 \\ 8 & -1 & 2 \\ -9 & 1 & -2 \end{vmatrix} = 0.$$

（3）$M_{11}-M_{12}+M_{13}-M_{14}=M_{11}+(-1)^3M_{12}+M_{13}+(-1)^5M_{14}=$
$A_{11}+A_{12}+A_{13}+A_{14}=\begin{vmatrix} 1 & 1 & 1 & 1 \\ -3 & 1 & 0 & -5 \\ 1 & 3 & 1 & 3 \\ -2 & -4 & -1 & -3 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & 3 & -2 \\ 0 & 2 & 0 & 2 \\ 0 & -2 & 1 & -1 \end{vmatrix}=$
$\begin{vmatrix} 4 & 3 & -2 \\ 2 & 0 & 2 \\ -2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} 4 & 3 & -2 \\ 1 & 0 & 1 \\ -2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 & -6 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 3 & -6 \\ 1 & 1 \end{vmatrix} =$
$-18.$

三、计算下列行列式：

1. $\begin{vmatrix} 0 & 2 & 0 & 0 \\ 1 & -1 & 5 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{vmatrix}.$

解：原式$=-2\begin{vmatrix} 1 & 5 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}=2\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}=-8.$

2. $\begin{vmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ y & 0 & 0 & 0 & x \end{vmatrix}.$

解：原式$=x\begin{vmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{vmatrix}+y\begin{vmatrix} y & 0 & 0 & 0 \\ x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \end{vmatrix}=x^2\begin{vmatrix} x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{vmatrix}+$
$y^2\begin{vmatrix} y & 0 & 0 \\ x & y & 0 \\ 0 & x & y \end{vmatrix}=x^5+y^5.$

3. $\begin{vmatrix} 1 & -1 & 1 & x-1 \\ 1 & -1 & x+1 & -1 \\ 1 & x-1 & 1 & -1 \\ x+1 & -1 & 1 & -1 \end{vmatrix}$.

解：原式$= \begin{vmatrix} x & -1 & 1 & x-1 \\ x & -1 & x+1 & -1 \\ x & x-1 & 1 & -1 \\ x & -1 & 1 & -1 \end{vmatrix} = x \begin{vmatrix} 1 & -1 & 1 & x-1 \\ 1 & -1 & x+1 & -1 \\ 1 & x-1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$

$= x \begin{vmatrix} 1 & -1 & 1 & x \\ 1 & -1 & x+1 & 0 \\ 1 & x-1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{vmatrix} = -x^2 \begin{vmatrix} 1 & -1 & x+1 \\ 1 & x-1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

$= -x^2 \begin{vmatrix} 1 & 0 & x+1 \\ 1 & x & 1 \\ 1 & 0 & 1 \end{vmatrix} = -x^3 \begin{vmatrix} 1 & x+1 \\ 1 & 1 \end{vmatrix} = x^4.$

4. $\begin{vmatrix} 1-a & a & 0 & 0 & 0 \\ -1 & 1-a & a & 0 & 0 \\ 0 & -1 & 1-a & a & 0 \\ 0 & 0 & -1 & 1-a & a \\ 0 & 0 & 0 & -1 & 1-a \end{vmatrix}$.

解：原式$= (1-a) \begin{vmatrix} 1-a & a & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ 0 & 0 & -1 & 1-a \end{vmatrix} + \begin{vmatrix} a & 0 & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ 0 & 0 & -1 & 1-a \end{vmatrix}$

$= (1-a) \left[(1-a) \begin{vmatrix} 1-a & a & 0 \\ -1 & 1-a & a \\ 0 & -1 & 1-a \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ -1 & 1-a & a \\ 0 & -1 & 1-a \end{vmatrix} \right] +$

$a \begin{vmatrix} 1-a & a & 0 \\ -1 & 1-a & a \\ 0 & -1 & 1-a \end{vmatrix}$

$= \left[(1-a)^2 + a \right] \begin{vmatrix} 1-a & a & 0 \\ -1 & 1-a & a \\ 0 & -1 & 1-a \end{vmatrix} + (1-a) \begin{vmatrix} a & 0 & 0 \\ -1 & 1-a & a \\ 0 & -1 & 1-a \end{vmatrix}$

$= \left[(1-a)^2 + a \right] \left[(1-a) \begin{vmatrix} 1-a & a \\ -1 & 1-a \end{vmatrix} + \begin{vmatrix} a & 0 \\ -1 & 1-a \end{vmatrix} \right] + a(1-a) \begin{vmatrix} 1-a & a \\ -1 & 1-a \end{vmatrix}$

$= (a^2-a+1)\left[(1-a)(a^2-a+1)+(a-a^2)\right]+a(1-a)(a^2-a+1)$
$= (a^2-a+1)(-a^3+a^2-a+1)+a(1-a)(a^2-a+1)$
$= -a^5+a^4-a^3+a^2-a+1 = -(a-1)(a+a^2+1)(-a+a^2+1)$.

对一般的n阶行列式，有$D_n=(1-a)D_{n-1}+aD_{n-2}\,(n\geqslant 3)$.

5. $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 5 \end{vmatrix}$.

解：原式$=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{vmatrix}+5\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix}=\begin{vmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix}-2\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix}-$
$5\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix}+20\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix}=-2\begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix}+15\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}+20\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}+$
$60\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=-34.$

四、解方程$\begin{vmatrix} \lambda-3 & -1 & 2 \\ 2 & \lambda-3 & -1 \\ 2 & -1 & \lambda-3 \end{vmatrix}=0.$

解：$\begin{vmatrix} \lambda-3 & -1 & 2 \\ 2 & \lambda-3 & -1 \\ 2 & -1 & \lambda-3 \end{vmatrix}=\begin{vmatrix} \lambda-3 & -1 & 2 \\ 0 & \lambda-2 & 2-\lambda \\ 2 & -1 & \lambda-3 \end{vmatrix}=(\lambda-3)\begin{vmatrix} \lambda-2 & 2-\lambda \\ -1 & \lambda-3 \end{vmatrix}+$
$2\begin{vmatrix} -1 & 2 \\ \lambda-2 & 2-\lambda \end{vmatrix}=(\lambda-3)(\lambda^2-6\lambda+8)-2\lambda+4=(\lambda-5)(\lambda-2)^2$
$=0, \lambda_1=\lambda_2=2, \lambda_3=5.$

3.5 行列式的计算(2) 3.6 克拉姆法则

一、利用范德蒙行列式解题：

1. 计算行列式$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{vmatrix}$.

解：$D_4 = (4-3)(4-2)(4-1)(3-2)(3-1)(2-1) = 12.$

2. 计算行列式$D_4 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \\ 5 & 4 & 3 & 2 \end{vmatrix}$.

解：$D_4 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \\ 6 & 6 & 6 & 6 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -6\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{vmatrix} =$
$-6 \times 12 = -72.$

3. 计算行列式$D_3 = \begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

解：$D_3 = \begin{vmatrix} b+c+a & c+a+b & a+b+c \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} =$
$(a+b+c)(c-b)(c-a)(b-a)$.

4. 证明：当a,b,c,d互不相同时，$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = 0$的充要条件是$a+$
$b+c+d=0$.

证明：因为$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = (a+b+c+d)(d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$,

又因为a,b,c,d互不相同，故$(a+b+c+d)(d-c)(d-b)(d-a)(c-b)(c-a)(b-a) =$

0得$a+b+c+d=0$,即$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}=0$的充要条件是$a+b+c+d=0$.

或添加一行一列构成范德蒙行列式. 比较同类项的系数.

二、计算下列n阶行列式：

1. $D_n=\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix}$.

解：$D_1=1, D_2=\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}=-2, n\geqslant 3$时，利用$-r_1+r_n, D_n=$
$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & n-2 \end{vmatrix}=(n-2)\,D_{n-1}.D_n=-2\cdot(n-2)!\,(n\geqslant 2)$

注：$D_n=\begin{vmatrix} a+a_1 & a & a & \cdots & a \\ a & a+a_2 & a & \cdots & a \\ a & a & a+a_3 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & a+a_n \end{vmatrix} \overset{-r_1+r_n}{=} \begin{vmatrix} a+a_1 & a & a & \cdots & a \\ a & a+a_2 & a & \cdots & a \\ a & a & a+a_3 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ -a_1 & 0 & 0 & \cdots & a_n \end{vmatrix}$

$=a_nD_{n-1}+(-1)^n\,a_1\begin{vmatrix} a & a & \cdots & a & a \\ a+a_2 & a & \cdots & a & a \\ a & a+a_3 & \cdots & a & a \\ \vdots & \vdots & & & \vdots \\ a & a & \cdots & a+a_{n-1} & a \end{vmatrix}=a_nD_{n-1}+$

$(-1)^n\,a_1\begin{vmatrix} a & a & \cdots & a & a \\ a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \end{vmatrix}=a_nD_{n-1}+aa_1\cdots a_{n-1}, D_1=a+$

a_1,故$D_n=\left(1+a\sum_{i=1}^{n}\frac{1}{a_i}\right)a_1a_2\cdots a_n$(可用数学归纳法证明).

2. $D_n=\begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & a_{n-2} & 0 \\ 1 & 0 & 0 & \cdots & 0 & a_{n-1} \end{vmatrix}, a_i\neq 0(i=1,2,\cdots,n-1)$.

解：$D_1=a_0, D_2=\begin{vmatrix} a_0 & 1 \\ 1 & a_1 \end{vmatrix}=a_0a_1-1=a_1\left(a_0-\sum_{i=1}^{1}\frac{1}{a_i}\right), n\geqslant$ 2时，按第一列展开：$D_n=a_{n-1}\begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & a_{n-2} \end{vmatrix}+(-1)^{n+1}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-2} & 0 \end{vmatrix}$
$=a_{n-1}D_{n-1}-a_1\cdots a_{n-2}$.故$D_1=a_0$时，$D_n=a_1a_2\cdots a_{n-1}\left(a_0-\sum_{i=1}^{n-1}\frac{1}{a_i}\right)$(可用数学归纳法证明).

3. $D_n=\begin{vmatrix} 2 & 0 & 0 & \cdots & 0 & 2 \\ -1 & 2 & 0 & \cdots & 0 & 2 \\ 0 & -1 & 2 & \cdots & 0 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}$.

解：$D_1=2$,按第一行展开：

$D_n=2\begin{vmatrix} 2 & 0 & \cdots & 0 & 2 \\ -1 & 2 & \cdots & 0 & 2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & \cdots & -1 & 2 \end{vmatrix}+(-1)^{n+1}2\begin{vmatrix} -1 & 2 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2 \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}=$
$2D_{n-1}+2$,故$D_n=2+2^2+\cdots+2^n=2^{n+1}-2$.

注：$D_n=\begin{vmatrix} x & 0 & 0 & \cdots & 0 & a_n \\ -1 & x & 0 & \cdots & 0 & a_{n-1} \\ 0 & -1 & x & \cdots & 0 & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & a_2 \\ 0 & 0 & 0 & \cdots & -1 & x+a_1 \end{vmatrix}, D_1=x+a_1$,按第一行

展开：$D_n = xD_{n-1} + a_n, D_n = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$. (可用数学归纳法证明).

4. $D_n = \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 & 1 \\ 1 & 1+a_2 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_{n-1} & 1 \\ 1 & 1 & \cdots & 1 & 1+a_n \end{vmatrix}, a_i \neq 0 (i = 1, 2, \cdots, n)$.

解：$D_1 = 1 + a_1, D_2 = \begin{vmatrix} 1+a_1 & 1 \\ 1 & 1+a_2 \end{vmatrix} = (a_1 + a_2 + a_1a_2) = a_2(1+a_1) + a_1, n \geqslant 3$时，$D_n = \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 & 1 \\ 1 & 1+a_2 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_{n-1} & 1 \\ -a_1 & 0 & \cdots & 0 & a_n \end{vmatrix}$

$= a_nD_{n-1} - a_1(-1)^{n+1} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ 1+a_2 & \cdots & 1 & 1 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 1+a_{n-1} & 1 \end{vmatrix} = a_nD_{n-1} + a_1(-1)^n \begin{vmatrix} 1 & \cdots & 1 & 1 \\ a_2 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & a_{n-1} & 0 \end{vmatrix}$

$= a_nD_{n-1} + a_1a_2\cdots a_{n-1}$,故$D_n = a_1a_2\cdots a_n\left(1 + \sum\limits_{i=1}^{n} a_i\right)$. (可用数学归纳法证明).

三、设$A = \begin{pmatrix} \lambda & 1 & 1 \\ 0 & \lambda-1 & 0 \\ 1 & 1 & \lambda \end{pmatrix}, b = \begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix}$,已知线性方程组$Ax = b$有两个不同的解，求$\lambda$和$a$的值.

解：$\begin{pmatrix} \lambda & 1 & 1 & a \\ 0 & \lambda-1 & 0 & 1 \\ 1 & 1 & \lambda & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1 \\ \lambda & 1 & 1 & a \end{pmatrix}$

$\xrightarrow{-\lambda r_1 + r_3} \begin{pmatrix} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1 \\ 0 & 1-\lambda & 1-\lambda^2 & a-\lambda \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 1 \\ 0 & 0 & 1-\lambda^2 & a+1-\lambda \end{pmatrix}$.由已知线性方程组$Ax = b$有两个不同的解，故$\lambda - 1 \neq 0$且$1 - \lambda^2 = a + 1 - \lambda = 0$,故$\lambda = -1, a = \lambda - 1 = -2$.

另解：由已知线性方程组$Ax = b$有两个不同的解，故$|A| = 0$（否则

由克拉姆法则方程有惟一解），即$\begin{vmatrix} \lambda & 1 & 1 \\ 0 & \lambda-1 & 0 \\ 1 & 1 & \lambda \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1 = 0,$
Solution is: $1, -1$以下同第一种解法前半部分，从而$\lambda = -1, a = \lambda - 1 = -2.$

习题课

一、填空题：

1. 行列式$D=\begin{vmatrix} 1 & x & 1 \\ 2x & x & 1 \\ 3 & 2 & x \end{vmatrix}$中$x$项的系数为__________.

2. 已知矩阵$A=\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$,$B=\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}$,则$\left|AB^T\right|=$______.

3. 排列$n,n-1,n-2,\cdots,2,1$的逆序数为__________.

4. 已知矩阵$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,$B=\begin{pmatrix} a_{11}-a_{12} & a_{13}+a_{11} & 2a_{12} \\ a_{21}-a_{22} & a_{23}+a_{21} & 2a_{22} \\ a_{31}-a_{32} & a_{33}+a_{31} & 2a_{32} \end{pmatrix}$,若$|A|=2$,则$|B|=$_____.

5. 已知3阶方阵A的行列式等于-2,则$|-2A^{-1}|=$__________.

6. 已知5阶行列式D的第3列元素为$2,1,3,4,2$,第5列元素对应的代数余子式为$-1,2,0,x,4$,则$x=$_____.

提示：1. $\begin{vmatrix} 1 & x & 1 \\ 2x & x & 1 \\ 3 & 2 & x \end{vmatrix}=-2x^3+x^2+4x-2$,$x$项的系数为4. 2. $\left|AB^T\right|=|A|\left|B^T\right|=|A|\,|B|=\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}\begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix}=5$. 3. 逆序数为$(n-1)+\cdots 2+1=\frac{n(n-1)}{2}$. 4. $|B|=\begin{vmatrix} a_{11} & a_{13}+a_{11} & 2a_{12} \\ a_{21} & a_{23}+a_{21} & 2a_{22} \\ a_{31} & a_{33}+a_{31} & 2a_{32} \end{vmatrix}+\begin{vmatrix} -a_{12} & a_{13}+a_{11} & 2a_{12} \\ -a_{22} & a_{23}+a_{21} & 2a_{22} \\ -a_{32} & a_{33}+a_{31} & 2a_{32} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{13} & 2a_{12} \\ a_{21} & a_{23} & 2a_{22} \\ a_{31} & a_{33} & 2a_{32} \end{vmatrix}=-2\,|A|=-4$. 5. $|-2A^{-1}|=(-2)^3\,|A^{-1}|=-8\,|A|^{-1}=4$. 6. $(2,1,3,4,2)\,(-1,2,0,x,4)^T=4x+8=0,x=-2$.

二、选择题：

1. 下列矩阵与矩阵$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$乘法可交换的是（ ）

A. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ C. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2. 已知A, B都是n阶方阵，且满足$AB = 0$,则必有（ ）

A. $A = 0, B = 0$ B. 若$|A| \neq 0$,则$|B| = 0$ C. $|A| = 0, |B| = 0$ D. 若$A \neq 0$,则$B = 0$

3. 设n阶方阵A经初等变换后所得方阵记为B,则必有（ ）

A. $|A| = |B|$ B. $|A| \neq |B|$ C. $|A| \cdot |B| > 0$ D. 若$|B| \neq 0$,则$|A| \neq 0$

4. 设$D = \begin{vmatrix} 1 & 2 & 4 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{vmatrix}$,则代数余子式$A_{21} =$（ ）

A. 6 B. 1 C. -6 D. -1

5. 已知3阶方阵$A = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & t \\ 3 & 5 & 3 \end{vmatrix}$,若齐次线性方程组$Ax = 0$有非零解，则$t =$（ ）

A. 3 B. 4 C. -4 D. 2

6. 设矩阵$A, B, A+B, A^{-1}+B^{-1}$均为n阶可逆阵，则$(A^{-1}+B^{-1})^{-1} =$（ ）

A. $A^{-1}+B^{-1}$ B. $A+B$ C. $A(A+B)^{-1}B$ D. $(A+B)^{-1}$

提示：1. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.B. 2. B. 3. D. 4. $A_{21} = (-1)^3\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = -6$,C. 5. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & t \\ 3 & 5 & 3 \end{vmatrix} = 4t-16 = 0, t = 4$.B. 6. $(A^{-1}+B^{-1})A(A+B)^{-1}B = (A^{-1}A+B^{-1}A)(B^{-1}(A+B))^{-1} = (E+B^{-1}A)(B^{-1}A+B^{-1}B)^{-1} = (E+B^{-1}A)(E+B^{-1}A)^{-1} = E$,C.

三、计算行列式$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 3 & 1 & 2 \\ 1 & 9 & 1 & 4 \\ -1 & 27 & 1 & 8 \end{vmatrix}$.

解：$D=\begin{vmatrix}1&1&1&1\\0&4&2&3\\0&8&0&3\\0&28&2&9\end{vmatrix}=\begin{vmatrix}4&2&3\\8&0&3\\28&2&9\end{vmatrix}=\begin{vmatrix}4&2&3\\4&-2&0\\16&-4&0\end{vmatrix}=3\begin{vmatrix}4&-2\\16&-4\end{vmatrix}=$ 48.

四、解线性方程组$\begin{cases}2x_1+x_2-x_3+x_4=1\\3x_1-2x_2+2x_3-3x_4=2\\5x_1+x_2-x_3+2x_4=-1\\2x_1-x_2+x_3-3x_4=4\end{cases}$.

解：$\begin{pmatrix}2x_1+x_2-x_3+x_4=1\\3x_1-2x_2+2x_3-3x_4=2\\5x_1+x_2-x_3+2x_4=-1\\2x_1-x_2+x_3-3x_4=4\end{pmatrix}$, Corresponding matrix: $\begin{pmatrix}2&1&-1&1&1\\3&-2&2&-3&2\\5&1&-1&2&-1\\2&-1&1&-3&4\end{pmatrix}$,

row echelon form: $\begin{pmatrix}1&0&0&0&0\\0&1&-1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$,原方程无解.

五、计算行列式$D=\begin{vmatrix}3&0&4&0\\2&2&2&2\\0&-7&0&0\\9&-8&7&5\end{vmatrix}$的第4行元素的余子式之和.

解：$M_{41}+M_{42}+M_{43}+M_{44}=+\begin{vmatrix}0&4&0\\2&2&2\\-7&0&0\end{vmatrix}+\begin{vmatrix}3&4&0\\2&2&2\\0&0&0\end{vmatrix}+\begin{vmatrix}3&0&0\\2&2&2\\0&-7&0\end{vmatrix}+$ $\begin{vmatrix}3&0&4\\2&2&2\\0&-7&0\end{vmatrix}=-28.$

或：$M_{41}+M_{42}+M_{43}+M_{44}=\begin{vmatrix}3&0&4&0\\2&2&2&2\\0&-7&0&0\\-1&1&-1&1\end{vmatrix}=7\begin{vmatrix}3&4&0\\2&2&2\\-1&-1&1\end{vmatrix}=$ $7\begin{vmatrix}3&4&0\\0&0&4\\-1&-1&1\end{vmatrix}=-28.$

六、设矩阵$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B = P^{-1}AP$,其中$P$为3阶可逆阵，求：

（1）A^4; （2）A^{2004}; （3）$B^{2004} - 2A^2$.

解：（1）$A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$;（2）$A^{2004} = (A^4)^{501} = (E)^{501} = E$;

（3）$B^2 = (P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$,

$B^4 = (P^{-1}A^2P)(P^{-1}A^2P) = P^{-1}A^4P = P^{-1}EP = E, B^{2004} - 2A^2 = E^{501} - 2A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

七、讨论k为何值时，线性方程组$\begin{cases} kx_1 + x_2 + x_3 = 1 \\ x_1 + kx_2 + x_3 = k \\ x_1 + x_2 + kx_3 = k^2 \end{cases}$

（1）有惟一解; （2）有无穷多个解; （3）无解.

解：$\begin{pmatrix} kx_1 + x_2 + x_3 = 1 \\ x_1 + kx_2 + x_3 = k \\ x_1 + x_2 + kx_3 = k^2 \end{pmatrix}$, Corresponding matrix: $\begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & k \\ 1 & 1 & k & k^2 \end{pmatrix}$

$\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 1 & k & k^2 \\ 1 & k & 1 & k \\ k & 1 & 1 & 1 \end{pmatrix} \xrightarrow[-kr_1 + r_2]{-r_1 + r_2,} \begin{pmatrix} 1 & 1 & k & k^2 \\ 0 & k-1 & 1-k & k-k^2 \\ 0 & 1-k & 1-k^2 & 1-k^3 \end{pmatrix} \xrightarrow{r_2 + r_3}$

$\begin{pmatrix} 1 & 1 & k & k^2 \\ 0 & k-1 & 1-k & k-k^2 \\ 0 & 0 & 2-k-k^2 & 1+k-k^2-k^3 \end{pmatrix}$.

（1）当$2 - k - k^2 \neq 0$且$k - 1 \neq 0$,即$k \neq 1$且$k \neq -2$时，原方程有惟一解：$\begin{cases} x_1 = -\frac{k+1}{k+2} \\ x_2 = \frac{1}{k+2} \\ x_3 = \frac{1}{k+2}(2k + k^2 + 1) \end{cases}$.

（2）当$k = 1$时，$k - 1 = 2 - k - k^2 = 1 + k - k^2 - k^3 = 0$,原方程有无穷多个解：$\begin{cases} x_1 = -x_2 - x_3 + 1 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$, $(x_2, x_3$是任意实数$)$;

（3）当$k = -2$时，$2 - k - k^2 = 0, 1 + k - k^2 - k^3 = 7 \neq 0$故原方程有无解.

八、计算$2n$阶行列式$\begin{vmatrix} a & 0 & 0 & \cdots & 0 & b \\ 0 & a & 0 & \cdots & b & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & b & 0 & \cdots & a & 0 \\ b & 0 & 0 & \cdots & 0 & a \end{vmatrix}$.

解：参见《线性代数》P.64练习13.令原式为D_{2n}.$D_2 = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2, D_{2n} = a\begin{vmatrix} a & 0 & \cdots & b & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ b & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & a \end{vmatrix} - b\begin{vmatrix} 0 & 0 & \cdots & 0 & b \\ a & 0 & \cdots & b & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ b & 0 & \cdots & a & 0 \end{vmatrix} = \left(a^2 - b^2\right) D_{2n-2}$,从而$D_{2n} = \left(a^2 - b^2\right)^n$.(可用数学归纳法证明).

九、设矩阵$B = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$,矩阵$X$满足$X(E - C^{-1}B)^T C^T = E$,其中$E$是4阶单位矩阵，计算$|X|$.

解：因为$(E-C^{-1}B)^T C^T = [C(E - C^{-1}B)]^T = (C - B)^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$,故由$X\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix} = E$得$|X|\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix} = 1, |X| = 1$.

4.1-4.3 分块矩阵 初等矩阵 矩阵的秩

一、填空题：

1. 设A,B均为方阵，则$\begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix}=$______；$\begin{vmatrix} A & C \\ 0 & B \end{vmatrix}=$______；$\begin{vmatrix} A & 0 \\ C & B \end{vmatrix}=$______；$\begin{vmatrix} 0 & A_n \\ B_m & 0 \end{vmatrix}=$______.

2. 若$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=5$,则$\begin{vmatrix} a_{12} & 2a_{11} & 0 \\ a_{22} & 2a_{21} & 0 \\ 0 & -2 & -1 \end{vmatrix}=$______.

3. 设分块矩阵$A=(\alpha_1,\gamma_1,\gamma_2,\gamma_3)$和$B=(\beta_1,\gamma_1,\gamma_2,\gamma_3)$均为四阶方阵，已知$|A|=4,|B|=1$,则$|A+B|=$______.

4. 若$A=\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$,则$A^3=$______.

5. 设$A_{m\times n},\overline{A}$分别为线性方程组$Ax=b$的系数矩阵与增广矩阵，则有解的充分必要条件是______；有惟一解的充分必要条件是______；有无穷多个解的充分必要条件是______.

6. 设$A=PBQ$,其中$A=\begin{pmatrix} 3 & 2 & 1 \\ -2 & -1 & 0 \\ 5 & 4 & 3 \end{pmatrix}$,$P=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,$Q=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,则$B=$______.

7. 设A,P均为3阶矩阵，P^T为P的转置矩阵，且$P^TAP=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$,若$P=(\alpha_1,\alpha_2,\alpha_3)$,$Q=(\alpha_1,\alpha_2-\alpha_1,2\alpha_3)$,则$Q^TAQ=$______.

8. 已知$P_1=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,$P_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,$A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$,则$P_1^{2016}AP_2^{2017}=$______.

提示：1. $|A||B|,|A||B|,|A||B|,(-1)^{mn}|A_n||B_m|$. 2. -10. 3. $A+B=(\alpha_1,\gamma_1,\gamma_2,\gamma_3)+(\beta_1,\gamma_1,\gamma_2,\gamma_3)=(\alpha_1+\beta_1,2\gamma_1,2\gamma_2,2\gamma_3)$, $|A+B|=8|A|+8|B|=40$. 4. 令$B=\begin{pmatrix}1&1\\2&2\end{pmatrix}$, $C=\begin{pmatrix}3&4\\0&3\end{pmatrix}$,则$A=\begin{pmatrix}B&0\\0&C\end{pmatrix}$, $A^3=\begin{pmatrix}B^3&0\\0&C^3\end{pmatrix}$.因为$B^3=\begin{pmatrix}9&9\\18&18\end{pmatrix}$, $C^3=\begin{pmatrix}27&108\\0&27\end{pmatrix}$,则$A^3=\begin{pmatrix}9&9&0&0\\18&18&0&0\\0&0&27&108\\0&0&0&27\end{pmatrix}$.

5. $\operatorname{rank}(A)=\operatorname{rank}(\overline{A})$;$\operatorname{rank}(A)=\operatorname{rank}(\overline{A})=n$;$\operatorname{rank}(A)=\operatorname{rank}(\overline{A})<n$.

6. 由$A=PBQ$,知$B=P^{-1}AQ^{-1}$.$P^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}$, $Q^{-1}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$,故$B=P^{-1}AQ^{-1}=\begin{pmatrix}1&2&3\\2&3&4\\3&4&5\end{pmatrix}$.逆矩阵计算：利用初等矩阵的逆矩阵，得到$P^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}$, $Q^{-1}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$.或$\begin{pmatrix}1&0&0&1&0&0\\-2&1&0&0&1&0\\0&0&1&0&0&1\end{pmatrix}\xrightarrow{2r_1+r_2}\begin{pmatrix}1&0&0&1&0&0\\0&1&0&2&1&0\\0&0&1&0&0&1\end{pmatrix}$, $\begin{pmatrix}0&0&1&1&0&0\\0&1&0&0&1&0\\1&0&0&0&0&1\end{pmatrix}\xrightarrow{r_1\leftrightarrow r_3}\begin{pmatrix}1&0&0&0&0&1\\0&1&0&0&1&0\\0&0&1&1&0&0\end{pmatrix}$.

7. 由$P=(\alpha_1,\alpha_2,\alpha_3)$, $Q=(\alpha_1,\alpha_2-\alpha_1,2\alpha_3)$及初等的列变换,知

$$Q=P\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix}=P\begin{pmatrix}1&-1&0\\0&1&0\\0&0&2\end{pmatrix},$$

$$故Q^TAQ=\begin{pmatrix}1&-1&0\\0&1&0\\0&0&2\end{pmatrix}^T\left(P^TAP\right)\begin{pmatrix}1&-1&0\\0&1&0\\0&0&2\end{pmatrix}$$

$$=\begin{pmatrix}1&-1&0\\0&1&0\\0&0&2\end{pmatrix}^T\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1&-1&0\\0&1&0\\0&0&2\end{pmatrix}=\begin{pmatrix}1&1&6\\3&0&6\\14&2&36\end{pmatrix}.$$

8. 由初等的行变换，由$P_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$, $P_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$,知$P_1^{2016}=$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P_2^{2017} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,再由$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$,则$P_1^{2016} A P_2^{2017} =$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{pmatrix}$.

二、求下列矩阵的秩：

1. $A = \begin{pmatrix} 6 & 1 & 1 & 7 \\ 4 & 0 & 4 & 1 \\ 1 & 2 & -9 & 0 \\ -1 & 3 & -16 & -1 \\ 2 & -4 & 22 & 3 \end{pmatrix}$.

解：$\begin{pmatrix} 6 & 1 & 1 & 7 \\ 4 & 0 & 4 & 1 \\ 1 & 2 & -9 & 0 \\ -1 & 3 & -16 & -1 \\ 2 & -4 & 22 & 3 \end{pmatrix}$, rank: 3.

$\begin{pmatrix} 6 & 1 & 1 & 7 \\ 4 & 0 & 4 & 1 \\ 1 & 2 & -9 & 0 \\ -1 & 3 & -16 & -1 \\ 2 & -4 & 22 & 3 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, rank: 3.

$$\begin{pmatrix} 6 & 1 & 1 & 7 \\ 4 & 0 & 4 & 1 \\ 1 & 2 & -9 & 0 \\ -1 & 3 & -16 & -1 \\ 2 & -4 & 22 & 3 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 2 & -9 & 0 \\ 4 & 0 & 4 & 1 \\ 6 & 1 & 1 & 7 \\ -1 & 3 & -16 & -1 \\ 2 & -4 & 22 & 3 \end{pmatrix} \xrightarrow[r_1+r_4, -2r_1+r_5]{-4r_1+r_2, -6r_1+r_3,}$$
$$\begin{pmatrix} 1 & 2 & -9 & 0 \\ 0 & -8 & 40 & 1 \\ 0 & -11 & 55 & 7 \\ 0 & 5 & -25 & -1 \\ 0 & -8 & 40 & 3 \end{pmatrix} \xrightarrow[2r_4+r_3]{-r_2+r_5,} \begin{pmatrix} 1 & 2 & -9 & 0 \\ 0 & -8 & 40 & 1 \\ 0 & -1 & 5 & 5 \\ 0 & 5 & -25 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow[-5r_5+r_3, -r_5+r_2]{\frac{1}{2}r_5, r_5+r_4,}$$

$$\begin{pmatrix} 1 & 2 & -9 & 0 \\ 0 & -8 & 40 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 5 & -25 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[8r_3+r_2,-5r_3+r_4]{-r_3,-2r_3+r_1,} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \leftrightarrow r_5]{r_2 \leftrightarrow r_3,}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathrm{rank}(A) = 3.$$

2. $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$

解：$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, rank: 5

$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, rank:
5.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{-r_1+r_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[r_3+r_2,-r_3+r_1]{\frac{1}{2}r_3,-r_3+r_4,} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{-r_4+r_5} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

rank: 5.

3. $A = \begin{pmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{pmatrix}.$

解：$a=b=0$时，$A=0$,$\mathrm{rank}(A)=0$;当$a=b\neq 0$时，$\mathrm{rank}(A)=1$;

当$a\neq b, a+4b=0, ab\neq 0$时，

$$\begin{pmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{pmatrix} \xrightarrow[i=1,2,3,4]{t_i+r_5,} \begin{pmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[i=1,2,3]{-\frac{b}{a}t_i+r_4,}$$

$$\begin{pmatrix} a & b & b & b & b \\ 0 & a-b & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\mathrm{rank}(A)=4;$$

$(a+4b)(a-b)^4\neq 0$时，

$$\begin{vmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{vmatrix} = \begin{vmatrix} a+4b & b & b & b & b \\ a+4b & a & b & b & b \\ a+4b & b & a & b & b \\ a+4b & b & b & a & b \\ a+4b & b & b & b & a \end{vmatrix} = (a+4b)\begin{vmatrix} 1 & b & b & b & b \\ 1 & a & b & b & b \\ 1 & b & a & b & b \\ 1 & b & b & a & b \\ 1 & b & b & b & a \end{vmatrix} =$$

$$(a+4b)\begin{vmatrix} 1 & b & b & b & b \\ 0 & a-b & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & 0 & a-b \end{vmatrix} = (a+4b)(a-b)^4\neq 0,$$即$\mathrm{rank}(A)=$ 5.

三、已知矩阵$A=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}$,求$A^n$.

解：$A^2=\begin{pmatrix}2&2&0&0\\2&2&0&0\\0&0&1&0\\0&0&2&1\end{pmatrix},A^3=\begin{pmatrix}4&4&0&0\\4&4&0&0\\0&0&1&0\\0&0&3&1\end{pmatrix},\cdots,A^n=\begin{pmatrix}2^{n-1}&2^{n-1}&0&0\\2^{n-1}&2^{n-1}&0&0\\0&0&1&0\\0&0&n&1\end{pmatrix}$.
（可用数学归纳法证明）

四、设矩阵$A=\begin{pmatrix}1&0&1\\-1&-1&1\\0&2&a\end{pmatrix},B=\begin{pmatrix}1&0&1\\0&-1&2\\0&0&0\end{pmatrix}$.

（1）问a为何值时，矩阵A和矩阵B等价？

（2）当A和B等价时，求可逆阵P，使得$PA=B$.

解：（1）$\begin{pmatrix}1&0&1\\-1&-1&1\\0&2&a\end{pmatrix}\xrightarrow{r_1+r_2}\begin{pmatrix}1&0&1\\0&-1&2\\0&2&a\end{pmatrix}\xrightarrow{2r_2+r_3}\begin{pmatrix}1&0&1\\0&-1&2\\0&0&a+4\end{pmatrix}$,当$a+4=0$即$a=-4$时，$\operatorname{rank}(A)=2,A$与$B$等价.

（2）当A与B等价，必有$a=-4$.由（1）及$PA=B$知$P=\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\2&2&1\end{pmatrix}$.

五、问a,b为何值时，线性方程组$\begin{cases}x_1+x_2+x_3+x_4=0\\x_2+2x_3+2x_4=1\\-x_2+(a-3)x_3-2x_4=b\\3x_1+2x_2+x_3+ax_4=-1\end{cases}$有惟一解、无解、无穷多个解？当有无穷多个解时求出其一般解.

解：$\begin{cases}x_1+x_2+x_3+x_4=0\\x_2+2x_3+2x_4=1\\-x_2+(a-3)\,x_3-2x_4=b\\3x_1+2x_2+x_3+ax_4=-1\end{cases}$, Corresponding matrix: $\begin{pmatrix}1&1&1&1&0\\0&1&2&2&1\\0&-1&a-3&-2&b\\3&2&1&a&-1\end{pmatrix}$,

row echelon form: $\begin{pmatrix}1&0&0&0&\frac{1}{a-1}\left(b-a+2\right)\\0&1&0&0&-\frac{1}{a-1}\left(2b-a+3\right)\\0&0&1&0&\frac{1}{a-1}\left(b+1\right)\\0&0&0&1&0\end{pmatrix}\left(a-1\neq0\right)$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{pmatrix} \xrightarrow{-3r_1+r_4} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 0 & -1 & -2 & a-3 & -1 \end{pmatrix} \xrightarrow[r_2+r_4]{r_2+r_3,}$$

$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{pmatrix}$.当$a \neq 1$时，原线性方程组有惟一解; 当$a = 1$且$b \neq -1$时，原线性方程组有无解; 当$a = 1$且$b = -1$时，原线性方程组有无穷多个解.

$$a = 1\text{且}b = -1\text{时，} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-r_2+r_1} \begin{pmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Corresponding equations: $\begin{cases} x_1 - x_3 - x_4 = -1 \\ x_2 + 2x_3 + 2x_4 = 1 \end{cases}$, Solution is: $[x_1 = x_3 + x_4 - 1, x_2 = 1 - 2x_4 - 2x_3]$（$x_3, x_4$为任意实数）.

4.4 矩阵可逆性的判定

一、填空题：

1. 设矩阵$A=\begin{pmatrix}1&1\\2&-1\end{pmatrix}$,则$A^{-1}=$______.

2. 设矩阵$A=\begin{pmatrix}0&2&0\\1&1&0\\0&0&2\end{pmatrix}$,则$A^{-1}=$______;伴随矩阵$A^*=$______; $(A^*)^{-1}=$ ______; $(A^T)^*=$______; $(2A)^*=$______.

3. 已知A和B均为可逆矩阵，则$\begin{pmatrix}A&0\\0&B\end{pmatrix}^{-1}=$______; $\begin{pmatrix}0&A\\B&0\end{pmatrix}^{-1}=$ ______; $\begin{pmatrix}A&C\\0&B\end{pmatrix}^{-1}=$______; $\begin{pmatrix}A&0\\C&B\end{pmatrix}^{-1}=$______.

4. 已知3阶矩阵A的秩为2, $B=\begin{pmatrix}0&2&0\\1&1&0\\0&0&2\end{pmatrix}$,则$AB$的秩为______.

5. 已知n阶矩阵A的行列式$|A|=2$,则$|A^*|=$______.

6. n阶矩阵A可逆的充要条件为$\mathrm{rank}(A)$ ______n.（填“<”“=”或“>”）

提示：1. $\begin{pmatrix}1&1&1&0\\2&-1&0&1\end{pmatrix}\xrightarrow{-2r_1+r_2}\begin{pmatrix}1&1&1&0\\0&-3&-2&1\end{pmatrix}\xrightarrow[-r_2+r_1]{-\frac{1}{3}r_2,}\begin{pmatrix}1&1&\frac{1}{3}&\frac{1}{3}\\0&1&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$,

$A^{-1}=\begin{pmatrix}\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$. 2. $A^{-1}=\begin{pmatrix}-\frac{1}{2}&1&0\\\frac{1}{2}&0&0\\0&0&\frac{1}{2}\end{pmatrix}$, $A^*=\begin{pmatrix}2&-4&0\\-2&0&0\\0&0&-2\end{pmatrix}$, $(A^*)^{-1}=$ $\begin{pmatrix}0&-\frac{1}{2}&0\\-\frac{1}{4}&-\frac{1}{4}&0\\0&0&-\frac{1}{2}\end{pmatrix}$, $(A^T)^*=\begin{pmatrix}2&-2&0\\-4&0&0\\0&0&-2\end{pmatrix}$, $(2A)^*=\begin{pmatrix}8&-16&0\\-8&0&0\\0&0&-8\end{pmatrix}$.

3. （1）设$\begin{pmatrix}X_1&X_2\\X_3&X_4\end{pmatrix}\begin{pmatrix}A&\mathbf{0}\\\mathbf{0}&B\end{pmatrix}=\begin{pmatrix}E&\mathbf{0}\\\mathbf{0}&E\end{pmatrix}$,则$X_1A=E, X_2B=\mathbf{0}, X_3A=\mathbf{0}, X_4B=E$,得$X_1=A^{-1}, X_2=\mathbf{0}, X_3=\mathbf{0}, X_4=B^{-1}$,从而$\begin{pmatrix}A&\mathbf{0}\\\mathbf{0}&B\end{pmatrix}^{-1}=\begin{pmatrix}A^{-1}&\mathbf{0}\\\mathbf{0}&B^{-1}\end{pmatrix}$.

（2）设$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix} = \begin{pmatrix} E & \mathbf{0} \\ \mathbf{0} & E \end{pmatrix}$,则$X_2B = E, X_1A = \mathbf{0}, X_4B = \mathbf{0}, X_3A = E$,得$X_2 = B^{-1}, X_1 = \mathbf{0}, X_4 = \mathbf{0}, X_3 = A^{-1}$‘,从而$\begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{0} & B^{-1} \\ A^{-1} & \mathbf{0} \end{pmatrix}$.

（3）设$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} A & C \\ \mathbf{0} & B \end{pmatrix} = \begin{pmatrix} E & \mathbf{0} \\ \mathbf{0} & E \end{pmatrix}$,则$X_1A = E, X_1C + X_2B = 0, X_3A = 0, X_3C + X_4B = E$,得$X_1 = A^{-1}, X_2 = -A^{-1}CB^{-1}, X_3 = \mathbf{0}, X_4 = B^{-1}$,从而$\begin{pmatrix} A & C \\ \mathbf{0} & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ \mathbf{0} & B^{-1} \end{pmatrix}$.

（4）设$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} A & \mathbf{0} \\ C & B \end{pmatrix} = \begin{pmatrix} E & \mathbf{0} \\ \mathbf{0} & E \end{pmatrix}$,则$X_1A + X_2C = E, X_2B = \mathbf{0}, X_3A + X_4C = \mathbf{0}, X_4B = E$,得$X_2 = \mathbf{0}, X_1 = A^{-1}, X_4 = B^{-1}, X_3 = -B^{-1}CA^{-1}$,从而$\begin{pmatrix} A & \mathbf{0} \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & \mathbf{0} \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix}$. 4. $B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,rank$(B) = 3$,故rank$(AB)$ =rank$(B) = 2$.

5. 因为$A^{-1} = \frac{1}{|A|}A^*, |A| = 2$,故$|A^*| = |2A^{-1}| = 2^n|A^{-1}| = 2^n|A|^{-1} = 2^{n-1}$.

6. “=”.

二、求矩阵$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 7 & 8 & 0 & 0 \\ 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 5 \end{pmatrix}$的逆矩阵$A^{-1}$.

解：$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 7 & 8 & 0 & 0 \\ 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 5 \end{pmatrix}$, inverse: $\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 3 & -4 & 0 & 0 \\ 0 & -\frac{5}{2} & \frac{7}{2} & 0 & 0 \\ 0 & 0 & 0 & -5 & 2 \\ 0 & 0 & 0 & 3 & -1 \end{pmatrix}$.

分块矩阵计算逆矩阵：

(2), inverse: $\frac{1}{2}$,

$\begin{pmatrix} 7 & 8 \\ 5 & 6 \end{pmatrix}$, inverse: $\begin{pmatrix} 3 & -4 \\ -\frac{5}{2} & \frac{7}{2} \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, inverse: $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$.

或整个矩阵计算逆矩阵：

$$\left(\begin{array}{ccccc|ccccc} 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & 6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{5}r_3,\frac{1}{3}r_5]{\frac{1}{2}r_1,\frac{1}{7}r_2,} \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{8}{7} & 0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 \\ 0 & 1 & \frac{6}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{5}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow[-r_4+r_5]{-r_2+r_3,} \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{8}{7} & 0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{35} & 0 & 0 & 0 & -\frac{1}{7} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -1 & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow[-3r_5,-r_5+r_4]{\frac{35}{2}r_3,-r_3+r_2,} \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{5}{2} & \frac{7}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -5 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & -1 \end{array}\right).$$

三、设$A=\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$,求$A^{-1}$.

解：$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, inverse: $\begin{pmatrix} 0 & 2 & -1 \\ \frac{1}{2} & -\frac{7}{2} & 2 \\ -\frac{1}{2} & \frac{5}{2} & -1 \end{pmatrix}$.

$$\begin{pmatrix} 3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_1\leftrightarrow r_2} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[-r_1+r_3]{-3r_1+r_2,}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -4 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{r_2\leftrightarrow r_3} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & -2 & -4 & 1 & -3 & 0 \end{pmatrix} \xrightarrow{2r_2+r_3}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 1 & -5 & 2 \end{pmatrix} \xrightarrow{-\frac{1}{2}r_3} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{5}{2} & -1 \end{pmatrix} \xrightarrow[-r_2+r_1]{-r_3+r_2,-r_3+r_1,}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{7}{2} & 2 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{5}{2} & -1 \end{pmatrix}, A^{-1} = \begin{pmatrix} 0 & 2 & -1 \\ \frac{1}{2} & -\frac{7}{2} & 2 \\ -\frac{1}{2} & \frac{5}{2} & -1 \end{pmatrix}.$$

四、已知A是3阶方阵，A^*是伴随矩阵，如果$|A| = \frac{1}{4}$,试证$\left|\left(\frac{2}{3}A\right)^{-1} - 8A^*\right|$.

解：$A^{-1} = \frac{1}{|A|}A^*, |A| = \frac{1}{4}, A^* = \frac{1}{4}A^{-1}, \left|\left(\frac{2}{3}A\right)^{-1} - 8A^*\right| = \left|\frac{3}{2}A^{-1} - 2A^{-1}\right| = \left|-\frac{1}{2}A^{-1}\right| = -\frac{1}{8}|A|^{-1} = -\frac{1}{2}$.

五、已知A, B均为n阶方阵，若$|A| = 3.|B| = 2, |A^{-1} + B| = 2$,试计算$|A + B^{-1}|$.

解：由$|A| = 3.|B| = 2, |A^{-1} + B| = 2$,知$|A + B^{-1}| = \frac{1}{2}|A + B^{-1}||B| = \frac{1}{2}|AB + E| = \frac{1}{2}|A||B + A^{-1}| = \frac{1}{2} \times 3 \times 2 = 3$.

六、已知3阶方阵A, B满足$A^2B - A - B = E$,其中E为单位矩阵. 若$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{pmatrix}$,求$|B|$.

解：由$A^2B-A-B = E$知$(A^2 - E)B = A+E, A^2-E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \\ 0 & 3 & 0 \\ -4 & 0 & -2 \end{pmatrix}, A+E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ -2 & 0 & 2 \end{pmatrix}$,

$$\begin{pmatrix} -2 & 0 & 2 & 2 & 0 & 1 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ -4 & 0 & -2 & -2 & 0 & 2 \end{pmatrix}, \text{row echelon form:} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 0 & 2 & 2 & 0 & 1 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ -4 & 0 & -2 & -2 & 0 & 2 \end{pmatrix} \xrightarrow[4r_1 + r_3]{-\frac{1}{2}r_1,} \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & -\frac{1}{2} \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & -6 & -6 & 0 & 0 \end{pmatrix} \xrightarrow[-\frac{1}{6}r_3, r_3 + r_1]{\frac{1}{3}r_2,} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, |B| = \frac{1}{2}.$$

七、设$A=\begin{pmatrix}1&2&-2\\2&-1&\lambda\\3&1&-1\end{pmatrix}$,若存在3阶非零方阵$B$满足$AB=0$,试求$\lambda$和$|B|$.

解：$A=\begin{pmatrix}1&2&-2\\2&-1&\lambda\\3&1&-1\end{pmatrix}\xrightarrow[-3r_1+r_3]{-2r_1+r_2,}\begin{pmatrix}1&2&-2\\0&-5&\lambda+4\\0&-5&5\end{pmatrix}\xrightarrow[-r_3]{-r_3+r_2,}$
$\begin{pmatrix}1&2&-2\\0&0&\lambda-1\\0&1&-1\end{pmatrix}\xrightarrow{r_2\leftrightarrow r_3}\begin{pmatrix}1&2&-2\\0&1&-1\\0&0&\lambda-1\end{pmatrix}\xrightarrow[\frac{1}{\lambda-1}r_3]{if\quad\lambda\neq1,}$
$\begin{pmatrix}1&2&-2\\0&1&1\\0&0&1\end{pmatrix}\xrightarrow[-2r_2+r_1]{-r_3+r_2,\ r_3+r_1,}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$,$A$可逆，故由$AB=\mathbf{0}$知$B=A^{-1}\mathbf{0}=\mathbf{0}$（零矩阵），与已知矛盾.故只能$\lambda=1$，从而$\begin{pmatrix}1&2&-2\\2&-1&\lambda\\3&1&-1\end{pmatrix}=\begin{pmatrix}1&2&-2\\2&-1&1\\3&1&-1\end{pmatrix}$.

若$|B|\neq0$,则B可逆，则由$AB=\mathbf{0}$知$A=\mathbf{0}B^{-1}=\mathbf{0}$（零矩阵），与$A=\begin{pmatrix}1&2&-2\\2&-1&\lambda\\3&1&-1\end{pmatrix}$矛盾，故$|B|=0$.

八、设矩阵$A=\begin{pmatrix}2&1&0\\1&2&0\\0&0&1\end{pmatrix}$,矩阵$B$满足$ABA^*=2BA^*+E$,其中$A^*$为伴随矩阵，$E$是单位矩阵，求矩阵$B$.

解：由$ABA^*=2BA^*+E$知$(AB-2B)\,A^*=E$,由$|A|=4-1=3\neq0$,故A^*可逆，从而$(A-2E)\,B=(A^*)^{-1}$, $B=(A-2E)^{-1}(A^*)^{-1}$, $A-2E=$
$\begin{pmatrix}2&1&0\\1&2&0\\0&0&1\end{pmatrix}-2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}$,

$(A-2E)^{-1}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}^{-1}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}$, $A^*=\begin{pmatrix}2&-1&0\\-1&2&0\\0&0&3\end{pmatrix}$,

$(A^*)^{-1}=\begin{pmatrix}2&-1&0\\-1&2&0\\0&0&3\end{pmatrix}^{-1}=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&0\\\frac{1}{3}&\frac{2}{3}&0\\0&0&\frac{1}{3}\end{pmatrix}$,

从而$B=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&0\\\frac{1}{3}&\frac{2}{3}&0\\0&0&\frac{1}{3}\end{pmatrix}=\begin{pmatrix}\frac{1}{3}&\frac{2}{3}&0\\\frac{2}{3}&\frac{1}{3}&0\\0&0&-\frac{1}{3}\end{pmatrix}$.

或：$B=(A-2E)^{-1}(A^*)^{-1}=(A^*A-2A^*)^{-1}=(|A|E-2A^*)^{-1}$,

$|A|E-2A^*=3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-2\begin{pmatrix}2&-1&0\\-1&2&0\\0&0&3\end{pmatrix}=\begin{pmatrix}-1&2&0\\2&-1&0\\0&0&-3\end{pmatrix}$,

$B=\begin{pmatrix}-1&2&0\\2&-1&0\\0&0&-3\end{pmatrix}^{-1}=\begin{pmatrix}\frac{1}{3}&\frac{2}{3}&0\\\frac{2}{3}&\frac{1}{3}&0\\0&0&-\frac{1}{3}\end{pmatrix}$.

手工计算逆矩阵：

$\begin{pmatrix}0&1&0&1&0&0\\1&0&0&0&1&0\\0&0&-1&0&0&1\end{pmatrix}\xrightarrow[-r_3]{r_1\leftrightarrow r_2,}\begin{pmatrix}1&0&0&0&1&0\\0&1&0&1&0&0\\0&0&1&0&0&-1\end{pmatrix}$,

$\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}^{-1}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}$.

$\begin{pmatrix}2&-1&0&1&0&0\\-1&2&0&0&1&0\\0&0&3&0&0&1\end{pmatrix}\xrightarrow[-r_1]{r_1\leftrightarrow r_2,}\begin{pmatrix}1&-2&0&0&-1&0\\2&-1&0&1&0&0\\0&0&3&0&0&1\end{pmatrix}\xrightarrow[\frac{1}{3}r_3]{-2r_1+r_2,}$
$\begin{pmatrix}1&-2&0&0&-1&0\\0&3&0&1&2&0\\0&0&1&0&0&\frac{1}{3}\end{pmatrix}\xrightarrow[2r_2+r_1]{\frac{1}{3}r_2,}\begin{pmatrix}1&0&0&\frac{2}{3}&\frac{1}{3}&0\\0&1&0&\frac{1}{3}&\frac{2}{3}&0\\0&0&1&0&0&\frac{1}{3}\end{pmatrix}$,

$\begin{pmatrix}2&-1&0\\-1&2&0\\0&0&3\end{pmatrix}^{-1}=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&0\\\frac{1}{3}&\frac{2}{3}&0\\0&0&\frac{1}{3}\end{pmatrix}$.

$\begin{pmatrix}-1&2&0&1&0&0\\2&-1&0&0&1&0\\0&0&-3&0&0&1\end{pmatrix}\xrightarrow[-2r_1+r_2,-\frac{1}{3}r_3]{-r_1,}\begin{pmatrix}1&-2&0&-1&0&0\\0&3&0&2&1&0\\0&0&1&0&0&-\frac{1}{3}\end{pmatrix}$
$\xrightarrow[2r_2+r_1]{\frac{1}{3}r_2,}\begin{pmatrix}1&0&0&\frac{1}{3}&\frac{2}{3}&0\\0&1&0&\frac{2}{3}&\frac{1}{3}&0\\0&0&1&0&0&-\frac{1}{3}\end{pmatrix}$, $\begin{pmatrix}-1&2&0\\2&-1&0\\0&0&-3\end{pmatrix}^{-1}=\begin{pmatrix}\frac{1}{3}&\frac{2}{3}&0\\\frac{2}{3}&\frac{1}{3}&0\\0&0&-\frac{1}{3}\end{pmatrix}$.

九、已知矩阵方程$X=AX+B$,其中$A=\begin{pmatrix}0&1&0\\-1&1&1\\-1&0&-1\end{pmatrix}$,$B=$

$\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 5 & -3 \end{pmatrix}$,求$X$.

解：由$X = AX + B$知$(E - A)X = B, X = (E - A)^{-1}B, E - A =$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$,

$(E - A)^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$,

$X = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$.手工计算逆矩阵：

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[-r_1+r_3]{-r_1+r_2,} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{pmatrix} \xrightarrow[\frac{1}{3}r_3]{-r_2+r_3,}$$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \xrightarrow[r_2+r_1]{r_3+r_2,} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

十、设矩阵$P = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, AP = P\Lambda$,求$A^n$.

解：由$AP = P\Lambda$知$A = P\Lambda P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}^{-1} =$
$\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^2 = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}, A^3 = A\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} =$
$\begin{pmatrix} -6 & 7 \\ -14 & 15 \end{pmatrix} A^4 = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^4 = \begin{pmatrix} -14 & 15 \\ -30 & 31 \end{pmatrix}, \cdots, A^n = \begin{pmatrix} 2 - 2^n & 2^n - 1 \\ 2 - 2^{n+1} & 2^{n+1} - 1 \end{pmatrix}$.
（可用数学归纳法证明）

或由$AP = P\Lambda$知$A = P\Lambda P^{-1}$知，$A^n = (P\Lambda P^{-1})^n = P\Lambda^n P^{-1} =$
$\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 2 - 2^n & -1 + 2^n \\ 2 - 2^{1+n} & -1 + 2^{1+n} \end{pmatrix}$.

十一、解线性方程组$\begin{cases} x_1+a_1x_2+a_1^2x_3=a_1^3 \\ x_1+a_2x_2+a_2^2x_3=a_2^3 \\ x_1+a_3x_2+a_3^2x_3=a_3^3 \\ x_1+a_4x_2+a_4^2x_3=a_4^3 \end{cases}$,其中$a_1,a_2,a_3,a_4$是互不相同的实数.

解：$\begin{cases} x_1+a_1x_2+a_1^2x_3=a_1^3 \\ x_1+a_2x_2+a_2^2x_3=a_2^3 \\ x_1+a_3x_2+a_3^2x_3=a_3^3 \\ x_1+a_4x_2+a_4^2x_3=a_4^3 \end{cases}$, Corresponding matrix: $\begin{pmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{pmatrix}$.

$\begin{vmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{vmatrix} = (a_4-a_3)(a_4-a_2)(a_4-a_1)(a_3-a_2)(a_3-a_1)(a_2-a_1) \neq 0$,故增广矩阵的秩为4,而系数矩阵的秩不超过3,小于4,从而原方程无解.

5.1 齐次线性方程组的解空间与向量空间 5.2 向量组的线性关系

一、判断题：

1. 齐次线性方程组$Ax=0$的解集$S=\{x|Ax=0\}$是一个向量空间.( )
2. 非齐次线性方程组$Ax=b$的解集$S=\{x|Ax=b\}$是一个向量空间.( )
3. 集合$V=\{x=(1,x_2,x_3,\cdots,x_n)^T\,|x_2,x_3,\cdots,x_n\in\mathbf{R}\}$是一个向量空间.( )
4. 集合$V=\{x=(0,x_2,x_3,\cdots,x_n)^T\,|x_2,x_3,\cdots,x_n\in\mathbf{R}\}$是一个向量空间.( )
5. 设$\alpha_1,\alpha_2,\alpha_3$为三个已知的$n$维向量，集合$V=\{x|x=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3,k_1,k_2,k_3\in\mathbf{R}\}$是一个向量空间.

提示：1. T.　2. F.　3. F.　4. T.　5. T.

二、填空题：

1. 已知$\alpha=(3,5,7,9)^T,\beta=(-1,5,2,0)^T,x$满足$2\alpha+3x=\beta$,则$x=$______.
2. 当$k=$______时，向量$\beta=(1,k,5)^T$能由$\alpha_1=(1,-3,2)^T,\alpha_2=(2,-1,1)^T$线性表示.
3. 若$\alpha_1=(1,0,5,2)^T,\alpha_2=(3,-2,3,-4)^T,\alpha_3=(-1,1,t,3)^T$线性相关，则$t=$______.
4. 若向量组$\alpha_1=\begin{pmatrix}1\\0\\-1\end{pmatrix},\alpha_2=\begin{pmatrix}k\\3\\0\end{pmatrix},\alpha_3=\begin{pmatrix}-1\\4\\k\end{pmatrix}$线性无关，则$k$满足______.
5. 填写“线性相关”或“线性无关”：

   (1) 单独一个零向量是______向量;

   (2) 任何一个非零向量是______向量;

   (3) 两个成比例的向量是______向量组;

   (4) 含有零向量的向量组必______;

   (5) 若一个向量组的部分组线性相关，则该向量组必______;

(6) 若一个向量组线性无关，则该向量组的任何部分组必_______;

(7) 若一个向量组线性无关，则延长分量后所得向量组必_______;

(8) 若一个向量组线性相关，则缩短分量后所得向量组必______;

(9) 当$m>n$时，m个n维列向量必______.

提示：1. 由$2\alpha+3x=\beta$知$x=\frac{1}{3}(\beta-2\alpha)=\begin{pmatrix} -\frac{7}{3} & -\frac{5}{3} & -4 & -6 \end{pmatrix}^T$.

2. 由$\beta=x\alpha_1+y\alpha_2$,即$(1,k,5)^T=x(1,-3,2)^T+y(2,-1,1)^T=\begin{pmatrix} x+2y \\ -3x-y \\ 2x+y \end{pmatrix}$,

即$\begin{cases} x+2y=1 \\ -3x-y=k \\ 2x+y=5 \end{cases}$, Solution is: $[k=-8,x=3,y=-1]$. 3. $x\alpha_1+y\alpha_2+z\alpha_3=0\,(x,y,z$不全为零$)$,即$x(1,0,5,2)^T+y(3,-2,3,-4)^T+z(-1,1,t,3)^T=$

$\begin{pmatrix} x+3y-z \\ z-2y \\ 5x+3y+tz \\ 2x-4y+3z \end{pmatrix}=0$,即$\begin{cases} x+3y-z=0 \\ z-2y=0 \\ 5x+3y+tz=0 \\ 2x-4y+3z=0 \end{cases}$.由$\begin{cases} x+3y-z=0 \\ z-2y=0 \\ 5x+3y+tz=0 \\ 2x-4y+3z=0 \end{cases}$,

Solution is: $\left[t=1,x=-\frac{1}{2}z,y=\frac{1}{2}z\right]$. 4. $\alpha_1=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} k \\ 3 \\ 0 \end{pmatrix}$, $\alpha_3=$

$\begin{pmatrix} -1 \\ 4 \\ k \end{pmatrix}$线性无关，则$x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}+y\begin{pmatrix} k \\ 3 \\ 0 \end{pmatrix}+z\begin{pmatrix} -1 \\ 4 \\ k \end{pmatrix}=0$只有零解.

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} k \\ 3 \\ 0 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \\ k \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & k & -1 \\ 0 & 3 & 4 \\ -1 & 0 & k \end{pmatrix}\xrightarrow[\frac{1}{3}r_2]{r_1+r_3,}\begin{pmatrix} 1 & k & -1 \\ 0 & 1 & \frac{4}{3} \\ 0 & k & k-1 \end{pmatrix}$

$\xrightarrow[-kr_2+r_3]{-kr_2+r_1,}\begin{pmatrix} 1 & 0 & -1-\frac{4}{3}k \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3}k-1 \end{pmatrix}$, $k\neq-3$. 5. (1)线性相关; (2)线性无关; (3)线性相关; (4)线性相关; (5)线性相关; (6)线性无关; (7)线性无关; (8)线性相关; (9)线性相关.

三、判断向量组$\alpha_1=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}$, $\alpha_3=\begin{pmatrix} 1 \\ 3 \\ 9 \\ 27 \end{pmatrix}$, $\alpha_4=$

$\begin{pmatrix} 1 \\ 4 \\ 16 \\ 64 \end{pmatrix}$是线性相关还是线性无关.

解：$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 = 0$, $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 9 \\ 27 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 16 \\ 64 \end{pmatrix}$,

concatenate: $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix} \xrightarrow[-r_1+r_4]{\substack{-r_1+r_2, \\ -r_1+r_3,}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 15 \\ 0 & 7 & 26 & 63 \end{pmatrix} \xrightarrow{\substack{-3r_2+r_3, \\ -7r_1+r_4}}$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 12 & 21 \end{pmatrix} \xrightarrow{\substack{\frac{1}{2}r_3, \\ -6r_3+r_4, \\ -\frac{1}{15}r_4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $r = 4$,方程组只有零解，故$\alpha_1, \alpha_2, \alpha_3, \alpha_4$线性无关.

四、判断向量组$\alpha_1 = \begin{pmatrix} 3 \\ 1 \\ -3 \\ -2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ -10 \\ 6 \\ -9 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 4 \\ -5 \\ 2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} -5 \\ -1 \\ -1 \\ 4 \end{pmatrix}$是线性相关还是线性无关.

解：$x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4 = 0$, $\begin{pmatrix} 3 \\ 1 \\ -3 \\ -2 \end{pmatrix}\begin{pmatrix} -1 \\ -10 \\ 6 \\ -9 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ -5 \\ 2 \end{pmatrix}\begin{pmatrix} -5 \\ -1 \\ -1 \\ 4 \end{pmatrix}$,

concatenate: $\begin{pmatrix} 3 & -1 & 2 & -5 \\ 1 & -10 & 4 & -1 \\ -3 & 6 & -5 & -1 \\ -2 & -9 & 2 & 4 \end{pmatrix}$

$$\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & -10 & 4 & -1 \\ 3 & -1 & 2 & -5 \\ -3 & 6 & -5 & -1 \\ -2 & -9 & 2 & 4 \end{pmatrix} \xrightarrow{\substack{-3r_1+r_2, \\ 3r_1+r_3, \\ 2r_1+r_4}} \begin{pmatrix} 1 & -10 & 4 & -1 \\ 0 & 29 & -10 & -2 \\ 0 & -24 & 7 & -4 \\ 0 & -29 & 10 & 2 \end{pmatrix} \xrightarrow{r_2+r_4}$$

$\begin{pmatrix} 1 & -10 & 4 & -1 \\ 0 & 29 & -10 & -2 \\ 0 & -24 & 7 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $r < 4$,方程组有非零解，故$\alpha_1, \alpha_2, \alpha_3, \alpha_4$线性相关.

五、试将$\beta = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$表示成$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$的线性组合.

解：$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 = \beta$,

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}$,

row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{5}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & -\frac{1}{4} \end{pmatrix}$, Corresponding equations:

$\left\{x_1 = \frac{5}{4}, x_2 = \frac{1}{4}, x_3 = -\frac{1}{4}, x_4 = -\frac{1}{4}\right\}$.$\beta = \frac{5}{4}\alpha_1 + \frac{1}{4}\alpha_2 - \frac{1}{4}\alpha_3 - \frac{1}{4}\alpha_4$

六、已知$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ t \end{pmatrix}$，试问：当$t$为何值时，向量组$\alpha_1, \alpha_2, \alpha_3$线性无关？当$t$为何值时，向量组$\alpha_1, \alpha_2, \alpha_3$线性相关？并将$\alpha_3$表示成$\alpha_1, \alpha_2$的线性组合.

解：$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = 0$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ t \end{pmatrix}$, concatenate:

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & 2 & t \end{pmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_3,} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -7 & t-6 \end{pmatrix} \xrightarrow[-7r_2+r_3]{\frac{1}{7}r_2,} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & t-5 \end{pmatrix},$$

当$t \neq 5$时，$r = 3, \alpha_1, \alpha_2, \alpha_3$线性无关; 当$t = 5$时，$r = 3, \alpha_1, \alpha_2, \alpha_3$线性相关，$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & \end{pmatrix} \xrightarrow{-3r_2+r_1} \begin{pmatrix} 1 & 0 & \frac{11}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & \end{pmatrix}$,

$\alpha_3 = \frac{11}{7}\alpha_1 + \frac{1}{7}\alpha_2$.

七、已知向量组$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 5 \\ -5 \\ t \\ 11 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ t \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 \\ -3 \\ 6 \\ 3 \end{pmatrix}$,问：（1）当$t$为何值时，向量组$\alpha_1, \alpha_2, \alpha_3, \alpha_4$线性相关？（2）当$t$为何值时，向量组$\alpha_1, \alpha_2, \alpha_3, \alpha_4$线性无关？（3）当$t$为何值时，$\alpha_3$能由$\alpha_1, \alpha_2, \alpha_4$线性表示.

解：$x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4 = 0, \begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ -5 \\ t \\ 11 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \\ t \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 6 \\ 3 \end{pmatrix}$,

concatenate: $\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & -5 & -1 & -3 \\ -3 & t & 3 & 6 \\ 1 & 11 & t & 3 \end{pmatrix} \xrightarrow[-r_1+r_4]{\substack{-2r_1+r_2, \\ 3r_1+r_3,}} \begin{pmatrix} 1 & 5 & 2 & 1 \\ 0 & -15 & -5 & -5 \\ 0 & t+15 & 9 & 9 \\ 0 & 6 & t-2 & 2 \end{pmatrix}$

$\xrightarrow[-6r_2+r_4]{\substack{-\frac{1}{15}r_2, \\ -(t+15)r_2+r_3,}} \begin{pmatrix} 1 & 5 & 2 & 1 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 4-\frac{1}{3}t & 4-\frac{1}{3}t \\ 0 & 0 & t-4 & 0 \end{pmatrix}$,当$t = 12$时，$r < 4, \alpha_1, \alpha_2, \alpha_3, \alpha_4$线性相关; 当$t \neq 12$时，$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 4-\frac{1}{3}t & 4-\frac{1}{3}t \\ 0 & 0 & t-4 & 0 \end{pmatrix} \xrightarrow[-(t-4)r_3+r_4]{\frac{3}{12-t}r_3,}$, $\begin{pmatrix} 1 & 5 & 2 & 1 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4-t \end{pmatrix}$,

当$t = 4$时，$r < 4, \alpha_1, \alpha_2, \alpha_3, \alpha_4$线性相关; 当$t \neq 4, r = 4, \alpha_1, \alpha_2, \alpha_3, \alpha_4$线性无关.

(1)当$t = 4, 12$时，$r < 4, \alpha_1, \alpha_2, \alpha_3, \alpha_4$线性相关;

(2)$t \neq 4, t \neq 12$时，$\alpha_1, \alpha_2, \alpha_3, \alpha_4$线性无关;

(3)$t = 4$时，$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[-r_3+r_1]{-\frac{1}{3}r_3+r_2,} \begin{pmatrix} 1 & 5 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{-5r_2+r_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \alpha_3 = \alpha_1 + 0\alpha_2 + \alpha_4.$

八、设向量组$\alpha_1, \alpha_2, \alpha_3$线性无关，证明：$\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3, \alpha_1 + 4\alpha_2 + 9\alpha_3$线性无关.

证明：设$x(\alpha_1+\alpha_2+\alpha_3) + y(\alpha_1+2\alpha_2+3\alpha_3) + z(\alpha_1+4\alpha_2+9\alpha_3) = 0$,即$(x+y+z)\alpha_1 + (x+2y+4z)\alpha_2 + (x+3y+9z) = 0$.因为$\alpha_1, \alpha_2, \alpha_3$线性无关，故$\begin{cases} x+y+z=0 \\ x+2y+4z=0 \\ x+3y+9z=0 \end{cases}$,系数矩阵$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \xrightarrow[-r_1+r_3]{-r_1+r_2,} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix}$

$\xrightarrow{-2r_2+r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix}$, $r = 3$,只有零解，即$\alpha_1+\alpha_2+\alpha_3, \alpha_1+2\alpha_2+3\alpha_3, \alpha_1+4\alpha_2+9\alpha_3$线性无关.

九、已知β可由$\alpha_1, \alpha_2, \cdots, \alpha_r$线性表示，证明：$\beta$能由$\alpha_1, \alpha_2, \cdots, \alpha_r$惟一线性表示的充分必要条件是$\alpha_1, \alpha_2, \cdots, \alpha_r$线性无关.

证明：由已知$\beta = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_r\alpha_r$,设$\beta = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_r\alpha_r$,从而$(x_1-y_1)\alpha_1 + (x_2-y_2) + \alpha_2 + \cdots + (x_r-y_r)\alpha_r$.则$\beta$能由$\alpha_1, \alpha_2, \cdots, \alpha_r$惟一线性表示的充分必要条件是$x_i - y_i = 0$,即$x_i = y_i\,(i = 1, 2, \cdots, r)$,即$\alpha_1, \alpha_2, \cdots, \alpha_r$.线性无关.

5.3 向量组的秩

一、填空题：

1. 设向量组$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 9 \\ 100 \\ 10 \\ 4 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -2 \\ -4 \\ 2 \\ -8 \end{pmatrix}$,由此向量组的秩为______,其中一个极大线性无关组为______.

2. 向量组$\alpha_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 9 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$的一个极大线性无关组为______.

3. 设向量组$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 3 \\ -x \\ -2x \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ -1 \\ 6 \\ 0 \end{pmatrix}$,若此向量组的秩为2，则$x =$ ______.

4. 设$A = \begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & -2 \\ 1 & 1 & x & 1 \end{pmatrix}$,且$A$的秩为2,则$x =$ ______.

提示：1. $\begin{pmatrix} 1 \\ 2 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} 9 \\ 100 \\ 10 \\ 4 \end{pmatrix} \begin{pmatrix} -2 \\ -4 \\ 2 \\ -8 \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & 9 & -2 \\ 2 & 100 & -4 \\ -1 & 10 & 2 \\ 4 & 4 & -8 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, rank: 2, α_1, α_2或α_2, α_3是极大线性无关组.

2. $\begin{pmatrix} 5 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 9 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, concatenate: $\begin{pmatrix} 5 & 6 & 9 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, row ech-

elon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha_1, \alpha_2, \alpha_3$是极大线性无关组. 3. $\begin{pmatrix} 1 \\ 1 \\ 2 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -x \\ -2x \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 6 \\ 0 \end{pmatrix}$,

concatenate: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 2 & -x & 6 \\ -2 & -2x & 0 \end{pmatrix} \xrightarrow[2r_1+r_2]{\substack{-r_1+r_2, \\ -r_1+r_2, \\ -2r_1+r_2,}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & -x-2 & 4 \\ 0 & -2x+2 & 2 \end{pmatrix}$

$\xrightarrow[(2x-2)r_2+r_4]{\substack{\frac{1}{2}r_2, \\ (x+2)r_2+r_3,}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -x+2 \\ 0 & 0 & -2x+4 \end{pmatrix}$,因此向量组的秩为2，故$-x+2 = -2x+4 = 0$,即$x = 2$. 4. $\begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & -2 \\ 1 & 1 & x & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 1 & x & 1 \\ 1 & x & 1 & -2 \\ x & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\substack{-r_1+r_2, \\ -xr_1+r_3}}$
$\begin{pmatrix} 1 & 1 & x & 1 \\ 0 & x-1 & 1-x & -3 \\ 0 & 1-x & 1-x^2 & 1-x \end{pmatrix} \xrightarrow{r_2+r_3} \begin{pmatrix} 1 & 1 & x & 1 \\ 0 & x-1 & 1-x & -3 \\ 0 & 0 & 2-x-x^2 & -2-x \end{pmatrix}$,因为$A$秩为2，故$2-x-x^2 = -(x+2)(x-1)$,且$x-1 \neq 0 2-x-x^2 = -2-x = 0$,从而$x = -2$.

二、设$A = \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix}$,试求$A$的秩.

解：$\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 1 & 1 & a \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ a & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\substack{-r_1+r_2, \\ -r_1+r_3, \\ -ar_1+r_4,}}$
$\begin{pmatrix} 1 & 1 & 1 & a \\ 0 & a-1 & 0 & 1-a \\ 0 & 0 & a-1 & 1-a \\ 0 & 1-a & 1-a & 1-a^2 \end{pmatrix} \xrightarrow{\substack{r_2+r_4, \\ r_3+r_4}} \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & a-1 & 0 & 1-a \\ 0 & 0 & a-1 & 1-a \\ 0 & 0 & 0 & 3-2a-a^2 \end{pmatrix}$,当$a = 1$时，$\mathrm{rank}(A) = 1$;当$a = -3$时，$\mathrm{rank}(A) = 3$;当

$3-2a-a^2 \neq 0$即$a \neq 1, a \neq -3$时，$\mathrm{rank}(A) = 4$.

三、求向量组$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 4 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -3 \\ 2 \\ 3 \\ -11 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 \\ 3 \\ 10 \\ 0 \end{pmatrix}$的一个极大线性无关组及此向量组的秩.

解：$\begin{pmatrix} 1 \\ 1 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \\ 4 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 3 \\ -11 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 10 \\ 0 \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & 1 & -3 & 1 \\ 1 & -1 & 2 & 3 \\ 4 & -2 & 3 & 10 \\ 2 & 4 & -11 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 1 & -\frac{5}{2} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, rank: 2,一个极大线性无关组为$\alpha_1, \alpha_2$.

四、设向量组$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 3 \\ 0 \\ 7 \\ 14 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 2 \\ 1 \\ 5 \\ 6 \end{pmatrix}$.

（1）求该向量组的秩;

（2）求$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$的一个极大线性无关组;

（3）将其余向量用该极大线性无关组表示.

解：（1）$\begin{pmatrix} 1 \\ -1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 7 \\ 14 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 5 \\ 6 \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ -1 & 3 & 0 & -1 & 1 \\ 2 & 1 & 7 & 2 & 5 \\ 4 & 2 & 14 & 0 & 6 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank: 3.

（2）$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$的一个极大线性无关组为$\alpha_1,\alpha_2,\alpha_4$.

（3）$\alpha_3=3\alpha_1+\alpha_2+0\alpha_4,\alpha_5=\alpha_1+\alpha_2+\alpha_4$.

五、已知向量组$\beta_1=\begin{pmatrix}0\\1\\-1\end{pmatrix},\beta_2=\begin{pmatrix}a\\2\\1\end{pmatrix},\beta_3=\begin{pmatrix}b\\1\\0\end{pmatrix}$与向量组$\alpha_1=\begin{pmatrix}1\\2\\-3\end{pmatrix},\alpha_2=\begin{pmatrix}3\\0\\1\end{pmatrix},\alpha_3=\begin{pmatrix}9\\6\\-7\end{pmatrix}$具有相同的秩，且$\beta_3$可由$\alpha_1,\alpha_2,\alpha_3$线性表示，求$a,b$的值.

解：$\begin{pmatrix}1\\2\\-3\end{pmatrix}\begin{pmatrix}3\\0\\1\end{pmatrix}\begin{pmatrix}9\\6\\-7\end{pmatrix}$, concatenate: $\begin{pmatrix}1&3&9\\2&0&6\\-3&1&-7\end{pmatrix}$, row echelon form: $\begin{pmatrix}1&0&3\\0&1&2\\0&0&0\end{pmatrix}$, rank: 2,因为$\beta_1,\beta_2,\beta_3$与$\alpha_1,\alpha_2,\alpha_3$具有相同的秩，故rank$(\beta_1,\beta_2,\beta_3)=2$. $\begin{pmatrix}0\\1\\-1\end{pmatrix}\begin{pmatrix}a\\2\\1\end{pmatrix}\begin{pmatrix}b\\1\\0\end{pmatrix}$, concatenate: $\begin{pmatrix}0&a&b\\1&2&1\\-1&1&0\end{pmatrix}\xrightarrow[r_1+r_3]{r_1\leftrightarrow r_2,}\begin{pmatrix}1&2&1\\0&a&b\\0&3&1\end{pmatrix}$

$\xrightarrow[-\frac{a}{3}r_2+r_3]{r_2\leftrightarrow r_3,}\begin{pmatrix}1&2&1\\0&3&1\\0&0&b-\frac{a}{3}\end{pmatrix}$,则$b=\frac{a}{3}$.又因为$\beta_3$可由$\alpha_1$,$\alpha_2$,$\alpha_3$线性表示，而$\begin{pmatrix}1&3&9&b\\2&0&6&1\\-3&1&-7&0\end{pmatrix}\xrightarrow[3r_1+r_2]{-2r_1+r_2,}\begin{pmatrix}1&3&9&b\\0&-6&-12&1-2b\\0&10&20&3b\end{pmatrix}\xrightarrow[-r_2+r_3]{-\frac{1}{6}r_2,\frac{1}{10}r_3,}$
$\begin{pmatrix}1&3&9&b\\0&1&2&\frac{2b-1}{6}\\0&0&0&\frac{3b}{10}-\frac{2b-1}{6}\end{pmatrix}$,故$\frac{3b}{10}-\frac{2b-1}{6}=0$,即$b=5$,从而$a=15$.

六、设$A_{m\times n}$及$B_{n\times s}$为两个矩阵，证明：rank$(AB)\leqslant\min($rank(A),rank$(B))$.

证明：参见《线性代数》P.89的例1.

令$A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$,$B=(b_{ij})_{n\times m}$,$AB=(\gamma_1,\gamma_2,\cdots,\gamma_m)$,则$\gamma_j=b_{1j}\alpha_1+b_{2j}\alpha_2+\cdots b_{nj}\alpha_n,j=1,2,\cdots m$.所以，$AB$的列向量组可以由$A$的列向量线性表示，而这两个向量组的秩分别是rank(AB)和rank(A)，于是由命题5.3.2即得rank$(AB)\leqslant$rank(A).同理可得rank$(AB)\leqslant$rank(B)，从而rank$(AB)\leqslant$

$\min(\text{rank}(A)\,,\text{rank}(B))$.

七、已知向量组：(I) $\alpha_1,\alpha_2,\alpha_3$;(II) $\alpha_1,\alpha_2,\alpha_3,\alpha_4$;(III) $\alpha_1,\alpha_2,\alpha_3,\alpha_5$.若各向量组的秩分别为rank(I)=rank(II)$=3$,rank(III)$=4$.证明：向量组$\alpha_1,\alpha_2,\alpha_3,\alpha_5-\alpha_4$的秩为4.

证明：因为rank(I)=rank(II)$=3$,故α_4可由$\alpha_1,\alpha_2,\alpha_3$线性表示，即存在$x_1,x_2,x_3$,使得$\alpha_4=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3$,$(\alpha_1,\alpha_2,\alpha_3,\alpha_5-\alpha_4)\xrightarrow{x_1c_1+c_4,x_2c_2+c_4,x_3c_3+c_4}(\alpha_1,\alpha_2,\alpha_3,\alpha_5)$,从而$r(\alpha_1,\alpha_2,\alpha_3,\alpha_5-\alpha_4)=r(\alpha_1,\alpha_2,\alpha_3,\alpha_5)=3$.

八、设$\beta_1=\alpha_1,\beta_2=\alpha_1+\alpha_2,\cdots,\beta_n=\alpha_1+\alpha_2+\cdots+\alpha_n$,且向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$的秩为$r$,证明：向量组$\beta_1,\beta_2,\cdots,\beta_n$的秩为$r$.

证明：由已知$(\beta_1\beta_2\cdots\beta_n)=(\alpha_1\alpha_2\cdots\alpha_n)\begin{pmatrix}1&1&\cdots&1\\0&1&\cdots&1\\\vdots&\vdots&&\vdots\\0&0&\cdots&1\end{pmatrix}$,rank$(\alpha_1\alpha_2\cdots\alpha_n)=r$,而$\begin{pmatrix}1&1&\cdots&1\\0&1&\cdots&1\\\vdots&\vdots&&\vdots\\0&0&\cdots&1\end{pmatrix}$的秩为$n$,故是可逆矩阵，从而rank$(\beta_1\beta_2\cdots\beta_n)$ =rank$(\alpha_1\alpha_2\cdots\alpha_n)=r$.

5.4 基、维数与坐标 5.5 线性方程组的解的结构

一、填空题：

1. 已知三维空间$\mathbf{R}^3$的基为$\alpha_1=\begin{pmatrix}1\\1\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\0\\1\end{pmatrix},\alpha_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$,则向量$\beta=\begin{pmatrix}2\\0\\0\end{pmatrix}$在此基下的坐标为______.

2. 从$\mathbf{R}^2$的基$\alpha_1=\begin{pmatrix}1\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\-1\end{pmatrix}$到基$\beta_1=\begin{pmatrix}1\\1\end{pmatrix},\beta_2=\begin{pmatrix}1\\2\end{pmatrix}$的过渡矩阵为______.

3. 设A为n阶方阵，且$\operatorname{rank}(A)=n-1,\alpha_1,\alpha_2$是齐次线性方程组$Ax=0$的两个不同的解向量，则$Ax=0$的通解为______.

4. 设$\alpha_1,\alpha_2,\alpha_3$是四元非齐次线性方程组$Ax=b$的三个解向量，且$\operatorname{rank}(A)=3,\alpha_1=(1,2,3,4)^T,\alpha_2+\alpha_3=(0,1,2,3)^T$,则线性方程组$Ax=b$的通解为______.

5. 设$\alpha_1,\alpha_2,\cdots,\alpha_r$是非齐次线性方程组$Ax=b$的解，若$c_1\alpha_1+c_2\alpha_2+\cdots+c_r\alpha_r$也是$Ax=b$的解，则$c_1+c_2+\cdots+c_r=$ ______.

6. 方程$x_1-2x_2+3x_3-4x_4=0$的通解是______________.

提示：1. $\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}2\\0\\0\end{pmatrix}$, concatenate: $\begin{pmatrix}1&1&0&2\\1&0&1&0\\0&1&1&0\end{pmatrix}$, row echelon form: $\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\end{pmatrix}$,此基下的坐标为$\begin{pmatrix}1, & 1, & -1\end{pmatrix}$. 2. 过渡矩阵$A=\begin{pmatrix}1&1\\0&-1\end{pmatrix}^{-1}\begin{pmatrix}1&1\\1&2\end{pmatrix}=\begin{pmatrix}1&1\\0&-1\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix}=\begin{pmatrix}2&3\\-1&-2\end{pmatrix}$. 3. 齐次线性方程组$Ax=0$解空间的维数为$n-\operatorname{rank}(A)=1,\alpha_1,\alpha_2$是齐次线性方程组$Ax=0$的两个不同的解向量，故$\alpha_1-\alpha_2$必是$Ax=$一个非零解，从而$Ax=0$的通解为$k(\alpha_1-\alpha_2)\,(k\in\mathbf{R})$. 4. α_1,α_2,α_3是四元非齐次线性方程组$Ax=b$的三个解向量，从而$2\alpha_1-(\alpha_2+\alpha_3)=2(1,2,3,4)^T-(0,1,2,3)^T=\begin{pmatrix}2&3&4&5\end{pmatrix}^T$是齐次线性方程组$Ax=0$的解，又由于$Ax=$

0解空间的维数为4$-\mathrm{rank}(A)=1$,故其基础解系为$\begin{pmatrix} 2 & 3 & 4 & 5 \end{pmatrix}^T$,从而$Ax=b$的通解为$k\begin{pmatrix} 2 & 3 & 4 & 5 \end{pmatrix}^T+(1,2,3,4)^T$.　　5. 提示：$(c_1\alpha_1+c_2\alpha_2+\cdots+c_r\alpha_r)A=b, c_1\alpha_1A+c_2\alpha_2A+\cdots+c_r\alpha_rA=c_1b+c_2b+\cdots+c_rb=(c_1+c_2+\cdots+c_r)b=b$,而$b\neq 0$,从而$c_1+c_2+\cdots+c_r=1$.　　6. $x_1=2x_2-3x_3+4x_4$.或$x=\begin{pmatrix} 2\\1\\0\\0 \end{pmatrix}x_2+\begin{pmatrix} -3\\0\\1\\0 \end{pmatrix}x_3+\begin{pmatrix} 4\\0\\0\\1 \end{pmatrix}x_4$, ($x_2,x_3,x_4$是任意实数).

二、设$\alpha_1=\begin{pmatrix} 1\\0\\1 \end{pmatrix}, \alpha_2=\begin{pmatrix} 1\\1\\-1 \end{pmatrix}, \alpha_3=\begin{pmatrix} 1\\-1\\1 \end{pmatrix}, \beta_1=\begin{pmatrix} 3\\0\\1 \end{pmatrix}, \beta_2=\begin{pmatrix} 2\\0\\0 \end{pmatrix}, \beta_3=\begin{pmatrix} 0\\2\\-2 \end{pmatrix}$

（**1**）　证明：$\alpha_1,\alpha_2,\alpha_3$与$\beta_1,\beta_2,\beta_3$都是$\mathbf{R}^3$的基;

（**2**）　求基$\alpha_1,\alpha_2,\alpha_3$到基$\beta_1,\beta_2,\beta_3$的过渡矩阵;

（**3**）　已知向量ξ在基β_1,β_2,β_3下的坐标为$(1,2,0)$,求ξ在基$\alpha_1,\alpha_2,\alpha_3$下的坐标;

（**4**）　求在基$\alpha_1,\alpha_2,\alpha_3$和基$\beta_1,\beta_2,\beta_3$下有相同坐标的非零向量.

解：（1）$\begin{pmatrix} 1\\0\\1 \end{pmatrix}\begin{pmatrix} 1\\1\\-1 \end{pmatrix}\begin{pmatrix} 1\\-1\\1 \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & -1\\ 1 & -1 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}$, rank: 3;

$\begin{pmatrix} 3\\0\\1 \end{pmatrix}\begin{pmatrix} 2\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\2\\-2 \end{pmatrix}$, concatenate: $\begin{pmatrix} 3 & 2 & 0\\ 0 & 0 & 2\\ 1 & 0 & -2 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}$, rank: 3.故$\alpha_1,\alpha_2,\alpha_3$与$\beta_1,\beta_2,\beta_3$都是线性无关的向量组，从而都是$\mathbf{R}^3$的基.

（2）过渡矩阵$A=\begin{pmatrix}1&1&1\\0&1&-1\\1&-1&1\end{pmatrix}^{-1}\begin{pmatrix}3&2&0\\0&0&2\\1&0&-2\end{pmatrix}$
$=\begin{pmatrix}0&1&1\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{2}&-1&-\frac{1}{2}\end{pmatrix}\begin{pmatrix}3&2&0\\0&0&2\\1&0&-2\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&1\\1&1&-1\end{pmatrix}$.

（3）ξ在基$\alpha_1,\alpha_2,\alpha_3$下的坐标$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=A\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&1\\1&1&-1\end{pmatrix}\begin{pmatrix}1\\2\\0\end{pmatrix}=$
$\begin{pmatrix}1\\3\\3\end{pmatrix}$,即$(1.3.3)$.

（4）由$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$知$(A-E)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0.A-E=\begin{pmatrix}1&0&0\\1&1&1\\1&1&-1\end{pmatrix}-$
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&0&0\\1&0&1\\1&1&-2\end{pmatrix}$, row echelon form: $\begin{pmatrix}1&0&1\\0&1&-3\\0&0&0\end{pmatrix}$, $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$
$c\begin{pmatrix}-1\\3\\1\end{pmatrix}$, ($c$是任意的非零实数).

或设$\alpha\neq0$,在基$\alpha_1,\alpha_2,\alpha_3$和基$\beta_1,\beta_2,\beta_3$下有相同坐标$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$,即$\alpha=$
$x_1\alpha_1+x_2\alpha_2+x_3\alpha_3=x_1\beta_1+x_2\beta_2+x_3\beta_3$,从而$x_1(\alpha_1-\beta_1)+x_2(\alpha_2-\beta_2)+$
$x_3(\alpha_3-\beta_3)=0$,即$\begin{cases}-2x_1-x_2+x_3=0\\x_2-3x_3=0\\-x_2+3x_3\end{cases}$, Solution is: $\{x_2=3x_3,x_1=-x_3,x_3=x_3\}$,
即$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=c\begin{pmatrix}-1\\3\\1\end{pmatrix}$, ($c$是任意的非零实数).

三、五元齐次线性方程组的系数矩阵的秩为2,已知方程组的解向量有$\xi_1=(1,-2,1,0,0)^T,\xi_2=(1,-2,0,1,0)^T,\xi_3=(1,-2,3,-2,0)^T,\xi_4=(5,-6,0,0,1)^T$,问这四个向量是否构成该方程组的基础解系？若不是，试求之.

解：五元齐次线性方程组的基础解系的向量个数为$5-2=3$,

$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & 1 & 1 & 5 \\ -2 & -2 & -2 & -6 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

row echelon form: $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, rank: 3,故$\xi_1, \xi_2, \xi_3, \xi_4$线性相关，不能构成该五元齐次线性方程组的基础解系.基础解系数为ξ_1, ξ_2, ξ_4或ξ_1, ξ_3, ξ_4.

四、求线性方程组$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$的导出组的一个基础解系，并求出线性方程组的通解.

解：$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$, Corresponding matrix: $\begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$,

row echelon form: $\begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank: 3,导出组的基础解系的向量个数$5-3=2$,由$\begin{cases} x_1 = -2x_2 + 2x_5 - 2 \\ x_2 = x_2 + 0x_5 + 0 \\ x_3 = 0x_2 + x_5 + 2 \\ x_4 = 0x_2 + 0x_5 + 1 \\ x_5 = 0x_2 + x_5 + 0 \end{cases}$故导出组的基础解系为$\xi_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,$\xi_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$,线性方程组的特解$\eta = \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$,从而通解为$x = c_1\xi_1 + c_2\xi_2 + \eta, (c_1, c_2$是任意的非零实数).

五、已知$\xi_1=\begin{pmatrix}0\\1\\0\end{pmatrix}$,$\xi_2=\begin{pmatrix}-3\\2\\2\end{pmatrix}$都是方程组$\begin{cases}x_1-x_2+2x_3=-1\\3x_1+x_2+4x_3=1\\ax_1+bx_2+cx_3=d\end{cases}$的解，求该方程组的通解.

解：$\begin{cases}x_1-x_2+2x_3=-1\\3x_1+x_2+4x_3=1\\ax_1+bx_2+cx_3=d\end{cases}$, Corresponding matrix: $\begin{pmatrix}1&-1&2&-1\\3&1&4&1\\a&b&c&d\end{pmatrix}$

$\xrightarrow[-ar_1+r_3]{-3r_1+r_2,}\begin{pmatrix}1&-1&2&-1\\0&4&-2&4\\0&b+a&c-2a&d+a\end{pmatrix}\xrightarrow[-(a+b)r_2+r_3]{\frac{1}{4}r_2,}\begin{pmatrix}1&-1&2&-1\\0&1&-\frac{1}{2}&1\\0&0&\frac{1}{2}b-\frac{3}{2}a+c&d-b\end{pmatrix}$.由

已知$\xi_1=\begin{pmatrix}0\\1\\0\end{pmatrix}$,$\xi_2=\begin{pmatrix}-3\\2\\2\end{pmatrix}$是原方程的解，代入原方程，得$b=d,-3a+2b+2c=d$,从而$-3a+b+2c=0$, $\begin{pmatrix}1&-1&2&-1\\0&1&-\frac{1}{2}&1\\0&0&\frac{1}{2}b-\frac{3}{2}a+c&d-b\end{pmatrix}=$

$\begin{pmatrix}1&-1&2&-1\\0&1&-\frac{1}{2}&1\\0&0&0&0\end{pmatrix}\xrightarrow{r_2+r_1}\begin{pmatrix}1&0&\frac{3}{2}&0\\0&1&-\frac{1}{2}&1\\0&0&0&0\end{pmatrix}$,方程组的通解为$x=t\begin{pmatrix}-\frac{3}{2}\\\frac{1}{2}\\1\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}$, ($t$是任意的非零实数).

六、已知线性方程组$\begin{cases}x_1+x_2-2x_3+3x_4=0\\2x_1+x_2-6x_3+4x_4=-1\\3x_1+2x_2+px_3+7x_4=-1\\x_1-x_2-6x_3-x_4=t\end{cases}$,讨论参数$p,t$取何值时，方程组无解、有解，有解时用导出组的基础解系表示出通解.

解：$\begin{cases}x_1+x_2-2x_3+3x_4=0\\2x_1+x_2-6x_3+4x_4=-1\\3x_1+2x_2+px_3+7x_4=-1\\x_1-x_2-6x_3-x_4=t\end{cases}$,

Solution is: $\begin{cases}\{[x_1=-x_4-1,x_2=-2x_4+1,x_3=0]\} & \text{if}\quad p\neq-8\wedge t=-2\\ \emptyset & \text{if}\quad t\neq-2\\ \{[x_1=4x_3-x_4-1,x_2=-2x_3-2x_4+1]\} & \text{if}\quad p=-8\wedge t=-2\\ \emptyset & \text{if}\quad t\neq-2\wedge p=-8\end{cases}$.

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & -6 & 4 & -1 \\ 3 & 2 & p & 7 & -1 \\ 1 & -1 & -6 & -1 & t \end{array}\right) \xrightarrow[-r_1+r_4]{\substack{-2r_1+r_2, \\ -3r_1+r_3}} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -2 & -2 & -1 \\ 0 & -1 & p+6 & -2 & -1 \\ 0 & -2 & -4 & -4 & t \end{array}\right)$$

$$\xrightarrow{-r_2} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & p+6 & -2 & -1 \\ 0 & -2 & -4 & -4 & t \end{array}\right) \xrightarrow[2r_2+r_4]{r_2+r_3} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & p+8 & 0 & 0 \\ 0 & 0 & 0 & 0 & t+2 \end{array}\right)$$

$$\xrightarrow{-r_2+r_1} \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & p+8 & 0 & 0 \\ 0 & 0 & 0 & 0 & t+2 \end{array}\right) \xrightarrow[\frac{1}{p+8}r_3]{if \quad p \neq -8,} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & t+2 \end{array}\right),$$

$t \neq -2$时，方程组无解；$t = -2$时，方程组有解.

$p = -8$且$t = -2$时，基础解系为$\xi_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$,

令$x_3 = x_4 = 0$,则特解为$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$,

通解为$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, ($c_1, c_2$是任意的非零实数).

$p \neq -8$时，$x_3 = 0$,基础解系为$\xi = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$,

令$x_3 = 0$,则特解为$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$,通解为$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, ($c$是任意的非零实数).

七、设线性方程组$\begin{cases} x_1+px_2+tx_3+x_4=0 \\ 2x_1+x_2+x_3+2x_4=0 \\ 3x_1+(2+p)x_2+(4+t)x_3+4x_4=1 \end{cases}$,已知$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$是该方程组的一个解，试求:（1）方程组的通解，并用导出组的基础解系表示出通解;（2）该方程组满足$x_2=x_3$的全部解.

解：因为$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$是方程组$\begin{cases} x_1+px_2+tx_3+x_4=0 \\ 2x_1+x_2+x_3+2x_4=0 \\ 3x_1+(2+p)x_2+(4+t)x_3+4x_4=1 \end{cases}$的一个解，故$t-p=0$,即$t=p$.

$\begin{cases} x_1+px_2+tx_3+x_4=0 \\ 2x_1+x_2+x_3+2x_4=0 \\ 3x_1+(2+p)x_2+(4+t)x_3+4x_4=1 \end{cases}$, Corresponding matrix:

$$\begin{pmatrix} 1 & p & t & 1 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & p+2 & t+4 & 4 & 1 \end{pmatrix} \xrightarrow[-3r_1+r_2]{-2r_1+r_2,} \begin{pmatrix} 1 & p & p & 1 & 0 \\ 0 & 1-2p & 1-2p & 0 & 0 \\ 0 & -2p+2 & -2p+4 & 1 & 1 \end{pmatrix} \xrightarrow[-r_2]{-r_3+r_2,}$$
$$\begin{pmatrix} 1 & p & p & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & -2p+2 & -2p+4 & 1 & 1 \end{pmatrix} \xrightarrow{(2p-2)\,r_2+r_3} \begin{pmatrix} 1 & p & p & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 4p-2 & 2p-1 & 2p-1 \end{pmatrix}.$$

（1）$p=\frac{1}{2}$时，$\begin{pmatrix} 1 & p & p & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 4p-2 & 2p-1 & 2p-1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}r_2+r_1}$

$\begin{pmatrix} 1 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,通解为$x=c_1\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}+c_2\begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $(c_1,c_2$是任意的非零实数).

$p\neq\frac{1}{2}$时，$\begin{pmatrix} 1 & p & p & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 4p-2 & 2p-1 & 2p-1 \end{pmatrix} \xrightarrow{\frac{1}{4p-2}r_3} \begin{pmatrix} 1 & p & p & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[-pr_3+r_1]{-3r_3+r_2,}$

$\begin{pmatrix} 1 & p & 0 & 1-\frac{p}{2} & -\frac{p}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow{-pr_2+r_1} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$,通解为$x=c\begin{pmatrix} -1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}+$

$\begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$, ($c$是任意的非零实数).

（2）$p=\frac{1}{2}$时，$x=c_1\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}+c_2\begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \\ 1 \end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} c_1-\frac{1}{2}c_2-\frac{1}{2} \\ 1-c_2-3c_1 \\ c_1 \\ c_2 \end{pmatrix}$,由$x_3=x_4$得，$1-c_2-3c_1=c_1, c_1=\frac{1}{4}-\frac{1}{4}c_2$,从而$x=\begin{pmatrix} c_1-\frac{1}{2}c_2-\frac{1}{2} \\ 1-c_2-3c_1 \\ c_1 \\ c_2 \end{pmatrix}=$

$\begin{pmatrix} -\frac{3}{4}c_2-\frac{1}{4} \\ \frac{1}{4}-\frac{1}{4}c_2 \\ \frac{1}{4}-\frac{1}{4}c_2 \\ c_2 \end{pmatrix}=c_2\begin{pmatrix} -\frac{3}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix}+\begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix}$.

$p\neq\frac{1}{2}$时，$x=c\begin{pmatrix} -1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}+\begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}=\begin{pmatrix} -1 \\ \frac{1}{2}c-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2}c \\ c \end{pmatrix}$,由$x_3=x_4$得，

$\frac{1}{2}c-\frac{1}{2}=\frac{1}{2}-\frac{1}{2}c$,从而$c=1$,方程组的解为$x=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

八、已知下列非齐次线性方程组：

(I)$\begin{cases} x_1+x_2-2x_4=-6 \\ 4x_1-x_2-x_3-x_4=1 \\ 3x_1-x_2-x_3=3 \end{cases}$,(II)$\begin{cases} x_1+mx_2-x_3-x_4=-5 \\ nx_2-x_3-2x_4=-11 \\ x_3-2x_4=-t+1 \end{cases}$.

求:（1）解方程组(I)，用其导出组的基础解系表示通解;（2）当方程组(II)中的参数m,n,t为何值时，方程组(I)与(II)同解.

解:（1）$\begin{pmatrix} x_1+x_2-2x_4=-6 \\ 4x_1-x_2-x_3-x_4=1 \\ 3x_1-x_2-x_3=3 \end{pmatrix}$, Corresponding matrix: $\begin{pmatrix} 1 & 1 & 0 & -2 & -6 \\ 4 & -1 & -1 & -1 & 1 \\ 3 & -1 & -1 & 0 & 3 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & -2 & -5 \end{pmatrix}$,导出组的基础解系为$\xi = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$,方程组的特解$\eta = \begin{pmatrix} -2 \\ -4 \\ -5 \\ 0 \end{pmatrix}$,通解为$x = c\xi + \eta$, ($c$是任意的非零实数).

（2）方程组(II)与方程组(I)同解，故η也是方程组(II)的解，$\begin{cases} -2 - 4m + 5 = -5 \\ -4n + 5 = -11 \\ -5 = -t + 1 \end{cases}$,

Solution is: $[m = 2, n = 4, t = 6]$.

6.1 矩阵的相似与对角化 6.2 特征值与特征向量

一、填空题：

1. 已知向量$\xi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$是矩阵$A = \begin{pmatrix} a & 1 & 1 \\ -2 & 0 & 1 \\ -1 & 2 & -2 \end{pmatrix}$的特征向量，则参数$a =$ ______.

2. 矩阵$\begin{pmatrix} a & 1 & b \\ 2 & 3 & 4 \\ -1 & 1 & -1 \end{pmatrix}$的特征值之和为3,特征值之积为$-24$,则$a =$ ______, $b =$ ______.

3. 设A为n阶矩阵且$Ax = 0$有非零解，则A必有特征值______.

4. 已知$\lambda_1 = 0$是3阶矩阵$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & a \end{pmatrix}$的特征值，则$a =$ ______, A的其他特征值$\lambda_2 =$ ______, $\lambda_3 =$ ______.

5. 设A是3阶方阵且$A - E, A - 2E, 2A + E$都不可逆，则$|A| =$ ______.

6. 已知3阶方阵A的特征值分别为$1, -1, 2$,则$A^* + 3A - 2E$的特征值为______, $|A^* + 3A - 2E| =$ ______, $|A^3 - 5A^2| =$ ______.

7. 已知3阶方阵A的特征值分别为$1, -1, 2$,则与A的伴随矩阵A^*相似的一个对角矩阵为______.

8. 设$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 4 & a \\ -3 & -3 & 5 \end{pmatrix}$, A的特征值分别为$6, 2, 2$,且A有三个线性无关的特征向量，则参数$a =$ ______.

提示：1. $\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} a & 1 & 1 \\ -2 & 0 & 1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} \lambda - a & -1 & -1 \\ 2 & \lambda & -1 \\ 1 & -2 & \lambda + 2 \end{pmatrix}$,

$\begin{pmatrix} \lambda - a & -1 & -1 \\ 2 & \lambda & -1 \\ 1 & -2 & \lambda + 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda - a - 2 \\ \lambda + 1 \\ \lambda + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\lambda = -1, a = -3$. 2. $\left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} a & 1 & b \\ 2 & 3 & 4 \\ -1 & 1 & -1 \end{pmatrix} \right| = 7a - 5b - 9\lambda + 2a\lambda + b\lambda -$

$2\lambda^2+\lambda^3-a\lambda^2+2=\lambda^3-(a+2)\lambda^2+(-9+2a+b)\lambda+7a-5b+2=0$,由已知得$a+2=3, 7a-5b+2=24$,即$a=1, b=-3$.　3. 0.　4. $\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&1\\0&2&0\\1&0&a\end{pmatrix}\right|=\lambda-2a+3a\lambda-3\lambda^2+\lambda^3-a\lambda^2+2=0$, $\lambda_1=0$是其根，则$-2a+2=0$,即$a=1$.这样特征方程为$4\lambda-4\lambda^2+\lambda^3=0$, Solution is: $2,0$,即$\lambda_2=\lambda_3=2$.　5. A是3阶方阵且$A-E, A-2E, 2A+E$都不可逆，则$|A-E|=|A-2E|=|2A+E|=0$从而$1,2,-\frac{1}{2}$是A的特征根，$|A|=\begin{vmatrix}1&0&0\\0&2&0\\0&0&-\frac{1}{2}\end{vmatrix}=-1$.　6. 因为3阶矩阵$A$的特征值为$1,-1,2$,故$A$可逆，$A^{-1}$的特征值为$1,-1,\frac{1}{2}$,又$A^*A=AA^*=|A|E$,故$A^*=|A|A^{-1}$, $P^{-1}(A^*+3A-2E)P=P^{-1}A^*P+3P^{-1}AP-2E=|A|P^{-1}A^{-1}P+3P^{-1}AP-2E$,又$P^{-1}AP=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&2\end{pmatrix}$,故$|A|=\begin{vmatrix}1&0&0\\0&-1&0\\0&0&2\end{vmatrix}=-2$, $P^{-1}(A^*+3A-2E)P=-2\begin{pmatrix}1&0&0\\0&-1&0\\0&0&\frac{1}{2}\end{pmatrix}+3\begin{pmatrix}1&0&0\\0&-1&0\\0&0&2\end{pmatrix}-2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-1&0&0\\0&-3&0\\0&0&3\end{pmatrix}$,故$A^*+3A-2E$的特征值为$-1,-3,3$.另法：因为3阶矩阵$A$的特征值为$1,-1,2$,故$A$可逆，$A^{-1}$的特征值为$1.-1,\frac{1}{2}$又$A^*A=|A|E$,故$A^*=|A|A^{-1}=-2A^{-1}$.设$f(x)=-2x^{-1}+3x-2$, $f(1)=-1$, $f(-1)=-3$, $f(2)=3$,令$B=f(A)=A^*+3A-2E$,则B的特征值为$-1,-3,3$. $|A^*+3A-2E|=\begin{vmatrix}-1&0&0\\0&-3&0\\0&0&3\end{vmatrix}=9$. $|A^3-5A^2|=|A|^2|A-5E|=4(\lambda-1)(\lambda+1)(\lambda-2)|_{\lambda=5}=288$,　7. 已知3阶方阵$A$的特征值分别为$1,-1,2$,则$|A|=\left|\begin{pmatrix}1&0&0\\0&-1&0\\0&0&2\end{pmatrix}\right|=-2$, $P^{-1}AP=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&2\end{pmatrix}$, $P^{-1}A^{-1}P=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&\frac{1}{2}\end{pmatrix}$, $P^{-1}\left(|A|A^{-1}\right)P=$

$$-2\begin{pmatrix}1&0&0\\0&-1&0\\0&0&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-2&0&0\\0&2&0\\0&0&-1\end{pmatrix},\text{即}P^{-1}A^*P=\begin{pmatrix}-2&0&0\\0&2&0\\0&0&-1\end{pmatrix}.$$

8. $\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&-1&1\\2&4&a\\-3&-3&5\end{pmatrix}\right|=34\lambda-6a+3a\lambda-10\lambda^2+\lambda^3-36=0$, $(34\lambda-6a+3a\lambda-10\lambda^2+\lambda^3-36)\,|_{\lambda=6}=12a+24=0, a=-2$,
$(34\lambda-6a+3a\lambda-10\lambda^2+\lambda^3-36)\,|_{\lambda=2}=0$.

二、求矩阵$A=\begin{pmatrix}1&0&0\\2&3&0\\4&5&6\end{pmatrix}$的特征值与特征向量.

解：$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\2&3&0\\4&5&6\end{pmatrix}\right|=\lambda^3-10\lambda^2+27\lambda-18=0$,
Solution is: $3,1,6$,

$$3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\2&3&0\\4&5&6\end{pmatrix}=\begin{pmatrix}2&0&0\\-2&0&0\\-4&-5&-3\end{pmatrix},\begin{pmatrix}2&0&0\\-2&0&0\\-4&-5&-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

, Solution is: $\begin{pmatrix}0\\-\frac{3}{5}\hat{t}_3\\\hat{t}_3\end{pmatrix}$, $\xi_1=c\begin{pmatrix}0\\-3\\5\end{pmatrix}$,（$c$为任意非零实数）；

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\2&3&0\\4&5&6\end{pmatrix}=\begin{pmatrix}0&0&0\\-2&-2&0\\-4&-5&-5\end{pmatrix},\begin{pmatrix}0&0&0\\-2&-2&0\\-4&-5&-5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

, Solution is: $\begin{pmatrix}5\hat{t}_3\\-5\hat{t}_3\\\hat{t}_3\end{pmatrix}$, $\xi_2=c\begin{pmatrix}5\\-5\\1\end{pmatrix}$,（$c$为任意非零实数）；

$$6\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\2&3&0\\4&5&6\end{pmatrix}=\begin{pmatrix}5&0&0\\-2&3&0\\-4&-5&0\end{pmatrix},\begin{pmatrix}5&0&0\\-2&3&0\\-4&-5&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

, Solution is: $\begin{pmatrix}0\\0\\\hat{t}_3\end{pmatrix}$, $\xi_3=c\begin{pmatrix}0\\0\\1\end{pmatrix}$,（$c$为任意非零实数）；

三、求矩阵$A=\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 3 & 9 \end{pmatrix}$的特征值与特征向量.

解：$\left|\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 3 & 9 \end{pmatrix}\right| = \lambda^3-13\lambda^2=0$, Solution is: $13, 0$.

$13\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 3 & 9 \end{pmatrix}=\begin{pmatrix} 11 & -1 & -3 \\ -4 & 11 & -6 \\ -6 & -3 & 4 \end{pmatrix}$, $\begin{pmatrix} 11 & -1 & -3 \\ -4 & 11 & -6 \\ -6 & -3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \frac{1}{3}\hat{t}_3 \\ \frac{2}{3}\hat{t}_3 \\ \hat{t}_3 \end{pmatrix}$, $\xi_1=c\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$,（$c$为任意非零实数）；

$0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 3 & 9 \end{pmatrix}=\begin{pmatrix} -2 & -1 & -3 \\ -4 & -2 & -6 \\ -6 & -3 & -9 \end{pmatrix}$, $\begin{pmatrix} -2 & -1 & -3 \\ -4 & -2 & -6 \\ -6 & -3 & -9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -\frac{1}{2}\hat{t}_2-\frac{3}{2}\hat{t}_3 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix}$, $\xi_2=c_1\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}+c_2\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$,（$c_1, c_2$是任意不全为零的实数）；

四、已知$A=\begin{pmatrix} -4 & -10 & 0 \\ 1 & 3 & 0 \\ 3 & 6 & 1 \end{pmatrix}$,判断$A$是否可以对角化.若能对角化，则求可逆矩阵$P$,使得$P^{-1}AP$为对角矩阵.

解：$\left|\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} -4 & -10 & 0 \\ 1 & 3 & 0 \\ 3 & 6 & 1 \end{pmatrix}\right| = \lambda^3-3\lambda+2=(\lambda+2)(\lambda-1)^2=0$, Solution is: $1, -2$.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} -4 & -10 & 0 \\ 1 & 3 & 0 \\ 3 & 6 & 1 \end{pmatrix}=\begin{pmatrix} 5 & 10 & 0 \\ -1 & -2 & 0 \\ -3 & -6 & 0 \end{pmatrix}$, $\begin{pmatrix} 5 & 10 & 0 \\ -1 & -2 & 0 \\ -3 & -6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -2\hat{t}_2 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$;

$$-2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}-4&-10&0\\1&3&0\\3&6&1\end{pmatrix}=\begin{pmatrix}2&10&0\\-1&-5&0\\-3&-6&-3\end{pmatrix},\begin{pmatrix}2&10&0\\-1&-5&0\\-3&-6&-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-\frac{5}{3}\hat{t}_3\\\frac{1}{3}\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix}-5\\1\\3\end{pmatrix}$.

$P=\begin{pmatrix}-2&0&-5\\1&0&1\\0&1&3\end{pmatrix}$, $P^{-1}AP=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$.

五、已知$\xi=\begin{pmatrix}1\\1\\-1\end{pmatrix}$是矩阵$A=\begin{pmatrix}2&-1&2\\5&a&3\\-1&b&-2\end{pmatrix}$的一个特征向量.

（1）试确定参数a,b及特征向量ξ所对应的特征值;

（2）问A是否相似于对角矩阵？并说明理由.

解：（1）$\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}2&-1&2\\5&a&3\\-1&b&-2\end{pmatrix}=\begin{pmatrix}\lambda-2&1&-2\\-5&\lambda-a&-3\\1&-b&\lambda+2\end{pmatrix}$,

$\begin{pmatrix}\lambda-2&1&-2\\-5&\lambda-a&-3\\1&-b&\lambda+2\end{pmatrix}\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}\lambda+1\\\lambda-a-2\\-b-\lambda-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\lambda=-1, a=-3, b=0$.

（2）$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}2&-1&2\\5&-3&3\\-1&0&-2\end{pmatrix}\right|=\lambda^3+3\lambda^2+3\lambda+1=(\lambda+1)^3=0$, Solution is: -1.

$$-1\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}2&-1&2\\5&-3&3\\-1&0&-2\end{pmatrix}=\begin{pmatrix}-3&1&-2\\-5&2&-3\\1&0&1\end{pmatrix},\begin{pmatrix}-3&1&-2\\-5&2&-3\\1&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-\hat{t}_3\\-\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha=\begin{pmatrix}-1\\-1\\1\end{pmatrix}$.由于基础解系只有一个解向量，故$A$不能对角化，不相似于对角矩阵.

六、设3阶矩阵A的特征值$\lambda_1=2,\lambda_2=-2,\lambda_3=1$,对应的特征向量依次为$p_1=\begin{pmatrix}0\\1\\1\end{pmatrix}$, $p_2=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $p_3=\begin{pmatrix}1\\1\\0\end{pmatrix}$,求$A$.

解：$Ap_i=\lambda_i p_i\,(i=1,2,3)\,, A\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}=\begin{pmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \lambda_3 p_3 \end{pmatrix}$,即$A\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}=$ $\begin{pmatrix} 0 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 0 \end{pmatrix}, A=\begin{pmatrix} 0 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}=\begin{pmatrix} 0 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}=$ $\begin{pmatrix} -2 & 3 & -3 \\ -4 & 5 & -3 \\ -4 & 4 & -2 \end{pmatrix}$.

七、已知$A=\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$,求$A^n(n\in\mathbf{N}^+)$.

解：$A=\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, A^2=\begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}, A^3=\begin{pmatrix} 9 & 9 \\ 18 & 18 \end{pmatrix}, A^4=\begin{pmatrix} 27 & 27 \\ 54 & 54 \end{pmatrix},\cdots,A^n=$ $\begin{pmatrix} 3^{n-1} & 3^{n-1} \\ 2\times 3^{n-1} & 2\times 3^{n-1} \end{pmatrix}$,（可用数学归纳法证明）.

或$\left|\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\right|=\lambda^2-3\lambda=0$, Solution is: $\{\lambda=0\},\{\lambda=3\}$.

$0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}=\begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -\hat{t}_1 \\ \hat{t}_1 \end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$;

$3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}=\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \hat{t}_1 \\ 2\hat{t}_1 \end{pmatrix}$,基础解系$\alpha_2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

令$P=\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$,则$P^{-1}AP=\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, A^n=P\begin{pmatrix} 0 & 0 \\ 0 & 3^n \end{pmatrix}P^{-1}=$ $\begin{pmatrix} 3^{-1+n} & 3^{-1+n} \\ 2\times 3^{-1+n} & 2\times 3^{-1+n} \end{pmatrix}$.

八、设A为3阶矩阵，$\alpha_1,\alpha_2,\alpha_3$是线性无关的三维列向量，且满足$A\alpha_1=\alpha_1+\alpha_2+\alpha_3, A\alpha_2=2\alpha_2+\alpha_3, A\alpha_3=2\alpha_2+3\alpha_3$,求：

（1）矩阵B,使得$A(\alpha_1\alpha_2\alpha_3)=(\alpha_1\alpha_2\alpha_3)B$;

（2）矩阵A的特征值;

（3）可逆矩阵P,使得$P^{-1}AP$为对角矩阵.

解：（1）由已知A为3阶矩阵，且$A\alpha_1=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $A\alpha_2=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}0\\2\\1\end{pmatrix}$, $A\alpha_3=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}0\\2\\3\end{pmatrix}$,故$A\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}$.又因为$\alpha_1,\alpha_2,\alpha_3$是线性无关的三维列向量，故$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}$是可逆矩阵，$B=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}^{-1}A\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}=\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}$.

（2）由（1）知A相似于B, A的特征值与B的特征值相同.$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}\right|=\lambda^3-6\lambda^2+9\lambda-4=(\lambda-4)(\lambda-1)^2=0$, Solution is: $1,4$,即A的特征值为$1,1,4$.

（3）$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}=\begin{pmatrix}0&0&0\\-1&-1&-2\\-1&-1&-2\end{pmatrix}$, $\begin{pmatrix}0&0&0\\-1&-1&-2\\-1&-1&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-\hat{t}_2-2\hat{t}_3\\\hat{t}_2\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-2\\0\\1\end{pmatrix}$;

$4\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}=\begin{pmatrix}3&0&0\\-1&2&-2\\-1&-1&1\end{pmatrix}$, $\begin{pmatrix}3&0&0\\-1&2&-2\\-1&-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}0\\\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$.

因为$\alpha_1,\alpha_2,\alpha_3$线性无关，有$P=\begin{pmatrix}-1&-2&0\\1&0&1\\0&1&1\end{pmatrix}$, $P^{-1}\begin{pmatrix}1&0&0\\1&2&2\\1&1&3\end{pmatrix}P=$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

6.3 内积与正交矩阵 6.4 实对称矩阵的对角化

一、填空题：

1. 设$\alpha=\begin{pmatrix}1\\2\\a\end{pmatrix}$,$\beta=\begin{pmatrix}b\\1\\-1\end{pmatrix}$,若$\alpha,\beta$正交，则$a,b$所满足的关系为______.

2. 与$\alpha_1=\begin{pmatrix}1\\-1\\0\\2\end{pmatrix}$,$\alpha_2=\begin{pmatrix}2\\3\\1\\1\end{pmatrix}$,$\alpha_3=\begin{pmatrix}0\\0\\1\\2\end{pmatrix}$都正交的单位向量是______.

3. 设$A=\begin{pmatrix}\frac{2}{3}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{18}}\\a&b&-\frac{4}{\sqrt{18}}\\\frac{2}{3}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{18}}\end{pmatrix}$是正交矩阵，则$a=$ ______,$b=$ ______.

4. 若A是正交矩阵，则行列式$|A^3A^T|=$ ______.

提示：1. $\alpha\cdot\beta=b-a+2=0$. 2. 设$\alpha=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$,由$\begin{cases}\alpha_1\cdot\alpha=0\\\alpha_2\cdot\alpha=0\\\alpha_3\cdot\alpha=0\end{cases}$得$\begin{cases}x_1-x_2+2x_4=0\\2x_1+3x_2+x_3+x_4=0\\x_3+x_4=0\end{cases}$, Corresponding matrix: $\begin{pmatrix}1&-1&0&2&0\\2&3&1&1&0\\0&0&1&1&0\end{pmatrix}$,

row echelon form: $\begin{pmatrix}1&0&0&\frac{6}{5}&0\\0&1&0&-\frac{4}{5}&0\\0&0&1&1&0\end{pmatrix}$,取$x_4=-5$,得$\alpha=\begin{pmatrix}-6\\4\\5\\-5\end{pmatrix}$,$\beta=\frac{\alpha}{\|\alpha\|}=\begin{pmatrix}-\frac{1}{17}\sqrt{102}\\\frac{2}{51}\sqrt{102}\\\frac{5}{102}\sqrt{102}\\-\frac{5}{102}\sqrt{102}\end{pmatrix}$.

3. $A^TA=\begin{pmatrix}a^2+\frac{8}{9}&ab&\frac{2}{9}\sqrt{2}-\frac{2}{3}\sqrt{2}a\\ab&b^2+1&-\frac{2}{3}\sqrt{2}b\\\frac{2}{9}\sqrt{2}-\frac{2}{3}\sqrt{2}a&-\frac{2}{3}\sqrt{2}b&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$,故$a^2+\frac{8}{9}=b^2+1=1,ab=-\frac{2}{3}\sqrt{2}b=\frac{2}{9}\sqrt{2}-\frac{2}{3}\sqrt{2}a=0$,从而$a=\frac{1}{3},b=0$.

4. 若A是正交矩阵，则$A^T = A^{-1}, |A^3A^T| = |A^3A^{-1}| = |A^2| = |A|^2$.

二、设$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,求$\alpha_2, \alpha_3$,使$\alpha_1, \alpha_2, \alpha_3$,相互正交.

解：取$\alpha_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$,则$\alpha_1 \cdot \alpha_2 = 0$.设$\alpha_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,由$\begin{cases} \alpha_1 \cdot \alpha_3 = 0 \\ \alpha_2 \cdot \alpha_3 = 0 \end{cases}$得$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \end{cases}$,

Corresponding matrix: $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$,取$x_3 = -1$,则$x_1 = 1, x_2 = -2$,即$\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

或取$\alpha_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$,则$\alpha_1 \cdot \alpha_2 = 0$.设$\alpha_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,由$\begin{cases} \alpha_1 \cdot \alpha_3 = 0 \\ \alpha_2 \cdot \alpha_3 = 0 \end{cases}$得$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$,

Corresponding matrix: $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{pmatrix}$,取$x_3 = -1$,则$x_1 = \frac{1}{2}, x_2 = \frac{1}{2}$,即$\alpha_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}$.

三、已知向量组$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$是线性无关的向量组，求与此向量组等价的正交向量组.

解：$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_2 = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$.

四、设$A = \begin{pmatrix} 3 & -2 & -4 \\ -2 & 6 & -2 \\ -4 & -2 & 3 \end{pmatrix}$,求正交矩阵$P$,使得$P^{-1}AP = \Lambda$（其

中Λ是对角矩阵）.

解：$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}3&-2&-4\\-2&6&-2\\-4&-2&3\end{pmatrix}\right|=\lambda^3-12\lambda^2+21\lambda+98=$ $(\lambda+2)(\lambda-7)^2=0$, Solution is: $7,-2$.

$7\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}3&-2&-4\\-2&6&-2\\-4&-2&3\end{pmatrix}=\begin{pmatrix}4&2&4\\2&1&2\\4&2&4\end{pmatrix}$, $\begin{pmatrix}4&2&4\\2&1&2\\4&2&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-\frac{1}{2}\hat{t}_2-\hat{t}_3\\\hat{t}_2\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}$;

$-2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}3&-2&-4\\-2&6&-2\\-4&-2&3\end{pmatrix}=\begin{pmatrix}-5&2&4\\2&-8&2\\4&2&-5\end{pmatrix}$,

$\begin{pmatrix}-5&2&4\\2&-8&2\\4&2&-5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}\hat{t}_3\\\frac{1}{2}\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_3=$ $\begin{pmatrix}1\\\frac{1}{2}\\1\end{pmatrix}$.

$\beta_1=\alpha_1=\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$,

$\beta_2=\alpha_2-\frac{\alpha_2\cdot\beta_1}{\beta_1\cdot\beta_1}\beta_1=\begin{pmatrix}-\frac{4}{5}\\-\frac{2}{5}\\1\end{pmatrix}$, $\beta_3=\alpha_3=\begin{pmatrix}1\\\frac{1}{2}\\1\end{pmatrix}$, $\epsilon_1=\frac{\beta_1}{\|\beta_1\|}=$ $\begin{pmatrix}-\frac{1}{5}\sqrt{5}\\\frac{2}{5}\sqrt{5}\\0\end{pmatrix}$, $\epsilon_2=\frac{\beta_2}{\|\beta_2\|}=\begin{pmatrix}-\frac{4}{15}\sqrt{5}\\-\frac{2}{15}\sqrt{5}\\\frac{1}{3}\sqrt{5}\end{pmatrix}$, $\epsilon_3=\frac{\beta_3}{\|\beta_3\|}=\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$. $P=\begin{pmatrix}-\frac{1}{5}\sqrt{5}&-\frac{4}{15}\sqrt{5}&\frac{2}{3}\\\frac{2}{5}\sqrt{5}&-\frac{2}{15}\sqrt{5}&\frac{1}{3}\\0&\frac{1}{3}\sqrt{5}&\frac{2}{3}\end{pmatrix}$,

$P^{-1}AP=\begin{pmatrix}7&0&0\\0&7&0\\0&0&-2\end{pmatrix}$

五、设$A=\begin{pmatrix}-1&2&2\\2&-1&-2\\2&-2&-1\end{pmatrix}$.

求：（1）A的特征值与特征向量;（2）$A^{-1}+A^{*}$的特征值与特征向量.

解：（1）$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}-1&2&2\\2&-1&-2\\2&-2&-1\end{pmatrix}\right|=\lambda^3+3\lambda^2-9\lambda+5=$ $(\lambda+5)(\lambda-1)^2=0$, Solution is: $1,-5$.

$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}-1&2&2\\2&-1&-2\\2&-2&-1\end{pmatrix}=\begin{pmatrix}2&-2&-2\\-2&2&2\\-2&2&2\end{pmatrix}$, $\begin{pmatrix}2&-2&-2\\-2&2&2\\-2&2&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}\hat{t}_2+\hat{t}_3\\\hat{t}_2\\\hat{t}_3\end{pmatrix}$,特征向量$\xi_1=c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\0\\1\end{pmatrix}$,（$c_1,c_2$是任意不全为零的实数）.

$-5\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}-1&2&2\\2&-1&-2\\2&-2&-1\end{pmatrix}=\begin{pmatrix}-4&-2&-2\\-2&-4&2\\-2&2&-4\end{pmatrix}$, $\begin{pmatrix}-4&-2&-2\\-2&-4&2\\-2&2&-4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-\hat{t}_3\\\hat{t}_3\\\hat{t}_3\end{pmatrix}$,特征向量$\xi_2=c\begin{pmatrix}-1\\1\\1\end{pmatrix}$,（$c$为任意非零实数）.

（2）$|A|=-5, A^{-1}+A^{*}=A^{-1}+A^{-1}|A|=(1+|A|)A^{-1}=-4A^{-1}, A^{-1}$的特征值为$A$的特征值的倒数：$1,-\frac{1}{5}$.$A^{-1}$的特征向量与$A$的特征向量相同.

六、设$A=\begin{pmatrix}1&-2&-2\\-2&1&a\\-2&a&1\end{pmatrix}$与$\Lambda=\begin{pmatrix}3&0&0\\0&3&0\\0&0&b\end{pmatrix}$相似.

求：（1）a,b的值;（2）A的所有特征值和特征向量;（3）可逆矩阵P,使得$P^{-1}AP=\Lambda$;（4）正交矩阵Q,使得$Q^TAQ=\Lambda$.

解：（1）因为A与Λ相似，故$\lambda=3,3,b$是A的特征值，从而

$\left|3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&-2&-2\\-2&1&a\\-2&a&1\end{pmatrix}\right|=-2a^2-8a-8=-2(a+2)^2=$ 0, Solution is: -2;

$\left|b\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{pmatrix}\right|=b^3-3b^2-9b+27=(b+3)(b-3)^2=$ 0, Solution is: $3,-3$.取$b=-3$.

（2）由（1）知，A的所有特征值为$3,3,-3$,

$$3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{pmatrix}=\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix},\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-\hat{t}_2-\hat{t}_3\\\hat{t}_2\\\hat{t}_3\end{pmatrix}$,特征向量$\xi_1=c_1\begin{pmatrix}-1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}-1\\0\\1\end{pmatrix}$,

（c_1,c_2是任意不全为零的实数）.

$$-3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{pmatrix}=\begin{pmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{pmatrix},\begin{pmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}\hat{t}_3\\\hat{t}_3\\\hat{t}_3\end{pmatrix}$,特征向量$\xi_2=c\begin{pmatrix}1\\1\\1\end{pmatrix}$,（$c$为任意非零实数）

.

（3）由（2）知$\alpha_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\alpha_2=\begin{pmatrix}-1\\0\\1\end{pmatrix},\alpha_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$,

可逆矩阵$P=\begin{pmatrix}-1&-1&1\\1&0&1\\0&1&1\end{pmatrix},P^{-1}AP=\begin{pmatrix}3&0&0\\0&3&0\\0&0&-3\end{pmatrix}$.

（4）$\beta_1=\alpha_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\beta_2=\alpha_2-\frac{\alpha_2\cdot\beta_1}{\beta_1\cdot\beta_1}\beta_1=\begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{pmatrix},\beta_3=$

$\alpha_3=\begin{pmatrix}1\\1\\1\end{pmatrix},\epsilon_1=\frac{\beta_1}{\|\beta_1\|}=\begin{pmatrix}-\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}\\0\end{pmatrix},\epsilon_2=\frac{\beta_2}{\|\beta_2\|}=\begin{pmatrix}-\frac{1}{6}\sqrt{6}\\-\frac{1}{6}\sqrt{6}\\\frac{1}{3}\sqrt{6}\end{pmatrix},\epsilon_3=$

$\frac{\beta_3}{\|\beta_3\|}=\begin{pmatrix}\frac{1}{3}\sqrt{3}\\\frac{1}{3}\sqrt{3}\\\frac{1}{3}\sqrt{3}\end{pmatrix}$,正交矩阵$P=\begin{pmatrix}-\frac{1}{2}\sqrt{2}&-\frac{1}{6}\sqrt{6}&\frac{1}{3}\sqrt{3}\\\frac{1}{2}\sqrt{2}&-\frac{1}{6}\sqrt{6}&\frac{1}{3}\sqrt{3}\\0&\frac{1}{3}\sqrt{6}&\frac{1}{3}\sqrt{3}\end{pmatrix},P^{-1}AP=$

$\begin{pmatrix}3&0&0\\0&3&0\\0&0&-3\end{pmatrix}$.

七、设$6,3,3$为实对称矩阵A的特征值，属于3的特征向量为$\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}$.

求：（1）对应于6的特征向量;（2）矩阵A.

解：（1）利用实对称矩阵属于不同特征值的向量一定正交，设$\alpha_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$为$A$的属于特征值$\lambda = 6$的特征向量，从而有$\begin{cases} -1 \times x_1 + 0 \times x_2 + 1 \times x_3 = 0 \\ 1 \times x_1 + 2 \times x_2 + 1 \times x_3 = 0 \end{cases}$，

即$\begin{cases} x_1 - x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases}$, Solution is: $[x_1 = x_3, x_2 = -x_3]$,即对应于6的特征向量为$c\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,（$c$为非零实数）.

（2）令$P = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$,则$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$,从而$A =$

$$P\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}P^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

八、设3阶实对称矩阵A的秩为2,并且$AB = C$,其中$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$,求$A^n (n \in \mathbf{N}^+)$.

解：由已知$A\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$,设$A\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & a \\ 1 & 0 & b \\ 0 & 2 & c \end{pmatrix}$,则$A = \begin{pmatrix} 0 & 2 & a \\ 1 & 0 & b \\ 0 & 2 & c \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a & 0 & 2-a \\ b & 1 & -b \\ c & 0 & 2-c \end{pmatrix}$.因为$A$是3阶实对称矩阵，故$b = 0, c = 2-a$,又$\begin{pmatrix} a & 0 & 2-a \\ b & 1 & -b \\ c & 0 & 2-c \end{pmatrix} = \begin{pmatrix} a & 0 & 2-a \\ 0 & 1 & 0 \\ 2-a & 0 & a \end{pmatrix} \xrightarrow[\frac{1}{2}r_3]{r_1 + r_2,}$

$\begin{pmatrix} a & 0 & 2-a \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow[-ar_1+r_3]{r_1 \leftrightarrow r_2,} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2-2a \end{pmatrix}$, A的秩为2,故$2-2a=0$,即$a=1$,从而$b=0, c=2-1=1$.$A=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.$A^2=\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$, $A^3=\begin{pmatrix} 4 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 4 \end{pmatrix}, \cdots,$

$A^n=\begin{pmatrix} 2^{n-1} & 0 & 2^{n-1} \\ 0 & 1 & 0 \\ 2^{n-1} & 0 & 2^{n-1} \end{pmatrix}$,（可用数学归纳法证明）.

或$\left|\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right|=\lambda^3-3\lambda^2+2\lambda=\lambda(\lambda-1)(\lambda-2)=0$, Solution is: $\{\lambda=0\},\{\lambda=1\},\{\lambda=2\}$.

$0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -\hat{t}_1 \\ 0 \\ \hat{t}_1 \end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$;

$1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} 0 \\ \hat{t}_1 \\ 0 \end{pmatrix}$,基础解系$\alpha_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$;

$2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \hat{t}_1 \\ 0 \\ \hat{t}_1 \end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$$令P=\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},则P^{-1}AP=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$A^n=P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}P^{-1}=\begin{pmatrix} 2^{-1+n} & 0 & 2^{-1+n} \\ 0 & 1 & 0 \\ 2^{-1+n} & 0 & 2^{-1+n} \end{pmatrix}.$$

习题课

一、填空题：

1. 已知4×3矩阵A的秩为$2, B=\begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 5 & 1 & 2 \end{pmatrix}$,则$\mathrm{rank}(AB)=$______.

2. 设$A=\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{pmatrix}$, $B_{3\times 3}$的列向量线性无关，则$\mathrm{rank}(AB)=$______.

3. 设3阶方阵$A\neq 0, B=\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & t \\ 3 & 5 & 3 \end{pmatrix}$,且$AB=0$,则$t=$______.

4. 假定线性方程组$\begin{pmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$有无穷多个解，则$\lambda=$______.

5. 设A为n阶可逆矩阵，其行向量可由$\beta_1,\beta_2,\cdots,\beta_s$线性表示，则$s$满足______.

6. 若向量组$\alpha_1=\begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_2=\begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \alpha_3=\begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \alpha_4=\begin{pmatrix} 1 \\ 8 \\ 3 \\ t \end{pmatrix}$线性相关，则$t=$______.

7. 已知$\alpha=(1,2,3)^T, \beta=(3,2,1)^T$,则当$k=$______, α与$k\alpha+\beta$正交.

8. 已知$\begin{pmatrix} x & -3 \\ y & -5 \end{pmatrix}$与$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$相似，则$x+y=$______.

9. 设3阶方阵A的特征值为$1,2,3$,则A的伴随矩阵A^*的特征值为______.

10. 设A是一行列式不为零的$n\times n$矩阵，λ是A的一个特征值，则$(A^*)^3+A^{-1}$应有特征值______.

提示：1. $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 5 & 1 & 2 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, rank: 3.

或$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 5 & 1 & 2 \end{pmatrix} \xrightarrow[-r_1+r_3]{\frac{1}{2}r_1,} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 5 & 0 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix} \xrightarrow[-r_2+r_3]{\frac{1}{5}r_2,} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$,秩为3.故$B$可逆，从而rank$(AB)$ =rank$(A) = 2$. 2. $B_{3\times 3}$的列向量线性无关，故rank$(B) = 3$,即B可逆，从而rank(AB) =rank(A),而
$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{-2r_2+r_3} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$（或$\begin{vmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{vmatrix} = -10 \neq 0$）,rank$(A) = 3$,即rank$(AB) = 3$. 3. 由条件知矩阵$B$不可逆，则$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & t \\ 3 & 5 & 3 \end{vmatrix} = 4t - 16 = 0$,即$t = 4$. 4. $\begin{pmatrix} 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 0 \\ \lambda & 1 & 1 & -1 \end{pmatrix}$
$\xrightarrow[-\lambda r_1+r_3]{-r_1+r_2,} \begin{pmatrix} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & -1 \\ 0 & 1-\lambda & 1-\lambda^2 & -1-\lambda \end{pmatrix} \xrightarrow{r_2+r_3}$
$\begin{pmatrix} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda-\lambda^2 & -2-\lambda \end{pmatrix}$, $2-\lambda-\lambda^2 = -2-\lambda = 0, \lambda = -2$.由已知得 5. 因为$A$可逆，故rank$(A) = n$,由条件rank$(A) \leqslant$rank$(\beta_1, \beta_2, \cdots, \beta_s) \leqslant s$,故$s \geqslant n$. 6. $\begin{vmatrix} -1 & 0 & 1 & 1 \\ 2 & 2 & 3 & 8 \\ 3 & -1 & 0 & 3 \\ 1 & 3 & 1 & t \end{vmatrix} = 33 - 11t = 0, t = 3$. 7. $(k\alpha + \beta) \cdot \alpha = k(\alpha \cdot \alpha) + \beta \cdot \alpha = 14k + 14 = 0, k = -1$. 8. . $\left| \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right| = (\lambda - 1)(\lambda - 2) = 0$, Solution is: $\{\lambda = 1\}, \{\lambda = 2\}$,则由已知得$\begin{pmatrix} x & -3 \\ y & -5 \end{pmatrix}$的特征值也为$1, 2$, $\left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} x & -3 \\ y & -5 \end{pmatrix} \right| = 3y - 6x + 6 = 0, x - y = 6$. $\left| 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} x & -3 \\ y & -5 \end{pmatrix} \right| = 14 - 7x + 3y = 0$,解$\begin{cases} x - y = 6 \\ 7x - 3y = 14 \end{cases}$, Solution is: $\{y = -7, x = -1\}$，从而$x + y = -8$. 9. $A^* = |A|A^{-1}$,由已知3阶方阵A的特征值为$1, 2, 3$,故A^{-1}的特征值为$1, \frac{1}{2}, \frac{1}{3}$, A^*的特征值为$6, 3, 2$. 10. $A^* = |A|A^{-1}, (A^*)^3 + A^{-1} = |A|^3 (A)^{-3} + A^{-1}$,令$\varphi(x) = |A|^3 x^{-3} +$

x^{-1},由λ是A的特征值，故$\varphi(\lambda)=|A|^3\lambda^{-3}+\lambda^{-1}$是$(A^*)^3+A^{-1}$的特征值.

二、设向量组$\alpha_1=\begin{pmatrix}1\\1\\1\\3\end{pmatrix},\alpha_2=\begin{pmatrix}-1\\-3\\5\\1\end{pmatrix},\alpha_3=\begin{pmatrix}3\\2\\-1\\t+2\end{pmatrix},\alpha_4=\begin{pmatrix}-2\\-6\\10\\t\end{pmatrix}$.问$t$为何值时，该向量组线性无关？并在此时将向量$\alpha=\begin{pmatrix}4\\1\\6\\10\end{pmatrix}$用$\alpha_1,\alpha_2,\alpha_3,\alpha_4$线性表示.

解：$\begin{pmatrix}1\\1\\1\\3\end{pmatrix}\begin{pmatrix}-1\\-3\\5\\1\end{pmatrix}\begin{pmatrix}3\\2\\-1\\t+2\end{pmatrix}\begin{pmatrix}-2\\-6\\10\\t\end{pmatrix}$, concatenate: $\begin{pmatrix}1&-1&3&-2\\1&-3&2&-6\\1&5&-1&10\\3&1&t+2&t\end{pmatrix}$

$$\xrightarrow[-3r_1+r_4]{-r_1+r_2,-r_1+r_3,}\begin{pmatrix}1&-1&3&-2\\0&-2&-1&-4\\0&6&-4&12\\0&4&t-7&t+6\end{pmatrix}\xrightarrow[2r_2+r_4]{3r_2+r_3,}\begin{pmatrix}1&-1&3&-2\\0&-2&-1&-4\\0&0&-7&0\\0&0&t-9&t-2\end{pmatrix}$$

$\xrightarrow{\frac{1}{7}(t-9)r_3+r_4}\begin{pmatrix}1&-1&3&-2\\0&-2&-1&-4\\0&0&-7&0\\0&0&0&t-2\end{pmatrix}$,当$t\neq 2$即$\mathrm{rank}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=4$时,$\alpha_1,\alpha_2,\alpha_3,\alpha_4$线性无关.当$t\neq 2$时，$\begin{pmatrix}1&-1&3&-2\\1&-3&2&-6\\1&5&-1&10\\3&1&t+2&t\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}4\\1\\6\\10\end{pmatrix}$, Solution is: $\begin{pmatrix}2\\\frac{3t-4}{t-2}\\1\\-\frac{t-1}{t-2}\end{pmatrix}$.从而$\alpha=2\alpha_1+\left(\frac{3t-4}{t-2}\right)\alpha_2+\alpha_3-\left(\frac{t-1}{t-2}\right)\alpha_4$.

三、设线性方程组为$\begin{cases}x_1+x_2+x_3+3x_4=0\\2x_1+x_2+3x_3+5x_4=1\\3x_1+2x_2+ax_3+7x_4=1\\x_1-x_2+3x_3-x_4=b\end{cases}$,问$a,b$各取何值时，此方程组无解、有惟一解、有无穷多个解？并在有无穷多个解时用导出组的基础解系表示出通解.

解：$\begin{cases} x_1+x_2+x_3+3x_4=0 \\ 2x_1+x_2+3x_3+5x_4=1 \\ 3x_1+2x_2+ax_3+7x_4=1 \\ x_1-x_2+3x_3-x_4=b \end{cases}$, Corresponding matrix: $\begin{pmatrix} 1 & 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 5 & 1 \\ 3 & 2 & a & 7 & 1 \\ 1 & -1 & 3 & -1 & b \end{pmatrix}$.设系数矩阵为$A$，常数矩阵为$b$.

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 5 & 1 \\ 3 & 2 & a & 7 & 1 \\ 1 & -1 & 3 & -1 & b \end{pmatrix} \xrightarrow[-r_1+r_4]{-2r_1+r_2,-3r_1+r_3,} \begin{pmatrix} 1 & 1 & 1 & 3 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & -1 & a-3 & -2 & 1 \\ 0 & -2 & 2 & -4 & b \end{pmatrix}$$

$\xrightarrow[-2r_1+r_4]{-r_2+r_3,} \begin{pmatrix} 1 & 1 & 1 & 3 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & a-4 & -1 & 0 \\ 0 & 0 & 0 & -2 & b-2 \end{pmatrix}$.当$a=4$且$b\neq 2$时，$\mathrm{rank}(A)=3\neq 4=\mathrm{rank}(A,b)$,方程组无解; 当$a\neq 4$时，$\mathrm{rank}(A)=\mathrm{rank}(A,b)=4$,方程组有惟一解; 当$a=4$且$b=2$时，$\mathrm{rank}(A)=\mathrm{rank}(A,b)=3<4$,方程组无穷多个解,且$\begin{pmatrix} 1 & 1 & 1 & 3 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix} \xrightarrow[-3r_3+r_1,-r_2+r_1]{-r_2,-r_3,-r_3+r_2,} \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

$\begin{cases} x_1+2x_3=1 \\ x_2-x_3=-1 \\ x_4=0 \end{cases}$,即$\begin{cases} x_1=-2x_3+1 \\ x_2=x_3-1 \\ x_3=x_3 \\ x_4=0 \end{cases}$,原线性方程组导出组的基础解系为$\xi=\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$特解$\eta=\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$,通解为$x=c\xi+\eta$,（$c$为任意实数）.

四、设3阶矩阵A的特征值$\lambda_1=1,\lambda_2=2,\lambda_3=3$,对应的特征向量依次为$p_1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,$p_2=\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$,$p_3=\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$,又向量$\beta=\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.（1）将$\beta$用$p_1,p_2,p_3$线性表示; （2）求$A^n\beta(n\in\mathbf{N}^+)$;（3）求矩阵$A$.

解：（1）$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, concatenate: $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 9 & 3 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\beta = 2p_1 - 2p_2 + p_3$.

（2）令$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$,则$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$,从而$A = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}P^{-1} =$
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$.

$A\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 13 \end{pmatrix}$, $A^2\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 13 \end{pmatrix} =$
$\begin{pmatrix} 3 \\ 13 \\ 51 \end{pmatrix}$,

$A^3\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 13 \\ 51 \end{pmatrix} = \begin{pmatrix} 13 \\ 51 \\ 181 \end{pmatrix}, \cdots, A^n\beta = \begin{pmatrix} 2 - 2^{n+1} + 3^n \\ 2 - 2^{n+2} + 3^{n+1} \\ 2 - 2^{n+3} + 3^{n+2} \end{pmatrix}$,
（可用数学归纳法证明）.

或$A^n = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}P^{-1} =$
$\begin{pmatrix} 3 - 3\times 2^n + 3^n & -\frac{5}{2} + 4\times 2^n - \frac{1}{2}3^{1+n} & \frac{1}{2} - 2^n + \frac{1}{2}3^n \\ 3 - 6\times 2^n + 3^{1+n} & -\frac{5}{2} + 8\times 2^n - \frac{9}{2}3^n & \frac{1}{2} - 2^{1+n} + \frac{1}{2}3^{1+n} \\ 3 - 12\times 2^n + 9\times 3^n & -\frac{5}{2} + 16\times 2^n - \frac{27}{2}3^n & \frac{1}{2} - 4\times 2^n + \frac{9}{2}3^n \end{pmatrix}$,

$A^n\beta = \begin{pmatrix} 3 - 3\times 2^n + 3^n & -\frac{5}{2} + 4\times 2^n - \frac{1}{2}3^{1+n} & \frac{1}{2} - 2^n + \frac{1}{2}3^n \\ 3 - 6\times 2^n + 3^{1+n} & -\frac{5}{2} + 8\times 2^n - \frac{9}{2}3^n & \frac{1}{2} - 2^{1+n} + \frac{1}{2}3^{1+n} \\ 3 - 12\times 2^n + 9\times 3^n & -\frac{5}{2} + 16\times 2^n - \frac{27}{2}3^n & \frac{1}{2} - 4\times 2^n + \frac{9}{2}3^n \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} =$
$\begin{pmatrix} 2 - 2^{1+n} + 3^n \\ 2 - 4\times 2^n + 3^{1+n} \\ 2 - 8\times 2^n + 3^{2+n} \end{pmatrix}$.

（3）由（2）知$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$.

五、设A为4×3矩阵，B为3×3矩阵，且$AB=0$,其中$A=\begin{pmatrix}1&1&-1\\1&2&1\\2&3&0\\0&-1&-2\end{pmatrix}$,证明：$B$的列向量线性相关.（一般地，若有$A_{m\times n}B_{n\times s}=0$,则$\mathrm{rank}(A)+\mathrm{rank}(B)\leqslant n$）

证明：$A=\begin{pmatrix}1&1&-1\\1&2&1\\2&3&0\\0&-1&-2\end{pmatrix}$, row echelon form: $\begin{pmatrix}1&0&-3\\0&1&2\\0&0&0\\0&0&0\end{pmatrix}$, rank: 2,由教材P.100例2知：$\mathrm{rank}(A)+\mathrm{rank}(B)\leqslant3$,故$\mathrm{rank}(B)\leqslant1<3$,从而$B$的列向量线性相关.

六、设A^*是$n(n\geqslant2)$阶方阵A的伴随矩阵，证明：（1）$\mathrm{rank}(A^*)=\begin{cases}n,\mathrm{rank}\,(A)=n\\1,\mathrm{rank}\,(A)=n-1\\0,\mathrm{rank}\,(A)<n-1\end{cases}$.（2）$|A^*|=|A|^{n-1}$.

证明：（1）略.

（2）因为$A^*A=|A|E$,故$|A^*A|=||A|E|$,即$|A^*||A|=|A|^n|E|=|A|^n$.又$|A|\neq0$,从而$|A^*|=|A|^{n-1}$.

七、设$\alpha_1,\alpha_2,\cdots,\alpha_n$是实数域上的向量空间$V$的一个基，向量组$\beta_1=\alpha_1,\beta_2=\alpha_1+\alpha_2,\cdots,\beta_n=\alpha_1+\alpha_2+\cdots+\alpha_n$.

（1）证明：$\beta_1,\beta_2,\cdots,\beta_n$也是$V$的一个基，并求出由$\alpha_1,\alpha_2,\cdots,\alpha_n$到$\beta_1,\beta_2,\cdots,\beta_n$的过渡矩阵$A$;

（2）设向量$\alpha=n\alpha_1+(n-1)\,\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$,求$\alpha$在基$\beta_1,\beta_2,\cdots,\beta_n$下的坐标.

证明（1）$(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}1&1&1&\cdots&1\\0&1&1&\cdots&1\\0&0&1&\cdots&1\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\end{pmatrix}$,由于$\begin{pmatrix}1&1&1&\cdots&1\\0&1&1&\cdots&1\\0&0&1&\cdots&1\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\end{pmatrix}$是$n$阶上三角矩阵，故是可逆矩阵，由此$\alpha_1,\alpha_2,\cdots,\alpha_n$与$\beta_1,\beta_2,\cdots,\beta_n$的秩相等. 由$\alpha_1,\alpha_2,\cdots,\alpha_n$是实数域上的向量空间$V$的一个基，知其线性无关，秩为$n$,从而$\beta_1,\beta_2,\cdots,\beta_n$的秩也为$n$,故线性无关，由此也是$V$的一个基.过

渡矩阵为$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

（2）$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)\begin{pmatrix} n \\ n-1 \\ \vdots \\ 2 \\ 1 \end{pmatrix}$,$\alpha$在基$\beta_1, \beta_2, \cdots, \beta_n$下的坐标为$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1}\begin{pmatrix} n \\ n-1 \\ \vdots \\ 2 \\ 1 \end{pmatrix}$.

八、设A为n阶实对称矩阵，λ_1, λ_2为A的互异特征值，ξ_1, ξ_2是分别对应属于λ_1, λ_2的特征向量，证明：ξ_1与ξ_2正交.

证明：参见《线性代数》P.118引理6.4.1.

设λ_1, λ_2为A的互异特征值为α_1, α_2,则$A\alpha_1 = \lambda_1\alpha_1, A\alpha_2 = \lambda_2\alpha_2$.于是$\lambda_1(\alpha_1, \alpha_2) = (\lambda_1\alpha_1, \alpha_2) = (A\alpha_1, \alpha_2) = (A\alpha_1)^T\alpha_2 = \left(\alpha_1^T A^T\right)\alpha_2 = \alpha_1^T(A\alpha_2) = (\alpha_1, A\alpha_2) = (\alpha_1, \lambda_2\alpha_2) = \lambda_2(\alpha_1, \alpha_2)$,而$\lambda_1 \neq \lambda_2$,则必有$(\alpha_1, \alpha_2) = 0$.

7.1 二次型的表示法 7.2 配方法化简二次型

一、填空题：

1. 二次型$f\left(x_1,x_2,x_3,x_4\right)=x_1^2-x_2^2+x_4^2-2x_1x_2+2x_1x_3+4x_2x_4+6x_3x_4$的矩阵是______.

2. 矩阵$A=\begin{pmatrix}1&2&4\\2&2&-1\\4&-1&3\end{pmatrix}$对应的二次型是______

3. 二次型$f\left(x_1,x_2,x_3\right)=x_1^2-x_2x_3$的规范形是______.

4. 二次型$f\left(x_1,x_2,x_3\right)=\left(ax_1+ax_2+ax_3\right)^2$的对应矩阵是______.

5. 已知二次型$f\left(x_1,x_2,x_3\right)=a\left(x_1^2+x_2^2+x_3^2\right)+4x_1x_2+4x_1x_3+4x_2x_3$经正交变换$x=Py$可化为标准形$f=6y_1^2$,则$a=$______.

提示：1. $\begin{pmatrix}1&-1&1&0\\-1&-1&0&2\\1&0&0&3\\0&2&3&1\end{pmatrix}$. 2. $\begin{pmatrix}x_1&x_2&x_3\end{pmatrix}\begin{pmatrix}1&2&4\\2&2&-1\\4&-1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=x_1^2+4x_1x_2+8x_1x_3+2x_2^2-2x_2x_3+3x_3^2$.或直接写出$f\left(x_1,x_2,x_3\right)=x_1^2+2x_2^2+3x_3^2+4x_1x_2+8x_1x_3-2x_2x_3$. 3. $f\left(x_1,x_2,x_3\right)=x_1^2-x_2x_3=\left(x_1-\frac{1}{2}x_2\right)^2-\frac{1}{4}x_2^2$,令$y_1=x_1-\frac{1}{2}x_2,y_2=\frac{1}{2}x_2^2$,则$f\left(x_1,x_2,x_3\right)=y_1^2-y_2^2$. 4. $f\left(x_1,x_2,x_3\right)=\left(ax_1+ax_2+ax_3\right)^2=a^2x_1^2+2a^2x_1x_2+2a^2x_1x_3+a^2x_2^2+2a^2x_2x_3+a^2x_3^2$,对应矩阵是$\begin{pmatrix}a^2&a^2&a^2\\a^2&a^2&a^2\\a^2&a^2&a^2\end{pmatrix}$. 5. 由配方法化简二次型知，$a=6$.

二、用配方法化二次型$f\left(x_1,x_2,x_3\right)=2x_1x_2-4x_1x_3+x_2^2+6x_2x_3+8x_3^2$为标准形，并写出变换矩阵.

解：$f\left(x_1,x_2,x_3\right)=2x_1x_2-4x_1x_3+x_2^2+6x_2x_3+8x_3^2$,因为$a_{11}=0$,故令$\begin{cases}x_1=y_1+y_2\\x_2=y_1-y_2\\x_3=y_3\end{cases}$,故$f\left(x_1,x_2,x_3\right)=2\left(y_1+y_2\right)\left(y_1-y_2\right)-4\left(y_1+y_2\right)y_3+\left(y_1-y_2\right)^2+6\left(y_1+y_2\right)y_3+8y_3^2=3y_1^2-2y_1y_2+2y_1y_3-y_2^2+2y_2y_3+8y_3^2=3\left(y_1-\frac{1}{3}y_2+\frac{1}{3}y_3\right)^2-\frac{4}{3}y_2^2+\frac{8}{3}y_2y_3+\frac{23}{3}y_3^2=3\left(y_1-\frac{1}{3}y_2+\frac{1}{3}y_3\right)^2-\frac{4}{3}\left(y_2-y_3\right)^2+\frac{27}{3}y_3^2=3z_1^2-\frac{4}{3}z_2^2+\frac{27}{3}z_3^2$,其中$\begin{cases}z_1=y_1-\frac{1}{3}y_2+\frac{1}{3}y_3\\z_2=y_2-y_3\\z_3=y_3\end{cases}$. $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$,变换矩阵

为$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} =$

$\begin{pmatrix} 1 & \frac{4}{3} & 1 \\ 1 & -\frac{2}{3} & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

三、用配方法化二次型$f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 - 6x_2x_3$为标准形，并写出变换矩阵.

解：$f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 - 6x_2x_3$,因为$a_{11} = 0$,故令$\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_3 = y_3 \end{cases}$,

$f(x_1, x_2, x_3) = 2(y_1 + y_2)(y_1 - y_2) + 2(y_1 + y_2)y_3 - 6(y_1 - y_2)y_3 = 2y_1^2 - 4y_3y_1 - 2y_2^2 + 8y_3y_2 = 2(y_1 - y_3)^2 + -2y_2^2 + 8y_3y_2 - 2y_3^2 = 2(y_1 - y_3)^2 - 2(y_2 - 2y_3)^2 + 6y_3^2 = 2z_1^2 - 2z_2^2 + 6z_3^2$,其中$\begin{cases} z_1 = y_1 - y_3 \\ z_2 = y_2 - 2y_3 \\ z_3 = y_3 \end{cases}$. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$,变换矩阵为

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} =$

$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

四、求一个正交变换$x = Py$将二次型$f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 - 6x_2x_3$化为标准形.

解：$f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 - 6x_2x_3 = X^TAX$,其中$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{pmatrix}$,

$\left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{pmatrix} \right| = \lambda^3 - 11\lambda + 6 = 0$, Solution is:

$\frac{1}{2}\sqrt{17}-\frac{3}{2}, -\frac{1}{2}\sqrt{17}-\frac{3}{2}, 3.$

$$\left(\frac{1}{2}\sqrt{17}-\frac{3}{2}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&1&1\\1&0&-3\\1&-3&0\end{pmatrix}=$$

$\begin{pmatrix}\frac{1}{2}\sqrt{17}-\frac{3}{2}&-1&-1\\-1&\frac{1}{2}\sqrt{17}-\frac{3}{2}&3\\-1&3&\frac{1}{2}\sqrt{17}-\frac{3}{2}\end{pmatrix}$,

$$\begin{pmatrix}\frac{1}{2}\sqrt{17}-\frac{3}{2}&-1&-1\\-1&\frac{1}{2}\sqrt{17}-\frac{3}{2}&3\\-1&3&\frac{1}{2}\sqrt{17}-\frac{3}{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$, Solution is: $\begin{pmatrix}\hat{t}_3\left(\frac{1}{2}\sqrt{17}+\frac{3}{2}\right)\\\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix}\frac{1}{2}\sqrt{17}+\frac{3}{2}\\1\\1\end{pmatrix}$;

$$\left(-\frac{1}{2}\sqrt{17}-\frac{3}{2}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&1&1\\1&0&-3\\1&-3&0\end{pmatrix}=$$

$\begin{pmatrix}-\frac{1}{2}\sqrt{17}-\frac{3}{2}&-1&-1\\-1&-\frac{1}{2}\sqrt{17}-\frac{3}{2}&3\\-1&3&-\frac{1}{2}\sqrt{17}-\frac{3}{2}\end{pmatrix}$,

$$\begin{pmatrix}-\frac{1}{2}\sqrt{17}-\frac{3}{2}&-1&-1\\-1&-\frac{1}{2}\sqrt{17}-\frac{3}{2}&3\\-1&3&-\frac{1}{2}\sqrt{17}-\frac{3}{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$, Solution is: $\begin{pmatrix}-\hat{t}_3\left(\frac{1}{2}\sqrt{17}-\frac{3}{2}\right)\\\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_2=\begin{pmatrix}-\frac{1}{2}\sqrt{17}+\frac{3}{2}\\1\\1\end{pmatrix}$;

$$3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&1&1\\1&0&-3\\1&-3&0\end{pmatrix}=\begin{pmatrix}3&-1&-1\\-1&3&3\\-1&3&3\end{pmatrix}, \begin{pmatrix}3&-1&-1\\-1&3&3\\-1&3&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}0\\-\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix}0\\-1\\1\end{pmatrix}$.

由于A是实对称矩阵，故不同特征值的特征向量$\alpha_1,\alpha_2,\alpha_3$是正交的，

$\varepsilon_1 = \frac{\alpha_1}{\|\alpha_1\|} = \begin{pmatrix} \frac{1}{2}\frac{\sqrt{17}+3}{\sqrt{\frac{3}{2}\sqrt{17}+\frac{17}{2}}} \\ \frac{1}{\sqrt{\frac{3}{2}\sqrt{17}+\frac{17}{2}}} \\ \frac{1}{\sqrt{\frac{3}{2}\sqrt{17}+\frac{17}{2}}} \end{pmatrix}, \varepsilon_2 = \frac{\alpha_2}{\|\alpha_2\|} = \begin{pmatrix} -\frac{1}{2}\frac{\sqrt{17}-3}{\sqrt{\frac{17}{2}-\frac{3}{2}\sqrt{17}}} \\ \frac{1}{\sqrt{\frac{17}{2}-\frac{3}{2}\sqrt{17}}} \\ \frac{1}{\sqrt{\frac{17}{2}-\frac{3}{2}\sqrt{17}}} \end{pmatrix}, \varepsilon_3 =$
$\frac{\alpha_3}{\|\alpha_3\|} = \begin{pmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$.令$P = \begin{pmatrix} \frac{1}{2}\frac{\sqrt{17}+3}{\sqrt{\frac{3}{2}\sqrt{17}+\frac{17}{2}}} & -\frac{1}{2}\frac{\sqrt{17}-3}{\sqrt{\frac{17}{2}-\frac{3}{2}\sqrt{17}}} & 0 \\ \frac{1}{\sqrt{\frac{3}{2}\sqrt{17}+\frac{17}{2}}} & \frac{1}{\sqrt{\frac{17}{2}-\frac{3}{2}\sqrt{17}}} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{\sqrt{\frac{3}{2}\sqrt{17}+\frac{17}{2}}} & \frac{1}{\sqrt{\frac{17}{2}-\frac{3}{2}\sqrt{17}}} & \frac{1}{2}\sqrt{2} \end{pmatrix}$,

则$P^TP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P^TAP = \begin{pmatrix} \frac{1}{2}\sqrt{17}-\frac{3}{2} & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{17}-\frac{3}{2} & 0 \\ 0 & 0 & 3 \end{pmatrix}$.即由正交变换$x = Py$,使$f(x_1, x_2, x_3) = 2x_1x_2+2x_1x_3-6x_2x_3$化为标准形$f(x_1, x_2, x_3) = \left(\frac{1}{2}\sqrt{17}-\frac{3}{2}\right)y_1^2 - \left(\frac{1}{2}\sqrt{17}+\frac{3}{2}\right)y_2^2 + 3y_3^2$.

五、已知二次型$f(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 + 3x_3^2 + 2ax_2x_3$（其中$a > 0$）通过正交变换化为标准形$f = y_1^2 + 2y_2^2 + 5y_3^2$,试求参数$a$及所用的正交变换矩阵.

解：由已知$f(x_1, x_2, x_3) = 2x_1^2+3x_2^2+3x_3^2+2ax_2x_3 = X^TAX$,其中$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & a \\ 0 & a & 3 \end{pmatrix}$,且$1, 2, 5$是$A$的特征值.

$\left|\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & a \\ 0 & a & 3 \end{pmatrix}\right| = -a^2\lambda+2a^2+\lambda^3-8\lambda^2+21\lambda-18 = 0, (-a^2\lambda + 2a^2 + \lambda^3 - 8\lambda^2 + 21\lambda - 18)|_{\lambda=1} = a^2 - 4 = 0$,再由$a > 0$得$a = 2$.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} 0 \\ -\hat{t}_3 \\ \hat{t}_3 \end{pmatrix}$,基础解系$\alpha_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$;

$2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \hat{t}_1 \\ 0 \\ 0 \end{pmatrix}$,基础解系$\alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;

基础解系$\alpha_2=\begin{pmatrix}1\\0\\0\end{pmatrix}$.

$5\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}2&0&0\\0&3&2\\0&2&3\end{pmatrix}=\begin{pmatrix}3&0&0\\0&2&-2\\0&-2&2\end{pmatrix}$, $\begin{pmatrix}3&0&0\\0&2&-2\\0&-2&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}0\\\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$.

由于A是实对称矩阵，故不同特征值的特征向量$\alpha_1,\alpha_2,\alpha_3$是正交的，

$\varepsilon_1=\frac{\alpha_1}{\|\alpha_1\|}=\begin{pmatrix}0\\-\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}\end{pmatrix}$, $\varepsilon_2=\frac{\alpha_2}{\|\alpha_2\|}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\varepsilon_3=\frac{\alpha_3}{\|\alpha_3\|}=\begin{pmatrix}0\\\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}\end{pmatrix}$.

令$P=\begin{pmatrix}0&1&0\\-\frac{1}{2}\sqrt{2}&0&\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}&0&\frac{1}{2}\sqrt{2}\end{pmatrix}$,则$P^TP=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $P^TAP=\begin{pmatrix}1&0&0\\0&2&0\\0&0&5\end{pmatrix}$.

即由正交变换$x=Py$,使$f(x_1,x_2,x_3)=2x_1^2+3x_2^2+3x_3^2+2ax_2x_3$化为标准形$f(x_1,x_2,x_3)=y_1^2+2y_2^2+5y_3^2$.

六、设二次型$f(x_1,x_2,x_3)=x^TAx=ax_1^2+2x_2^2-2x_3^2+2bx_1x_2\,(b>0)$,其中二次型的矩阵$A$的特征值之和为$-1$,特征值之积为12.

（1）求a,b的值;

（2）利用正交变换将二次型f化为标准形，并写出所利用的正交变换和对应的正交矩阵.

解:（1）$A=\begin{pmatrix}a&b&0\\b&2&0\\0&0&-2\end{pmatrix}$,

$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}a&b&0\\b&2&0\\0&0&-2\end{pmatrix}\right|=-b^2\lambda-2b^2+\lambda^3-a\lambda^2-4\lambda+4a=$ $\lambda^3+-a\lambda^2-(b^2+4)\lambda+4a-2b^2=0$,由已知$a=-1,2b^2-4a=12$,从而$b^2=4$,又$b>0$得$b=2$.

（2）$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}-1&2&0\\2&2&0\\0&0&-2\end{pmatrix}\right|=\lambda^3+\lambda^2-8\lambda-12=$ $(\lambda-3)(\lambda+2)^2=0$, Solution is: $-2,3$.

$$-2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}-1&2&0\\2&2&0\\0&0&-2\end{pmatrix}=\begin{pmatrix}-1&-2&0\\-2&-4&0\\0&0&0\end{pmatrix},\begin{pmatrix}-1&-2&0\\-2&-4&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-2\hat{t}_2\\\hat{t}_2\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix}-2\\1\\0\end{pmatrix}$,$\alpha_2=\begin{pmatrix}0\\0\\1\end{pmatrix}$;

$$3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}-1&2&0\\2&2&0\\0&0&-2\end{pmatrix}=\begin{pmatrix}4&-2&0\\-2&1&0\\0&0&5\end{pmatrix},\begin{pmatrix}4&-2&0\\-2&1&0\\0&0&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}\frac{1}{2}\hat{t}_2\\\hat{t}_2\\0\end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix}1\\2\\0\end{pmatrix}$.

$\beta_1=\alpha_1=\begin{pmatrix}-2\\1\\0\end{pmatrix}$,$\beta_2=\alpha_2-\frac{\alpha_2\cdot\beta_1}{\beta_1\cdot\beta_1}\beta_1=\begin{pmatrix}0\\0\\1\end{pmatrix}$,$\beta_3=\alpha_3=$ $\begin{pmatrix}1\\2\\0\end{pmatrix}$,$\varepsilon_1=\frac{\beta_1}{\|\beta_1\|}=\begin{pmatrix}-\frac{2}{5}\sqrt{5}\\\frac{1}{5}\sqrt{5}\\0\end{pmatrix}$,$\varepsilon_2=\frac{\beta_2}{\|\beta_2\|}=\begin{pmatrix}0\\0\\1\end{pmatrix}$,$\varepsilon_3=\frac{\beta_3}{\|\beta_3\|}=$ $\begin{pmatrix}\frac{1}{5}\sqrt{5}\\\frac{2}{5}\sqrt{5}\\0\end{pmatrix}$,

令$P=\begin{pmatrix}-\frac{2}{5}\sqrt{5}&0&\frac{1}{5}\sqrt{5}\\\frac{1}{5}\sqrt{5}&0&\frac{2}{5}\sqrt{5}\\0&1&0\end{pmatrix}$,则$P^TP=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$,$P^TAP=\begin{pmatrix}-2&0&0\\0&-2&0\\0&0&3\end{pmatrix}$.即由正交变换$x=Py$,使$f(x_1,x_2,x_3)=-x_1^2+2x_2^2-2x_3^2+4x_1x_2$化为标准形$f(x_1,x_2,x_3)=-2y_1^2-2y_2^2+3y_3^2$.

七、已知二次型$f(x_1,x_2,x_3)=(1-a)x_1^2+(1-a)x_2^2+2x_3^2+2(1+a)x_1x_2$的秩为2,求：

（1）a的值;

（2）正交变换$x=Py$将$f(x_1,x_2,x_3)$化成标准形;

（3）方程$f(x_1,x_2,x_3)=0$的解.

解：（1）$A=\begin{pmatrix}1-a&1+a&0\\1+a&1-a&0\\0&0&2\end{pmatrix}\xrightarrow[\frac{1}{2}r_1]{r_2+r_1,}\begin{pmatrix}1&1&0\\1+a&1-a&0\\0&0&2\end{pmatrix}\xrightarrow{-(1+a)r_1+r_2}$

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -2a & 0 \\ 0 & 0 & 2 \end{pmatrix}$,因为$f\left(x_1,x_2,x_3\right)$的秩为2,故$A$的秩为2,从而$a=0$.

（2）$\left|\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right|=\lambda^3-4\lambda^2+4\lambda=\lambda\left(\lambda-2\right)^2=0$, Solution is: $2,0$.

$2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}=\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \hat{t}_2 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$;

$0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}=\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -\hat{t}_2 \\ \hat{t}_2 \\ 0 \end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

$\beta_1=\alpha_1=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\beta_2=\alpha_2-\frac{\alpha_2\cdot\beta_1}{\beta_1\cdot\beta_1}\beta_1=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\beta_3=\alpha_3=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\varepsilon_1=\frac{\beta_1}{\|\beta_1\|}=\begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix}$, $\varepsilon_2=\frac{\beta_2}{\|\beta_2\|}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\varepsilon_3=\frac{\beta_3}{\|\beta_3\|}=\begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix}$.

令$P=\begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$,则$P^TP=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P^TAP=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.即由正交变换$x=Py$,使$f\left(x_1,x_2,x_3\right)=x_1^2+x_2^2+2x_3^2+2x_1x_2$化为标准形$f\left(x_1,x_2,x_3\right)=2y_1^2+2y_2^2+0y_3^2$.

（3）$f\left(x_1,x_2,x_3\right)=x_1^2+x_2^2+2x_3^2+2x_1x_2=\left(x_1+x_2\right)^2+x_3^2=0, x_1=-x_2, x_3=0$.即解为$c\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$,（$c$为任意实数）.

八、已知二次方程$x_1^2+ax_2^2+x_3^2+2bx_1x_2+2x_1x_3+2x_2x_3=4$可以经过正交变换$x=Py$化为方程$y_2^2+4y_3^2=4$,求$a,b$的值和正交矩阵$P$.

解：令$f\left(x_1,x_2,x_3\right)=x_1^2+ax_2^2+x_3^2+2bx_1x_2+2x_1x_3+2x_2x_3$,则可以经过正交变换$x=Py$化为方程$y_2^2+4y_3^2$.从而$0,1,4$是$A=\begin{pmatrix}1&b&1\\b&a&1\\1&1&1\end{pmatrix}$的特征值.

$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&b&1\\b&a&1\\1&1&1\end{pmatrix}\right|=2a\lambda-\lambda-2b-2\lambda^2+\lambda^3-a\lambda^2-b^2\lambda+b^2+1=0,$

$\left(2a\lambda-\lambda-2b-2\lambda^2+\lambda^3-a\lambda^2-b^2\lambda+b^2+1\right)|_{\lambda=0}=b^2-2b+1=0,$ Solution is: 1.即$b=1$.

$\left(2a\lambda-\lambda-2b-2\lambda^2+\lambda^3-a\lambda^2-b^2\lambda+b^2+1\right)|_{\lambda=1}=a-2b-1=0$,从而$a=3$.

$\left(2a\lambda-\lambda-2b-2\lambda^2+\lambda^3-a\lambda^2-b^2\lambda+b^2+1\right)|_{\lambda=4}=-3b^2-2b-8a+29=0.$

$0\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&1&1\\1&3&1\\1&1&1\end{pmatrix}=\begin{pmatrix}-1&-1&-1\\-1&-3&-1\\-1&-1&-1\end{pmatrix}$, $\begin{pmatrix}-1&-1&-1\\-1&-3&-1\\-1&-1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}-\hat{t}_3\\0\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}$;

$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&1&1\\1&3&1\\1&1&1\end{pmatrix}=\begin{pmatrix}0&-1&-1\\-1&-2&-1\\-1&-1&0\end{pmatrix}$, $\begin{pmatrix}0&-1&-1\\-1&-2&-1\\-1&-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}\hat{t}_3\\-\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_2=\begin{pmatrix}1\\-1\\1\end{pmatrix}$;

$4\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&1&1\\1&3&1\\1&1&1\end{pmatrix}=\begin{pmatrix}3&-1&-1\\-1&1&-1\\-1&-1&3\end{pmatrix}$, $\begin{pmatrix}3&-1&-1\\-1&1&-1\\-1&-1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, Solution is: $\begin{pmatrix}\hat{t}_3\\2\hat{t}_3\\\hat{t}_3\end{pmatrix}$,基础解系$\alpha_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$.

由于A是实对称矩阵，故不同特征值的特征向量$\alpha_1,\alpha_2,\alpha_3$是正交的，

$\varepsilon_1 = \frac{\alpha_1}{\|\alpha_1\|} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix}, \varepsilon_2 = \frac{\alpha_2}{\|\alpha_2\|} = \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix}, \varepsilon_3 = \frac{\alpha_3}{\|\alpha_3\|} = \begin{pmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}$,令$P = \begin{pmatrix} -\frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & -\frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{6} \\ \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{6} \end{pmatrix}$,则

$$P^TP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P^TAP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

7.3 正定二次型　习题课

一、填空题：

1. 若二次型$f(x_1,x_2,x_3)=x_1^2+x_2^2+5x_3^2+2\lambda x_1x_2-2x_1x_3+4x_2x_3$是正定的，则$\lambda$满足的条件是______.

2. 若对称矩阵$A=\begin{pmatrix}1&1&0\\1&k&0\\0&0&k^2\end{pmatrix}$是正定矩阵，则$k$满足的条件是______.

3. 当$a=$______时，二次型$f(x_1,x_2,x_3)=5x_1^2+5x_2^2+ax_3^2-2x_1x_2+6x_1x_3-6x_2x_3$的秩为2.

4. 二次型$f(x_1,x_2,x_3)=x_1^2-x_2^2+3x_3^2$的秩为______,正惯性指数为______,负惯性指数为______.

提示：1. $A=\begin{pmatrix}1&\lambda&-1\\\lambda&1&2\\-1&2&5\end{pmatrix}$, $1>0$, $\begin{vmatrix}1&\lambda\\\lambda&1\end{vmatrix}=1-\lambda^2>0$, $\begin{vmatrix}1&\lambda&-1\\\lambda&1&2\\-1&2&5\end{vmatrix}=-5\lambda^2-4\lambda>0$,得$\begin{cases}1-\lambda^2>0\\-5\lambda^2-4\lambda>0\end{cases}$, Solution is: $\left(-\frac{4}{5},0\right)$.　　2. $1>0$, $\begin{vmatrix}1&1\\1&k\end{vmatrix}=k-1>0$, $\begin{vmatrix}1&1&0\\1&k&0\\0&0&k^2\end{vmatrix}=k^3-k^2>0$,得$\begin{cases}k-1>0\\k^3-k^2>0\end{cases}$, Solution is: $(1,+\infty)$.　　3. $A=\begin{pmatrix}5&-1&3\\-1&5&-3\\3&-3&a\end{pmatrix}\xrightarrow[-r_1]{r_1\leftrightarrow r_2,}\begin{pmatrix}1&-5&3\\5&-1&3\\3&-3&a\end{pmatrix}$ $\xrightarrow[3r_1+r_3]{-5r_1+r_2,}\begin{pmatrix}1&-5&3\\0&24&-12\\0&12&a-9\end{pmatrix}\xrightarrow[-6r_2+r_3]{\frac{1}{12}r_2,}\begin{pmatrix}1&-5&3\\0&2&-1\\0&0&a-3\end{pmatrix}$,由已知得$a-3=0$,即$a=3$.　　4. 3,正惯性指数$p=2$,负惯性指数$r-p=1$.

二、判断二次型$f(x_1,x_2,x_3)=6x_1^2+5x_2^2+7x_3^2-4x_1x_2+4x_1x_3$的正定性.

解：$A=\begin{pmatrix}6&-2&2\\-2&5&0\\2&0&7\end{pmatrix}$, $6>0$, $\begin{vmatrix}6&-2\\-2&5\end{vmatrix}=26>0$, $\begin{vmatrix}6&-2&2\\-2&5&0\\2&0&7\end{vmatrix}=162>0$,故$f(x_1,x_2,x_3)=6x_1^2+5x_2^2+7x_3^2-4x_1x_2+4x_1x_3$是正定的.

三、判断二次型$f(x_1,x_2,x_3)=-5x_1^2-6x_2^2-4x_3^2+4x_1x_2+4x_1x_3$的正定性.

解：$A=\begin{pmatrix}-5&2&2\\2&-6&0\\2&0&-4\end{pmatrix}$，$-5<0$，故$f(x_1,x_2,x_3)=-5x_1^2-6x_2^2-4x_3^2+4x_1x_2+4x_1x_3$不是正定的.

四、已知二次型$f(x_1,x_2,x_3)=x_1^2+2x_2^2+(1-k)x_3^2+2kx_1x_2+2x_1x_3$，其中$k$为参数，求使$f$为正定二次型的$k$的取值范围.

解：$A=\begin{pmatrix}1&k&1\\k&2&0\\1&0&1-k\end{pmatrix}$，$1>0$，$\begin{vmatrix}1&k\\k&2\end{vmatrix}=2-k^2>0$，$\begin{vmatrix}1&k&1\\k&2&0\\1&0&1-k\end{vmatrix}=k^3-k^2-2k>0$，得$\begin{cases}2-k^2>0\\k^3-k^2-2k>0\end{cases}$，Solution is: $(-1,0)$.

五、用配方法化二次型$f(x_1,x_2,x_3)=x_1^2-x_2^2+4x_3^2-2x_1x_2+2x_1x_3-6x_2x_3$为标准形，并写出变换矩阵.

解：$f(x_1,x_2,x_3)=(x_1-x_2+x_3)^2-2x_2^2-4x_2x_3+3x_3^2=(x_1-x_2+x_3)^2-2(x_2+x_3)^2+5x_3^2$，$\begin{cases}y_1=x_1-x_2+x_3\\y_2=x_2+x_3\\y_3=x_3\end{cases}$，$y=\begin{pmatrix}1&-1&1\\0&1&1\\0&0&1\end{pmatrix}x$，

$x=\begin{pmatrix}1&-1&1\\0&1&1\\0&0&1\end{pmatrix}^{-1}y=\begin{pmatrix}1&1&-2\\0&1&-1\\0&0&1\end{pmatrix}y$，变换矩阵为$P=\begin{pmatrix}1&1&-2\\0&1&-1\\0&0&1\end{pmatrix}$.

六、已知二次型$f(x_1,x_2,x_3)=5x_1^2+5x_2^2+ax_3^2-2x_1x_2+6x_1x_3-6x_2x_3$的秩为2，求参数$a$及此二次型对应矩阵的特征值.

解：$A=\begin{pmatrix}5&-1&3\\-1&5&-3\\3&-3&a\end{pmatrix}\xrightarrow[-r_1]{r_1-r_2,}\begin{pmatrix}1&-5&3\\5&-1&3\\3&-3&a\end{pmatrix}\xrightarrow[-3r_1+r_3]{-5r_1+r_2,}\begin{pmatrix}1&-5&3\\0&24&-12\\0&12&a-9\end{pmatrix}$

$\xrightarrow[-6r_2+r_3]{\frac{1}{12}r_2,}\begin{pmatrix}1&-5&3\\0&2&-1\\0&0&a-3\end{pmatrix}$，由已知得$\operatorname{rank}(A)=2$，从而$a=3$.

$\left|\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}5&-1&3\\-1&5&-3\\3&-3&3\end{pmatrix}\right|=\lambda^3-13\lambda^2+36\lambda=\lambda(\lambda-4)(\lambda-9)=0$, Solution is: $4,0,9$.

七、已知二次型$f(x_1,x_2,x_3)=3x_1^2+3x_2^2+5x_3^2+4x_1x_3-4x_2x_3$

（1）写出f对应的矩阵A;

（2）求f的秩;

（3）写出A^{-1}的特征值;

（4）求正交变换$x = Py$化二次型f为标准形.

解：（1）$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 5 \end{pmatrix}$;

（2）$\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 5 \end{pmatrix} \xrightarrow{-\frac{2}{3}r_1 + r_2} \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 0 & -2 & \frac{11}{3} \end{pmatrix} \xrightarrow{\frac{2}{3}r_1 + r_3} \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & \frac{7}{3} \end{pmatrix}$,rank$(A) =$ 3,故f的秩3.

（3）$\left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 5 \end{pmatrix} \right| = \lambda^3 - 11\lambda^2 + 31\lambda - 21 = 0,$

Solution is: $7, 1, 3. A^{-1}$的特征值为$\frac{1}{7}, 1, \frac{1}{3}$.

（4）

$7\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & 2 \\ -2 & 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & 2 \\ -2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \frac{1}{2}\hat{t}_3 \\ -\frac{1}{2}\hat{t}_3 \\ \hat{t}_3 \end{pmatrix}$,基础解系$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$;

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -2 & 2 \\ -2 & 2 & -4 \end{pmatrix}$, $\begin{pmatrix} -2 & 0 & -2 \\ 0 & -2 & 2 \\ -2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -\hat{t}_3 \\ \hat{t}_3 \\ \hat{t}_3 \end{pmatrix}$,基础解系$\alpha_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$;

$3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & -2 \\ 2 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ -2 & 2 & -2 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ -2 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \hat{t}_2 \\ \hat{t}_2 \\ 0 \end{pmatrix}$,基础解系$\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$\varepsilon_1 = \frac{\alpha_1}{\|\alpha_1\|} = \begin{pmatrix} \frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \end{pmatrix}$, $\varepsilon_2 = \frac{\alpha_2}{\|\alpha_2\|} = \begin{pmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix}$, $\varepsilon_3 = \frac{\alpha_3}{\|\alpha_3\|} =$

$\begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix}$,

令$P = \begin{pmatrix} \frac{1}{6}\sqrt{6} & -\frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} \\ \frac{1}{3}\sqrt{6} & \frac{1}{3}\sqrt{3} & 0 \end{pmatrix}$,则$P^TP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P^TAP = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

八、已知二次型$f(x_1, x_2, x_3) = 7x_1^2 + 7x_2^2 + 7x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

（1）将f表示为矩阵形式;

（2）用正交变换$x = Py$化二次型f为标准形，并写出所用正交变换矩阵及二次型的标准形;

（3）将f的对应矩阵A表示成$A = WW^T$.

解：（1）$f(x_1, x_2, x_3) = (x_1, x_2, x_3) \begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

（2）$\left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix} \right| = \lambda^3 - 21\lambda^2 + 144\lambda - 324 =$ $(\lambda - 9)(\lambda - 6)^2 = 0$, Solution is: $6, 9$.

$6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -\hat{t}_2 - \hat{t}_3 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix}$,基础解系$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$;

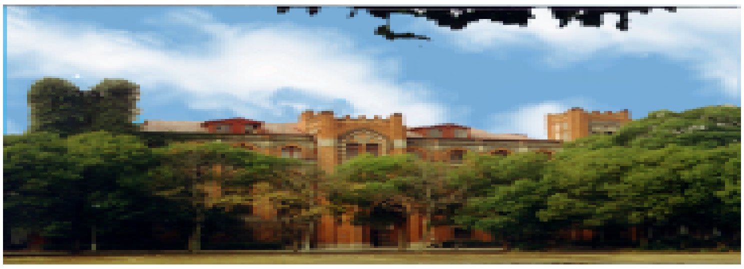
$9 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \hat{t}_3 \\ \hat{t}_3 \\ \hat{t}_3 \end{pmatrix}$,基础解系$\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$, $\beta_3 = \alpha_3 =$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \varepsilon_1 = \frac{\beta_1}{\|\beta_1\|} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix}, \varepsilon_2 = \frac{\beta_2}{\|\beta_2\|} = \begin{pmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \end{pmatrix}, \varepsilon_3 = \frac{\beta_3}{\|\beta_3\|} = \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix}.$

令$P = \begin{pmatrix} -\frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{6} & \frac{1}{3}\sqrt{3} \end{pmatrix}$,则$P^TP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P^TAP = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.即用正交变换$x = Py$化二次型$f$为标准形$6y_1^2 + 6y_2^2 + 9y_3^2$.

（3）$A = P \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix} P^T = P \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 3 \end{pmatrix} P^T = \begin{pmatrix} -\frac{1}{2}\sqrt{2}\sqrt{6} & -1 & \sqrt{3} \\ \frac{1}{2}\sqrt{2}\sqrt{6} & -1 & \sqrt{3} \\ 0 & 2 & \sqrt{3} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\sqrt{2}\sqrt{6} & -1 & \sqrt{3} \\ \frac{1}{2}\sqrt{2}\sqrt{6} & -1 & \sqrt{3} \\ 0 & 2 & \sqrt{3} \end{pmatrix}^T.$



苏州大学精正楼（数学楼）