四川大学期末考试试题参考答案

(2020——2021学年第 I 学期)

一、 填空题(每题4分，共24分)

1. $\lim\limits_{n\to+\infty}\left[\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2n-1)\times(2n+1)}\right]=\underline{\frac{1}{2}}.$

2. 若方程$y=e^{x+y}-1$确定隐函数$y=y(x)$，则$y'(x)=\underline{-\frac{1+y}{y}, y\neq0 \quad or \quad \frac{e^{x+y}}{1-e^{x+y}}, e^{x+y}\neq1}.$

3. $\int e^{\sin^2 x}\sin 2x\mathrm{d}x=\underline{e^{\sin^2 x}+C}.$

4. $\lim\limits_{x\to-\infty}e^{2x}\left(1-\frac{1}{x}\right)^{2x^2}=\underline{e^{-1}}.$

5. 已知$y=g(x)$和$y=f(x)$互为反函数，满足$f(0)=1, g'(1)=2$, $f''(0)=8$，则$g''(1)=\underline{-1}$.

6. 已知$f(x)=x^2\cos x$，则$f^{(2020)}(0)=\underline{-2\mathrm{C}_{2020}^2\text{ 或}-4078380 \quad or \quad -2020\times2019}.$

二、 (8分)计算不定积分$\int\frac{x\ln x}{\sqrt{x^2+1}}\mathrm{d}x$.

解1:令$u=\sqrt{x^2+1}>1, u\mathrm{d}u=x\mathrm{d}x,\cdots\cdots②$

$$\int\frac{x\ln x}{\sqrt{x^2+1}}\mathrm{d}x=\frac{1}{2}\int\frac{\ln(u^2-1)}{u}u\mathrm{d}u=\frac{1}{2}\int\ln(u^2-1)\mathrm{d}u\cdots\cdots②$$

$$=\frac{u}{2}\ln(u^2-1)-\int\frac{u^2}{u^2-1}\mathrm{d}u\cdots\cdots②$$

$$=\frac{u}{2}\ln(u^2-1)-\int(1+\frac{1}{u^2-1})\mathrm{d}u\cdots\cdots①$$

$$=\frac{u}{2}\ln(u^2-1)-u-\frac{1}{2}\ln(\frac{u-1}{u+1})+C\cdots\cdots②$$

$$=\sqrt{x^2+1}\ln x-\sqrt{x^2+1}-\frac{1}{2}\ln(\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1})+C\cdots\cdots②$$

解2:令$x=\tan t,-\frac{\pi}{2}<t<\frac{\pi}{2},\mathrm{d}x=\sec^2 t\mathrm{d}t,$

$$\int\frac{x\ln x}{\sqrt{x^2+1}}\mathrm{d}x=\int\frac{\tan t\ln\tan t}{\sec t}\sec^2 t\mathrm{d}t=\int\tan t\ln\tan t\sec t\mathrm{d}t$$

$$=\int\ln\tan t\mathrm{d}\sec t$$

$$=\sec t\ln\tan t-\int\frac{\sec^3 t}{\tan t}\mathrm{d}t$$

$$=\sec t\ln\tan t-\int\frac{\sec^2 t}{\sec^2 t-1}\mathrm{d}\sec t$$

$$=\sec t\ln\tan t-\int\frac{\sec^2 t}{\sec^2 t-1}\mathrm{d}\sec t$$

$$=\sqrt{x^2+1}\ln x-\sqrt{x^2+1}-\frac{1}{2}\ln(\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1})+C$$

三、 (9分)求曲线$\boldsymbol{f(x)=\dfrac{(x^2+x+1)\ln x}{x\ln(x-1)}}$的斜渐近线.

解:

$$k=\lim_{k\to+\infty}\frac{f(x)}{x}=\lim_{k\to+\infty}\frac{(x^2+x+1)\ln x}{x^2\ln(x-1)}\cdots\cdots②$$

$$=\lim_{k\to+\infty}\frac{\ln x}{\ln(x-1)}\lim_{k\to+\infty}\frac{x^2+x+1}{x^2}=1,\cdots\cdots②$$

$$b=\lim_{k\to+\infty}(f(x)-kx)=\lim_{k\to+\infty}\frac{(x^2+x+1)\ln x}{x\ln(x-1)}-x$$

$$=\lim_{k\to+\infty}\frac{(x^2+x+1)\ln x-x^2\ln(x-1)}{x\ln(x-1)}\cdots\cdots②$$

$$=\lim_{k\to+\infty}[\frac{x\ln x}{\ln(x-1)}-x]+\lim_{k\to+\infty}\frac{\ln x}{\ln(x-1)}-\lim_{k\to+\infty}\frac{\ln x}{x\ln(x-1)}$$

$$=\lim_{k\to+\infty}\frac{\ln x}{\ln(x-1)}=1,\cdots\cdots②$$

故斜渐近线为$y=x+1.\cdots\cdots①$

四、 (9分)计算$\boldsymbol{\lim\limits_{x\to0}\dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{\sin(\tan x)-\tan x}}$.

解:

$$\lim_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{\sin(\tan x)-\tan x}=\frac{1}{2}\lim_{x\to0}\frac{\tan x-\sin x}{\sin(\tan x)-\tan x}\cdots\cdots②$$

注意到，当$x\to0$时，有

$$\sin(\tan x)-\tan x\sim-\frac{1}{6}\tan^3x\sim-\frac{1}{6}x^3,\cdots\cdots②$$

$$\tan x-\sin x\sim\frac{1}{2}x^3,\cdots\cdots②$$

故

$$\lim_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{\sin(\tan x)-\tan x}=\lim_{x\to0}\frac{\frac{1}{2}x^3+o(x^3)}{-\frac{1}{6}x^3+o(x^3)}=-3,\cdots\cdots②$$

从而原题的极限为$-\dfrac{3}{2}\cdots\cdots①$.

五、 (9分)证明$\boldsymbol{f(x)=x\ln(1+x)-(1+x)\ln x}$在$\boldsymbol{(0,+\infty)}$内有唯一零点.

证：因为

$$\lim_{x\to0^+}(x\ln(1+x)-(1+x)\ln x)=+\infty,\cdots\cdots②$$

当$x>e$，$\dfrac{\ln x}{x}$单调递减，故存在$\gamma>e$

$$\gamma\ln(1+\gamma)-(1+\gamma)\ln\gamma<0,\cdots\cdots②$$

从而根据连续函数的零点存在定理可知$f(x)$在$(0,+\infty)$有零点存在$\cdots\cdots①$.

下证单调性：当$x>0$时有

$$f'(x)=\ln(1+x)+\frac{x}{1+x}-\ln x-\frac{1+x}{x}=[\ln(1+\frac{1}{x})-\frac{1}{x}]-\frac{1}{1+x}<0,\cdots\cdots②$$

故$f(x)$严格单减，从而零点唯一.$\cdots\cdots$②

或

$$\lim_{x\to+\infty} f'(x)=0,$$

$$f''(x)=\frac{x^2+x+1}{(x+1)^2x^2)}>0$$

故$f'(x)<0, x\in(0,+\infty)$.

六、 (9分)求曲线$\left\{\begin{array}{lcl} \boldsymbol{x} & = & \boldsymbol{t^3+3t} \\ \boldsymbol{y} & = & \boldsymbol{t^2+2t} \end{array}\right.$ 的极值点和拐点.

解：

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2t+2}{3t^2+3},$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow t=-1, x=-4, \cdots\cdots ①$$

当$t<-1, x<-4$时,$\dfrac{\mathrm{d}y}{\mathrm{d}x}<0, \cdots\cdots$①

当$t>-1, x>-4$时,$\dfrac{\mathrm{d}y}{\mathrm{d}x}>0, \cdots\cdots$①

故$x=-4$极小值点.$\cdots\cdots$①

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{2}{3}\frac{t^2+2t-1}{(t^2+1)^3},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0\Rightarrow t=\pm\sqrt{2}-1, \cdots\cdots ①$$

当$t<-\sqrt{2}-1, x<-8\sqrt{2}-10$时,$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}<0, \cdots\cdots$①

当$-\sqrt{2}-1<t<\sqrt{2}-1, -8\sqrt{2}-10<x<8\sqrt{2}-10$时,$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}>0, \cdots\cdots$①

当$t>\sqrt{2}-1, x>8\sqrt{2}-10$时,$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}>0,$

故$(8\sqrt{2}-10,1), (-8\sqrt{2}-10,1)$是拐点$\cdots\cdots$①.

七、 (9分) 已知函数$\boldsymbol{f(x)}$的导函数$\dfrac{\mathbf{d}\boldsymbol{f(x)}}{\mathbf{d}\boldsymbol{x}}=\boldsymbol{x^2|\sin x|}$.

(1)求使得$\boldsymbol{f^{(n)}(0)}$存在的$\boldsymbol{n}$的最大值.

(2)若$\boldsymbol{x\in[-\pi,\pi]}$，且$\boldsymbol{f(\pi)=0}$，求$\boldsymbol{f(x)}$.

解：(1)

$$f'(x)=\left\{\begin{array}{ll} x^2\sin x, & x\geq 0, \\ -x^2\sin x, & x<0. \end{array}\right.$$

$$f''(x)=\left\{\begin{array}{ll} 2x\sin x+x^2\cos x, & x>0, \\ -2x\sin x-x^2\cos x, & x<0. \end{array}\right. \quad \cdots\cdots ①$$

由$f'(x)$的连续性以及$\lim\limits_{x\to0} f''(x)$存在，可知$f''(0)$存在.

$$f'''(x)=\left\{\begin{array}{ll} 2\sin x+4x\cos x-x^2\sin x, & x>0, \\ -2\sin x-4x\cos x+x^2\sin x, & x<0. \end{array}\right. \quad \cdots\cdots ①$$

由$f''(x)$的连续性以及$\lim\limits_{x\to0} f'''(x)$存在，可知$f'''(0)$存在.

又

$$f_{+}^{(4)}(0)=\lim_{x\to 0^{+}}\frac{2\sin x+4x\cos x-x^{2}\sin x}{x}=6,\cdots\cdots ①$$

$$f_{-}^{(4)}(0)=\lim_{x\to 0^{+}}\frac{-2\sin x-4x\cos x+x^{2}\sin x}{x}=-6,\cdots\cdots ①$$

故$n=3.\cdots\cdots$ ①

(2)当$\pi\ge x\ge 0$,

$$f(x)=\int x^{2}\sin x=2x\sin x+(2-x^{2})\cos x+C_{1},\cdots\cdots ①$$

当$-\pi\le x<0$,

$$f(x)=\int -x^{2}\sin x=-2x\sin x-(2-x^{2})\cos x+C_{2},\cdots\cdots ①$$

$$f(\pi)=0\Rightarrow C_{1}=2-\pi^{2},\cdots\cdots ①$$

$$f(0^{+})=f(0^{-})\Rightarrow C_{2}=6-\pi^{2}.\cdots\cdots ①$$

八、 (9分)设$\boldsymbol{f(x)\ne 0}$在$\boldsymbol{[a,b]}$上连续，$\boldsymbol{(a,b)}$内可导.证明：存在$\boldsymbol{\xi\in(a,b)}$，使得$\boldsymbol{\dfrac{f'(\xi)}{f(\xi)}=\dfrac{1}{b-\xi}-\dfrac{2}{\xi-a}}$.

证：

$$F(x)=f(x)(b-x)(x-a)^{2},\cdots\cdots ③$$

显然，

$$F(a)=F(b)=0,\cdots\cdots ③$$

根据Rolle中值定理可知，$\exists\xi\in(a,b)$，使得

$$F'(\xi)=0\Rightarrow\frac{f'(\xi)}{f(\xi)}=\frac{1}{b-\xi}-\frac{2}{\xi-a}.\cdots\cdots ③$$

九、 (7分)设$\boldsymbol{f(x)}$在$\boldsymbol{(0,+\infty)}$上有定义，满足$\boldsymbol{y^{2}f(\frac{x}{y})=yf(x)-xf(y)}$，且$\boldsymbol{f'(1)}$存在. 证明：$\boldsymbol{f(x)}$在$\boldsymbol{(0,+\infty)}$上可导，且有$\boldsymbol{f'(x)=\dfrac{f(x)}{x}+f'(1)}$.

证：令$x=y\in(0,\infty)$，则有

$$y^{2}f(1)=xf(x)-xf(x)=0\Rightarrow f(1)=0.\cdots\cdots ①$$

因为

$$f(\frac{x+\Delta x}{x}\cdot x)-f(x)=xf(\frac{x+\Delta x}{x})-x(x+\Delta x)f(\frac{1}{x})-f(x),\cdots\cdots ②$$

$$f(\frac{1}{x})=\frac{f(1)}{x}-\frac{f(x)}{x^{2}}=-\frac{f(x)}{x^{2}},\cdots\cdots ②$$

故

$$f(\frac{x+\Delta x}{x}\cdot x)-f(x)=xf(\frac{x+\Delta x}{x})+\Delta x\frac{f(x)}{x},\cdots\cdots ②$$

从而

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(\frac{x+\Delta x}{x}\cdot x)-f(x)}{\Delta x},$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(\frac{x+\Delta x}{x})-f(1)}{\frac{\Delta x}{x}}+\frac{f(x)}{x}=f'(1)+\frac{f(x)}{x}.\cdots\cdots ②$$

十、 (7分)$\forall \boldsymbol{x_0 \in (1, \sqrt[4]{e})}$，$\boldsymbol{x_{n+1} = x_n^2 \ln x_n - x_n + 2, n = 0, 1, 2, \cdots}$.证明：

(1)数列$\boldsymbol{\{x_n\}}$是收敛的，并计算$\boldsymbol{a = \lim\limits_{n\to+\infty} x_n}$的值.

证：设$f(x) = x^2 \ln x - x + 2$, 先证有界性：

$$f'(x) = 2x\ln x + x - 1, \cdots\cdots ①$$

$$f'(1) = 0, f'(\sqrt[4]{e}) = \frac{3}{2}\sqrt[4]{e} - 1 > 0,$$

$$f''(x) = 2\ln x + 3,$$

$$f''(x) = 2\ln x + 3 > 0, \forall x \in (1, \sqrt[4]{e}),$$

故$\forall x \in (1, \sqrt[4]{e})$，有

$$f'(x) > f'(1) = 0,$$

$$1 = f(1) < f(x) < f(\sqrt[4]{e}) = \frac{1}{4}(\sqrt[4]{e} - 2)^2 + 1 < \sqrt[4]{e},$$

从而，$\{x_n\}$有界，且$x_n \in (1, \sqrt[4]{e})$.$\cdots\cdots$①

证单调性：考虑$x_{n+1} - x_n = x_n^2 \ln x_n - 2x_n + 2$，令$g(x) = x^2 \ln x - 2x + 2$,

$$g'(x) = 2x\ln x + x - 2,$$

$$g'(1) = -1, g'(\sqrt[4]{e}) = \frac{3}{2}\sqrt[4]{e} - 2 < 0,$$

$$g''(x) = 2\ln x + 3,$$

$$g''(x) = 2\ln x + 3 > 0, \forall x \in (1, \sqrt[4]{e}),$$

故$\forall x \in (1, \sqrt[4]{e})$，有

$$g'(x) < 0 \Rightarrow g(x) < g(1) = 0,$$

从而$x_{n+1} < x_n$，数列单减.$\cdots\cdots$①

根据单调有界，必有极限可知数列收敛，并有

$$a = a^2 \ln a - a + 2 \Rightarrow a = 1. \cdots\cdots ①$$

(2)$\boldsymbol{(x_n - a)^2 < \frac{2}{3}(x_{n+1} - a)}$.

证：由(1)可知

$$f(1) = 1, f'(1) = 0, \cdots\cdots ①$$

则有

$$x_{n+1} = f(x_n) = f(1) + f'(1)(x_n - 1) + \frac{f''(\xi_n)}{2}(x_n - 1)^2, \xi_n \in (1, x_n), \cdots\cdots ①$$

$$x_{n+1} - 1 = \frac{f''(\xi_n)}{2}(x_n - 1)^2 = \frac{2\ln \xi_n + 3}{2}(x_n - 1)^2 > \frac{3}{2}(x_n - 1)^2, \xi_n \in (1, x_n). \cdots\cdots ①$$