四川大学半期考试试题答案

(2020——2021学年第 II 学期)

课程号: **201075030** 课序号: 课程名称: **微积分II-2** 任课教师: 成绩:
适用专业年级: **2020级** 学生人数: 印题份数: 学号: 姓名:

一、 填空题(每题4分，共20分)

1. $\int_{-1}^{1} x\cos x \mathrm{d}x = \underline{0}$.

2. $\int_{0}^{+\infty} \frac{1}{x^2+2x+2}\mathrm{d}x = \underline{\frac{\pi}{4}}$.

3. 曲线的极坐标方程为$\rho = \sin\theta, \theta \in [0,\pi]$，则曲线长度为$\underline{\pi}$.

4. $\lim\limits_{n\to+\infty} n\cdot\sum_{k=1}^{n}\frac{1}{n^2+k^2} = \underline{\frac{\pi}{4}}$.

5. $\lim\limits_{(x,y)\to(0,0)} \frac{e^{x^4+y^4}-1}{x^2+y^2} = \underline{0}$.

二、 解答题(共80分)

1. (10分)已知函数$F(x)$在$(-1,1)$上有定义，$F'(0)$存在. 当$x\neq 0$时，$F(x) = \frac{1}{x}\int_0^x (e^t - t - 1)\ln|t|\mathrm{d}t$. 求$F'(0)$.

解：注意到$\lim\limits_{t\to 0}(e^t-t-1)\ln|t| = 0$，故

$$F(0) = \lim_{x\to 0}\frac{1}{x}\int_0^x (e^t-t-1)\ln|t|\mathrm{d}t = 0 \ldots\ldots 2$$

$$F'(0) = \lim_{x\to 0}\frac{F(x)-F(0)}{x-0} \ldots\ldots 2$$

$$= \lim_{x\to 0}\frac{\int_0^x (e^t-t-1)\ln|t|\mathrm{d}t}{x^2} \ldots\ldots 2$$

$$= \lim_{x\to 0}\frac{(e^x-x-1)\ln|x|}{2x} \ldots\ldots 2$$

$$= 0 \ldots\ldots 2$$

注：直接求导算极限要扣5分.

2. (12分)计算$\int_0^{\pi} x\cdot|\cos x|\cdot e^{\sin x}\mathrm{d}x$.

解:

$$\int_0^{\pi} x\cdot|\cos x|\cdot e^{\sin x}\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}|\cos x|\cdot e^{\sin x}\mathrm{d}x\ldots\ldots 3$$

$$=\frac{\pi}{2}[\int_0^{\frac{\pi}{2}}|\cos x|\cdot e^{\sin x}\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}|\cos x|\cdot e^{\sin x}\mathrm{d}x]\ldots\ldots 3$$

$$=\frac{\pi}{2}[\int_0^{\frac{\pi}{2}}\cos x\cdot e^{\sin x}\mathrm{d}x+\int_0^{\frac{\pi}{2}}\cos x\cdot e^{\sin x}\mathrm{d}x]=\pi\int_0^{\frac{\pi}{2}}\cos x\cdot e^{\sin x}\mathrm{d}x\ldots\ldots 3$$

$$=\pi\, e^{\sin x}\Big|_0^{\frac{\pi}{2}}=\pi(e-1)\ldots\ldots 3$$

3. (12分)函数$f(x,y)$的偏导数分别为$\frac{\partial f}{\partial x}=y,\frac{\partial f}{\partial y}=x+y$，求$f(x,y)$.

解:

$$\int\frac{\partial f}{\partial x}\mathrm{d}x=\int y\mathrm{d}x$$
$$=xy+\varphi(y)\ldots\ldots 3$$

$$\frac{\partial f}{\partial y}=\frac{\partial(xy+\varphi(y))}{\partial y}=x+\varphi'(y)=x+y\Rightarrow\ldots\ldots 3$$

$$\varphi'(y)=y\Rightarrow\varphi(y)=\frac{y^2}{2}+C\ldots\ldots 3$$

故$f(x,y)=xy+\frac{y^2}{2}+C$.……3

4. (12分)函数$z(x,y)=(2x+y)^{x+2y}$，求$\frac{\partial z}{\partial x}\Big|_{x=0,y=1},\frac{\partial^2 z}{\partial x\partial y}\Big|_{x=0,y=1}$.

解:

$$z(x,y)=(2x+y)^{x+2y}=e^{(x+2y)\ln(2x+y)},$$

$$\frac{\partial z}{\partial x}=e^{(x+2y)\ln(2x+y)}[\ln(2x+y)+2\frac{x+2y}{2x+y}],\ldots\ldots 3$$

$$\frac{\partial z}{\partial x}\Big|_{x=0,y=1}=4.\ldots\ldots 3$$

$$\frac{\partial^2 z}{\partial x\partial y}\Big|_{x=0,y=1}=\frac{\mathrm{d}z'_x(0,y)}{\mathrm{d}y}\Big|_{y=1}\ldots\ldots 3$$

$$=e^{2y\ln y}[(2\ln y+2)(\ln y+4)+\frac{1}{y}]|_{y=1}=9.\ldots\ldots 3$$

5. (12分)已知$z(x,y)=\begin{cases}\dfrac{x^2\sin y^2}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$

(1)计算$f_x'(0,0),f_y'(0,0)$;

(2)分析$z(x,y)$在点$(0,0)$处的可微性.

解：(1)

$$f_x'(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}$$

$$=\lim_{x\to0}\frac{0}{x}=0\ldots\ldots3$$

$$f_y'(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}$$

$$=\lim_{y\to0}\frac{0}{y}=0\ldots\ldots3$$

(2)

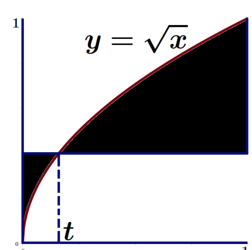
$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-(f_x'(0,0)x+f_y'(0,0)y)}{\sqrt{x^2+y^2}}\ldots\ldots3$$

$$a=\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{(x^2+y^2})}=\lim_{(x,y)\to(0,0)}\frac{\frac{x^2\sin y^2}{x^2+y^2}}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{x^2\sin y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$0\le\frac{x^2\sin y^2}{(x^2+y^2)^{\frac{3}{2}}}\le\frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}\le\frac{1}{4}\sqrt{x^2+y^2},$$

故$a=0$，因此在$(0,0)$处可微……3

6. (12分)如下图所示，在$[0,1]$中选取合适的t，使得由曲线$y=\sqrt{x}$，直线$x=1$，$y=\sqrt{t}$以及$x=0$围成的黑色区域绕x轴旋转一周所成的旋转体的体积最小.



解：如图，以t为分界点计算体积$V_1(t),V_2(t)$，

$$V_1(t)=\pi\int_0^t[(\sqrt{t})^2-(\sqrt{x})^2]\mathrm{d}x=\pi\frac{t^2}{2},\ldots\ldots3$$

$$V_2(t)=\pi\int_t^1[(\sqrt{x})^2-(\sqrt{t})^2]\mathrm{d}x=\pi(\frac{t^2}{2}-t+\frac{1}{2}),\ldots\ldots3$$

$$V(t)=V_1(t)+V_2(t)=\pi[t^2-t+\frac{1}{2}],$$

$$V'(t)=\pi(2t-1)=0\Rightarrow t=\frac{1}{2},\ldots\ldots3$$

$$V''(t)=2\pi>0,$$

故$t=\frac{1}{2}$时，体积取得最小值……3

7. (10分)假设$f(x)$是$(-\infty,+\infty)$上的连续函数，满足$\lim\limits_{x\to\infty}\frac{f(x)}{x}=1$，且

$$\frac{1}{2}\int_x^{3x}f(t-x)\mathrm{d}t-\int_0^{2x}f\left(\frac{t}{2}\right)\mathrm{d}t+2x=0,$$

求$f(x)$.

解：

$$\frac{1}{2}\int_x^{3x}f(t-x)\mathrm{d}t-\int_0^{2x}f\left(\frac{t}{2}\right)\mathrm{d}t+2x$$

$$=\frac{1}{2}\int_0^{2x}f(u)\mathrm{d}u-2\int_0^{x}f(u)\mathrm{d}u+2x=0,\ldots\ldots2$$

两边同时求导，得

$$f(2x)-2f(x)+2=0\Rightarrow$$

$$f(x)=\frac{1}{2}f(2x)+1,\ldots\ldots2$$

当$x=0$时，有

$$f(0)=\frac{1}{2}f(0)+1\Rightarrow f(0)=2.\ldots\ldots2$$

当$x\neq0$时，有

$$f(x)=\frac{1}{2^2}f(2^2x)+1+\frac{1}{2}=\frac{1}{2^n}f(2^nx)+\sum_{k=0}^{n-1}\frac{1}{2^k}$$

$$=x\lim_{n\to\infty}\frac{f(2^nx)}{2^nx}+2=x+2.\ldots\ldots2$$

所以，$f(x)=x+2.\ldots\ldots2$