四川大学微积分II-(1)期末考试试题（A卷）-参考答案

(2019——2020学年 第 1 学期)

一、填空题(每题4分，共24分)

1. $\lim\limits_{x\to0}\dfrac{x+\ln(1-x)}{x^2}=\underline{-\dfrac{1}{2}}$.

2. $\displaystyle\int\cos x\cdot\ln(\sin x)\mathrm{d}x=\underline{\sin x\ln\sin x-\sin x+C}$.

3. 若$x^2+y^2-4xy=0$确定隐函数$y=y(x)$,则$\frac{\mathrm{d}y}{\mathrm{d}x}=\underline{\dfrac{2y-x}{y-2x},y\neq 2x}$.

4. 已知$y(x)=\sin x\cdot\ln(1+x^2)$,则$y^{(5)}(0)=\underline{-80}$.

5. 函数$y(x)=(x-1)(x-2)^2(x-4)$的拐点个数为$\underline{2}$.

6. 函数$y(x)=\dfrac{\sqrt{x+4}\cdot e^x}{1+x}$,则$\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=\underline{\dfrac{1}{4}}$.

二、(9分)求$y=\dfrac{(x+1)\cdot e^x}{e^x-1}$的所有渐近线.

解：因为

$$\lim_{x\to-\infty}y(x)=0,$$

故有水平渐近线：$y=0.\cdots\cdots\cdots\cdots(\mathbf{2})$

又

$$\lim_{x\to0}y(x)=\infty,$$

故有垂直渐近线：$x=0.\cdots\cdots\cdots\cdots(\mathbf{2})$

因为

$\lim\limits_{x\to+\infty}\frac{y}{x}=\lim\limits_{x\to+\infty}\frac{e^x}{e^x-1}=1,\cdots\cdots\cdots\cdots(\mathbf{2})$

$\lim\limits_{x\to+\infty}(y-x)=\lim\limits_{x\to+\infty}\frac{x+e^x}{e^x-1}=1,\ \cdots\cdots\cdots\cdots(\mathbf{2})$

故有渐近线：$y=x+1.\cdots\cdots\cdots\cdots(\mathbf{1})$

三、(9分)设$F(x)$是$f(x)$的一个原函数,并且当$x>0$时,有$f(x)F(x)=\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$.如果$F(1)=2\sqrt{e}$,求$f(x)$.

解:$f(x)=F'(x)$,有

$F(x)F'(x)=\dfrac{e^{\sqrt{x}}}{\sqrt{x}}\Rightarrow\cdots\cdots\cdots(\mathbf{2})$

$\displaystyle\int F(x)F'(x)\mathrm{d}x=\int\frac{e^{\sqrt{x}}}{\sqrt{x}}\mathrm{d}x\Rightarrow\cdots\cdots\cdots(\mathbf{2})$

$\dfrac{F^2(x)}{2}=2e^{\sqrt{x}}+C\Rightarrow\cdots\cdots\cdots(\mathbf{2})$

由$F(1)=2\sqrt{e}$,可知$C=0,F(x)=2e^{\frac{\sqrt{x}}{2}},\cdots\cdots\cdots(\mathbf{2})$

可得$f(x)=F'(x)=\dfrac{e^{\frac{\sqrt{x}}{2}}}{2\sqrt{x}}.\cdots\cdots\cdots(\mathbf{1})$

四、(9分)$\displaystyle\int\frac{\sqrt{x^4+1}}{x}\mathrm{d}x.$

解1:$\displaystyle\int\frac{\sqrt{x^4+1}}{x}\mathrm{d}x=\int\frac{\sqrt{x^4+1}x}{x^2}\mathrm{d}x=\frac{1}{2}\int\frac{\sqrt{x^4+1}}{x^2}\mathrm{d}x^2$

令$\sinh t=x^2,\cosh t=\sqrt{x^4+1},t>0,$

$$=\frac{1}{2}\int\frac{\sqrt{\sinh^2 t+1}}{\sinh t}\mathrm{d}\sinh t$$

$$=\frac{1}{2}\int\frac{\cosh^2 t}{\sinh t}\mathrm{d}t=\frac{1}{2}\int\frac{1+\sinh^2 t}{\sinh t}\mathrm{d}t$$

$$=\frac{1}{2}\int(\frac{1}{\sinh t}+\sinh t)\mathrm{d}t=\frac{1}{2}\cosh t+\frac{1}{2}\int\frac{1}{\sinh t}\mathrm{d}t$$

$$=\frac{1}{2}\sqrt{x^4+1}-\frac{1}{2}\int\frac{1}{1-\cosh^2 t}\mathrm{d}\cosh t$$

$$=\frac{1}{2}\sqrt{x^4+1}-\frac{1}{2}\tanh^{-1}(\frac{1}{\cosh t})+C$$

$= \frac{1}{2}\sqrt{x^4+1} - \frac{1}{2}\tanh^{-1}(\frac{1}{\sqrt{x^4+1}}) + C$,写成$\tanh^{-1}(\sqrt{x^4+1})$要扣分.

解2：$\int \frac{\sqrt{x^4+1}}{x}\mathrm{d}x = \int \frac{\sqrt{x^4+1}x}{x^2}\mathrm{d}x = \frac{1}{2}\int \frac{\sqrt{x^4+1}}{x^2}\mathrm{d}x^2$

令$\tan t = x^2, \sec t = \sqrt{x^4+1}, 0 < t < \frac{\pi}{2}, \csc t = \frac{\sqrt{x^4+1}}{x^2}$,

$= \frac{1}{2}\int \frac{\sqrt{\tan^2 t+1}}{\tan t}\mathrm{d}\tan t = \frac{1}{2}\int \frac{\sec t}{\tan t}\mathrm{d}\tan t$,分部积分

$= \frac{1}{2}\sec t + \frac{1}{2}\int \frac{1}{\sin t}\mathrm{d}t$

$= \frac{1}{2}\sqrt{x^4+1} + \frac{1}{2}\ln|\csc t - \cot t| + C$

$= \frac{1}{2}\sqrt{x^4+1} + \frac{1}{2}\ln\frac{\sqrt{x^4+1}-1}{x^2} + C$

解3：令$u = \sqrt{x^4+1}, \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2x^3}{\sqrt{x^4+1}}, \cdots\cdots\cdots(\mathbf{2})$

原式$=\frac{1}{2}\int \frac{u^2}{u^2-1}\mathrm{d}u \cdots\cdots\cdots(\mathbf{1})$

$= \frac{1}{4}\int \frac{1}{u-1} - \frac{1}{2}\int \frac{1}{u+1} + \frac{1}{2}u \cdots\cdots\cdots(\mathbf{2})$

$= \frac{1}{2}\ln\frac{u-1}{u+1} + \frac{1}{2}u + C \cdots\cdots\cdots(\mathbf{2})$

$= \frac{1}{2}\ln\frac{\sqrt{x^4+1}-1}{\sqrt{x^4+1}+1} + \frac{1}{2}\sqrt{x^4+1} + C \cdots\cdots\cdots(\mathbf{2})$

五、(9分)已知$\begin{cases} x = t\cdot e^{-t}, \\ y = t\cdot e^{2t}. \end{cases}$ 在$t<1$时确定函数$y=y(x)$.求函数$y(x)$的极值点,并判断是极大值还是极小值.

解：$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)} = \frac{(1+2t)e^{2t}}{(1-t)e^{-t}}, \cdots\cdots\cdots(\mathbf{2})$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow t = -\frac{1}{2}, \cdots\cdots\cdots(\mathbf{1})$

此时,$x(-\frac{1}{2})=-\frac{e^{1/2}}{2}, y(-\frac{1}{2})=-\frac{e^{-1}}{2}$为驻点$\cdots\cdots\cdots(\mathbf{1})$

注意到：

$t<-\frac{1}{2}, x'(t)>0, x(t)\uparrow,\cdots\cdots\cdots(\mathbf{1})$

$t<-\frac{1}{2}, y'(t)<0, y(t)\downarrow,\cdots\cdots\cdots(\mathbf{1})$

$-\frac{1}{2}<t<1, x'(t)>0, x(t)\uparrow,\cdots\cdots\cdots(\mathbf{1})$

$-\frac{1}{2}<t<1, y'(t)>0, y(t)\uparrow,\cdots\cdots\cdots(\mathbf{1})$

故$-\frac{e^{1/2}}{2}$为极小值点，极小值为$-\frac{e^{-1}}{2}.\cdots\cdots\cdots(\mathbf{1})$

也可用二阶导数判断：

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3\,\frac{\mathrm{e}^{4\,t}\left(2\,t^2-t-2\right)}{\left(-1+t\right)^3}, \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{-1/2}=\frac{8e^{-2}}{9}>0.$$

六、(9分)若函数$f(x)=\begin{cases}x^\alpha, & x>0\\ \sqrt{1+x^3}-1, & x\le 0.\end{cases}$,问

(1)当α取何值时,函数$f(x)$连续;

(2)当α取何值时,导函数$f'(x)$连续.

解：(1)$f(0-)=\lim\limits_{x\to 0^-}(\sqrt{1+x^3}-1)=0,\cdots\cdots\cdots(\mathbf{2})$

故$f(0+)=\lim\limits_{x\to 0^+}x^\alpha=0\Rightarrow\alpha>0.$

当$\alpha>0$取何值时,函数$f(x)$连续$.\cdots\cdots\cdots(\mathbf{2})$

(2)$x<0, f'(x)=\frac{3}{2}\frac{x^2}{\sqrt{x^3+1}},\cdots\cdots\cdots(\mathbf{1})$

$f'_-(0)=0,\cdots\cdots\cdots(\mathbf{1})$

$f'_+(0)=\lim\limits_{x\to 0^+}x^{\alpha-1},\cdots\cdots\cdots(\mathbf{1})$

$x>0, f'(x)=\alpha x^{\alpha-1},\cdots\cdots\cdots(\mathbf{1})$

要导函数连续，必须满足$f'_+(0)=0$

$\Rightarrow\alpha>1.\cdots\cdots\cdots(\mathbf{1})$

当$\alpha > 1$取何值时,导函数$f'(x)$连续.

七、(9分) 已知$\lim\limits_{x\to 0}\dfrac{\arcsin x - \arcsin\left(\frac{x}{1+x}\right)}{x^k} = c \neq 0$,求$k$和$c$.

解：用Lagrange中值定理有：

$$\lim_{x\to 0}\frac{\arcsin x - \arcsin\left(\frac{x}{1+x}\right)}{x^k} = \lim_{x\to 0}\frac{1}{\sqrt{1-\xi^2}}\frac{x-\frac{x}{1+x}}{x^k}, \cdots\cdots\cdots (\mathbf{2})$$

ξ位于$x, \dfrac{x}{1+x}$之间.$\cdots\cdots\cdots(\mathbf{1})$

因为,$x \to 0 \Rightarrow \xi \to 0, \cdots\cdots\cdots (\mathbf{2})$

可知$\lim\limits_{x\to 0}\frac{1}{\sqrt{1-\xi^2}}\frac{x-\frac{x}{1+x}}{x^k} = \lim\limits_{x\to 0}\frac{x-\frac{x}{1+x}}{x^k} = \lim\limits_{x\to 0}\frac{x^2}{x^k(1+x)} \Rightarrow \cdots\cdots\cdots (\mathbf{2})$

$c = 1, k = 2. \cdots\cdots\cdots (\mathbf{2})$

八、(9分)已知$f_n(x) = x^n - 2x + 1$,其中$n > 2$是正整数.证明：

(1)$f_n(x) = 0$在$(\dfrac{1}{2}, 1)$内有唯一实数解β_n..

(2)证明数列$\{\beta_n\}$是收敛的,并计算$\lim\limits_{n\to+\infty}\sqrt[n]{2\beta_n - 1}$.

证:(1)显然,

$f_n(\dfrac{1}{2}) = \dfrac{1}{2^n} > 0, \cdots\cdots\cdots (\mathbf{1})$

$f_n(\dfrac{4}{5}) = (\dfrac{4}{5})^n - \dfrac{3}{5} < 0, n > 2, \cdots\cdots\cdots (\mathbf{1})$

根据连续函数零点存在定理知$(\frac{1}{2}, \frac{4}{5})$内存在零点$\beta_n, \cdots\cdots\cdots (\mathbf{1})$

又$f_n'(x) = nx^{n-1} - 2, f_n''(x) = n(n-1)x^{n-2} > 0, \forall x \in (0, 1)$,

从而$f_n(x)$在$(\dfrac{1}{2}, 1)$内有唯一实数解$\beta_n. \cdots\cdots\cdots (\mathbf{1})$

(2)因为$\frac{1}{2} < \beta_n < \frac{4}{5}$,由夹逼定理知$\lim\limits_{n\to+\infty}\beta_n^n = 0. \cdots\cdots\cdots (\mathbf{1})$

又$f_n(\beta_n) = \beta_n^n - 2\beta_n + 1 = 0 \Rightarrow \lim\limits_{n\to+\infty}\beta_n = \frac{1}{2}, \cdots\cdots\cdots (\mathbf{2})$

$\lim\limits_{n\to+\infty}(2\beta_n - 1)^{1/n} = \lim\limits_{n\to+\infty}\beta_n = \frac{1}{2}. \cdots\cdots\cdots (\mathbf{2})$

九、(7分)(1)判断函数$f(t)=\dfrac{\ln t}{1-t}$的单调性.

(2)$\forall y>x>0, x\neq 1, y\neq 1$,比较$y\cdot x^y$与$x\cdot y^x$的大小.

证明：(1)$f'(t)=\dfrac{t\ln t-t+1}{t(t-1)^2}, f'(t)=0\Rightarrow t=1,$

令$g(t)=t\ln t-t+1, g'(t)=\ln t,\cdots\cdots\cdots\cdot(\mathbf{1})$

当$t>1$时,

$g'(t)>0, g(t)>g(1)=0, f'(t)>0$,此时函数$f(t)=\dfrac{\ln t}{1-t}$单增,$\cdots\cdots\cdots\cdot(\mathbf{1})$

当$0<t<1$时,

$g'(t)<0, g(t)>g(1)=0, f'(t)>0$,此时函数$f(t)=\dfrac{\ln t}{1-t}$单增,$\cdots\cdots\cdots\cdot(\mathbf{1})$.

(2)由(1)的结论可知：

$y>x>1$时,有

$f(y)>f(x)\Rightarrow \frac{\ln y}{1-y}>\frac{\ln x}{1-x}\Rightarrow y\cdot x^y>x\cdot y^x.\cdots\cdots\cdots\cdot(\mathbf{1})$

$1>y>x>0$时,有

$f(y)>f(x)\Rightarrow \frac{\ln y}{1-y}>\frac{\ln x}{1-x}\Rightarrow y\cdot x^y>x\cdot y^x.\cdots\cdots\cdots\cdot(\mathbf{1})$

$y>1>x>0$时,有

$f(y)>\lim\limits_{x\to 1}f(x)=-1>f(x)\Rightarrow \frac{\ln y}{1-y}>\frac{\ln x}{1-x}\Rightarrow y\cdot x^y<x\cdot y^x.\cdots\cdots\cdot(\mathbf{2})$

注：若第三种情形没有考虑$\lim\limits_{x\to 1}f(x)=-1$不得分.

十、(6分)已知$f(x)\in C^4[0,1]$,三次多项式$p_3(x)$满足$p_3(0)=f(0)$,$p_3'(0)=f'(0)$,$p_3(1)=f(1)$,$p_3'(1)=f'(1)$. 证明：$|f(x)-p_3(x)|\leq \dfrac{1}{384}\max\limits_{0\leq x\leq 1}|f^{(4)}(x)|$.

证明：构造函数$F(t)=f(t)-p_3(t)-\lambda t^2(t-1)^2$,

显然,$F(0)=F'(0)=F(1)=F'(1)=0$.

选择合适$\lambda, 0<x_0<1$,使得x_0是$F(t)$另一个零点.为此有

$0=F(x_0)=f(x_0)-p_3(x_0)-\lambda x_0^2(x_0-1)^2\Rightarrow$

$\lambda = \frac{f(x_0)-p_3(x_0)}{x_0^2(x_0-1)^2}$,

从而$F(t) = f(t) - p_3(t) - \frac{f(x_0)-p_3(x_0)}{x_0^2(x_0-1)^2}t^2(t-1)^2. \cdots\cdots\cdots (\mathbf{2})$

应用Rolle中值定理有

$\exists\xi \in (0,1), F^{(4)}(\xi) = 0 \Rightarrow$

$$r^{(4)}(\xi) - 24\frac{r(x)}{x^2(x-1)^2} = 0 \Rightarrow \cdots\cdots\cdots (\mathbf{2})$$

$$r(x) = \frac{f^{(4)}(\xi)}{24}x^2(x-1)^2 \Rightarrow$$

$$|r(x)| \leq \max\frac{f^{(4)}(\xi)}{24}x^2(x-1)^2 \Rightarrow$$

$$|r(x)| \leq \frac{1}{384}\max_{0\leq x\leq 1}|f^{(4)}(x)|. \cdots\cdots\cdots (\mathbf{2})$$