四川大学期末考试试题（闭卷）
(2020——2021学年第 2 学期) A卷答案

一、 填空题(每题5分，共30分)

1. $\displaystyle\int_0^{+\infty}\frac{1}{1+4x^2}\mathrm{d}x=\underline{\frac{\pi}{4}}.$

2. $\displaystyle\lim_{(x,y)\to(+\infty,+\infty)}\left(1+\frac{1}{x+y}\right)^{x+y}=\underline{e}.$

3. $\displaystyle\iint\limits_{x^2+y^2\le 1}\mathrm{d}x\mathrm{d}y=\underline{\pi}.$

4. 若$z(x,y)=xe^y$，则$\mathrm{d}z|_{(1,0)}=\underline{\mathrm{d}x+\mathrm{d}y}.$

5. 已知$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}x}=2xy,\\ y(0)=1,\end{cases}$，则$y(x)=\underline{e^{x^2}}.$

6. 若二阶线性常系数齐次常微分方程的两个特解分别是e^x和xe^x，则此二阶常微分方程为$\underline{y''-2y'+y=0}.$

二、 计算题(每小题8分，共32分)

1. 计算$\displaystyle\int_0^1 x^2\cdot(1-x^2)^{3/2}\mathrm{d}x$

解：令$x=\sin t, 0\le t\le\frac{\pi}{2}$......2,

$\displaystyle\int_0^1 x^2\cdot(1-x^2)^{3/2}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\cos^4 t\mathrm{d}t$......2

$\displaystyle=\int_0^{\frac{\pi}{2}}\cos^4 t\mathrm{d}t-\int_0^{\frac{\pi}{2}}\cos^6 t\mathrm{d}t$......2

$\displaystyle=\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$\displaystyle=\frac{\pi}{32}$.......2

2. 计算二重积分$\displaystyle\iint\limits_D xy\mathrm{d}x\mathrm{d}y$，

其中D是由曲线$y^2=x$, $y=1$, $x=0$所围成的平面区域.

解：$\displaystyle\iint\limits_D xy\mathrm{d}x\mathrm{d}y=\int_0^1 x\mathrm{d}x\int_{\sqrt{x}}^1 y\mathrm{d}y$......2

$\displaystyle=\int_0^1 x\mathrm{d}x\frac{y^2}{2}\Big|_{\sqrt{x}}^1$......2

$\displaystyle=\int_0^1(\frac{x}{2}-\frac{x^2}{2})\mathrm{d}x$......2

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$.......②

或者

$$\iint_D xy\mathrm{d}x\mathrm{d}y = \int_0^1 y\mathrm{d}y \int_0^{y^2} x\mathrm{d}x$$

3. 求$\boldsymbol{f(x,y)=e^x(x+y^2)}$的极值，并判断是极大还是极小.

解：

$f_x' = e^x(x+y^2+1) = 0$,......①

$f_y' = 2ye^x = 0$,......①

解得：$x=-1, y=0$,......①

$A = f_{xx}''(-1,0) = e^x(x+y^2+2)|_{(-1,0)} = e^{-1}$, ...①

$B = f_{xy}''(-1,0) = 2ye^x|_{(-1,0)} = 0$, ...①

$C = f_{yy}''(-1,0) = 2e^x|_{(-1,0)} = 2e^{-1}$, ...①

$B^2 - AC = -2e^{-2} < 0, A > 0$......①

故$(-1,0)$是极小值点，极小值为$-e^{-1}$.......①

4. 设$\boldsymbol{F(x,y)}$具有连续偏导，且$\boldsymbol{F(0,0)=1}$，$\boldsymbol{F_1'(0,0)=1}$，$\boldsymbol{F_2'(0,0)=-1}$.

对$\boldsymbol{\forall x,y}$，$\boldsymbol{F(xz,yz)=z-y}$确定隐函数$\boldsymbol{z=z(x,y)}$，求$\boldsymbol{(z_x'+z_y')\Big|_{(0,0)}}$.

解：令$G(x,y,z) = F(xz,yz) - z + y, z(0,0) = F(0,0) - 0 = 1$, ...①

$G_x' = F_1'(xz,yz)\cdot z$,......①

$G_y' = F_2'(xz,yz)\cdot z + 1$,......①

$G_z' = F_1'(xz,yz)\cdot x + F_2'(xz,yz)\cdot y - 1$,......②

$z_x'(0,0) = -\frac{G_x'}{G_z'}\Big|_{x=0,y=0} = -\frac{F_1'(0,0)\cdot z(0,0)}{-1} = 1$,......①

$z_y'(0,0) = -\frac{G_y'}{G_z'}\Big|_{x=0,y=0} = -\frac{F_2'(0,0)\cdot z(0,0)+1}{-1} = 0$,......①

$z_x'(0,0) + z_y'(0,0) = 1$.......①

三、 (8分) 已知二元函数$\boldsymbol{F(x,y) = \begin{cases} \dfrac{\sin(x^2)\cdot\sin(x+y)}{x^2+y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0). \end{cases}}$

(1)计算$\boldsymbol{F_x'(0,0), F_y'(0,0)}$.

(2)分析$\boldsymbol{F(x,y)}$在点$\boldsymbol{(0,0)}$处的可微性.

解：(1)$F_x'(0,0) = \lim\limits_{x\to0}\frac{F(x,0)-F(0,0)}{x} = \lim\limits_{x\to0}\frac{\sin(x^3)\sin x}{x^2} = 1$, ...①

$F_y'(0,0) = \lim\limits_{y\to0}\frac{F(0,y)-F(0,0)}{y} = \lim\limits_{y\to0}\frac{0}{y} = 0$, ...①

(2)$\Delta F(x,y)|_{(0,0)} = F(\Delta x,\Delta y) - F(0,0) = \frac{\sin((\Delta x)^2)\cdot\sin(\Delta x+\Delta y)}{(\Delta x)^2+(\Delta y)^2}$, ...①

考察$\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\Delta F(x,y)|_{(0,0)} - F_x'(0,0)\Delta x - F_y'(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$......②

$= \lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\sin((\Delta x)^2)\cdot\sin(\Delta x+\Delta y) - (\Delta x)^3 - \Delta x(\Delta y)^2}{[(\Delta x)^2+(\Delta y)^2]^{3/2}}$

沿$\Delta y=-\Delta x, \Delta x\to 0^-$，上面的极限为$2^{-1/2}$,

沿$\Delta y=\Delta x$，上面的极限为$0, \ldots\ldots$②

故极限不存在，从而不可微.......①

四、 (8分)设$\boldsymbol{f(x)=\int_x^1 e^{t^2}\mathrm{d}t}$.求曲线$\boldsymbol{y=f(x)}$与坐标轴围成的区域的面积.

解：因为$f(1)=0$，故$x=1$为函数$f(x)$的零点.......②

又$f'(x)=-e^{x^2}<0$ ，故$f(x)$单减且与x轴有唯一交点$(1,0)$.

因为$f(0)>0$，故$y=f(x)$与y交点在x轴的上方.......②

从而面积为$A=\int_0^1 \mathrm{d}x\int_x^1 e^{t^2}\mathrm{d}t$,......②

交换积分顺序，有$A=\int_0^1 \mathrm{d}t\int_0^t e^{t^2}\mathrm{d}x=\dfrac{e-1}{2}$.......②

五、 (8分)设$\boldsymbol{f(x)}$是$\boldsymbol{(-\infty,+\infty)}$上的连续函数，满足

$$\boldsymbol{\sin x+\int_0^x t^2\cdot f(x-t)\mathrm{d}t=\int_0^x f(t)\mathrm{d}t}.$$

求$\boldsymbol{f(x)}$的表达式.

解：令$u=x-t, \mathrm{d}u=-\mathrm{d}t$,从而原始变为

$\sin x+\int_0^x (x-u)^2 f(u)\mathrm{d}u=\int_0^x f(t)\mathrm{d}t$,......②

两边求导有

$\cos x+2x\int_0^x f(u)\mathrm{d}u-2\int_0^x uf(u)\mathrm{d}u-f(x)=0\Rightarrow f(0)=1$,......②

由上式可知$f(x)$可导，继续求导得

$-\sin x+2\int_0^x f(u)\mathrm{d}u-f'(x)=0\Rightarrow f'(0)=0$,

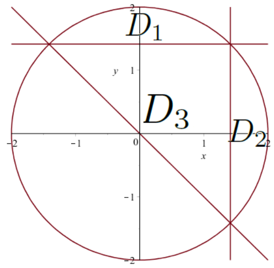
$f''(x)-2f(x)=-\cos x$,......②

解得$f(x)=\dfrac{1}{3}e^{\sqrt{2}x}+\dfrac{1}{3}e^{-\sqrt{2}x}+\dfrac{\cos x}{3}$......②

六、 (8分)设$\boldsymbol{f(x,y)=x^2\cos^2 y+x^5 y+y^2\sin^2 x}$，计算下面的二次积分：

$$\boldsymbol{\int_{-\sqrt{2}}^{\sqrt{2}}\mathrm{d}x\int_{\sqrt{2}}^{\sqrt{4-x^2}} f(x,y)\mathrm{d}y+\int_{\sqrt{2}}^{2}\mathrm{d}x\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\mathrm{d}y}.$$

证：积分区域$D=D_1\cup D_2$如图所示：



等式左边的积分$I=\iint\limits_D f(x,y)\mathrm{d}\sigma=\frac{1}{2}\iint\limits_D[f(x,y)+f(y,x)]\mathrm{d}\sigma$......②

$=\frac{1}{2}\iint\limits_D[x^2+x^5y+y^5x+y^2]\mathrm{d}\sigma=\frac{1}{2}\iint\limits_D[x^2+y^2]\mathrm{d}\sigma$,......②

$I=\frac{1}{2}\iint\limits_{D\cup D_3}[x^2+y^2]\mathrm{d}\sigma-\frac{1}{2}\iint\limits_{D_3}[x^2+y^2]\mathrm{d}\sigma$......②

$=\frac{1}{2}\int_{-\pi/4}^{3\pi/4}\mathrm{d}\theta\int_0^2 r^3\mathrm{d}r-\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\mathrm{d}x\int_{-x}^{\sqrt{2}}(x^2+y^2)\mathrm{d}y$......①

$=2\pi-\frac{8}{3}=\frac{6\pi-8}{3}$.......①

七、 (6分)设$\boldsymbol{f(x)\in C^1[0,1]}$，$\boldsymbol{f(0)=0}$，$\boldsymbol{f(1)=1}$.证明：存在$\boldsymbol{0\le\xi\le1}$，使得$\boldsymbol{f(\xi)+f'(\xi)=\frac{e}{e-1}}$.

证：$\int_0^1[e^xf(x)]'\mathrm{d}x=e$,......③

又由积分中值定理知，存在$0\le\xi\le1$，使得

$\int_0^1[e^xf(x)]'\mathrm{d}x=\int_0^1e^x[f(x)+f'(x)]\mathrm{d}x=[f(\xi)+f'(\xi)]\int_0^1e^x\mathrm{d}x$,......③

可知$[f(\xi)+f'(\xi)]\int_0^1e^x\mathrm{d}x=e\Rightarrow f(\xi)+f'(\xi)=\frac{e}{e-1}$.......②