一、简答题**(每题 5 分，共 20 分. 请简要写出计算过程)**

1. 设 A_i 表示第 i 各元件正常工作，则 $P(A_i) = 0.9$. 所求概率为

$$P(A_1\cup A_2\cup A_3\cup A_4)=1-P(\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4)=1-\prod_{i=1}^{4}P(\bar{A}_i)=1-(1-0.9)^4=0.9999$$

2. $E(X) = 100\times0.1 = 10$, $D(X) = 100\times0.1\times0.9 = 9$, $E(Y) = 2$, $D(Y) = 4$,

$\mathrm{Cov}(X, Y) = R(X, Y)\sqrt{D(X)}\sqrt{D(Y)} = 0.5\times3\times2 = 3$

$D(X-2Y+5) = D(X) + 4D(Y) - 4\mathrm{Cov}(X, Y) = 9 + 4\times4 - 4\times3 = 13$

3. $E(X) = 2$, $D(X) = 1$, $E(Y) = 1$, $D(Y) = 3$.
$E(X^2) = D(X) + [E(X)]^2 = 5$, $E(Y^2) = D(Y) + [E(Y)]^2 = 4$
$D(XY) = E(X^2Y^2) - [E(XY)]^2 = E(X^2)\,E(Y^2) - [E(X)]^2\,[E(Y)]^2 = 5\times4-4\times1 = 16$

4. $P(X < 0.5) = F(0.5) = 0.25$，所求概率为 $C_4^3\cdot\left(\dfrac{1}{4}\right)^3\cdot\dfrac{3}{4}=\dfrac{3}{64}$

二、解答题

1. (15 分) 设 A_i 表示“第 i 台探测器认为有目标”，$i = 1, 2$; B 表示“探测区域有目标”. 则

$$P(A_i\mid B) = 0.98,\quad P(A_i\mid\bar{B}) = 0.01,\ P(B) = 0.2$$

(1) $P(A_1\cup A_2|B) = 1-P(\bar{A}_1\bar{A}_2\mid B)=1-P(\bar{A}_1\mid B)P(\bar{A}_2\mid B)$

$= 1- [1- P(A_1\mid B)]\,[1- P(A_1\mid B)] = 1-(1-0.98)\,(1-0.98) = 0.9996$

(2) 记 $A = A_1\cup A_2$，则所求概率为 $P(\bar{B}\mid A)$.

$$P(A\mid\bar{B})=P(A_1\cup A_2\mid\bar{B})=1-P(\bar{A}_1\mid\bar{B})P(\bar{A}_2\mid\bar{B})$$

$$=1-[1-P(A_1\mid\bar{B})][1-P(A_2\mid\bar{B})]=1-(1-0.01)\,(1-0.01)=0.0199$$

$$P(\bar{B}\mid A)=\frac{P(A\bar{B})}{P(A)}=\frac{P(\bar{B})P(A\mid\bar{B})}{P(\bar{B})P(A\mid\bar{B})+P(B)P(A\mid B)}$$

$$=\frac{0.8\times0.0199}{0.8\times0.0199+0.2\times0.9996}=0.074$$

2. (10 分) X 的分布律为

X	-1	1	2
P	0.2	0.3	0.5

于是 $Y=\cos\dfrac{\pi X}{3}$ 的分布律为

Y	$-1/2$	$1/2$
P	1/2	1/2

所以 Y 的分布函数为 $F_Y(y)=\begin{cases} 0, & y<-\frac{1}{2} \\ 0.5, & -\frac{1}{2}\le y<\frac{1}{2} \\ 1, & y\ge\frac{1}{2} \end{cases}$

3. (15 分)

(1) $\lim\limits_{x\to 2^+}F(x)=F(2),\ \lim\limits_{x\to 2^+}F(x)=\lim\limits_{x\to 2^+}[a+b(x-2)^3]=a$，所以 $a=F(2)=0$;

$\lim\limits_{x\to 4^+}F(x)=F(4),\ \lim\limits_{x\to 4^+}F(x)=1, F(4)=8b$, 所以 $b=\dfrac{1}{8}$.

(2) $F_Y(y)=P\{Y\le y\}=\ P\{\sqrt{X}\le y\}$

当 $y<\sqrt{2}$ 时, $F_Y(y)=0$; 当 $y>2$ 时, $F_Y(y)=1$;

当 $\sqrt{2}<y\le 2$ 时, $F_Y(y)=P\{X\le y^2\}=F(y^2)=\ \dfrac{1}{8}(y^2-2)^3$.

于是 $f_Y(y)=F_Y^{'}(y)=\begin{cases} \dfrac{3}{4}y(y^2-2)^2, & \sqrt{2}<y\le 2 \\ 0, & \text{其他} \end{cases}$

4. (20 分)

(1) $f_X(x)=\begin{cases} xe^{-x}, & x>0 \\ 0, & x\le 0 \end{cases}$; 当 $x>0$ 时，$f_{Y|X}(y|x)=\begin{cases} \frac{1}{x}, & 0<y<x \\ 0, & \text{其他} \end{cases}$

所以 $f(x,y)=f_X(x)f_{Y|X}(y|x)=\begin{cases} e^{-x}, & 0<y<x \\ 0, & \text{其他} \end{cases}$

(2) $f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx$.

当 $y\le 0$ 时, $f_Y(y)=0$

当 $y>0$ 时，$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\int_{y}^{+\infty}e^{-x}dx=e^{-y}$

所以 $f_Y(y)=\begin{cases} e^{-y}, & y>0 \\ 0, & y\le 0 \end{cases}$

(3) 当 $y>0$ 时有

$f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(x)}=\begin{cases} e^{-x+y}, & x>y \\ 0, & x\le y \end{cases}$

(4) $f_{X|Y=1}(x|y=1)=\begin{cases} e^{-x+1}, & x>1 \\ 0, & x\le 1 \end{cases}$,

$P\{X>2|Y=1\}=\int_{2}^{+\infty}e^{-x+1}dx\ =\mathrm{e}^{-1}$.

5. (10 分) 区域 G 的面积为$\dfrac{\pi}{2}$, 故(X, Y)的联合密度函数为

$$f(x,y)=\begin{cases}\dfrac{2}{\pi}, & (x,y)\in G\\ 0, & \text{其他}\end{cases}$$

先求 R 的分布函数 $F_R(r) = P\{R \le r\}$:
当 $r < 0$ 时, $F_R(r) = 0$; 当 $r > 1$ 时, $F_R(r) = 1$;
当 $0 \le R \le 1$ 时:

$$F_R(r) = P\{R \le r\} = P\{\sqrt{X^2+Y^2} \le r\} = \iint_{\sqrt{x^2+y^2}\le r, y\ge 0} \frac{2}{\pi} dxdy = \frac{2}{\pi}\cdot\frac{\pi r^2}{2} = r^2$$

所以 R 的概率密度

$$f_R(r) = F_R^{'}(r) = \begin{cases}2r, & 0\le r\le 1\\ 0, & \text{其他}\end{cases}$$

6. (10 分) 记参数为λ的指数分布的分布函数为 $F(x)$, 则 $F(x)=\begin{cases}1-e^{-\lambda x}, & x>0\\ 0, & x\le 0\end{cases}$.

$$\begin{aligned}F_Z(z) &= P\{Z \le z\} = P\{\min\{X_1, X_2, \ldots, X_n\}\le z\} = 1 - P\{\min\{X_1, X_2, \ldots, X_n\} > z\}\\ &= 1 - P\{X_1 > z, X_2 > z, \ldots, X_n > z\} = 1 - P\{X_1 > z\}P\{X_2 > z\}\ldots P\{X_n > z\}\\ &= 1 - (1 - F(z))^n\\ &= \begin{cases}1-e^{-n\lambda z}, & z>0\\ 0, & z\le 0\end{cases}\end{aligned}$$

即 $Z \sim e(n\lambda)$, $E(Z)=\dfrac{1}{n\lambda}, D(Z)=\dfrac{1}{(n\lambda)^2}$