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The pulse treatment of computer viruses: a modeling study

Lu-Xing Yang · Xiaofan Yang

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Abstract Unlike new medical procedures, new antivirus software can be disseminated rapidly through the Internet and takes effect immediately after it is run. As a result, a considerable number of infected computers can be cured almost simultaneously. Consequently, it is of practical importance to understand how pulse treatment affects the spread of computer viruses. For this purpose, an impulsive malware propagation model is proposed. To the best of our knowledge, this is the first computer virus model that takes into account the effect of pulse treatment. The dynamic properties of this model are investigated comprehensively. Specifically, it is found that (a) the virus-free periodic solution is globally asymptotically stable when the basic reproduction ratio (BRR) is less than unity, (b) infections are permanent when the BRR exceeds unity, and (c) a locally asymptotically stable viral periodic solution bifurcates from the virus-free periodic solution when the BRR goes through unity. A close inspection of the influence of different model parameters on the BRR allows us to suggest some feasible measures of eradicating electronic infections.

L.-X. Yang
College of Computer Science, Chongqing University, Chongqing 400044, China
e-mail:ylx910920@gmail.com

X. Yang (✉)
College of Mathematics and Statistics, Chongqing University, Chongqing 400044, China
e-mail:xfyang1964@gmail.com

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1 Introduction

Computer viruses, ranging from email viruses to network worms, are loosely defined as malware that can spread over the Internet [1]. In the past few decades, computer viruses have caused enormous financial losses and, hence, have posed a grave threat to human society. Although antivirus software is adept to detecting and sweeping away malware staying in individual computers, it is incompetent to contain virus spreading on the Internet. Also, the ability of firewall to suppress infections is very limited, because of its inevitable false alarm rate or missing report rate.

For the purpose of effectively inhibiting virus spreading, it is vitally important to understand the way that malware propagate over the Internet. Due to the appealing analogy between electronic infections and biological infectious diseases, Cohen [2] and Murray [3] suggested to exploit the compartment modeling technique developed in the epidemiology for infectious diseases to study the laws governing the propagation of computer infections. Following this idea, Kephart and White [4] introduced the first epidemiological model for computer viruses. From then on, multifarious malware spreading models, ranging

from conventional models, such as the SIS models [4,5], SIR models [6,7], SIRS models [8–10], SEIR models [11,12], SEIRS models [13], SEIQRS models [14], SLBS models [15–24], SICS models [25,26], and some other models [27–30], to unconventional models such as the delayed models [31–38] and the stochastic models [15,39], have been proposed.

Vaccination is widely regarded as an effective means of controlling infectious diseases. Motivated by the success in applying pulse vaccination strategies to the control of poliomyelitis and measles throughout Central and South America [40,41], Agur et al. [42] initiated the theoretical study of pulse vaccination. From then on, infectious disease models with pulse vaccination have received considerable attention [43–49]. Recently, the idea of pulse vaccination has been applied to the vaccination of computer viruses; Yao et al. [50] and Zhang et al. [51] examined the impact of pulse quarantine and pulse vaccination on virus spreading, respectively, and Yang and Yang [15] proposed some interesting virus-spreading models with impulses.

Usually, there is a considerable interval from the time an efficacious treatment against a new infectious disease comes into service to the time a substantial number of patients get recovered due to receiving this treatment, because (a) there is no way to rapidly disseminate the new treatment to a large number of patients, and (b) it usually takes a long period of time to finish one or a few courses of the treatment. As a result, epidemic models of infectious diseases with pulse treatment are unrealistic and, hence, have never been studied. In contrast, as new antivirus software can be disseminated rapidly over the Internet and takes effect immediately after it is run, a considerable number of infected computers can be cured almost simultaneously. Consequently, it is of practical importance to understand how pulse treatment has influence on the spread of computer viruses. To the best of our knowledge, however, all previous models except the one suggested in Ref. [15] ignore this impulsive effect completely.

This paper attempts to reveal the impact of pulse treatment on computer infections. For this purpose, an impulsive computer virus model is proposed. To the best of our knowledge, this is the first computer virus model that takes into account the effect of pulse treatment. The dynamic properties of this model are investigated theoretically. Specifically, it is found that (a) the virus-free periodic solution is globally asymptotically stable when the basic reproduction ratio (BRR) is less than unity, (b) viruses are permanent when the BRR exceeds unity, and (c) a locally asymptotically stable viral periodic solution bifurcates from the virus-free periodic solution when the BRR goes through unity. A close look at the impact of different model parameters on the BRR allows us to suggest some feasible measures of eradicating electronic viruses.

The subsequent materials are organized in this fashion: the new model is elaborated and theoretically studied in Sects. 2 and 3, respectively; the impact of model parameters is addressed in Sect. 4; finally, Sect. 5 concludes the contributions of this work and points out some future directions of research.

2 Model description

For any real number x and any nonzero real number y, let $y \mid x$ denote that x is a multiple of y, and let $y \nmid x$ denote that x is not a multiple of y.

We shall call computers as *nodes*. A node on the Internet is called *internal*, while a node outside the Internet is called *external*. As usual, we shall neglect the nature of computer viruses as well as the details of infections, and simply assume that every node is in one of three possible states: *susceptible*, *infected*, and *recovered*. Susceptible nodes and recovered nodes are both uninfected, but the former are not installed with the newest antivirus software, whereas the latter are installed with the newest antivirus software. Let $S(t)$ [respectively, $I(t)$, $R(t)$] denote at time t the density of the susceptible (respectively, infected, recovered) internal nodes in all internal nodes.

For the modeling purpose, the following assumptions are imposed as usual.

(A1) External nodes enter the Internet at constant rate $\delta > 0$, and every internal node leaves the Internet with probability per unit time δ. Thus, the total number of internal nodes is unvaried.

(A2) All external nodes are susceptible.

(A3) Due to possible communications with infected internal nodes, at time t every susceptible internal node gets infected with probability per unit time $\beta I(t)$, where $\beta > 0$ is a constant.

(A4) Due to possible run of old antivirus software, every infected internal node becomes susceptible with constant probability per unit time $\gamma > 0$ or

gets recovered with constant probability per unit time $\alpha > 0$.

(A5) Due to possible loss of efficacy of the antivirus software, every recovered internal node becomes susceptible with constant probability per unit time $\eta > 0$.

Furthermore, the following additional assumptions are incorporated.

(A6) New antivirus software is released periodically at discrete times $t = nT$, $n \in \mathbb{N}$, where $T > 0$ is a constant.

(A7) Due to rapid dissemination of new antivirus software, a θ_1 (resp. θ_2) fraction of susceptible (resp. infected) internal nodes get recovered at time instant nT, where $\theta_1 > 0$ and $\theta_2 > 0$ are constants, $\theta_2 \geq \theta_1$, because the owner of an infected computer is more likely to choose to install new antivirus software than the owner of an uninfected computer.

Collecting the above assumptions, we get a new computer virus epidemic model, which can be formulated in terms of the impulsive differential system

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \delta - \beta S(t)I(t) + \gamma I(t) + \eta R(t) - \delta S(t), \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta S(t)I(t) - \gamma I(t) - \alpha I(t) - \delta I(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \alpha I(t) - \eta R(t) - \delta R(t), \end{aligned}\right\} & \text{if } T \nmid t, \\ \left.\begin{aligned} S(t^+) &= (1-\theta_1)S(t), \\ I(t^+) &= (1-\theta_2)I(t), \\ R(t^+) &= R(t) + \theta_1 S(t) + \theta_2 I(t), \end{aligned}\right\} & \text{if } T \mid t, \end{cases} \tag{1}$$

with initial condition $(S(0^+), I(0^+), R(0^+)) \in \{(S, I, R) \in \mathbb{R}^3_+ : S + I + R = 1\}$.

As $S(t) + I(t) + R(t) \equiv 1$, system (1) can be simplified as

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta\big(1 - I(t) - R(t)\big)I(t) - (\gamma + \alpha + \delta)I(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \alpha I(t) - (\eta + \delta)R(t), \end{aligned}\right\} & \text{if } T \nmid t, \\ \left.\begin{aligned} I(t^+) &= (1-\theta_2)I(t), \\ R(t^+) &= (1-\theta_1)R(t) + (\theta_2 - \theta_1)I(t) + \theta_1, \end{aligned}\right\} & \text{if } T \mid t, \end{cases} \tag{2}$$

with initial condition $(I(0^+), R(0^+)) \in \Omega = \{(I, R) \in \mathbb{R}^2_+ : I + R \leq 1\}$. It is easily verified that Ω is positively invariant for the system. The right-hand side of this system ensures the existence, uniqueness, and piecewise continuity of its solution [52].

A solution $(I(t), R(t))$ to system (2) is *virus-free* or *viral* depending on whether $I(t) \equiv 0$ or not. A solution $(I(t), R(t))$ to system (2) is *virus persistent* if there is $m > 0$ such that $I(t) \geq m$ for all large t. System (2) is *virus permanent* if there is $m > 0$ such that every solution $(I(t), R(t))$ to system (2) with $I(0^+) > 0$ is virus persistent. Similar notions apply to system (1).

3 Model analysis

The analysis of the new model consists of three parts: the existence and global stability of a virus-free periodic solution, the viral permanence, and the emergence of a viral periodic solution via bifurcation, which will be presented sequentially as follows.

3.1 Existence and global stability of a virus-free periodic solution

We begin by showing the existence of a virus-free T-periodic solution to system (2), i.e., a solution $(0, \tilde{R}(t))$ to system (2) such that $\tilde{R}(t)$ is a T-period solution to the following system.

$$\begin{cases} \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = -(\eta + \delta)R(t), & \text{if } T \nmid t, \\ R(t^+) = (1-\theta_1)R(t) + \theta_1, & \text{if } T \mid t. \end{cases} \tag{3}$$

Lemma 1 *System (3) has a globally asymptotically stable T-period solution*

$$\tilde{R}(t) = \frac{\theta_1 e^{-(\eta+\delta)(t-nT)}}{1 - (1-\theta_1)e^{-(\eta+\delta)T}}, \quad \text{if } nT < t \leq (n+1)T, n \in \mathbb{N} \cup \{0\}. \tag{4}$$

Proof Solving the first equation in system (3), we get

$$R(t) = R(nT^+)e^{-(\eta+\delta)(t-nT)}, \quad \text{if } nT < t \leq (n+1)T, n \in \mathbb{N} \cup \{0\}. \tag{5}$$

Let $R_n = R(nT^+)$. Then, the stroboscopic mapping F for system (3) is given by

$$R_{n+1} = F(R_n) = (1-\theta_1)e^{-(\eta+\delta)T}R_n + \theta_1. \tag{6}$$

Clearly, the mapping F has a unique (positive) fixed point $R_* = \frac{\theta_1}{1-(1-\theta_1)e^{-(\eta+\delta)T}}$, which implies that system (3) has $\tilde{R}(t)$ as its unique T-period solution. As

$$R_n - R_* = (1-\theta_1)e^{-(\eta+\delta)T}(R_{n-1} - R_*) = \ldots$$
$$= \left[(1-\theta_1)e^{-(\eta+\delta)T}\right]^n (R_0 - R_*),$$

R_* is globally asymptotically stable for Eq. (6). The global asymptotic stability of $\tilde{R}(t)$ follows naturally.

From this lemma, we immediately have the following results. □

Theorem 1 *System* (2) *has a unique virus-free* T*-period solution* $(0, \tilde{R}(t))$.

Corollary 1 *System* (1) *has a unique virus-free* T*-period solution* $(\tilde{S}(t), 0, \tilde{R}(t))$*, where* $\tilde{S}(t) = 1 - \tilde{R}(t)$.

Now, let us examine the global stability of the virus-free periodic solution to system (2). For this purpose, we define

$$\mathfrak{R}_o = \frac{\beta \int_0^T \tilde{S}(t)\mathrm{d}t}{(\gamma+\alpha+\delta)T + \ln\frac{1}{1-\theta_2}} = \frac{\beta T - \frac{\beta\theta_1}{\eta+\delta}\frac{1-e^{-(\eta+\delta)T}}{1-(1-\theta_1)e^{-(\eta+\delta)T}}}{(\gamma+\alpha+\delta)T + \ln\frac{1}{1-\theta_2}}. \tag{7}$$

Theorem 2 *The virus-free periodic solution* $(0, \tilde{R}(t))$ *to system* (2) *is locally asymptotically stable or unstable according to* $\mathfrak{R}_o < 1$ *or* $\mathfrak{R}_o > 1$.

Proof The linearized system of system (2) at $(0, \tilde{R}(t))$ is

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = (\beta\tilde{S}(t) - \gamma - \alpha - \delta)v(t), \\ \dfrac{\mathrm{d}w(t)}{\mathrm{d}t} = \alpha v(t) - (\eta+\delta)w(t), \end{array}\right\} \text{ if } T \nmid t,$$
$$\left\{\begin{array}{l} v(t^+) = (1-\theta_2)v(t), \\ w(t^+) = (\theta_2-\theta_1)v(t) + (1-\theta_1)w(t), \end{array}\right\} \text{ if } T \mid t. \tag{8}$$

Let

$$A(t) = \begin{pmatrix} \beta\tilde{S}(t) - \gamma - \alpha - \delta & 0 \\ \alpha & -(\eta+\delta) \end{pmatrix},$$
$$B = \begin{pmatrix} 1-\theta_2 & 0 \\ \theta_2-\theta_1 & 1-\theta_1 \end{pmatrix}.$$

Then,

$$\Phi(t) = e^{\int_0^t A(\mathrm{d}t} = \begin{pmatrix} e^{\beta\int_0^t \tilde{S}(t)\mathrm{d}t - (\gamma+\alpha+\delta)t} & 0 \\ * & e^{-(\eta+\delta)t} \end{pmatrix},$$

where the exact form of the $*$ entry is not necessary. The monodromy matrix M for system (8) is

$$M = B\Phi(T) = \begin{pmatrix} (1-\theta_2)e^{\beta\int_0^T \tilde{S}(t)\mathrm{d}t - (\gamma+\alpha+\delta)T} & 0 \\ * & (1-\theta_1)e^{-(\eta+\delta)T} \end{pmatrix}.$$

The eigenvalues of M are

$$\lambda_1 = (1-\theta_1)e^{-(\eta+\delta)T} < 1$$

and

$$\lambda_2 = (1-\theta_2)e^{\beta\int_0^T \tilde{S}(t)\mathrm{d}t - (\gamma+\alpha+\delta)T}.$$

It follows from the Floquet theory [53] that (a) $(0, \tilde{R}(t))$ is locally asymptotically stable if $\lambda_2 < 1$ or, equivalently, $\mathfrak{R}_o < 1$, and (b) $(0, \tilde{R}(t))$ is unstable if $\lambda_2 > 1$ or, equivalently, $\mathfrak{R}_o > 1$. The proof is complete. □

Remark 1 It can be seen from Theorem 2 that $\mathfrak{R}_o$ is exactly the basic reproduction ratio for system (2), i.e., the average number of previously uninfected nodes that are infected by a single infected node during its life cycle.

Theorem 3 *The virus-free periodic solution* $(0, \tilde{R}(t))$ *to system* (2) *is globally asymptotically stable if* $\mathfrak{R}_o < 1$.

Proof As $\mathfrak{R}_o < 1$ can be rewritten as

$$(1-\theta_2)e^{\beta\int_0^T \tilde{S}(t)\mathrm{d}t - (\gamma+\alpha+\delta)T} < 1,$$

we can choose a small $\varepsilon > 0$ such that

$$\sigma = (1-\theta_2)e^{\beta\int_0^T (\tilde{S}(t)+\varepsilon)\mathrm{d}t - (\gamma+\alpha+\delta)T} < 1.$$

Let $(I(t), R(t))$ be an arbitrary solution to system (2). In view of Theorem 2, it suffices to show that

$$\lim_{t\to+\infty} I(t) = 0, \qquad \lim_{t\to+\infty} R(t) = \tilde{R}(t).$$

From system (2), we have

$$\left\{\begin{array}{ll} \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} \geq -(\eta+\delta)R(t), & \text{if } T \nmid t, \\ R(t^+) \geq (1-\theta_1)R(t) + \theta_1, & \text{if } T \mid t. \end{array}\right.$$

Now, consider the comparison system

$$\left\{\begin{array}{ll} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -(\eta+\delta)x(t), & \text{if } T \nmid t, \\ x(t^+) = (1-\theta_1)x(t) + \theta_1, & \text{if } T \mid t, \end{array}\right.$$

with initial condition $x(0^+) = R(0^+)$. According to Lemma 1, this system has a globally asymptotically stable periodic solution $\tilde{x}(t) = \tilde{R}(t)$. It follows from

the comparison theorem for impulsive differential systems [52] that there is $T_1 > 0$ such that, for all $t \geq T_1$, we have

$$R(t) \geq x(t) > \tilde{R}(t) - \varepsilon. \tag{9}$$

Plugging this equation into system (2), we get that

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} \leq \left[\beta(\tilde{S}(t)+\varepsilon) - \gamma - \alpha - \delta\right] I(t), & \text{if } t \geq T_1, T \nmid t, \\ I(t^+) = (1-\theta_2) I(t), & \text{if } t \geq T_1, T \mid t. \end{cases}$$

Let $N_1 = \lceil \frac{T_1}{T} \rceil$ and consider the comparison system

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \left[\beta(\tilde{S}(t)+\varepsilon) - \gamma - \alpha - \delta\right] y, & \text{if } t \geq T_1, T \nmid t, \\ y(t^+) = (1-\theta_2) y(t), & \text{if } t \geq T_1, T \mid t, \end{cases}$$

with initial condition $y(N_1 T^+) = I(N_1 T^+)$. Then,

$$y(nT^+) = I(N_1 T^+)\sigma^{n-N_1} \quad \text{for } n \geq N_1,$$

which implies that $\lim_{n\to\infty} y(nT^+) = 0$. Furthermore, if $n \geq N_1$, $nT < t \leq (n+1)T$, then

$$\begin{aligned} y(t) &= y(nT^+) e^{\int_{nT}^{t} [\beta(\tilde{S}(t)+\varepsilon) - \gamma - \alpha - \delta - \frac{1}{T} \ln \frac{1}{1-\theta_2}] \mathrm{d}t} \\ &\leq I(N_1 T^+) e^{\beta(1+\varepsilon)T} \sigma^{n-N_1}, \end{aligned}$$

implying that $\lim_{t\to+\infty} y(t) = 0$. It follows from the comparison theorem [52] that

$$\lim_{t\to+\infty} I(t) = 0.$$

Thus, there is $T_2 > T_1$ such that $I(t) < \varepsilon$ for $t \geq T_2$. Plugging into system (2), we get

$$\begin{cases} \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} \leq \alpha\varepsilon - (\eta+\delta) R(t), & \text{if } t \geq T_2, T \nmid t, \\ R(t^+) \leq (1-\theta_1) R(t) + (\theta_2 - \theta_1)\varepsilon + \theta_1, & \text{if } t \geq T_2, T \mid t. \end{cases}$$

Let $N_2 = \lceil \frac{T_2}{T} \rceil$ and consider the comparison system

$$\begin{cases} \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = \alpha\varepsilon - (\eta+\delta) z, & \text{if } t \geq T_2, T \nmid t, \\ z(t^+) = (1-\theta_1) z(t) + (\theta_2 - \theta_1)\varepsilon + \theta_1, & \text{if } t \geq T_2, T \mid t, \end{cases}$$

with initial condition $z(N_2 T^+) = R(N_2 T^+)$. An argument analogous to that for Lemma 1 shows that this system has a globally asymptotically stable periodic solution

$$\tilde{z}(t) = \frac{\alpha\varepsilon}{\eta+\delta} + \left(z_* - \frac{\alpha\varepsilon}{\eta+\delta}\right) e^{-(\eta+\delta)(t-nT)}, \quad \text{if } nT < t \leq (n+1)T, n \in \mathbb{N} \cup \{0\},$$

where

$$z_* = \frac{(\theta_2 - \theta_1)\varepsilon + \theta_1 + \frac{\alpha(1-\theta_1)\varepsilon}{\eta+\delta}(1 - e^{-(\eta+\delta)T})}{1 - (1-\theta_1) e^{-(\eta+\delta)T}}.$$

According to the comparison theorem [52], there is $T_3 > T_2$ such that

$$R(t) \leq z(t) < \tilde{z}(t) + \varepsilon, \quad t \geq T_3. \tag{10}$$

Looking at Eqs. 9 and 10, considering the arbitrariness of ε, and noting that $\lim_{\varepsilon\to 0^+} \tilde{z}(t) = \tilde{R}(t)$, we get

$$\lim_{t\to+\infty} R(t) = \tilde{R}(t)$$

The proof is complete. □

As a direct consequence of Theorem 3, we have

Theorem 4 *The virus-free periodic solution* $(\tilde{S}(t), 0, \tilde{R}(t))$ *to system* (1) *is globally asymptotically stable if* $\mathfrak{R}_o < 1$.

This theorem has the following corollary.

Corollary 2 *The virus-free periodic solution* $(0, \tilde{R}(t))$ *to system* (2) *is globally asymptotically stable if* $\beta < \gamma + \alpha + \delta$.

Example 1 For system (1) with $\beta = 0.2$, $\alpha = 0.03$, $\gamma = 0.04$, $\eta = 0.2$, $\delta = 0.06$, $\theta_1 = 0.3$, $\theta_2 = 0.5$, and $T = 10$, we have $\mathfrak{R}_o = 0.8904 < 1$. So, Theorem 5 guarantees the global asymptotic stability of its virus-free periodic solution. Figure 1 displays the time plots of $S(t)$, $I(t)$, and $R(t)$ for the initial condition $(S(0), I(0), R(0)) = (0.5, 0.1, 0.4)$, from which it can be seen that the state of this system is approaching a virus-free periodic solution, in accordance with the theoretical prediction.

3.2 Viral permanence

Now, let us examine the condition under which computer viruses are permanent on the Internet.

Theorem 5 *System* (2) *is virus permanent if* $\mathfrak{R}_o > 1$.

Proof As $\mathfrak{R}_o > 1$, we can choose small $m_1 > 0$ and small $\epsilon_1 > 0$ such that

$$\begin{aligned} \Delta &= (1-\theta_2) \\ &\quad \times e^{[\beta(1-m_1-\epsilon_1-\frac{\alpha m_1}{\eta+\delta}) - (\gamma+\alpha+\delta)]T - \frac{\beta}{\eta+\delta}(u_* - \frac{\alpha m_1}{\eta+\delta})[1-e^{-(\eta+\delta)T}]} \\ &\quad > 1, \end{aligned} \tag{11}$$

where

$$u_* = \frac{(\theta_2 - \theta_1) m_1 + \theta_1 + \frac{\alpha(1-\theta_1) m_1}{\eta+\delta}\left[1 - e^{-(\eta+\delta)T}\right]}{1 - (1-\theta_1) e^{(\eta+\delta)T}}. \tag{12}$$

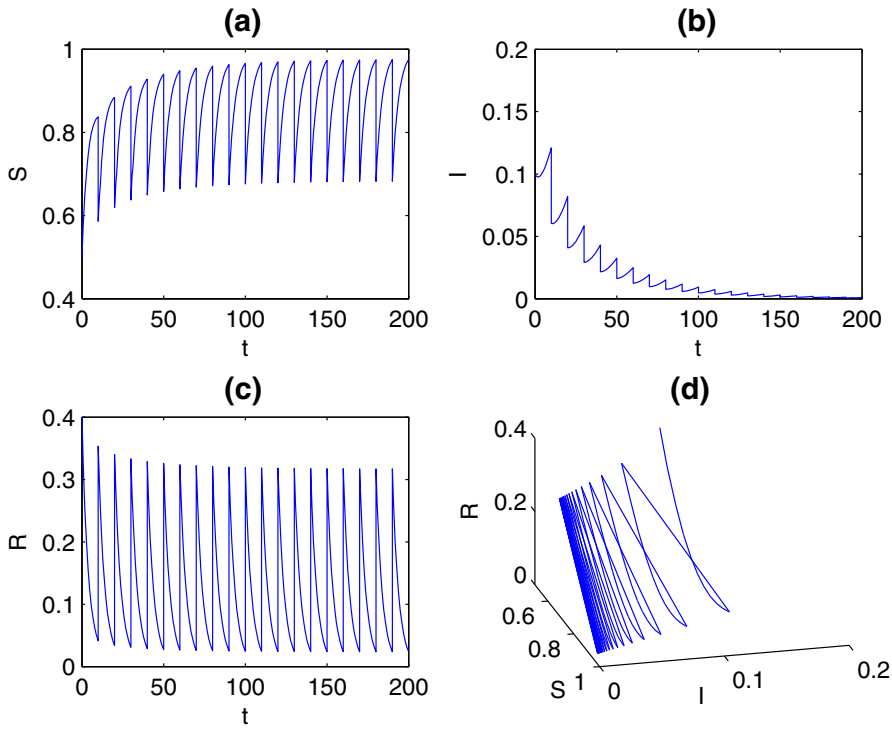

Fig. 1 Dynamic behavior of the system given in Example 1: **a** the time plot for the density of susceptible internal nodes, **b** the time plot for the density of infected internal nodes, **c** the time plot for the density of recovered internal nodes, and **d** the phase portrait

Let

$$n_2 = \left\lceil \frac{\ln \frac{1+u_*}{\epsilon_1}}{(\eta+\delta)T} \right\rceil + 1, \tag{13}$$

$$n_3 = \left\lceil \frac{(\beta m_1 + \gamma + \alpha + \delta) n_2 T + n_2 \ln \frac{1}{1-\theta_2}}{\ln \Delta} \right\rceil + 1, \tag{14}$$

$$m_2 = m_1 (1-\theta_2)^{n_2+n_3} e^{-(n_2+n_3)(\beta m_1+\gamma+\alpha+\delta)T}, \tag{15}$$

$$m_3 = m_1 (1-\theta_2)^{n_2+n_3+1} e^{-(n_2+n_3+1)(\beta m_1+\gamma+\alpha+\delta)T}. \tag{16}$$

Let $(I(t), R(t))$ be an arbitrary solution to system (2) with $I(0^+) > 0$.

Claim 1 *$I(t) \geq m_1$ for some $t > 0$.*

Proof of Claim 1 On the contrary, suppose that $I(t) < m_1$ for all $t > 0$. Then, we have

$$\begin{cases} \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} \leq \alpha m_1 - (\eta+\delta)R(t), & \text{if } T \nmid t, \\ R(t^+) \leq (1-\theta_1)R(t), & \text{if } T \mid t. \\ \qquad + (\theta_2-\theta_1)m_1 + \theta_1 \end{cases}$$

Consider the comparison system

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = \alpha m_1 - (\eta+\delta)u(t), & \text{if } T \nmid t, \\ u(t^+) = (1-\theta_1)u(t) + (\theta_2-\theta_1)m_1 & \text{if } T \mid t, \\ \qquad + \theta_1, \end{cases}$$

with initial condition $u(0^+) = R(0^+)$. As was shown previously, this system has a globally asymptotically stable periodic solution

$$\tilde{u}(t) = \frac{\alpha m_1}{\eta+\delta} + \left(u_* - \frac{\alpha m_1}{\eta+\delta}\right) e^{-(\eta+\delta)(t-nT)}$$
$$\text{for } nT < t \leq (n+1)T, n \in \mathbb{N} \cup \{0\}.$$

According to the comparison theorem [52], there is $\hat{T} > 0$ such that

$$R(t) \leq u(t) < \tilde{u}(t) + \epsilon_1, \qquad t \geq \hat{T}.$$

Plugging into system (2), we get

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} \geq \left[\beta(1-m_1-\tilde{u}(t)-\epsilon_1) \right. & \text{if } t \geq \hat{T}, T \nmid t, \\ \qquad \left. -\gamma-\alpha-\delta\right] I(t) & \\ I(t^+) = (1-\theta_2)I(t) & \text{if } t \geq \hat{T}, T \mid t. \end{cases}$$

Letting $\hat{N} = \lceil \frac{\hat{T}}{T} \rceil$, integrating the first equation in this system on $(nT, (n+1)T]$, $n \geq \hat{N}$, and substituting $I(nT^+) = (1-\theta_2)I(nT)$ into the resulting equation, we get

$$I((n+1)T) \geq (1-\theta_2)I(nT) \times e^{\int_{nT}^{(n+1)T}[\beta(1-m_1-\tilde{u}(t)-\epsilon_1)-\gamma-\alpha-\delta]\mathrm{d}t} = I(nT)\Delta.$$

Thus, $I(nT) \geq I(\hat{N}T)\Delta^{n-\hat{N}}$ for $n \geq \hat{N}$. Noting that $I(\hat{N}T) > 0$, we have $\lim_{n\to\infty} I(nT) = +\infty$. This, however, contradicts the fact that $I(t) \leq 1$. Hence, Claim 1 is proven. We may assume that $I(T_0) \geq m_1$.

If $I(t) \geq m_1$ for all $t \geq T_0$, the claimed result is already proven. Now, suppose $I(t) < m_1$ for some $t > T_0$. Let

$$T_1 = \inf_{t>T_0}\{t : I(t) < m_1\}.$$

At this point, there are two possibilities.

Case 1. $T \mid T_1$. Then, $I(t) \geq m_1$ for $t \in [T_0, T_1]$, and $I(T_1^+) = (1-\theta_2)I(T_1) < m_1$. Let $T_1 = n_1 T$, and let

$$T_2 = (n_1+n_2)T, \qquad T_3 = (n_1+n_2+n_3)T.$$

Claim 2 *$I(t) \geq m_1$ for some $t \in (T_1, T_3]$.*

Proof of Claim 2 On the contrary, suppose $I(t) < m_1$ for $T_1 < t \leq T_3$. Then, we have

$$\begin{cases} \frac{\mathrm{d}R(t)}{\mathrm{d}t} \leq \alpha m_1 - (\eta+\delta)R(t), & \text{if } T_1 < t < T_3,\ T \nmid t, \\ R(t^+) \leq (1-\theta_1)R(t) + (\theta_2-\theta_1)m_1 + \theta_1, & \text{if } T_1 < t \leq T_3,\ T \mid t. \end{cases}$$

Consider the comparison system

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = \alpha m_1 - (\eta+\delta)u(t), & \text{if } T_1 < t < T_3,\ T \nmid t, \\ u(t^+) = (1-\theta_1)u(t) + (\theta_2-\theta_1)m_1 + \theta_1, & \text{if } T_1 < t \leq T_3,\ T \mid t, \end{cases}$$

with initial condition $u(T_1^+) = R(T_1^+)$. Solving the system, we get

$$u(t) = \tilde{u}(t) + \left[R(T_1^+) - u_*\right] e^{-(\eta+\delta)(t-T_1)}, \quad T_1 < t \leq T_3.$$

So,

$$|u(t) - \tilde{u}(t)| < (1+u_*)e^{-(\eta+\delta)(t-T_1)} < \epsilon_1, \quad T_1 < t \leq T_3.$$

By the comparison theorem [52] and the definitions of n_2 and n_3, we get

$$R(t) \leq u(t) < \tilde{u}(t) + \epsilon_1, \quad T_2 \leq t \leq T_3.$$

Thus, we have

$$\begin{cases} \frac{\mathrm{d}I(t)}{\mathrm{d}t} \geq \left[\beta(1-m_1-\tilde{u}(t)-\epsilon_1)-\gamma-\alpha-\delta\right] I(t), & \text{if } T_2 < t < T_3,\ T \nmid t, \\ I(t^+) = (1-\theta_2)I(t), & \text{if } T_2 \leq t \leq T_3,\ T \mid t. \end{cases}$$

Repeatedly integrating the first equation in this system and substituting the second equation into the resulting equation, we get

$$I(T_3) \geq I(T_2)\Delta^{n_3}. \tag{17}$$

In addition, we have

$$\begin{cases} \frac{\mathrm{d}I(t)}{\mathrm{d}t} \geq -(\beta m_1 + \gamma + \alpha + \delta)I, & \text{if } T_1 < t < T_2,\ T \nmid t, \\ I(t^+) = (1-\theta_2)I(t), & \text{if } T_1 \leq t \leq T_2,\ T \mid t. \end{cases}$$

Repeatedly integrating the first equation in this system and substituting the second equation into the resulting equation, and noting that $I(T_1) \geq m_1$, we get

$$I(T_2) \geq m_1(1-\theta_2)^{n_2} e^{-(\beta m_1+\gamma+\alpha+\delta)n_2 T}. \tag{18}$$

Combining Eqs. 17 and 18, we get

$$I(T_3) \geq m_1(1-\theta_2)^{n_2} e^{-(\beta m_1+\gamma+\alpha+\delta)n_2 T}\Delta^{n_3} > m_1.$$

A contradiction occurs. Hence, Claim 2 is proven.

Let

$$\bar{T} = \inf_{t>T_1}\{t : I(t) \geq m_1\}.$$

Then, $I(\bar{T}) = m_1$, and $I(t) < m_1$ for $t \in (T_1, \bar{T})$.

Claim 3 *$I(t) \geq m_2$ for all $t \in (T_1, \bar{T}]$.*

Proof of Claim 3 For any $t \in (T_1, \bar{T})$, we may assume that $t \in (kT, (k+1)T]$, $n_1 \leq k < n_1+n_2+n_3$. Then, we have

$$\begin{cases} \frac{\mathrm{d}I(t)}{\mathrm{d}t} \geq -(\beta m_1 + \gamma + \alpha + \delta)I, & \text{if } T_1 < t < \bar{T},\ T \nmid t, \\ I(t^+) = (1-\theta_2)I(t), & \text{if } T_1 < t < \bar{T},\ T \mid t. \end{cases}$$

Repeatedly integrating the first equation in this system and substituting the second equation into the resulting equation, we get that, for all $T_1 < t < \bar{T}$,

$$\begin{aligned} I(t) &\geq I(T_1^+)(1-\theta_2)^{k-n_1} e^{-(k-n_1)(\beta m_1+\gamma+\alpha+\delta)T} \\ &\quad \times e^{-(\beta m_1+\gamma+\alpha+\delta)(t-kT)} \\ &\geq m_1(1-\theta_2)^{k-n_1} e^{-(k+1-n_1)(\beta m_1+\gamma+\alpha+\delta)T} \\ &\geq m_1(1-\theta_2)^{n_2+n_3} e^{-(n_2+n_3)(\beta m_1+\gamma+\alpha+\delta)T} = m_2. \end{aligned}$$

Hence, Claim 3 is proven.

By alternatively repeating the arguments similar to those for Claims 2 and 3, we get

Claim 4 *$I(t) \geq m_2$ for all $t > T_1$.*

Case 2. $T \nmid T_1$. Then, $I(t) \geq m_1$ for $t \in [T_0, T_1]$, and $I(T_1^+) = m_1$. Now, assume $T_1 \in (n_1 T, (n_1+1)T)$. We proceed by treating two subcases, respectively.
Case 2.1. $I(t) < m_1$ for all $t \in (T_1, (n_1+1)T]$. By an argument analogous to that for Claim 2, we get

Claim 5 *$I(t) \geq m_1$ for some $t \in ((n_1+1)T, (n_1+n_2+n_3+1)T]$.*

Let

$$\hat{T} = \inf_{t>(n_1+1)T} \{t : I(t) \geq m_1\}.$$

Then, $I(t) < m_1$ for $t \in ((n_1+1)T, \hat{T})$, and $I(\hat{T}) = I(\hat{T}^+) = m_1$. By an argument analogous to that for Claim 3, we get

Claim 6 *$I(t) \geq m_3$ for all $t \in (T_1, \hat{T})$.*

By alternatively repeating the arguments similar to those for Claims 5 and 6, we get

Claim 7 *$I(t) \geq m_3$ for all $t > T_1$.*

Case 2.2. $I(t) \geq m_1$ for some $t \in (T_1, (n_1+1)T]$. Let

$$\check{T} = \inf_{t>T_1} \{t : I(t) \geq m_1\}.$$

Then, $I(t) < m_1$ for $t \in (T_1, \check{T})$, and $I(\check{T}) = I(\check{T}^+) = m_1$. By an argument analogous to that for Claim 3, we get

Claim 8 *$I(t) \geq m_3$ for all $t \in (T_1, \check{T})$.*

Again, by alternatively repeating the arguments similar to those for Claims 5 and 6, we get

Claim 9 *$I(t) \geq m_3$ for all $t > T_1$.*

Combining the above discussions, we get that $I(t) \geq \min\{m_2, m_3\}$ for all $t > T_1$. The proof is complete.

As a direct consequence of Theorem 5, we have

Theorem 6 *System (1) is virus permanent if $\mathfrak{R}_o > 1$.*

This theorem has the following corollary.

Corollary 3 *System (1) is virus permanent if $(\beta-\gamma-\alpha-\delta)T > \frac{\beta\theta_1}{\eta+\delta} + \ln\frac{1}{1-\theta_2}$.*

Example 2 Consider system (1) with $\beta = 0.2$, $\alpha = 0.03$, $\gamma = 0.04$, $\eta = 0.2$, $\delta = 0.06$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, and $T = 10$. Then, $\mathfrak{R}_o = 1.1158 > 1$. It follows from Theorem 6 that computer virus cannot be eradicated. Figure 2 displays the time plots of $S(t)$, $L(t)$, and $R(t)$ for the system with initial condition $L(0) = 0.1$, $R(0) = 0.4$, fitting the theoretical prediction.

Example 3 Consider system (1) with parameters $\beta = 0.2$, $\alpha = 0.003$, $\gamma = 0.004$, $\eta = 0.2$, $\delta = 0.06$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, and $T = 15$. Then, $\mathfrak{R}_o = 2.3802 > 1$. By Theorem 6, system (1) is virus permanent. Figure 3 displays the time plots of $S(t)$, $L(t)$, and $R(t)$ for the system with initial condition $L(0) = 0.1$, $R(0) = 0.4$, again in consistent with the theoretical prediction.

3.3 Supercritical bifurcation and the appearance of viral periodic solution

Now, let us study the existence and local stability of a viral periodic solution to system (2). For this purpose, we present a fundamental theorem in impulsive bifurcation theory [54] as follows.

Consider the impulsive differential system

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = f_1(x_1(t), x_2(t)), & \text{if } T \nmid t, \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = f_2(x_1(t), x_2(t)), & \text{if } T \nmid t, \\ x_1(t^+) = \theta_1(x_1(t), x_2(t)), & \text{if } T \mid t, \\ x_2(t^+) = \theta_2(x_1(t), x_2(t)), & \text{if } T \mid t. \end{cases} \tag{19}$$

where f_1, f_2, θ_1, and θ_2 are sufficiently smooth, and there is a trivial T-period solution $\zeta(t) = (x_p(t), 0)^T$. Let $\Phi = (\phi_1, \phi_2)^T$ denote the flow associated with this system. Let $\mathbf{x}(t) = (x_1(t), x_2(t))^T$, $\mathbf{x}_0 = \mathbf{x}(0)$.

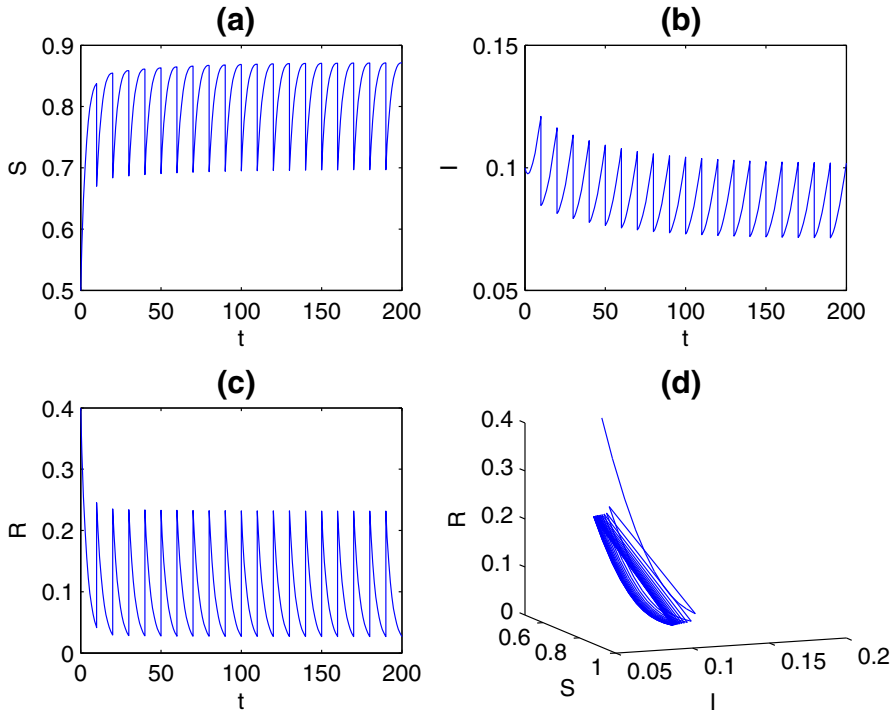

Fig. 2 Dynamic behavior of the system given in Example 2: **a** the time plot for the density of susceptible internal nodes, **b** the time plot for the density of infected internal nodes, **c** the time plot for the density of recovered internal nodes, and **d** the phase portrait

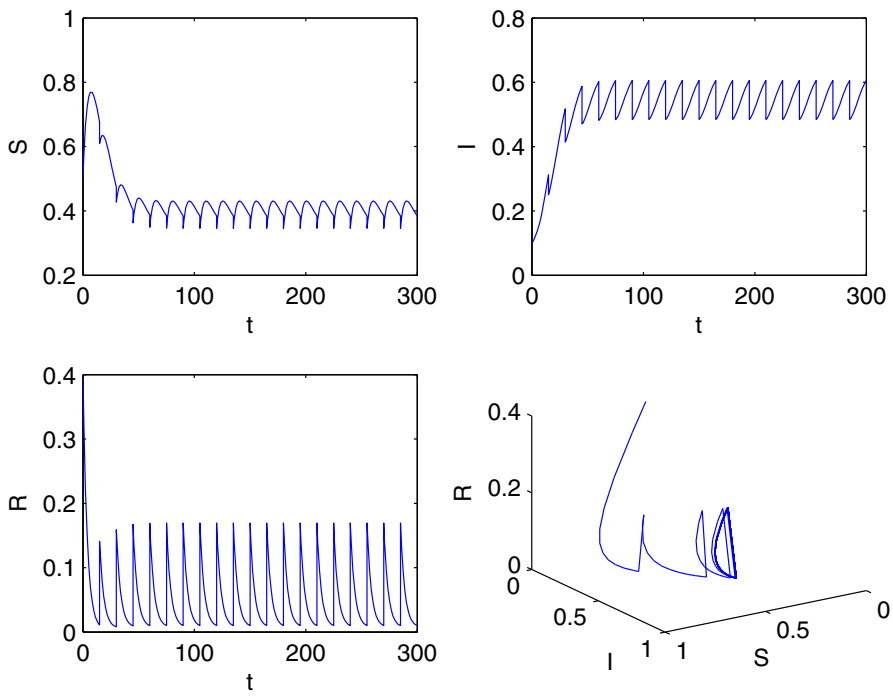

Fig. 3 Dynamical behavior of the system given in Example 3: **a** the time plot for the density of susceptible internal nodes, **b** the time plot for the density of infected internal nodes, **c** the time plot for the density of recovered internal nodes, and **d** the phase portrait

Then, $\mathbf{x}(t) = \Phi(t, \mathbf{x}_0)$, $0 < t \le T$. On introducing the following notations [54], we get

$$d_0 = 1 - \left(\frac{\partial\theta_2}{\partial x_2}\frac{\partial\phi_2}{\partial x_2}\right)_{(T,\mathbf{x}_0)}, \quad a_0 = 1 - \left(\frac{\partial\theta_1}{\partial x_1}\frac{\partial\phi_1}{\partial x_1}\right)_{(T,\mathbf{x}_0)},$$

$$b_0 = -\left(\frac{\partial\theta_1}{\partial x_1}\frac{\partial\phi_1}{\partial x_2} + \frac{\partial\theta_1}{\partial x_2}\frac{\partial\phi_2}{\partial x_2}\right)_{(T,\mathbf{x}_0)},$$

$$\begin{aligned} B = &-\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\left(\frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial t} + \frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial x_1}\frac{1}{a_0}\right.\\ &\left.\times\frac{\partial\theta_1}{\partial x_1}\frac{\phi_1(T,\mathbf{x}_0)}{\partial t}\right)\frac{\partial\phi_2(T,\mathbf{x}_0)}{\partial x_2} - \frac{\partial\theta_2}{\partial x_2}\\ &\times\left(\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial t\partial x_2} + \frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_1\partial x_2}\frac{1}{a_0}\right.\\ &\left.\times\frac{\partial\theta_1}{\partial x_1}\frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial t}\right), \end{aligned}$$

$$\begin{aligned} C = &-2\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\left(-\frac{b_0}{a_0}\frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial x_1} + \frac{\phi_1(T,\mathbf{x}_0)}{\partial x_2}\right)\\ &\times\frac{\phi_2(T,\mathbf{x}_0)}{\partial x_2} - \frac{\partial^2\theta_2}{\partial x_2^2}\left(\frac{\partial\phi_2(T,\mathbf{x}_0)}{\partial x_2}\right)^2\\ &+2\frac{\partial\theta_2}{\partial x_2}\frac{b_0}{a_0}\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_1\partial x_2} - \frac{\partial\theta_2}{\partial x_2}\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_2^2}, \end{aligned}$$

where

$$\frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial t} = \frac{\mathrm{d}x_p(T)}{\mathrm{d}t}, \quad \frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial x_1} = e^{\int_0^T \frac{\partial f_1(\zeta(r))}{\partial x_1}\mathrm{d}r},$$

$$\frac{\partial\phi_2(T,\mathbf{x}_0)}{\partial x_2} = e^{\int_0^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r},$$

$$\begin{aligned} \frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial x_2} &= \int_0^T e^{\int_u^T \frac{\partial f_1(\zeta(r))}{\partial x_1}\mathrm{d}r}\frac{\partial f_1(\zeta(u))}{\partial x_2}\\ &\quad\times e^{\int_0^u \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}\mathrm{d}u,\\ \frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_1\partial x_2} &= \int_0^T e^{\int_u^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}\frac{\partial^2 f_2(\zeta(u))}{\partial x_1\partial x_2}\\ &\quad\times e^{\int_0^u \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}\mathrm{d}u,\\ \frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_2^2} &= \int_0^T e^{\int_u^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}\frac{\partial^2 f_2(\zeta(u))}{\partial x_2^2}\\ &\quad\times e^{\int_0^u \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}\mathrm{d}u\\ &\quad+\int_0^T e^{\int_u^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}\frac{\partial^2 f_2(\zeta(u))}{\partial x_1\partial x_2}\\ &\quad\times\int_0^u\left(e^{\int_p^u \frac{\partial f_1(\zeta(r))}{\partial x_1}\mathrm{d}r}\frac{\partial f_1(\zeta(p))}{\partial x_2}\right.\\ &\quad\left.\times\ e^{\int_0^p \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}\mathrm{d}p\right)\mathrm{d}u, \end{aligned}$$

$$\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial t\partial x_2} = \frac{\partial f_2(\zeta(T))}{\partial x_2}e^{\int_0^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r}.$$

Theorem 7 [54] *Consider system* (19) *with* $0 < a_0 < 2$.

(a) *If* $BC < 0$, *a supercritical bifurcation occurs at* $d_0 = 0$.
(b) *If* $BC > 0$, *a subcritical bifurcation occurs at* $d_0 = 0$.
(c) *If* $BC = 0$, *there is an undetermined case.*

Theorem 8 *For system* (2), *a supercritical bifurcation occurs at* $\mathfrak{R}_o = 1$. *Specifically, a locally asymptotically stable viral periodic solution bifurcates from the virus-free periodic solution when* $\mathfrak{R}_o$ *increasingly goes across unity.*

Proof Let $x_1(t) = R(t), x_2(t) = I(t)$,

$$\begin{aligned} f_1(x_1(t), x_2(t)) &= \alpha x_2(t) - (\eta+\delta)x_1(t),\\ f_2(x_1(t), x_2(t)) &= \beta(1 - x_1(t) - x_2(t))x_2(t)\\ &\quad - (\gamma+\alpha+\delta)x_2(t),\\ \theta_1(x_1(t), x_2(t)) &= (1-\theta_1)x_1(t) + (\theta_2-\theta_1)x_2(t) + \theta_1,\\ \theta_2(x_1(t), x_2(t)) &= (1-\theta_2)x_2(t). \end{aligned}$$

Direct calculations yield

$$\begin{aligned} d_0 &= 1 - (1-\theta_2)e^{\int_0^T[\beta\tilde{S}(r)-\gamma-\alpha-\delta]\mathrm{d}r},\\ a_0 &= 1 - (1-\theta_1)e^{-(\eta+\delta)T},\\ b_0 &= -(1-\theta_1)\alpha\int_0^T e^{(\eta+\delta)(u-T)+\int_0^u[\beta\tilde{S}(t)-(\gamma+\alpha+\delta)]\mathrm{d}t}\mathrm{d}u\\ &\quad-(\theta_2-\theta_1)e^{\int_0^T[\beta\tilde{S}(t)-(\gamma+\alpha+\delta)]\mathrm{d}t} < 0,\\ \frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial t} &= \frac{-\theta_1(\eta+\delta)e^{-(\eta+\delta)T}}{1-(1-\theta_1)e^{-(\eta+\delta)T}} < 0,\\ \frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial x_1} &= e^{-(\eta+\delta)T},\\ \frac{\partial\phi_2(T,\mathbf{x}_0)}{\partial x_2} &= e^{\int_0^T[\beta\tilde{S}(t)-(\gamma+\alpha+\delta)]\mathrm{d}t},\\ \frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial x_2} &= \alpha\int_0^T e^{(\eta+\delta)(u-T)+\int_0^u[\beta\tilde{S}(t)-(\gamma+\alpha+\delta)]\mathrm{d}t}\\ &\quad\times\mathrm{d}u > 0, \end{aligned}$$

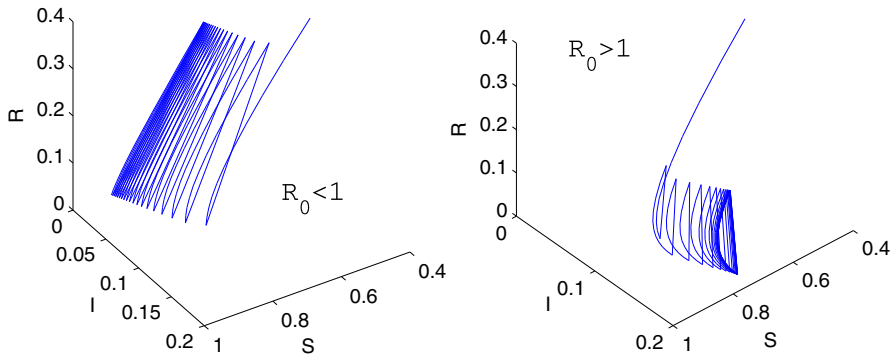

Fig. 4 Bifurcation diagram for system (1) with $\Re_o$ as the bifurcation parameter

$$\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_1\partial x_2} = -\beta T e^{\int_0^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r} < 0,$$

$$\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_2^2} = -2\beta T e^{\int_0^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r} - \alpha\beta\int_0^T e^{\int_u^T \frac{\partial f_2(\zeta(r))}{\partial x_2}\mathrm{d}r} \times\left(\int_0^u e^{\int_p^u \frac{\partial f_1(\zeta(r))}{\partial x_1}\mathrm{d}r+\int_0^p \frac{\partial f_1(\zeta(r))}{\partial x_2}\mathrm{d}r}\mathrm{d}p\right)\mathrm{d}u < 0,$$

$$\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial t\partial x_2} = (\beta\tilde{S}(T)-\gamma-\alpha-\delta) \times e^{\int_0^T(\beta\tilde{S}(t)-\gamma-\alpha-\delta)\mathrm{d}r},$$

$$B = -(1-\theta_2)\left(\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_1\partial x_2}\frac{1}{a_0}\frac{\partial\theta_1}{\partial x_1}\frac{\partial\phi_1(T,\mathbf{x}_0)}{\partial t} + \frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial t\partial x_2}\right),$$

$$C = 2(1-\theta_1)\frac{b_0}{a_0}\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_1\partial x_2} -(1-\theta_2)\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial x_2^2} > 0.$$

Clearly, $|1-a_0|<1$, and $\Re_o = 1$ implies $d_0 = 0$. As $\frac{\mathrm{d}\tilde{S}(t)}{\mathrm{d}t} > 0$ and

$$\int_0^T [\beta\tilde{S}(t)-\gamma-\alpha-\delta]\mathrm{d}t = \ln\frac{1}{1-\theta_2} > 0,$$

we have $\beta\tilde{S}(T)-\gamma-\alpha-\delta > 0$, implying that $\frac{\partial^2\phi_2(T,\mathbf{x}_0)}{\partial t\partial x_2} > 0$. So, $B<0$ and $BC<0$. Hence, the claimed result follows from Theorem 7.

Examples 2 and 3 illustrate the existence and local stability of a viral periodic solution for system (1) with $\Re_o > 1$. Figure 4 presents the bifurcation diagram for system (1) with $\Re_o$ as the bifurcation parameter.

4 Further discussions

The previous discussions suggest that, to eradicate electronic viruses, practicable measures should be taken so that the basic reproduction ratio $\Re_o$ is well below unity. This section examines the impact of different model parameters on $\Re_o$.

Theorem 9 *$\Re_o$ drops with the increasing θ_1, θ_2, α, and γ, and $\Re_o$ rises with the increasing β, η, and T.*

Proof First, it can be observed from Eq. (7) that $\Re_o$ drops with the increasing θ_2, α, and γ, and that $\Re_o$ rises with the increasing β.

Second, straightforward calculations show that

$$\frac{\partial\Re_o}{\partial\theta_1} = -\frac{\beta}{\eta+\delta}\frac{(1-e^{-(\eta+\delta)T})^2}{\left((\gamma+\alpha+\delta)T+\ln\frac{1}{1-\theta_2}\right)\left(1-(1-\theta_1)e^{-(\eta+\delta)T}\right)^2} < 0.$$

Next,

$$\frac{\partial\Re_o}{\partial\eta} = \frac{\beta\theta_1\left((1-e^{-(\eta+\delta)T})(1-(1-\theta_1)e^{-(\eta+\delta)T})-\theta_1(\eta+\delta)Te^{-(\eta+\delta)T}\right)}{(\eta+\delta)^2\left((\gamma+\delta+\alpha)T+\ln\frac{1}{1-\theta_2}\right)\left(1-(1-\theta_1)e^{-(\eta+\delta)T}\right)^2}$$

It suffices to show that

$$\left(1-e^{-(\eta+\delta)T}\right)\left(1-(1-\theta_1)e^{-(\eta+\delta)T}\right)-\theta_1(\eta+\delta)T \times e^{-(\eta+\delta)T} > 0.$$

For this purpose, define an auxiliary function as follow.

$$g(x) = (1-e^{-x})\left(1-(1-\theta_1)e^{-x}\right)-\theta_1 x e^{-x}, \quad x>0.$$

As

$$g'(x) = 2 - 2\theta_1 + \theta_1 x + (\theta_1 - 1)e^{-x} \quad \text{and}$$
$$g''(x) = \theta_1 + (1 - \theta_1)e^{-x} > 0,$$

it follows from $g'(0) = 1 - \theta_1 > 0$ that $g'(x) > 0$. Thus, $g(x) > g(0) = 0$.

Finally, noting that $1 - e^{-x} - xe^{-x} > 0$ for $x > 0$, we have

$$\frac{\partial \mathfrak{R}_o}{\partial T} = \frac{\beta \ln \frac{1}{1-\theta_2} \frac{(1-e^{-(\eta+\delta)T})(1-(1-\theta_1)^2 e^{-(\eta+\delta)T})}{(1-(1-\theta_1)e^{-(\eta+\delta)T})^2} + \frac{\beta\theta_1(\gamma+\alpha+\delta)}{1-(1-\theta_1)e^{-(\eta+\delta)T}} \frac{(1-e^{-(\eta+\delta)T})^2 + \theta_1 e^{-(\eta+\delta)T}(1-(\eta+\delta)T - e^{-(\eta+\delta)T})}{(\eta+\delta)(1-(1-\theta_1)e^{-(\eta+\delta)T})}}{\left((\gamma+\alpha+\delta)T + \ln \frac{1}{1-\theta_2}\right)^2}$$
$$> \frac{\beta \ln \frac{1}{1-\theta_2} \frac{(1-e^{-(\eta+\delta)T})(1-(1-\theta_1)^2 e^{-(\eta+\delta)T})}{(1-(1-\theta_1)e^{-(\eta+\delta)T})^2} + \frac{\beta\theta_1(\gamma+\alpha+\delta)}{1-(1-\theta_1)e^{-(\eta+\delta)T}} \frac{1-e^{-(\eta+\delta)T} - (\eta+\delta)e^{-(\eta+\delta)T}}{(\eta+\delta)(1-(1-\theta_1)e^{-(\eta+\delta)T})}}{\left((\gamma+\alpha+\delta)T + \ln \frac{1}{1-\theta_2}\right)^2} > 0.$$

The proof is complete.

Lemma 2 *Let*

$$h(T) = \mathfrak{R}_o - 1 = \frac{\beta T - \frac{\beta\theta_1}{\eta+\delta} \frac{1-e^{-(\eta+\delta)T}}{1-(1-\theta_1)e^{-(\eta+\delta)T}}}{(\gamma+\alpha+\delta)T + \ln \frac{1}{1-\theta_2}} - 1.$$

Then, $h(T)$ has a unique positive zero, denoted as T_*.

Proof By Theorem 9, $h(T)$ is strictly increasing. As $h(0) = -1$ and $\lim_{T\to+\infty} h(T) = \frac{\beta}{\gamma+\delta+\alpha} > 0$, it follows that $h(T)$ has a unique positive zero.

This lemma plus Theorem 8 implies

Theorem 10 *For system (1), a supercritical bifurcation occurs at $T = T_*$. Specifically, a locally asymptotically stable viral periodic solution bifurcates from the virus-free periodic solution when T increasingly goes across T_*.*

Example 4 Consider system (1) with $\beta = 0.2$, $\alpha = 0.03$, $\gamma = 0.04$, $\eta = 0.2$, $\delta = 0.06$, $\theta_1 = 0.2$, and $\theta_2 = 0.3$. Then, $T_* = 7.215$. For each $T \in [5, 10]$, Fig. 5 demonstrates the final left limits of $I(t)$ and $R(t)$ at impulsive points, from which it can be seen that a viral periodic solution bifurcates from the virus-free periodic solution when T goes across T_*, in accordance with the theoretical prediction.

The following conclusions can be drawn from the above discussions.

(1) Shortening the period T of pulse treatment is conducive to the inhibition of virus prevalence (Fig. 6). So, it is highly rewarding for antivirus software manufacturers to accelerate the development cycle of new antivirus software at a higher cost.
(2) Raising the pulse vaccination rate θ_1 and the pulse cure rate θ_2 contributes to the containment of virus propagation (Fig. 7). As thus, a major concern for antivirus software developers is how to detect/eliminate the currently popular viruses accurately.
(3) Enhancing the two continuous cure probabilities, α and γ, always conduces to the eradication of viruses (Fig. 8). Hence, computer users should run current antivirus software termly so as to purge viruses timely.
(4) Reducing the efficacy-losing probability η of antivirus software helps suppress virus spreading (Fig. 9). In practice, antivirus software developers should pay considerable attention to the ability of their products to detect/clear future viruses.

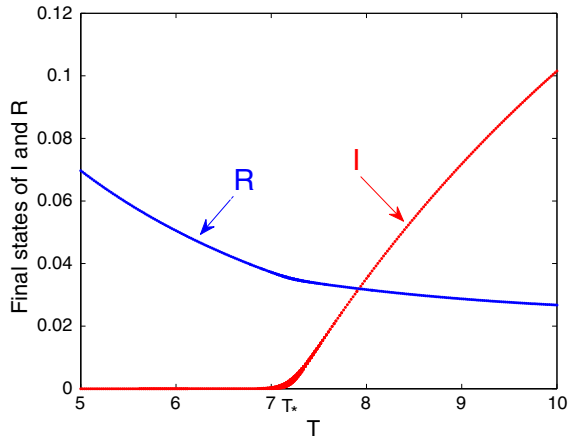

Fig. 5 The final left limits of I and R at impulsive points for the system given in Example 4

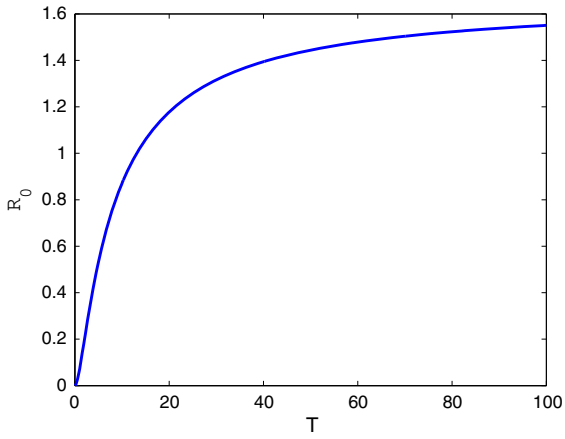

Fig. 6 R_0 versus T in the case that $\beta = 0.2, \alpha = 0.05, \gamma = 0.03, \eta = 0.06, \delta = 0.04, \theta_1 = 0.3$, and $\theta_2 = 0.4$

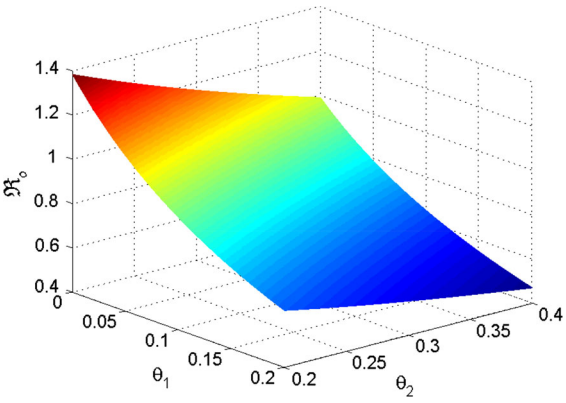

Fig. 7 R_0 versus θ_1 and θ_2 in the case that $\beta = 0.2, \alpha = 0.05, \gamma = 0.03, \eta = 0.02, \delta = 0.02$, and $T = 5$

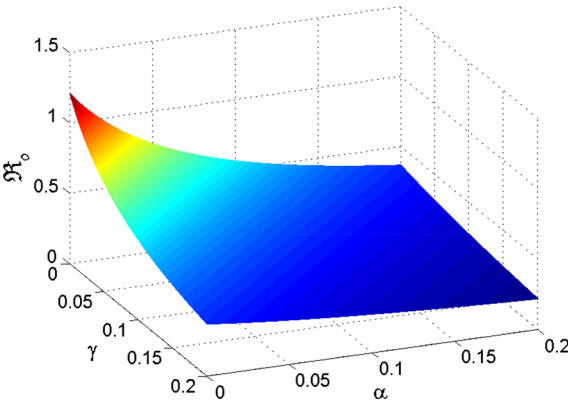

Fig. 8 R_0 versus α and γ in the case that $\beta = 0.2, \eta = 0.02, \delta = 0.03, \theta_1 = 0.2, \theta_2 = 0.25$, and $T = 5$

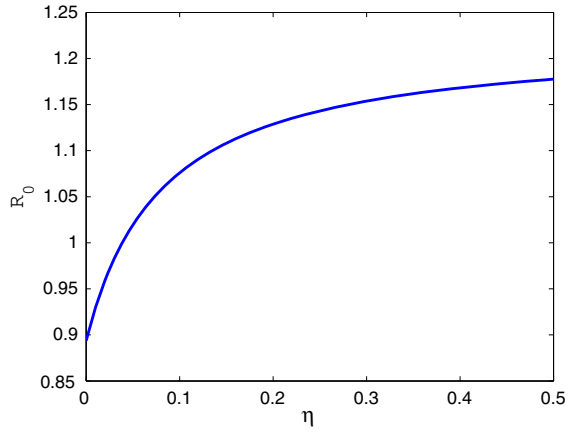

Fig. 9 R_0 versus η in the case where $\beta = 0.3, \alpha = 0.05, \gamma = 0.1, \delta = 0.06, \theta_1 = 0.2, \theta_2 = 0.3$ and $T = 10$

5 Conclusions and prospects

To understand the impact of pulse treatment on computer viruses, a virus-spreading model with pulse treatment has been proposed. The qualitative properties of this model have been studied in depth. The influence of different model parameters on the BRR has been examined. Thereby, several measures for containing the propagation of computer infections have been suggested. In particular, it has been found that accelerating the antivirus software development cycle contributes much to virus inhibition.

Toward this direction, numerous work is yet to be done. First, the proposed model is based on the overly simplified SIRS model. To better understand the effect of pulse treatment, other kinds of virus epidemic models, ranging from SEIRS and SEIQRS models [13,14] to SLBS models [17,18,20,22,24] to SICS models [25,26], should be modified by introducing pulse treatment. Second, as a new virus can spread rapidly through the Internet and, hence, can bring about a sharp rise in the number of infected computers, it is essential to develop virus-spreading models with pulse infection. Third, pulse treatment and pulse infection should be combined to form new virus spreading models. Last, but not least, as empirical data show that, in either the router level or the domain level, the Internet topology approximately follows a power law degree distribution rather than a Poisson distribution [55,56], the proposed model needs to be adapted to scale-free networks [21,57–59].

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