Introduction to Probability and Statistics

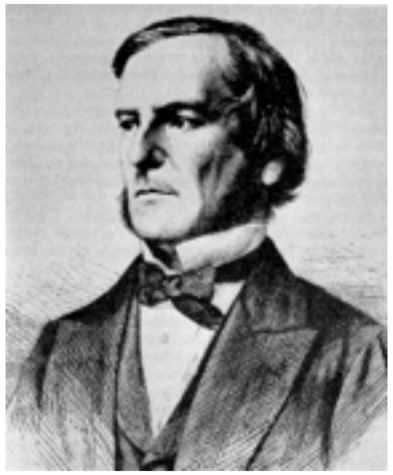
January 23, 2014

Last time ...

1) Formulate a *model* of pairs of sentences.
2) *Learn* an instance of the model from *data*.
3) Use it to *infer* translations of new inputs.

Why Probability?

- Probability formalizes ...
  - the concept of ***models***
  - the concept of ***data***
  - the concept of ***learning***
  - the concept of ***inference*** (prediction)



Probability is expectation founded upon partial knowledge.

$$p(x \mid \text{partial knowledge})$$

"Partial knowledge" is an apt description of what we know about language and translation!

Probability Models

- Key components of a probability model
  - The space of events (Ω or $\boldsymbol{S}$)
  - The assumptions about conditional independence / dependence among events
  - Functions assigning probability (density) to events
  - *We will assume discrete distributions.*

Events and Random Variables

A **random variable** is a function from a random event from a set of possible outcomes (Ω) and a probability distribution (*p*), a function from outcomes to probabilities.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$X(\omega) = \omega$$

$$\rho_X(x) = \begin{cases} \frac{1}{6} & \text{if } x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

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$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$Y(\omega) = \begin{cases} 0 & \text{if } \omega \in \{2, 4, 6\} \\ 1 & \text{otherwise} \end{cases}$$

$$\rho_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

What is our event space?

What are our random variables?

Probability Distributions

A probability distribution (p_X) assigns probabilities to the values of a random variable (X).

There are a couple of philosophically different ways to define probabilities, but we will give only the invariants in terms of **random variables**.

$$\sum_{x \in \mathcal{X}} \rho_X(x) = 1$$

$$\rho_X(x) \geq 0 \quad \forall x \in \mathcal{X}$$

Probability distributions of a random variable may be specified in a number of ways.

Specifying Distributions

- Engineering/mathematical convenience
- Important techniques in this course
  - Probability mass functions
    - Tables (“stupid multinomials”)
    - Log-linear parameterizations (maximum entropy, random field, multinomial logistic regression)
  - Construct random variables from other r.v.’s with known distributions

Sampling Notation

$$x = 4 \times z + 1.7$$

Variable

Expression

Sampling Notation

$$x = 4 \times z + 1.7$$

$$y \sim \mathrm{Distribution}(\boldsymbol{\theta})$$

Distribution

Random variable

Parameter

Sampling Notation

$$y \sim \text{Distribution}(\boldsymbol{\theta})$$

$$y' = y \times x$$

Multivariate r.v.'s

Probability theory is particularly useful because it lets us reason about (cor)related and dependent events.

A **joint probability distribution** is a probability distribution over r.v.'s with the following form:

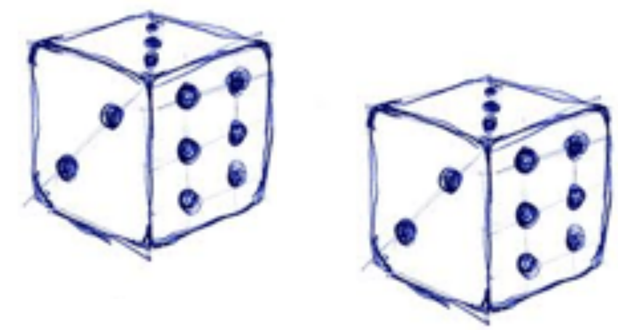
$$Z = \begin{bmatrix} X(\omega) \\ Y(\omega) \end{bmatrix}$$

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 1 \qquad \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \geq 0 \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$X(\omega) = \omega$$

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),\}\end{aligned}$$



$$X(\omega) = \omega_1 \quad Y(\omega) = \omega_2$$

$$\rho_{X,Y}(x,y) = \begin{cases} \frac{1}{36} & \text{if } (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

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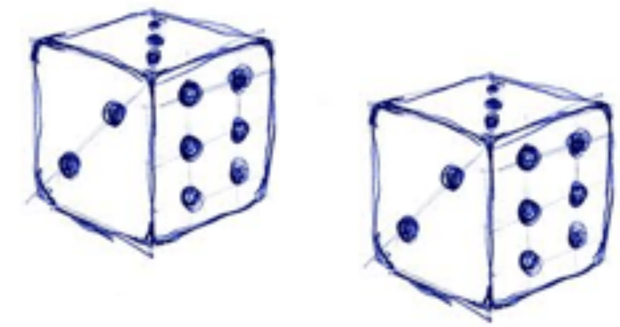

$$X(\omega) = \omega_1 \quad Y(\omega) = \omega_2$$

$$\rho_{X,Y}(x, y) = \begin{cases} \frac{x+y}{252} & \text{if } (x, y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Marginal Probability

$$p(X = x, Y = y) = \rho_X(x, y)$$

$$p(X = x) = \sum_{y' = \mathcal{Y}} p(X = x, Y = y')$$

$$p(Y = y) = \sum_{x' = \mathcal{X}} p(X = x', Y = y)$$

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),\}\end{aligned}$$

$$p(X = 4) = \sum_{y' \in [1,6]} p(X = 4, Y = y')$$

$$p(Y = 3) = \sum_{x' \in [1,6]} p(X = x', Y = 3)$$

$$\rho_{X,Y}(x,y) = \begin{cases} \frac{1}{36} & \text{if } (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

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$$\frac{6}{36} = \frac{1}{6}$$

$$\rho_{X,Y}(x,y) = \begin{cases} \frac{x+y}{252} & \text{if } (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

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$$\frac{4+1+4+2+4+3+4+4+4+5+4+6}{252} = \frac{45}{252}$$

Conditional Probability

The **conditional probability** of one random variable given another is defined as follows:

$$p(X = x \mid Y = y) = \frac{p(X = x, Y = y)}{p(Y = y)} = \frac{\text{joint probability}}{\text{marginal}}$$

Given that $p(y) \neq 0$

Conditional probability distributions are useful for specifying joint distributions since:

$$p(x \mid y)p(y) = p(x, y) = p(y \mid x)p(x)$$

Why might this be useful?

Conditional Probability Distributions

A **conditional probability distribution** is a probability distribution over r.v.'s X and Y with the form $\rho_{X|Y=y}(x)$.

$$\sum_{x \in \mathcal{X}} \rho_{X|Y=y}(x) = 1 \; \forall y \in \mathcal{Y}$$

Chain rule

The **chain rule** is derived from a repeated application of the definition of conditional probability:

$$p(a, b, c, d)$$

Use as many times as necessary!

Bayes' Rule

Likelihood

Posterior Prior

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} \quad \left(= \frac{p(y \mid x)p(x)}{\sum_{x'} p(y \mid x')p(x')} \right)$$

Evidence

Independence

Two random variables are independent iff

$$p(X = x, Y = y) = p(X = x)p(Y = y)$$

Equivalently, (use def. of cond. prob to prove)

$$p(X = x \mid Y = y) = p(X = x)$$

Equivalently again:

$$p(Y = y \mid X = x) = p(Y = y)$$

"Knowing about X doesn't tell me about Y"

$$\rho_{X,Y}(x,y) = \begin{cases} \frac{1}{36} & \text{if } (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

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Independence

Independence has **practical benefits.** Think about how many parameters you need for a naive parameterization of $\rho_{X,Y}(x, y)$ vs $\rho_X(x)$ and $\rho_Y(y)$

$$O(xy) \text{ vs } O(x + y)$$

Conditional Independence

Two equivalent statements of conditional independence:

$$p(a, c \mid b) = p(a \mid b)p(c \mid b)$$

and:

$$p(a \mid b, c) = p(a \mid b)$$

"If I know B, then C doesn't tell me about A"

Conditional Independence

$$\begin{aligned} p(a, b, c) &= p(a \mid b, c) p(b, c) \\ &= p(a \mid b, c) p(b \mid c) p(c) \end{aligned}$$

"If I know B, then C doesn't tell me about A"

$$p(a \mid b, c) = p(a \mid b)$$

$$\begin{aligned} p(a, b, c) &= p(a \mid b, c) p(b, c) \\ &= p(a \mid b, \cancel{c}) p(b \mid c) p(c) \\ &= p(a \mid b) p(b \mid c) p(c) \end{aligned}$$

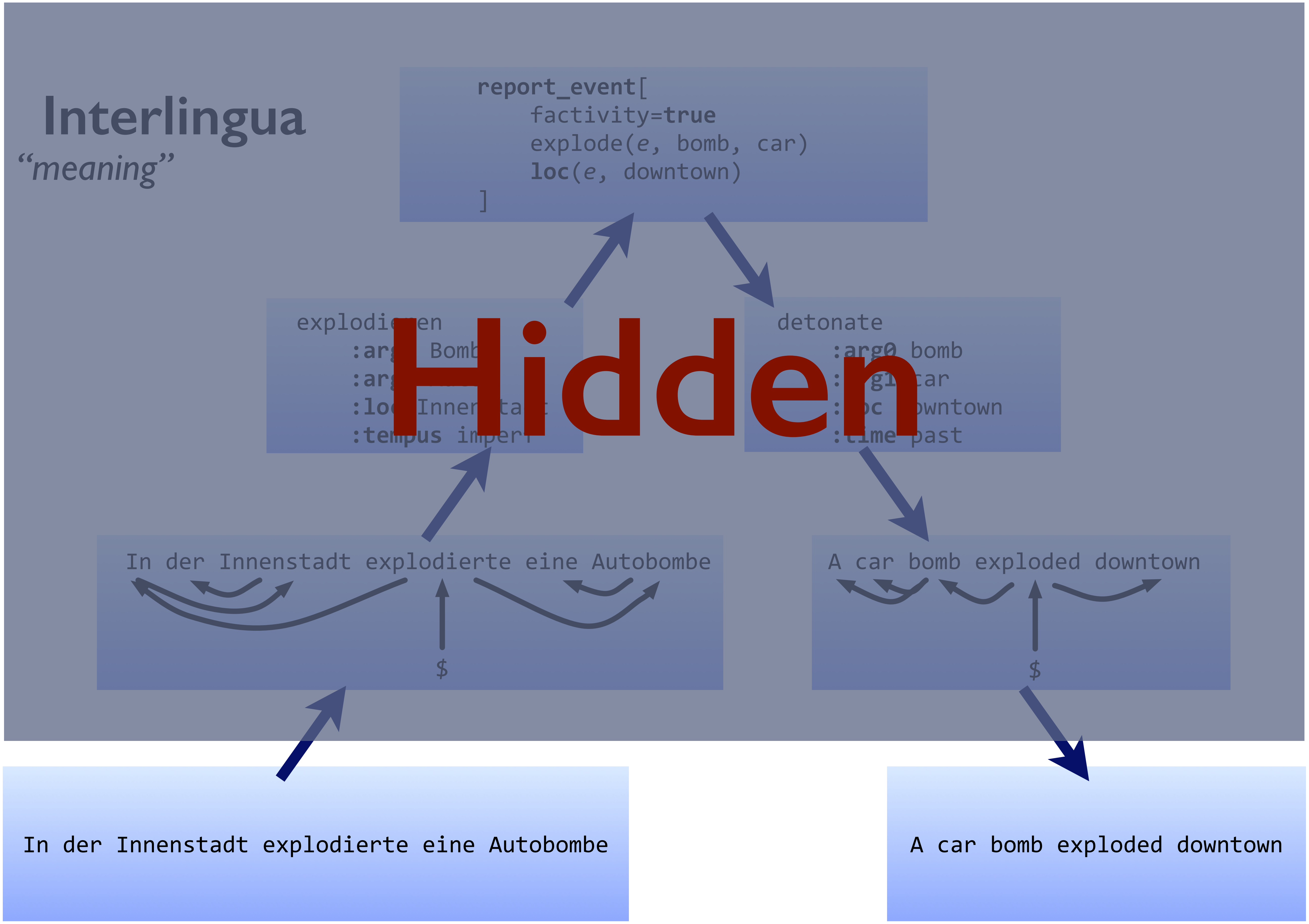
Do we need more parameters or fewer parameters in conditional independence?

Independence

- Some variables are independent In Nature
    - How do we know?
- Some variables we *pretend* are independent for computational convenience
    - Examples?
- Assuming independence is equivalent to letting our model "forget" something that happened in its past
    - What should we forget in language?

A Word About Data

- When we formulate our models there will be two kinds of random variables: observed and latent
  - Observed: words, sentences(?), parallel corpora, web pages, formatting...
  - Latent: parameters, syntax, “meaning”, word alignments, translation dictionaries...


Interlingua
"meaning"
report_event[
factivity=true
explode(e, bomb, car)
loc(e, downtown)
]
Hidden
detonate
:arg0 bomb
:time past
:tempus
In der Innenstadt explodierte eine Autobombe
$
A car bomb exploded downtown
$
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A car bomb exploded downtown

Observed

Garcia and associates .

Garcia y asociados .

Carlos Garcia has three associates .

Carlos Garcia tiene tres asociados .

his associates are not strong .

sus asociados no son fuertes .

Garcia has a company also .

Garcia tambien tiene una empresa .

its clients are angry .

sus clientes estan enfadados .

the associates are also angry .

los asociados tambien estan enfadados .

the clients and the associates are enemies .

los clientes y los asociados son enemigos .

the company has three groups .

la empresa tiene tres grupos .

its groups are in Europe .

sus grupos estan en Europa .

the modern groups sell strong pharmaceuticals .

los grupos modernos venden medicinas fuertes .

the groups do not sell zanzanine .

los grupos no venden zanzanina .

the small groups are not modern .

los grupos pequenos no son modernos .

Hidden

Garcia and associates .
Garcia y asociados .

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Learning

- Let's say we have formulated a model of a phenomenon
  - Made independence assumptions
  - Figured out what kinds of parameters we want
- Let's say we have collected data we assume to be generated by this model
  - E.g. some parallel data

What do we do now?

Parameter Estimation

- Inputs
  - Given a model with unspecified parameters
  - Given some data
- Goal: learn model parameters
- How?
  - Find parameters that make the model make predictions that look like the data do
  - What do we mean "look like the data?"
    - Probability (other options: accuracy, moment matching)

Strategies

- **Maximum likelihood estimation**
  - What is the *probability* of generating the data?
- **Accuracy**
  - Using an auxiliary similarity function, find parameters that maximize the (expected?) accuracy of data
- **Bayesian techniques**

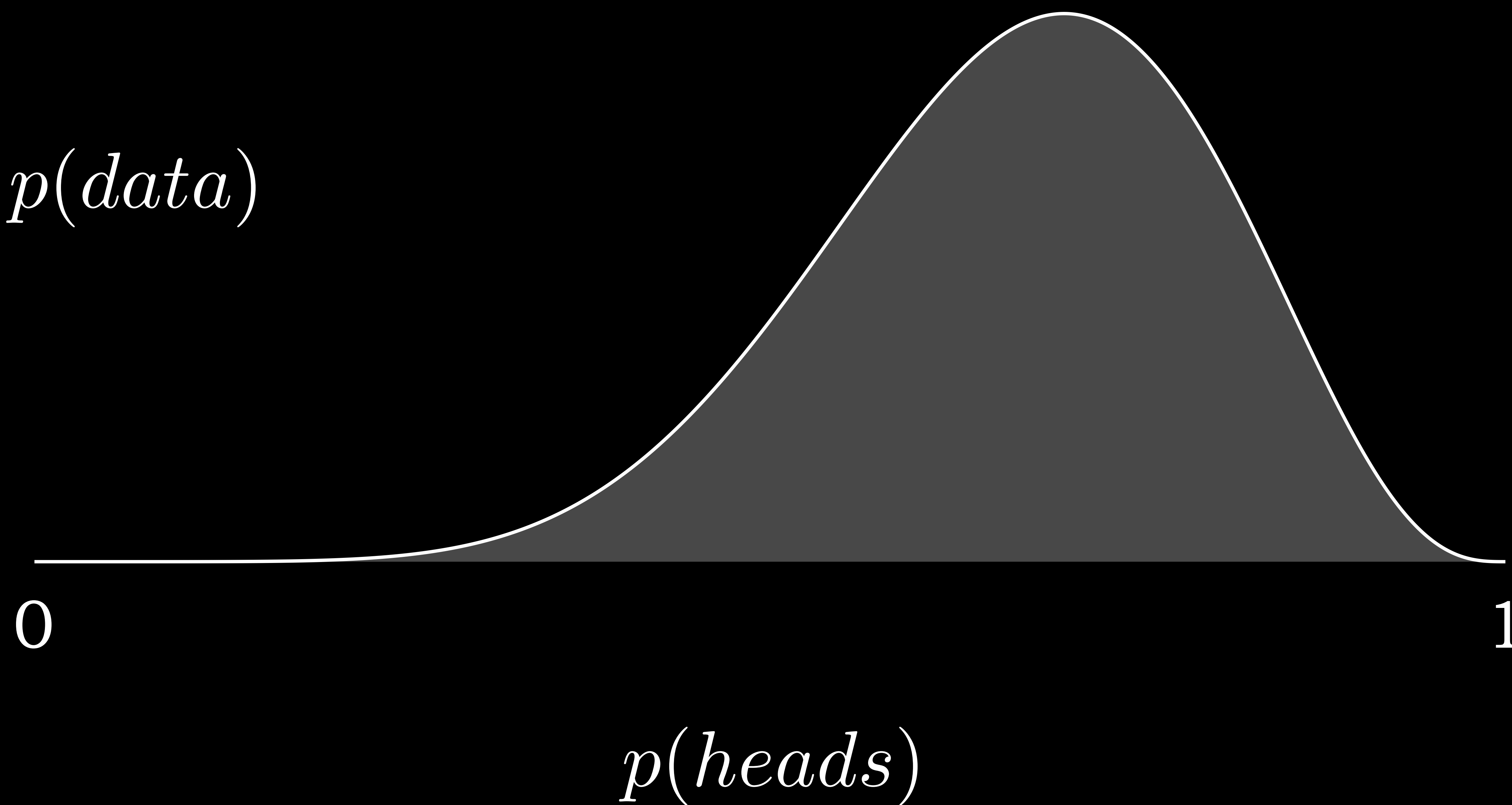
$p(heads)$

$1 - p(heads)$

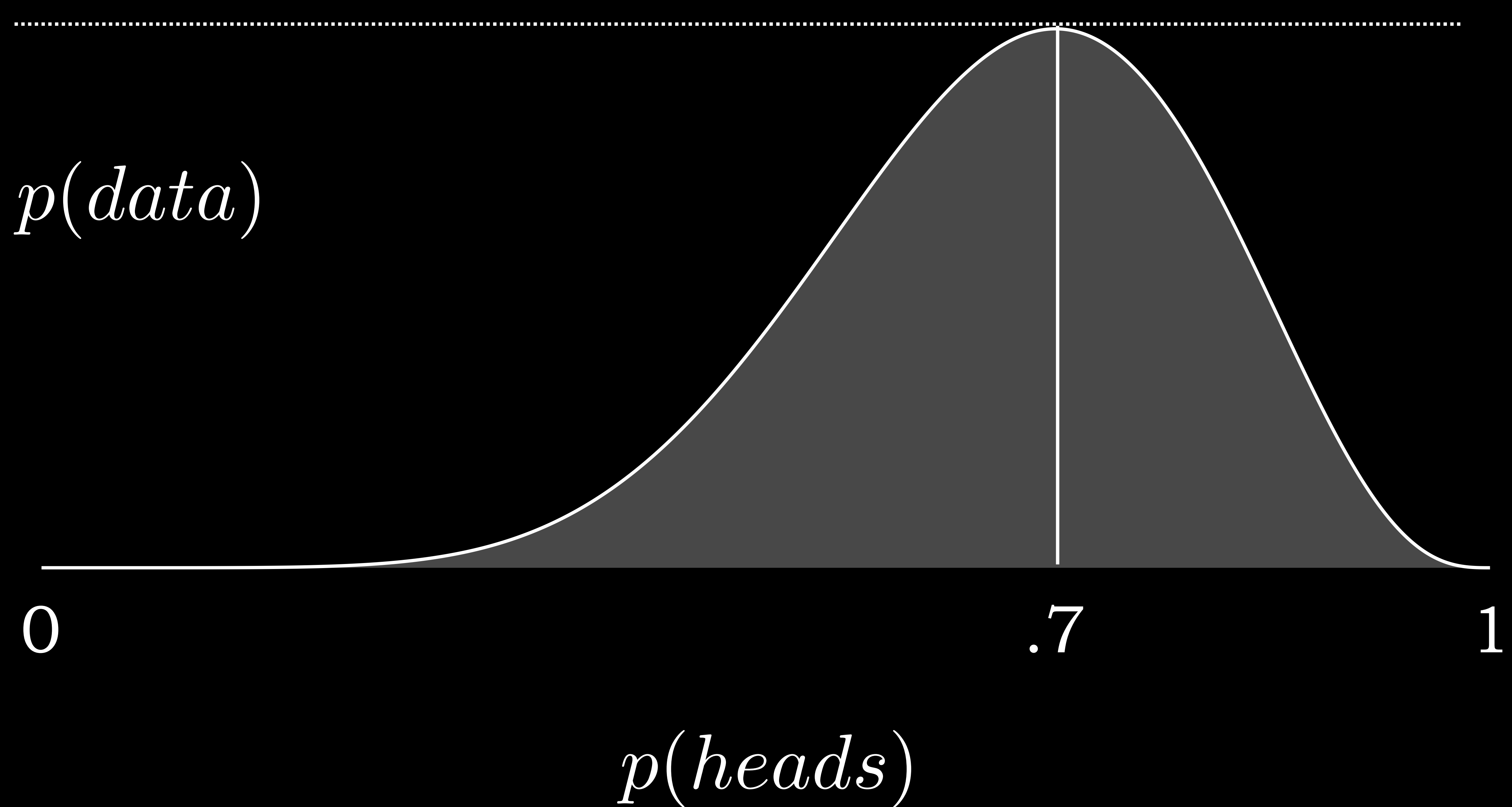
$$p(heads)\,?$$

$$p(data) = p(heads)^7 \times p(tails)^3$$

$$p(data) = p(heads)^7 \times [1 - p(heads)]^3$$



$p(data)$
0
1
$p(heads)$



$p(data)$
0
.7
1
$p(heads)$

Optimization

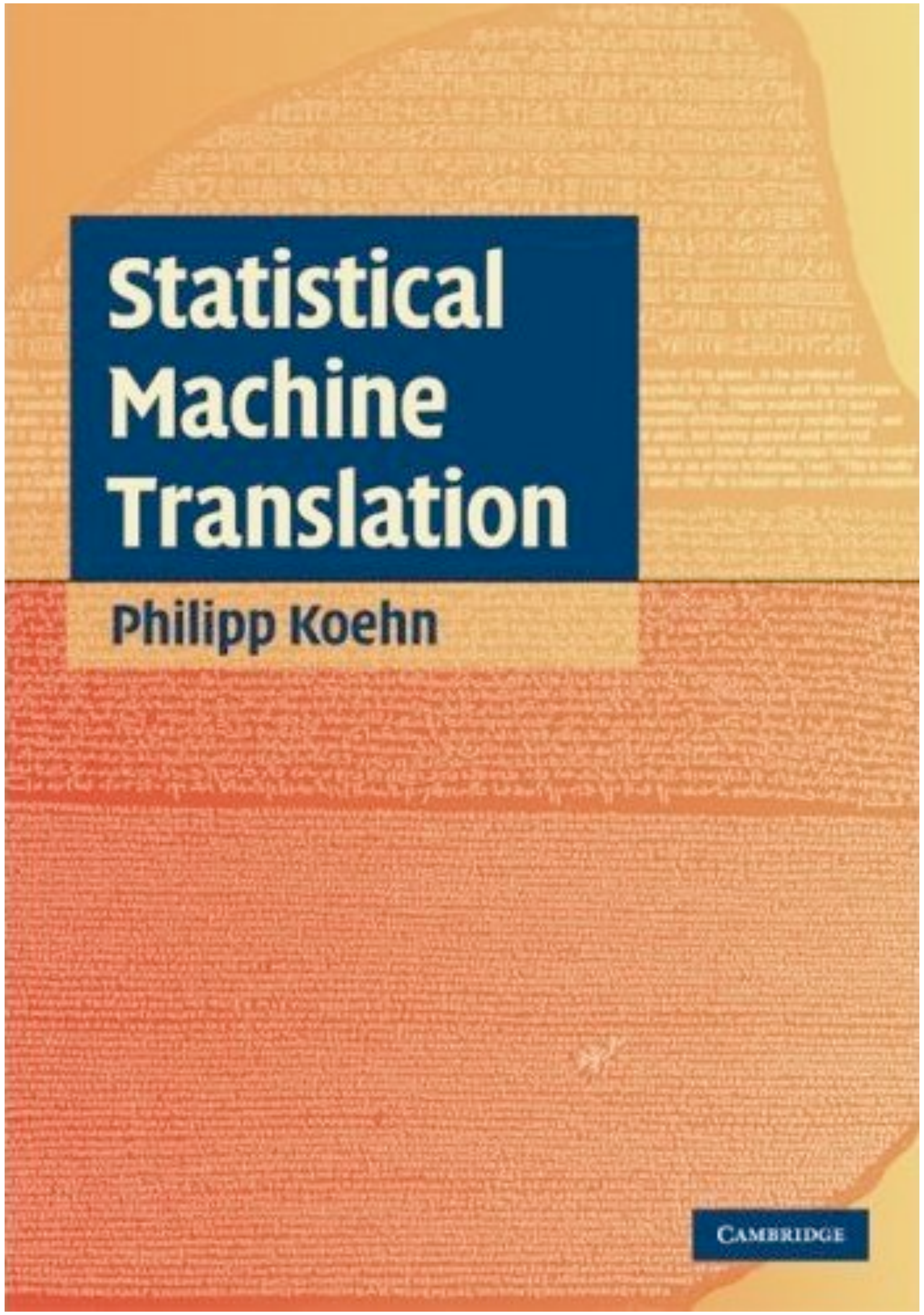
- For the most part, we will be working with maximum likelihood estimation
- The general recipe is:
    - Come up with an expression of the likelihood of your probability model, as a function of **data** and the model **parameters**
    - Set the parameters to maximize the likelihood
    - This optimization is generally difficult
        - You must respect any constraints on the parameters (>0, sum to 1, etc)
        - There may not be analytical solutions (log-linear models)

Probability lets us

1) Formulate a *model* of pairs of sentences.
2) *Learn* an instance of the model from *data*.
3) Use it to *infer* translations of new inputs.

Supplemental Reading

- If this was unfamiliar to you, then please read Chapter 3 from the textbook "Statistical Machine Translation" by Philipp Koehn



Announcements

- If you haven't done so already, complete HW 0 by today at11:59pm.
- Office hours are set. Jonny: Wednesdays at 2pm (Levine 5th floor bump space), Chris: Mondays at 10:30am (Levine 506)
- HW1 will be released this weekend, and due on Thursday Feb 6. I strongly encourage you to do it before the end of the course selection period (Feb 3).

Announcements

- Grading has been set
- 7 homework assignments, 10 points each
- 1 in-class presentation, 10 points
- 1 final project with writeup and code 20 points
- I'll post a description of the requirements on the web page, and then send a note to piazza