Assignment 4: Rectifying Your Ellipse

2021 Fall EECS205002 Linear Algebra

Due: 2022/1/17

In assignment 3, we fitted the ellipses using the general quadratic form,

$$ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 = 1. \tag{1}$$

This approaches has many drawbacks. First, the quadratic form is not only for ellipse. It is for all kinds of conic sections, including circle, ellipse, parabola, and hyperbola. How to tell the shape of the drawing? Second, it is not the standard ellipse we are familar with, as shown in Figure 1. From the coefficients in (1), we do not know what the parameters of the ellipse, such as the width and the height.

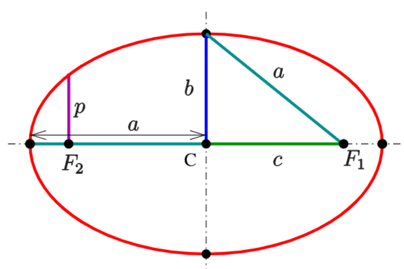


Figure 1: The standard ellipse centered at the origin.

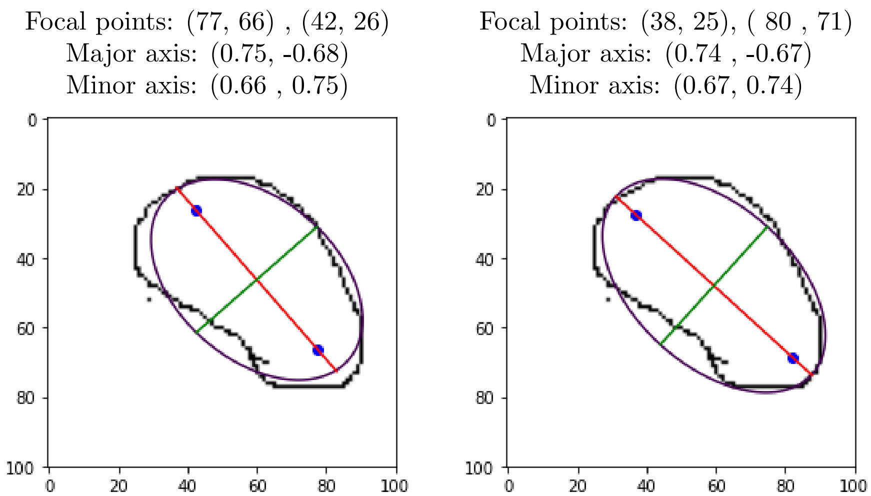
In this assignment, we will study two algorithms to find the standard form of the ellipse best approxmating to the drawing.

The first algorithm follows what we have done in assignment 3, as shown in Algorithm 1. The first step is to fit the drawing with the quadratic model, $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 = 1$. Next, from the model parameters, we can find the center of the ellipse. Let us assume its center is (z, w). The goal is to eliminate the linear term, so the form after translation becomes

$$a'(x_1 - z)^2 + b'(x_1 - z)(x_2 - w) + c'(x_2 - w)^2 = 1.$$

The above equation has five variables: a', b', c', z and w. We need five equations to solve them. We can expand the above equation and have

$$a'x_1^2 + b'x_1x_2 + c'x_2^2 + (-2a'z - b'w)x_1 + (-b'z - 2c'w)x_2 = 1 - a'z^2 - b'zw - c'w^2.$$



(a) Algorithm 1 (b) Algorithm 2

Figure 2: The results of algorithm1 and algorithm 2.

Let $\rho = 1 - a'z^2 - b'zw - c'w^2$. We have

$$\frac{a'}{\rho}x_1^2 + \frac{b'}{\rho}x_1x_2 + \frac{c'}{\rho}x_2^2 + \frac{-2a'z - b'w}{\rho}x_1 + \frac{-b'z - 2c'w}{\rho}x_2 = 1.$$

Comparing to the original model, we have the following equations

$$a' = a\rho \tag{2}$$
$$b' = b\rho \tag{3}$$
$$c' = c\rho \tag{4}$$
$$(-2a'z - b'w) = d\rho \tag{5}$$
$$(-b'z - 2c'w) = e\rho \tag{6}$$

Puting (2), (3), (4) to (5) and (6), we can cancel out ρ, and have the following linear system.

$$\begin{bmatrix} -2a & -b \\ -b & -2c \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix}.$$

Solving it, we can have the center (z, w). Plugging the result of z, w back to (2), (3), (4), we can compute a', b', and c' as follows

$$\begin{bmatrix} z^2 + 1/a & zw & w^2 \\ z^2 & zw + 1/b & w^2 \\ z^2 & zw & w^2 + 1/c \end{bmatrix} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Last, let $x' = x - z$ and $y' = y - w$. We can have

$$a'x'^2 + b'x'y' + c'y'^2 = 1,$$

Algorithm 1 Algorithm 1: quadratic formulation to rectify ellipse

1. Solve the least squart problem to fit the model
$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 = 1.$$
2. Find the center of the ellipse (z, w) so the above model becomes
$$a'(x_1 - z)^2 + b'(x_1 - z)(x_2 - w) + c'(x_2 - w)^2 = 1.$$
3. Find the rotation matrix to transform the above equation to
$$\frac{x_1'^2}{\alpha^2} + \frac{x_2'^2}{\beta^2} = 1$$
4. Use the standard equation to find two focal points of ellipse, and use inverse rotation and translation to put them on the original figure, and output the parameters, α and β.

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which can be expressed as $\vec{x}^T A\vec{x} = 1$. Use the eigenvectors of A, we can obtain the rotation matrix to further transform it to the standard form. The details are in textbook 6.6.

The second algorithm is based on PCA (principal component analysis), whose derivation can be found in last year's assignment 4. Suppose there are N points, whose coordinates are $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$. The basic idea is to translate and rotate them into the standard ellipse. The first step is to find the center of those points, which is the mean, and then translate them so the center will be the new origin. The second step is to form a special matrix, called *covariant matrix*, whose definition is

$$S = \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix} (x_i - \bar{x})^2 & (x_i - \bar{x})(y_i - \bar{y}) \\ (x_i - \bar{x})(y_i - \bar{y}) & (y_i - \bar{y})^2 \end{bmatrix}.$$

The orthgonal basis of the eigenvectors of S forms a rotation matrix for those points, which can rectify them. Moreover, the eigenvector of the largest eigenvalue is the direction of the major-axis, the red line in Figure 2, which is called the Principal Component; the eigenvector of the smallest eigenvalue is the direction of minor-axis, the green line in Figure 2.

The fourth step is to fit the transformed points with the standard equation of ellipses,
$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$
For the parameters, you can easily find where the focal points, the width and the height of the approximate ellipse.

Algorithm 2 Algorithm 2: PCA to rectify ellipse

1. Compute the center of the points:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

2. Compute the covariance matrix and its eigenvalues/eigenvectors decomposition.

$$S = \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix} (x_i-\bar{x})^2 & (x_i-\bar{x})(y_i-\bar{y}) \\ (x_i-\bar{x})(y_i-\bar{y}) & (y_i-\bar{y})^2 \end{bmatrix} = U\Sigma U^T$$

3. Translate the points to the center and rotate them with the rotation matrix U.

$$W = U\begin{bmatrix} x_1-\bar{x} & x_2-\bar{x} & \cdots & x_N-\bar{x} \\ y_1-\bar{y} & y_2-\bar{y} & \cdots & y_N-\bar{y} \end{bmatrix}$$

4. Fit the points in W using the standard form of ellipse

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2} = 1.$$

5. Use the standard equation to find two focal points of ellipse, and use inverse rotation and translation to put them on the original figure, and output the parameters, α and β.

The last step is to express the fitted ellipse back into the general quadratic form, as in (1). Since we already know the translation and rotation, we can reverse the process. Let

$$A = \begin{bmatrix} \alpha^{-2} & 0 \\ 0 & \beta^{-2} \end{bmatrix},$$

$S = U\Sigma U^T$ be the eigen-decomposition of the covariant matrix, and $(\bar{x}, \bar{y})$ be the center of data. The quadratic form that fits the original data will be

$$\begin{bmatrix} x-\bar{x} & y-\bar{y} \end{bmatrix} U^T A U \begin{bmatrix} x-\bar{x} \\ y-\bar{y} \end{bmatrix} = 1.$$

After expanding the above equation, we can have the general quadratic form.

1 Assignments

1. (20%) Read textbook 6.6, and express (1) in the matrix form

$$\vec{x}^T A\vec{x} + B\vec{x} = 1.$$

where $\vec{x} = [x_1 \ x_2]^T$.

- What are matrix A and B?
- How to use A to determine which conic section the quadratic form represent?
  - Circle: $x_1^2 + x_2^2 = r^2$.
  - Ellipse: $\dfrac{x_1^2}{\alpha^2} + \dfrac{x_2^2}{\beta^2} = 1$
  - Parabola: $x_2^2 = 4\alpha x_1$
  - Hyperbola: $\dfrac{x_1^2}{\alpha^2} - \dfrac{x_2^2}{\beta^2} = 1$

2. (20%) Implement algorithm 1.
   - For the translation, solve the system in (2)-(6). You can use `numpy.linalg.solve` to solve the linear systems.
   - For the rotation, read textbook 6.6 to find the rotation matrix. The computation of eigenvalues/eigenvectors of a matrix can use `numpy.linalg.eig`.
3. (20%) Implement algorithm 2.
   - Complete the code of PCA.
   - Complete the code of `standard_to_general`.
4. (20%) Give one or two examples to compare algorithm 1 and algorithm 2. From the results, can you say which algorithm is more accurate?
5. (20%) In `numpy.linalg.lstsq`, what is the purpose to return SVD? Explain its relation with rank. If `numpy.linalg.lstsq` already returns rank, why it still needs to returen SVD?

2 Submissions

1. Write a report in a PDF file that includes (1), (2), (3), (4), and (5).
2. Python code of your implementation of (2) and (3).
3. Zip them and submit to eeclass system.