

1. For a group of 7 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.

Let  $A_i$  be the event that there are no birthdays in the  $i$ th season. The probability that all seasons occur at least once is  $1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$ . Note that  $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$ . Using the inclusion-exclusion principle and the symmetry of the seasons,

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) &= \sum_{i=1}^4 P(A_i) - \sum_{i=1}^3 \sum_{j>i} P(A_i \cap A_j) \\ &\quad + \sum_{i=1}^2 \sum_{j>i} \sum_{k>j} P(A_i \cap A_j \cap A_k) \\ &= 4P(A_1) - 6P(A_1 \cap A_2) + 4P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

We have  $P(A_1) = (3/4)^7$ . Similarly,

$$P(A_1 \cap A_2) = \frac{1}{2^7} \text{ and } P(A_1 \cap A_2 \cap A_3) = \frac{1}{4^7}.$$

Therefore,  $P(A_1 \cup A_2 \cup A_3 \cup A_4) = 4(\frac{3}{4})^7 - \frac{6}{2^7} + \frac{4}{4^7}$ . So the probability that all 4 seasons occur at least once is  $1 - (4(\frac{3}{4})^7 - \frac{6}{2^7} + \frac{4}{4^7}) \approx 0.513$ .

2. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday?

*Direct Method:* There are two general ways that Alice can have class every day: either she has 2 days with 2 classes and 3 days with 1 class, or she has 1

day with 3 classes, and has 1 class on each of the other 4 days. The number of possibilities for the former is  $\binom{5}{2} \binom{6}{2}^2 6^3$  (choose the 2 days when she has 2 classes, and then select 2 classes on those days and 1 class for the other days). The number of possibilities for the latter is  $\binom{5}{1} \binom{6}{3} 6^4$ . So the probability is

$$\frac{\binom{5}{2} \binom{6}{2}^2 6^3 + \binom{5}{1} \binom{6}{3} 6^4}{\binom{30}{7}} = \frac{114}{377} \approx 0.302.$$

*Inclusion-Exclusion Method:* we will use inclusion-exclusion to find the probability of the complement, which is the event that she has at least one day with no classes. Let  $B_i = A_i^c$ . Then

$$P(B_1 \cup B_2 \cup \dots \cup B_5) = \sum_i P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \sum_{i<j<k} P(B_i \cap B_j \cap B_k)$$

(terms with the intersection of 4 or more  $B_i$ 's are not needed since Alice must have classes on at least 2 days). We have

$$P(B_1) = \frac{\binom{24}{7}}{\binom{30}{7}}, P(B_1 \cap B_2) = \frac{\binom{18}{7}}{\binom{30}{7}}, P(B_1 \cap B_2 \cap B_3) = \frac{\binom{12}{7}}{\binom{30}{7}}$$

and similarly for the other intersections. So

$$P(B_1 \cup \dots \cup B_5) = 5 \frac{\binom{24}{7}}{\binom{30}{7}} - \binom{5}{2} \frac{\binom{18}{7}}{\binom{30}{7}} + \binom{5}{3} \frac{\binom{12}{7}}{\binom{30}{7}} = \frac{263}{377}.$$

Therefore,

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \frac{114}{377} \approx 0.302.$$