1 Preface

This report describes a physical derivation of the nonlinear single-track model adopted from [1] and implemented in our research group SDS RC, FEE, CTU in Prague in Matlab/Simulink framework.

2 Nonlinear Single-Track Model

The described nonlinear single-track model (Fig. 1) has 3 degrees of freedom and is used to represent the planar translation and yaw motion of a vehicle. This model can be augmented with load transfer (pitch motion) and roll motion. According to [1], the single-track model can be analyzed on the $0 - 2\,\mathrm{Hz}$ frequency spectrum.

The following assumptions are assumed:

- All lifting, rolling, and pitching motion are neglected.
- Vehicle mass is assumed to be concentrated at the center of gravity.
- Front and rear tires are represented as one single tire on each axle. Imaginary contact points of tires and surface are assumed to lie along the center of axles.
- Pneumatic trail and aligning torque resulting from a side-slip angle of a tire are neglected.
- Mass distribution on the axles is assumed to be constant.
- Vehicle dynamics is controlled by the steering angles and angular velocities of wheels.

The derived model consists of 3 states and 4 inputs, which are listed in Table 1. All parameters used in the model are presented in Table 2. Presented values are used as benchmark parameters.

Table 1: States and inputs of the model

State/Input	Symbol	Units
Velocity of CG	v	$\mathrm{m\,s^{-1}}$
Side-slip angle	β	rad
Yaw rate	r	$\mathrm{rad\,s^{-1}}$
Steering angle of the front axle	δ_f	rad
Steering angle of the rear axle	δ_r	rad
Angular velocity of the front wheel	ω_f	$\mathrm{rad\,s^{-1}}$
Angular velocity of the rear wheel	ω_r	$\mathrm{rad\,s^{-1}}$

Table 2: Parameters of the model

Parameter	Symbol	Value	Units
Vehicle mass	m	1200	kg
Yaw moment of inertia	I	2688	$\mathrm{kg\,m^{-2}}$
Front axle-CG distance	l_f	1.4	m
Rear axle-CG distance	l_r	1.6	m
Radius of wheels	p	0.33	m
Shaping coefficients for lateral dynamics	$c_{\mathrm{D,y}}$	1	-
	$c_{\mathrm{B,y}}$	6.9	-
	$c_{\mathrm{C,y}}$	1.8	-
	$c_{\mathrm{E,y}}$	0.1	-
Shaping coefficients for longitudinal dynamics	$c_{\mathrm{D,x}}$	1	-
	$c_{\mathrm{B,x}}$	15	-
	$c_{\mathrm{C,x}}$	1.7	-
	$c_{\mathrm{E,x}}$	-0.5	-

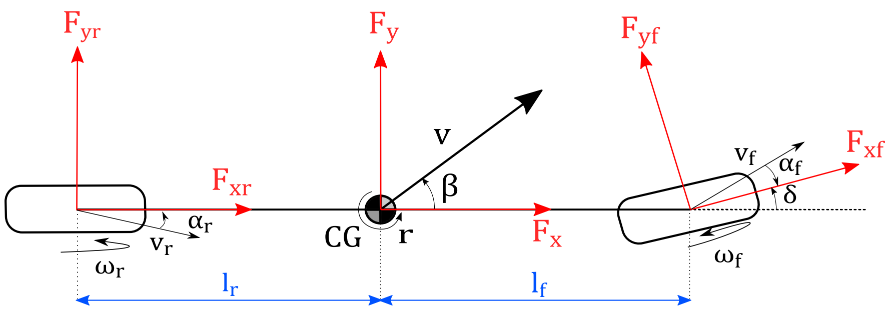


Figure 1: The single-track model.

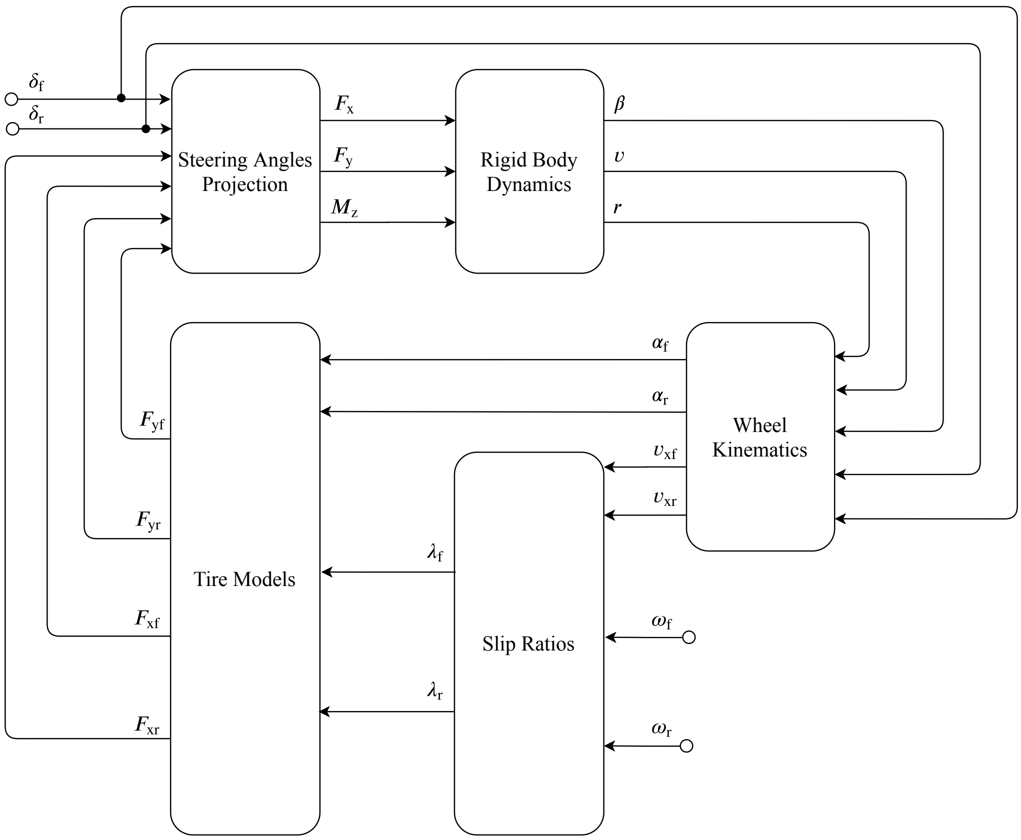


Figure 2: The block diagram of the single-track model

3 Block representation

The model consists of:

- **rigid body dynamics**, which represents a chassis;
- **steering angles projection** block computes longitudinal and lateral forces and angular momentum acting on the rigid body;
- **tire models** block calculates tire forces from slip variables and restrictions made by traction ellipse;
- **wheel kinematics**, which calculates side-slip angles of the tires in the particular coordinate frame;

- **slip ratios** calculation part, which computes how each wheel spins.

The block diagram of the nonlinear singe-track model is shown in Fig. 2. Each block has its own section in this report.

4 Rigid Body Dynamics

The rigid body dynamics consists of three degrees of freedom, which are translation motion represented by velocity of CG v and side-slip angle β and rotation motion modeled by rotation rate (yaw rate) r.

The longitudinal motion is derived as

$$F_{\mathrm{x}} = F_{\mathrm{x,tr}} + F_{\mathrm{x,rot}} = ma_{\mathrm{x}} - mrv_{\mathrm{y}} = m\frac{\mathrm{d}(v\cos\beta)}{\mathrm{d}t} - mvr\sin\beta = m(\dot{v}\cos\beta - v\sin\beta(\dot{\beta} + r)). \quad (1)$$

The lateral motion is derived as

$$F_{\mathrm{y}} = F_{\mathrm{y,tr}} + F_{\mathrm{y,rot}} = ma_{\mathrm{y}} + mrv_{\mathrm{x}} = m\frac{\mathrm{d}(v\sin\beta)}{\mathrm{d}t} + mvr\cos\beta = m(\dot{v}\sin\beta + v\cos\beta(\dot{\beta} + r)). \quad (2)$$

The rotational motion is derived as

$$M_{\mathrm{z}} = I\dot{r}. \quad (3)$$

Whole rigid body dynamics can be written in a matrix form as

$$\begin{pmatrix} \dot{\beta} \\ \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{mv} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{I} \end{pmatrix} \begin{pmatrix} -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_{\mathrm{x}} \\ F_{\mathrm{y}} \\ M_{\mathrm{z}} \end{pmatrix} - \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}. \quad (4)$$

5 Steering Angles Projection

The steering angle projection translates forces acting on tires in wheel coordinate frames into forces and rotation momentum acting on a rigid body via transformation matrix as follows

$$\begin{pmatrix} F_{\mathrm{x}} \\ F_{\mathrm{y}} \\ M_{\mathrm{z}} \end{pmatrix} = \begin{pmatrix} \cos\delta_{\mathrm{f}} & -\sin\delta_{\mathrm{f}} & \cos\delta_{\mathrm{r}} & -\sin\delta_{\mathrm{r}} \\ \sin\delta_{\mathrm{f}} & \cos\delta_{\mathrm{f}} & \sin\delta_{\mathrm{r}} & \cos\delta_{\mathrm{r}} \\ l_{\mathrm{f}}\sin\delta_{\mathrm{f}} & l_{\mathrm{f}}\cos\delta_{\mathrm{f}} & -l_{\mathrm{r}}\sin\delta_{\mathrm{r}} & -l_{\mathrm{r}}\cos\delta_{\mathrm{r}} \end{pmatrix} \begin{pmatrix} F_{\mathrm{xf}} \\ F_{\mathrm{yf}} \\ F_{\mathrm{xr}} \\ F_{\mathrm{yr}} \end{pmatrix}. \quad (5)$$

6 Tire Models

The block representation of the tire models can be seen in the Figure [3]. The block contains two tire models, which calculate via Simplified Pacejka Magic formula raw lateral and longitudinal forces for each wheel, then that forces are scaled in blocks named "Traction Ellipse".

The tire model is the main part of any nonlinear car model because tires are primary system actuators of a vehicle dynamics. There are a couple of different ways to model a tire [1], each with a bunch of various modeling techniques. This project uses Simplified Pacejka Magic [2] formula to map tire generated force on slip variable in both lateral and longitudinal directions defined as

$$F_{\mathrm{xf,raw}}(\lambda_{\mathrm{f}}) = c_{\mathrm{D,x}}F_{\mathrm{z,f}}\sin(c_{\mathrm{C,x}}\arctan(c_{\mathrm{B,x}}\lambda_{\mathrm{f}} - c_{\mathrm{E,x}}(c_{\mathrm{B,x}}\lambda_{\mathrm{f}} - \arctan(c_{\mathrm{B,x}}\lambda_{\mathrm{f}})))), \quad (6)$$

$$F_{\mathrm{xr,raw}}(\lambda_{\mathrm{r}}) = c_{\mathrm{D,x}}F_{\mathrm{z,r}}\sin(c_{\mathrm{C,x}}\arctan(c_{\mathrm{B,x}}\lambda_{\mathrm{r}} - c_{\mathrm{E,x}}(c_{\mathrm{B,x}}\lambda_{\mathrm{r}} - \arctan(c_{\mathrm{B,x}}\lambda_{\mathrm{r}})))), \quad (7)$$

$$F_{\mathrm{yf,raw}}(\alpha_{\mathrm{f}}) = c_{\mathrm{D,y}}F_{\mathrm{z,f}}\sin(c_{\mathrm{C,y}}\arctan(c_{\mathrm{B,y}}\alpha_{\mathrm{f}} - c_{\mathrm{E,y}}(c_{\mathrm{B,y}}\alpha_{\mathrm{f}} - \arctan(c_{\mathrm{B,y}}\alpha_{\mathrm{f}})))), \quad (8)$$

$$F_{\mathrm{yr,raw}}(\alpha_{\mathrm{r}}) = c_{\mathrm{D,y}}F_{\mathrm{z,r}}\sin(c_{\mathrm{C,y}}\arctan(c_{\mathrm{B,y}}\alpha_{\mathrm{r}} - c_{\mathrm{E,y}}(c_{\mathrm{B,y}}\alpha_{\mathrm{r}} - \arctan(c_{\mathrm{B,y}}\alpha_{\mathrm{r}})))), \quad (9)$$

where load forces are constant and are calculated from the car parameters as follows

$$F_{\mathrm{z,f}} = gm\frac{l_{\mathrm{r}}}{l_{\mathrm{f}} + l_{\mathrm{r}}}, \quad F_{\mathrm{z,r}} = gm\frac{l_{\mathrm{f}}}{l_{\mathrm{f}} + l_{\mathrm{r}}}, \quad (10)$$

where $g = 9.81\,\mathrm{m\,s^{-2}}$ is a gravity coefficient of the Earth.

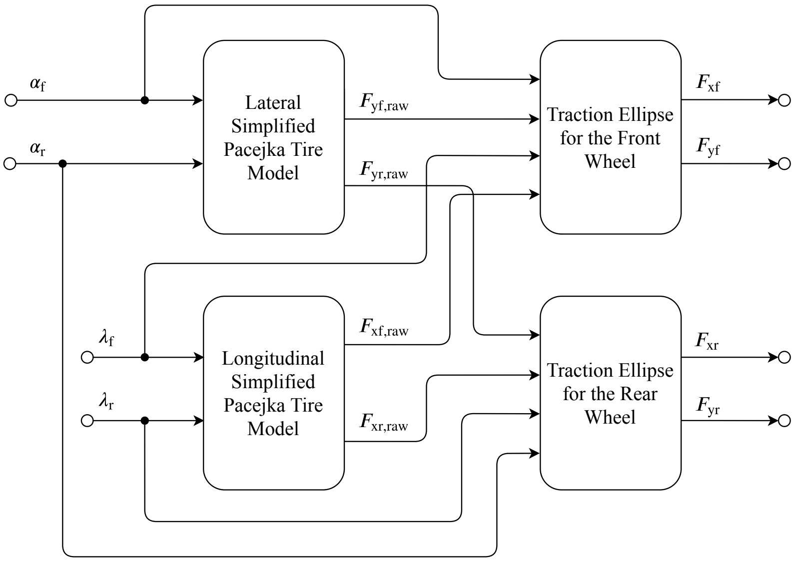
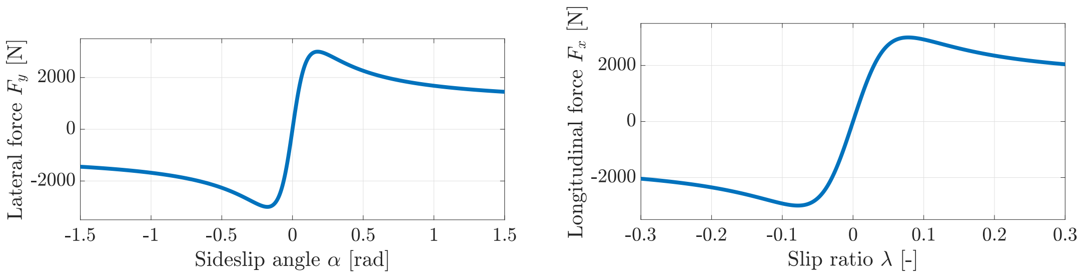


Figure 3: The block diagram of the tire models



(a) Lateral force (b) Longitudinal force

Figure 4: Example of lateral and longitudinal forces acting on a tire.

6.1 Traction ellipse

A tire cannot generate combined force (lateral and longitudinal together) grater than the vertical force F_z acting on the wheel by the vehicle. Combined slip occurs when the vehicle is accelerating or braking in a cornering maneuver. That restriction is guaranteed by the traction ellipse

$$F_{\text{tot}} = \sqrt{\frac{F_x^2}{c_{D,x}^2} + \frac{F_y^2}{c_{D,y}^2}} \leq \mu F_z, \tag{11}$$

where μ is a friction coefficient of a road; $c_{D,x}$ and $c_{D,y}$ are parameters from Pacejka (in general, they are friction coefficients of the road in different directions).

Figure [5] presents an example of the traction ellipse.

The implementation of the traction ellipse can be adopted from [3]. The following algorithm (Eq. (12) - (16)) is applied to scale (if it is needed) the resulted force:

$$\beta^* = \arccos\left(\frac{|\lambda|}{\sqrt{\lambda^2 + \sin^2(\alpha)}}\right), \tag{12}$$

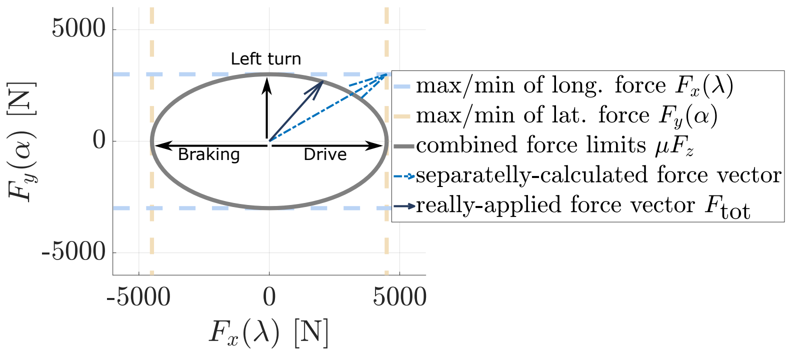


Figure 5: Traction ellipse and example of scaling of the force vector.

$$\mu_{\text{x,act}} = \frac{F_{\text{x,raw}}}{F_{\text{z}}}, \qquad \mu_{\text{y,act}} = \frac{F_{\text{y,raw}}}{F_{\text{z}}}, \tag{13}$$

$$\mu_{\text{x,max}} = c_{\text{D,x}}, \qquad \mu_{\text{y,max}} = c_{\text{D,y}}, \tag{14}$$

$$\mu_{\text{x}} = \frac{1}{\sqrt{(\frac{1}{\mu_{\text{x,act}}})^2 + (\frac{\tan(\beta^*)}{\mu_{\text{y,max}}})^2}}, \qquad F_{\text{x}} = |\frac{\mu_{\text{x}}}{\mu_{\text{x,act}}}|F_{\text{x,raw}}, \tag{15}$$

$$\mu_{\text{y}} = \frac{\tan(\beta^*)}{\sqrt{(\frac{1}{\mu_{\text{x,max}}})^2 + (\frac{\tan(\beta^*)}{\mu_{\text{y,act}}})^2}}, \qquad F_{\text{y}} = |\frac{\mu_{\text{y}}}{\mu_{\text{y,act}}}|F_{\text{y,raw}}. \tag{16}$$

7 Wheel Kinematics

The wheel kinematics of the single-track model includes calculation of the velocity vector of each wheel and side-slip angle of each tire in the particular wheel coordinate frame. Velocity vectors for the front and rear wheels are calculated as

$$\begin{pmatrix} v_{\text{xf}} \\ v_{\text{yf}} \end{pmatrix} = \begin{pmatrix} \cos\delta_{\text{f}} & \sin\delta_{\text{f}} \\ -\sin\delta_{\text{f}} & \cos\delta_{\text{f}} \end{pmatrix} \begin{pmatrix} v_{\text{x}} \\ v_{\text{y}} + l_{\text{f}} r \end{pmatrix} = \begin{pmatrix} \cos\delta_{\text{f}} & \sin\delta_{\text{f}} \\ -\sin\delta_{\text{f}} & \cos\delta_{\text{f}} \end{pmatrix} \begin{pmatrix} v\cos\beta \\ v\sin\beta + l_{\text{f}} r \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} v_{\text{xr}} \\ v_{\text{yr}} \end{pmatrix} = \begin{pmatrix} \cos\delta_{\text{r}} & \sin\delta_{\text{r}} \\ -\sin\delta_{\text{r}} & \cos\delta_{\text{r}} \end{pmatrix} \begin{pmatrix} v\cos\beta \\ v\sin\beta - l_{\text{r}} r \end{pmatrix}. \tag{18}$$

Thus, the side-slip angles of each wheel can be calculated using the definition as

$$\alpha_{\text{f}} = -\arctan\frac{v_{\text{yf}}}{|v_{\text{xf}}|}, \tag{19}$$

$$\alpha_{\text{r}} = -\arctan\frac{v_{\text{yr}}}{|v_{\text{xr}}|}. \tag{20}$$

8 Slip Ratios

This block is used to calculate slip ratio per each wheel, which can be calculated using following definition:

$$\lambda_{\text{f}} = \frac{\omega_{\text{f}} p - v_{\text{xf}}}{\max(|\omega_{\text{f}} p|, |v_{\text{xf}}|)}, \quad \lambda_{\text{r}} = \frac{\omega_{\text{r}} p - v_{\text{xr}}}{\max(|\omega_{\text{r}} p|, |v_{\text{xr}}|)}. \tag{21}$$

Notice that according to the definition, slip ratios are bounded on the interval $[-1, 1]$.

References

[1] D. Schramm, M. Hiller, and R. Bardini, *Vehicle dynamics, Modeling and Simulation.* Springer, 2014.

[2] H. Pacejka, *Tire and vehicle dynamics.* Elsevier, 2005.

[3] D. Efremov, T. Haniš, and M. Hromčík, "Introduction of driving envelope and full-time-full-authority control for vehicle stabilization systems," *22nd International Conference on Process Control (PC19) 2019*, pp. 173–178, 2019.