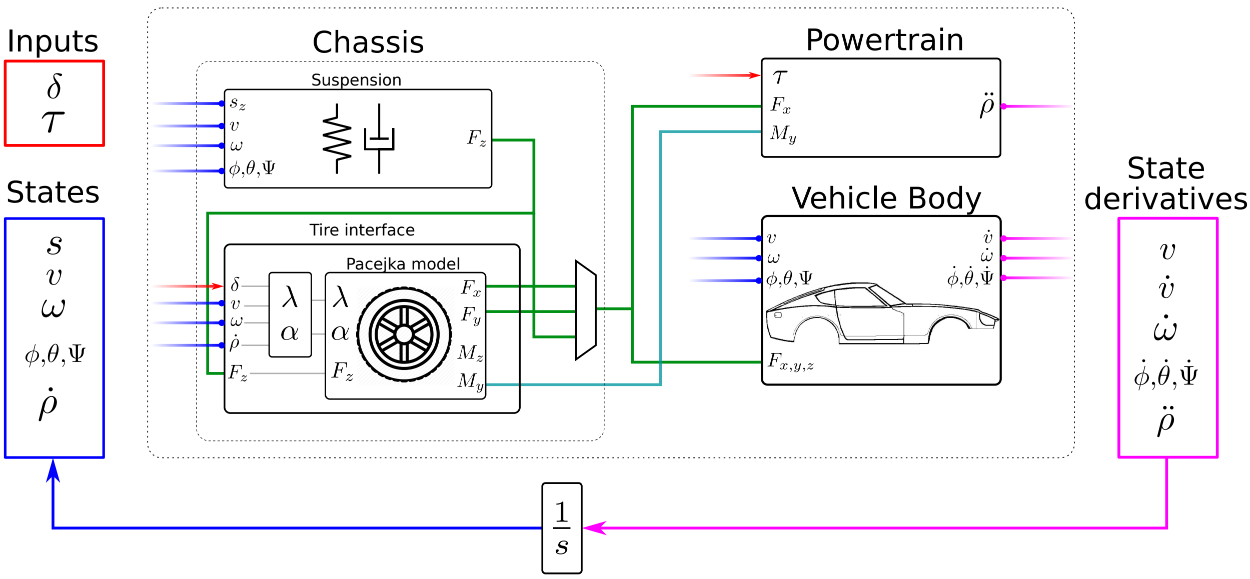
# 1 Preface

This report describes nonlinear twin-track vehicle model developed according to [1] at CTU-FEE in Prague. The model is implemented in MATLAB/Simulink.

# 2 Scheme




The model can be divided into 3 main parts: Chassis, Powertrain and Vehicle Body.

Chassis contains tires and suspension, the suspension then supports the Vehicle Body (sprung mass). Driving torque is generated in the Powertrain and distributed among the wheels (it is up to the user, which wheels will be driven).

# 3 States and inputs

The vehicle model has a total of 16 states and 8 inputs as described in Table 1 and Table 2.

| Symbol | Unit | Dimension | Description |
|---|---|---|---|
| $\mathbf{v}$ | $\mathrm{m\,s^{-1}}$ | $[3 \times 1]$ | Velocity of the vehicle body in body-fixed coordinate system. |
| $\boldsymbol{\Omega}$ | $\mathrm{rad\,s^{-1}}$ | $[3 \times 1]$ | Angular velocitites the of the vehicle body in the body-fixed coordinate system. |
| $\phi, \Theta, \Psi$ | rad | $[3 \times 1]$ | Euler angles (earth-fixed coordinate system). |
| $\mathbf{s}_E$ | m | $[3 \times 1]$ | Position of the vehicle body in earth-fixed coordinate system. |
| $\omega_i$ | $\mathrm{rad\,s^{-1}}$ | $[4 \times 1]$ | Wheel angular velocity in $i^{\mathrm{th}}$ wheel coordinate system for $i = 1..4$. |

Table 1: Vehicle states

| Symbol | Unit | Dimension | Description |
|---|---|---|---|
| $\boldsymbol{\delta}$ | rad | $[4 \times 1]$ | Steering angles of all four wheels. |
| $\boldsymbol{\tau}$ | $\mathrm{N\,m^{-1}}$ | $[4 \times 1]$ | Input torques of all four wheels. |

Table 2: Vehicle inputs

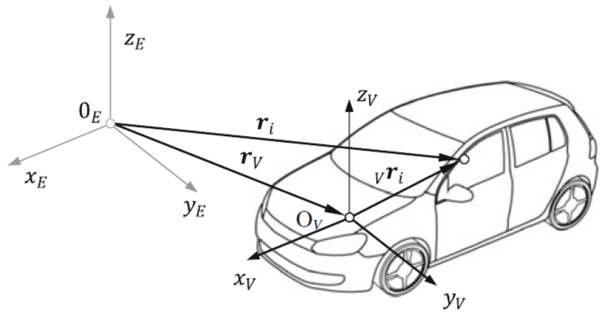



Figure 2: Inertial (earth-fixed) and Vehicle (body-fixed) coordinate systems.
Source: [1]

# 4 Parameters

The parameters of the vehicle model are described in Table 3. Note that the parameters for tire models are not listed, because the implementation considers general tire model so the tire parameters will differ per each user. The implemented tire models are listed in subsection 7.2.

| Symbol | Unit | Description |
|---|---|---|
| m | kg | Mass of the vehicle. |
| J | $\mathrm{kg\,m^2}$ | Inertia matrix. |
| h | m | Distance of CG to the spring anchor point. |
| wheelbase | m | Wheelbase, distance between front and rear tires. |
| $\mathrm{f_r}$ | - | Rolling resistance constant. |
| $\mathrm{C_d}$ | - | Drag coefficient. |
| $\rho$ | $\mathrm{kg\,m^{-3}}$ | Air density. |
| A | $\mathrm{m^2}$ | Drag area. |
| $\mathrm{J_{wheel}}$ | $\mathrm{kg\,m^2}$ | Wheel moment of inertia. |
| r | m | Wheel radius. |
| $\mathrm{c_i}$ | m | Front/rear spring stiffness. |
| $\mathrm{d_i}$ | m | Front/rear damping coefficient. |
| $\mathrm{S_{l/r}}$ | m | Distance of CG from center along $y$-axis (left/right). |
| $\mathrm{L_{f/r}}$ | m | Distance of CG from center along $x$-axis (front/rear). |

Table 3: Vehicle parameters

# 5 Coordinate transformations

The various sections of this document describe the model compoments in either body-fixed or wheel-fixed coordinate system. The following rotation matrices are used for transformation between various coordinate systems.

Wheel-fixed to body-fixed:

$$^{V}\mathbf{T}_{\mathrm{w_i}} = \begin{bmatrix} \cos(\delta_i)\cos(\Theta) & -\sin(\delta_i)\cos(\Theta) & -\sin(\Theta) \\ \sin(\phi)\sin(\Theta)\cos(\delta_i)+\cos(\phi)\sin(\delta_i) & -\sin(\phi)\sin(\Theta)\sin(\delta_i)+\cos(\phi)\cos(\delta_i) & \sin(\phi)\cos(\Theta) \\ \cos(\phi)\sin(\Theta)\cos(\delta_i)-\sin(\phi)\sin(\delta_i) & -\cos(\phi)\sin(\Theta)\sin(\delta_i)-\sin(\phi)\cos(\delta_i) & \cos(\phi)\cos(\Theta) \end{bmatrix} \tag{1}$$

where $\delta_i$ is the steering angle of wheel $i$, $\phi$ and $\Theta$ are Euler angles (roll and pitch).

Body-fixed to inertial:

$$^{E}\mathbf{T}_{V} = \begin{bmatrix} \cos(\Theta)\cos(\Psi) & \sin(\phi)\sin(\Theta)\cos(\Psi)-\cos(\phi)\sin(\Psi) & \cos(\phi)\sin(\Theta)\cos(\Psi)+\sin(\phi)\sin(\Psi) \\ \cos(\Theta)\sin(\Psi) & \sin(\phi)\sin(\Theta)\sin(\Psi)+\cos(\phi)\cos(\Psi) & \cos(\phi)\sin(\Theta)\sin(\Psi)-\sin(\phi)\cos(\Psi) \\ -\sin(\Theta) & \sin(\phi)\cos(\Theta) & \cos(\phi)\cos(\Theta) \end{bmatrix} \tag{2}$$

where $\phi$, $\Theta$ and $\Psi$ are the Euler angles. For more clarification on how these matrices were derived, see [1].

# 6 Vehicle body

The vehicle body is modeled as a typical rigid body with Newton-Euler equations. The body is moved by suspension forces $\mathbf{F}_i$ at points $\mathbf{r}_i$. The vectors $\mathbf{r}_i$ have the physical meaning of the anchors of the suspension springs and all the suspension forces are applied only at these points.

The translational dynamics is described by

$$\mathrm{m}\left(\dot{\mathbf{v}} + \mathbf{\Omega} \times \mathbf{v}\right) = \sum_{\mathrm{i}=1}^{4} \begin{bmatrix} \mathbf{F}_{i,x} \\ \mathbf{F}_{i,y} \\ \mathbf{F}_{i,z} \end{bmatrix} - \frac{1}{2}\mathrm{c_w}\rho\mathrm{A}\sqrt{\mathrm{v_x^2} + \mathrm{v_y^2}} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} + {}^{V}\mathbf{T}_\mathrm{E} \begin{bmatrix} 0 \\ 0 \\ -\mathrm{mg} \end{bmatrix} \tag{3}$$

The forces $\mathbf{F}_i$ are in body-fixed coordinates, the matrix ${}^{V}\mathbf{T}_E$ transforms the earth-fixed gravitational acceleration to vehicle-fixed coordinates. The rest of the variables are vehicle-fixed. The term $-\frac{1}{2}c_w\rho A\sqrt{v_x^2 + v_y^2} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$ described aerodynamics resistance.

The rotational dynamics is described by

$$\mathbf{J}\dot{\mathbf{\Omega}} + \mathbf{\Omega} \times (\mathbf{J}\mathbf{\Omega}) = \sum_{i=1}^{4} \mathbf{r}_i \times \mathbf{F}_i + \mathbf{r}_\mathrm{a} \times \mathbf{F}_\mathrm{a} \tag{4}$$

where the vector $\mathbf{F}_\mathrm{a}$ describes aerodynamic forces acting at the center of aerodynamics pressure $\mathbf{r}_\mathrm{a}$, the vector is w.r.t. center of gravity in vehicle coordinates. The vector $\mathbf{r}_i$ is the point of application of the force $\mathbf{F}_i$, its values are determined from the dimensions of the body, eg for the first, front-left, wheel the value would be $\mathbf{r}_1 = \begin{bmatrix} L_\mathrm{f} \\ S_\mathrm{l} \\ -h \end{bmatrix}$.

# 7 Chassis

## 7.1 Suspension

The suspension is modeled as spring-damper systems acting on each wheel individually.

$$\mathbf{F}_{i,z} = -\left(c_i \Delta l_i + d_i \dot{\Delta l_i}\right) {}^{V}\mathbf{T}_E \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

where $\Delta l_i$ is the compression of spring $i$ and the multiplication by ${}^{V}\mathbf{T}_E \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ means that the force acts only along the (inertial) $z_E$-axis (the spring is assumed to always point upwards with respect to the inertial coordinates).

## 7.2 Tire models

The model comes with 2 tire model: Pacejka2002 and simplified Pacejka model with constant coefficients. Both models use slipratio $\lambda$ and slipangle $\alpha$ as their inputs.

The notation in this section will differ from [1] at some places, most importantly the slip angle will be denoted by $\lambda$ instead of $s$.

### 7.2.1 Slip variables

Longitudinal (circumferential) slip is calculated as

$$\lambda_i = \frac{\dot{x}_i - r\omega_i}{\max\left(|r\omega_i|, |\dot{x}_i|\right)} \tag{6}$$

and the slip angle as

$$\alpha_i = -\arctan\left(\frac{\dot{y}_i}{\max\left(|r\omega_i|, |\dot{y}_i|\right)}\right) \tag{7}$$

where $\dot{x}_i/\dot{y}_i$ are the velocities of the wheel center points along $x$ and $y$ axes in the wheel-fixed coordinate system.

The velocities of the wheel center point are obtained by

$$\mathbf{v}_i = \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{bmatrix} = {}^{\mathrm{w_i}}\mathbf{T}_V \mathbf{v}_{\mathrm{w_i}} \tag{8}$$

where ${}^{\mathrm{w_i}}\mathbf{T}_V$ is a rotation matrix transforming the wheel center point from vehicle coordinates $\mathbf{v}_{\mathrm{w_i}}$ to wheel coordinates $\mathbf{v}_i$. The velocity $\mathbf{v}_{\mathrm{w_i}}$ is calculated as

$$\mathbf{v}_{\mathrm{w_i}} = \begin{bmatrix} \dot{x}_{\mathrm{w_i}} \\ \dot{y}_{\mathrm{w_i}} \\ \dot{z}_{\mathrm{w_i}} \end{bmatrix} = \mathbf{v} + \mathbf{\Omega} \times \mathbf{r}_{\mathrm{w_i}} + {}^{V}\mathbf{T}_E \begin{bmatrix} 0 \\ 0 \\ -\dot{l}_{F_i} \end{bmatrix}, \forall i \in \{1,2,3,4\} \tag{9}$$

where $\mathbf{r}_{\mathrm{w_i}}$ is the position of the $i^{\mathrm{th}}$ wheel center with respect to the center of gravity, in vehicle coordinate system. The matrix ${}^{V}\mathbf{T}_E$ transforms earth-fixed coordinates into vehicle-fixed. The vector $\mathbf{r}_{\mathrm{w_i}}$ can be obtained as

$$\mathbf{r}_{\mathrm{w_i}} = \mathbf{r}_{A_i} + {}^{V}\mathbf{T}_E \begin{bmatrix} 0 \\ 0 \\ -l_{F_i} \end{bmatrix}, \forall i \in \{1,2,3,4\} \tag{10}$$

where $\mathbf{r}_{A_i}$ is the position of the spring anchor with respect to the vehicle coordinates.

### 7.2.2 Simplified Pacejka

The simplified model was modelled according to [2]. The model uses constant coefficients $B, C, D, E$ for the Magic formula:

$$F = D\cos\left(C\arctan\left(Bx - E\left(Bx - \arctan\left(Bx\right)\right)\right)\right) \tag{11}$$

where is either sideslip angle $\alpha$ or longitudinal slip $\lambda$. F is either $F_y$,$M_z$ or $F_x$, depending on the input argument.

Coefficients $B, C, D, E$ are generally time-variant and dependant on what F means. In this model, $B, C, D, E$ are constant for given F. So for calculating $F_y$,$M_z$ and $F_x$, one would need 3 sets of these parameters.

This approach would make forces $F_x$ and $F_y$ independent, which is never the case in the real world. This dependancy is often expressed with traction ellipse (also called friction ellipse,Kamm's circle)

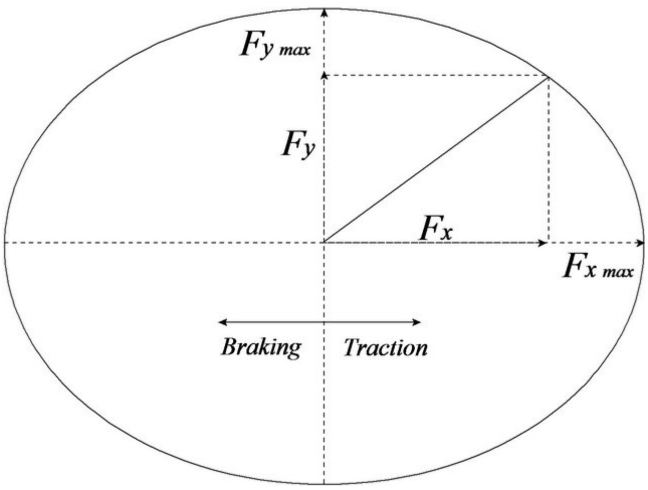



Figure 3: Traction ellipse for constant $B, C, D, E$.
x-axis is for $\lambda$, y-axis is for $\alpha$. Note that maximum force can be achieved only when either $\lambda$ or $\alpha$ are zero.
Source: http://www.racingfr.com/forum/lofiversion/index.php/t47367-450.html

To capture this dependancy, a method called "Combined Slip with friction ellipse" from [2] is used.

#### Combined Slip with friction ellipse

Let's call the forces calculated from Equation 11 $F_{x,max}$ and $F_{y,max}$ as in Figure 3.

$$\alpha^* = \sin\left(\alpha\right) \tag{12}$$

$$\beta = \arccos\left(\frac{|\lambda|}{\sqrt{\lambda^2 + {\alpha^*}^2}}\right) \tag{13}$$

$$\mu_{x,act} = \frac{F_{x,max}}{F_z} \qquad \mu_{y,act} = \frac{F_{y,max}}{F_z} \tag{14}$$

$$\mu_{x,max} = \frac{D_x}{F_z} \qquad \mu_{y,max} = \frac{D_y}{F_z} \tag{15}$$

$$\mu_x = \frac{1}{\sqrt{\left(\frac{1}{\mu_{x,act}}\right)^2 + \left(\frac{\tan(\beta)}{\mu_{y,max}}\right)^2}} \tag{16}$$

$$\mu_y = \frac{\tan(\beta)}{\sqrt{\left(\frac{1}{\mu_{x,max}}\right)^2 + \left(\frac{\tan(\beta)}{\mu_{y,act}}\right)^2}} \tag{17}$$

$$F_x = \frac{\mu_x}{\mu_{x,act}} F_{x,max} \tag{18}$$

$$F_y = \frac{\mu_y}{\mu_{y,act}} F_{y,max} \tag{19}$$

Forces $F_x$ and $F_y$ are now respecting the traction ellipse from Figure 3. The force $F_z$ is load on the tire, the resultant force from the spring-damper system.

Note that the shape of the ellipse is determined by $F_{x,max}$ and $F_{y,max}$.

#### 7.2.3 Pacejka2002

The other model is Pacejka2002, implemented according to [3] and [2]. The model is too complex and beyond the scope of this article.

The input for this model is about 120 constant coefficients, describing the behaviour of the tire. The dynamics of the tire (or the dynamics of $\lambda$ and $\alpha$ to be more precise) is not implemented, only steady state behaviour.

The current implementation uses coefficients from the Automotive challenge 2018 (`http://www.med-control.org/med2018/?page_id=634`) organized by Rimac Automobili.

## 8 Powetrain

The powertrain model for wheel $i$ is

$$J_{\mathrm{w_i}} \ddot{\rho}_{\mathrm{w_i}} = \tau_i - r F_{\mathrm{w_i,x}} \tag{20}$$

where $J_{\mathrm{w_i}}$ is the wheel moment of inertia, $\omega_{\mathrm{w_i}}$ is the angular velocity of the wheel. $M_a$ and $M_b$ are input and braking torques respectively, $r$ is the wheel radius and $F_{\mathrm{w_i,x}}$ is the longitudinal force of the wheel (In wheel-fixed coordinate system!).

To further enhance the model, the motor power

$$\mathrm{Power} = \tau_{\mathrm{i}} \cdot \omega_{\mathrm{i}} \tag{21}$$

and maximum torque are limited. This is implemented in Simulink.

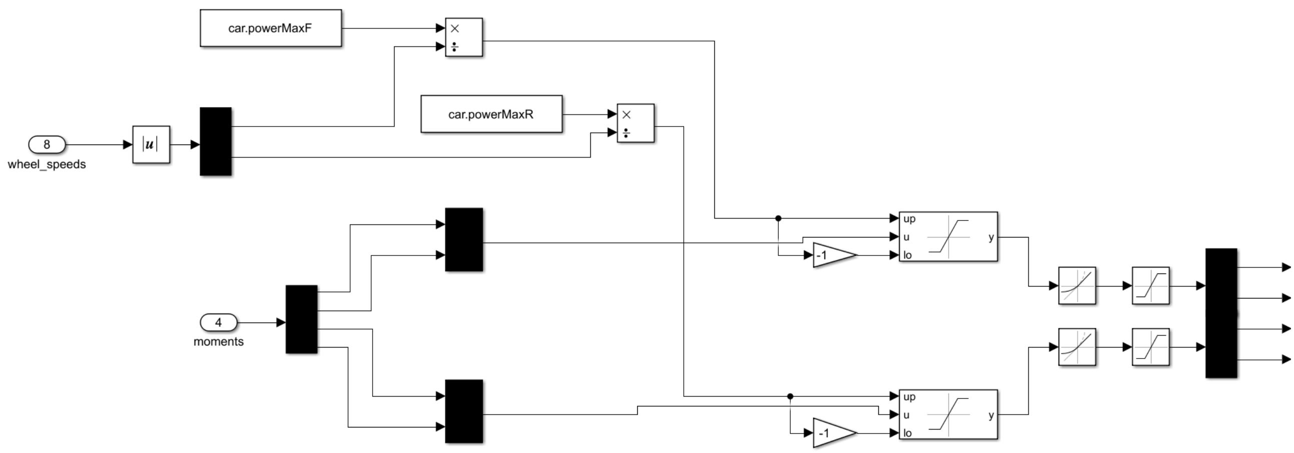



Figure 4: Maximum power to front and rear is limited by Dynamic saturation blocks, the resulting torque is then limited by Rate limiter and Saturation blocks.

## References

[1] Dieter Schramm; Manfred Hiller, Roberto Bardini *Vehicle Dynamics Modeling and Simulation*, Springer-Verlag Berlin Heidelberg 2014

[2] *Using the PAC2002Tire Model*, downloaded from http://mech.unibg.it/~lorenzi/VD&S/Matlab/Tire/tire_models_pac2002.pdf

[3] H.B. Pacejka, *Tyre and Vehicle Dynamics*, Butterworth-Heinemann 2002