# Ch. 22 - Infinite Sequences

**Def** An infinite sequence of real numbers is a fn whose domain is $\mathbb{N}$.

**Notation** we could denote a sequence by $a$, and its particular values by $a(1), a(2), \dots$
Instead, a sequence is usually denoted by $\{a_n\}$ and values by $a_1, a_2, \dots$

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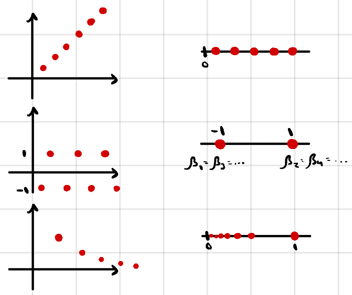
## Examples

| sequence | definition | written as |
|---|---|---|
| $\alpha$ | $\alpha_n = n$ | $\{n\}$ |
| $\beta$ | $\beta_n = (-1)^n$ | $\{(-1)^n\}$ |
| $\gamma$ | $\gamma_n = 1/n$ | $\{1/n\}$ |

graph

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**Def**: A seq. $\{a_n\}$ converges to $\ell$ ($\lim_{n\to\infty} a_n = \ell$) iff for every $\epsilon > 0$ there is $N \in \mathbb{N}$ s.t. $\forall n, n \in \mathbb{N} \wedge n > N \to |a_n - \ell| < \epsilon$.

Alternatively, $\{a_n\}$ approaches $\ell$
has the limit $\ell$

A seq. is said to converge if it converges to some $\ell$.

A seq. diverges if it does not converge.

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## Example

Let $a_n = \sqrt{n+1} - \sqrt{n}$. Let's prove that $a_n$ converges to 0, ie $\lim_{n\to\infty} a_n = 0$

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} < \epsilon \to n > \frac{1}{4\epsilon^2}$$

Thus, $\forall \epsilon > 0, \exists N = \frac{1}{4\epsilon^2}$ s.t. $\forall n, n \in \mathbb{N} \wedge n > N \to 0 < \sqrt{n+1} - \sqrt{n} < \epsilon \to |a_n| < \epsilon$.

Thus, $\lim_{n\to\infty} a_n = 0$.

Alternatively, let $f(x) = \sqrt{x}$ on $[n, n+1]$

$$\text{MVT} \to \exists x \in (n, n+1) \text{ s.t. } f'(x) = \frac{\sqrt{n+1} - \sqrt{n}}{1} = \frac{1}{2\sqrt{x}} < \frac{1}{2\sqrt{n}} < \epsilon \to n > \frac{1}{4\epsilon^2}$$

Thus, $\lim_{n\to\infty} a_n = 0$.

**Example** $\lim\limits_{n\to\infty} \dfrac{3n^3+7n^2+1}{4n^3-8n+63}$

$$= \lim_{n\to\infty} \frac{3+\frac{7}{n}+\frac{1}{n^3}}{4-\frac{8}{n^2}+\frac{63}{n^3}}$$