Ch 13 Appendix - Riemann Sums

$P = \{t_0, \dots, t_n\}$ a partition of $[a,b]$

For each i, choose $x_i \in [t_{i-1}, t_i]$.

Then

$$L(f,P) \le \sum_i^n f(x_i)\Delta t_i \le U(f,P)$$

Any sum $\sum_i^n f(x_i)\Delta t_i$ is a **Riemann sum**

Theorem 1: Suppose f is integrable on $[a,b]$. Then for every $\epsilon > 0$, there is $\delta > 0$ s.t. if $P = \{t_0, \dots, t_n\}$ is partition of $[a,b]$ with $t_i - t_{i-1} < \delta$ for all i then

$$\left| \sum_i^n f(x_i)\Delta t_i - \int_a^b f(x)dx \right| < \epsilon$$

for any Riemann sum formed by choosing $x_i \in [t_{i-1}, t_i]$

Proof

f integr. $\to$ f bounded $\to \exists M,\ |f| \le M$

Let $P^* = \{u_0, \dots, u_k\}$ s.t. $U(f,P^*) - L(f,P^*) < \epsilon/2$

choose δ s.t. $\delta < \dfrac{\epsilon}{4Mk}$

For any P s.t. $\Delta t_i < \delta$ we can break $U(f,P) - L(f,P) = \sum_{i=1}^n (M_i - m_i)\Delta t_i$ into two sums

Some of the Δt_i's are completely contained in $[u_{j-1}, u_j]$ for some j.

The part of the sum for such Δt_i's is $< \epsilon/2$.

The other remaining Δt_i's have $t_{i-1} < u_j < t_i$ for some $j = 1, \dots, k-1$. Thus, there are at most $k-1$ of such Δt_i's.

The sum for these Δt_i's is smaller than $(k-1)\cdot(2M\delta) = \epsilon/2$.

Thus, $U(f,P) - L(f,P) < \epsilon$

But we know that

$$L(f,P) \le \sum_i^n f(x_i)\Delta t_i,\ \int_a^b f \le U(f,P)$$

Therefore

$$\left| \sum_i^n f(x_i)\Delta t_i - \int_a^b f(x)dx \right| < \epsilon$$