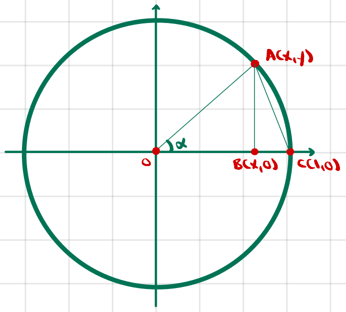
Ch.16 - π is Irrational

1. (a)



$\alpha \le \frac{\pi}{4}$

$$\text{Area}(OAC) = \frac{1}{2}\sqrt{\frac{1-\sqrt{1-16(A(OAB))^2}}{2}}$$

Proof

$$\begin{cases} xy = 2A_{OAB} & \text{area equals } 2A_{OAB} \\ x^2+y^2=1 \to x=\sqrt{1-y^2} & \text{on the unit circle} \end{cases}$$

$$y\sqrt{1-y^2} = 2A_{OAB}$$

$$y^2 - y^4 = 4A_{OAB}^2$$

$$y^4 - y^2 + 4A_{OAB}^2 = 0$$

$$(y^2)^2 - (y^2) + 4A_{OAB}^2 = 0$$

$$\Delta = 1 - 4\cdot 4 \cdot A_{OAB}^2 = 1-16A_{OAB}^2$$

Note that for $\alpha \le \pi/4$ we have $A_{OAB} \le \dfrac{\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{2} = \dfrac{1}{4}$

$\Delta \ge 0 \to 1-16A_{OAB}^2 \ge 0 \to A_{OAB} \le \dfrac{1}{4}$, so this condition is met for $\alpha \le \pi/4$.

$$y^2 = \frac{1 \pm \sqrt{1-16A_{OAB}^2}}{2}$$

$$0 \le \alpha \le \frac{\pi}{4} \to 0 \le \sin(\alpha) = y \le \frac{\sqrt{2}}{2} \to y^2 \le \frac{1}{2}$$

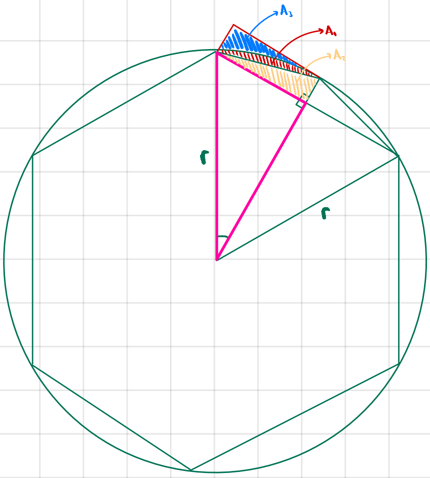
$$\to y^2 = \frac{1-\sqrt{1-16A_{OAB}^2}}{2}$$

$$\text{T.F. } A_{OAC} = \frac{y}{2} = \frac{1}{2}\sqrt{\frac{1-\sqrt{1-16A_{OAB}^2}}{2}}$$

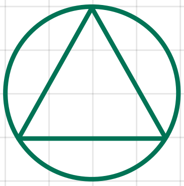
(b) P_m = regular polygon, m sides, inscribed in the unit circle

A_m = area of P_m

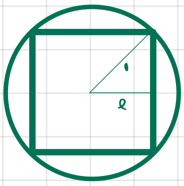
$$A_{2m} = \frac{m}{2}\sqrt{2-2\sqrt{1-\left(\frac{2A_m}{m}\right)^2}}$$

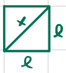


P_3



P_4





$x^2 = 2\ell^2 \to x = \ell\sqrt{2}$

$1 = \ell\sqrt{2} \to \ell = \frac{\sqrt{2}}{2}$

$A_4 = \sqrt{2}\cdot\sqrt{2} = 2$

From part (a), $\text{Area}(OAC) = \frac{1}{2}\sqrt{\frac{1-\sqrt{1-16(A(OAB))^2}}{2}}$

$2m \cdot OAB$ represents the area of P_m, $2m \cdot OAC$ represents the area of P_{2m}.

Ie, $A_m = 2m\, A_{OAB}$

$A_{2m} = 2m\, A_{OAC}$

Thus,

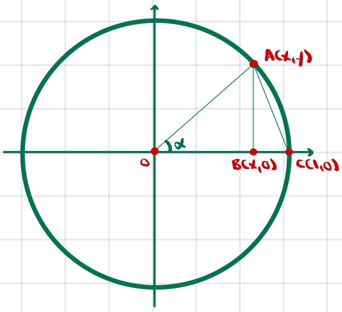
$$A_{2m} = 2m\,\frac{1}{2}\sqrt{\frac{1-\sqrt{1-16(A_m/2m)^2}}{2}} = m\sqrt{\frac{1-\sqrt{1-16(A_m/2m)^2}}{2}} = \frac{m}{2}\sqrt{\frac{4-4\sqrt{1-16(A_m/2m)^2}}{2}}$$

$$= \sqrt{2-2\sqrt{1-4(A_m/m)^2}}$$

2. (a) $\dfrac{A_{OAB}}{A_{OAC}} = OB \quad \rightarrow \quad \dfrac{A_m}{A_{2m}} = \alpha_m$

α_m = distance from O to one side of P_m

Proof



$$\alpha_m = OB = \frac{A_{OAB}}{A_{OAC}} = \frac{A_m}{A_{2m}}$$

(b) $\dfrac{2}{A_{2^n}} = \alpha_4 \cdot \alpha_8 \cdot \alpha_{16} \cdot \dots \cdot \alpha_{2^{n-1}}$

Proof

$A_4 = 2$

$$\frac{2}{A_{2^n}} = \frac{2}{A_8} \cdot \frac{A_8}{A_{16}} \cdot \frac{A_{16}}{A_{32}} \cdot \dots \cdot \frac{A_{2^{n-1}}}{A_{2^n}} = \frac{A_4}{A_8} \cdot \frac{A_8}{A_{16}} \cdot \frac{A_{16}}{A_{32}} \cdot \dots \cdot \frac{A_{2^{n-1}}}{A_{2^n}} = \alpha_4 \cdot \alpha_8 \cdot \dots \cdot \alpha_{2^{n-1}}$$

(c) $\alpha_m = \cos\left(\dfrac{\pi}{m}\right)$

$\cos(x/2) = \sqrt{\dfrac{1+\cos x}{2}}$

$\rightarrow$

$\alpha_4 = \sqrt{1/2}$

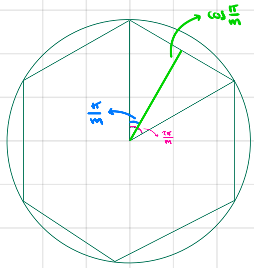
$\alpha_8 = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{1}{2}}}$

$\alpha_{16} = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{1}{2} + \sqrt{\dfrac{1}{2}}}}$

etc

Proof

$\alpha_4 = \dfrac{A_4}{A_8} = \cos(\pi/4) = \dfrac{\sqrt{2}}{2} = \sqrt{\dfrac{1}{2}}$



$\alpha_8 = \cos(\pi/8)$

$\cos(\pi/8) = \sqrt{\dfrac{1+\cos(\pi/4)}{2}}$

$= \sqrt{\dfrac{2+\sqrt{2}}{4}} = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{1}{2}}}$

$$\alpha_{2^n} = \cos(\pi/2^n) = \left[\frac{1+\cos(\pi/2^{n-1})}{2}\right]^{1/2}$$

Recap: what have we done?

We're looking at polygons inscribed in a circle. In particular with m sides and $2m$ sides.

We found the following

$$A_{2m} = \frac{m}{2}\sqrt{2-2\sqrt{1-\left(\frac{2A_m}{m}\right)^2}}$$

→ As we know from Problem 8-11, $A_{circle} - A_{2m} < \dfrac{A_{circle} - A_m}{2}$

→ $\forall \epsilon > 0\ \exists m$ s.t. $A_{circle} - A_m < \epsilon$

$$\frac{A_m}{A_{2m}} = \alpha_m$$

$$\frac{A_4}{A_{2^n}} = \alpha_4 \cdot \alpha_8 \cdot \alpha_{16} \cdots \alpha_{2^{n-1}}$$

product of distances from O to P_m for $m = 4$ to $m = 2^{n-1}$

Since π is the limit of A_{2^n} when $n \to \infty$, then

$$\frac{2}{\pi} = \prod_{i=2}^{\infty} \alpha_{2^i} \quad \to \quad \pi = \frac{2}{\prod_{i=2}^{\infty} \alpha_{2^i}}$$

$$= \frac{2}{\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{(\ldots}}$$

Since $\pi = 2\int_{-1}^{1}(1-x^2)^{1/2}\,dx = A_{circle}$

we can make A_m arbitrarily close to π.

The expression we found for A_m is just a way to actually measure area of the inscribed polygons used to approx π.