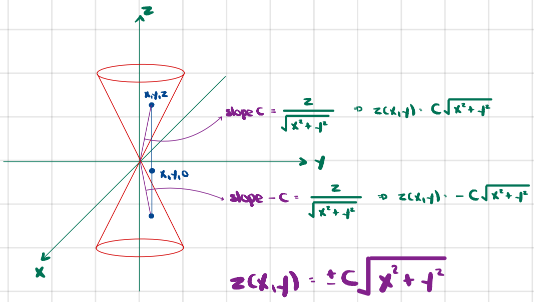
Ch4 Appendix 2: The Conic Sections

→ 3D space

→ Infinite cone

subset of 3D space

specified by single point in 3D, or by 2D point and a slope C

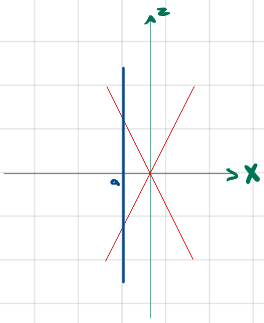


$z(x,y) = \pm C\sqrt{x^2+y^2}$

→ intersect cone w/ plane

case 1: vertical plane, $x = a$

$z = \pm C\sqrt{a^2+y^2}$



$z^2 = C^2(a^2+y^2)$

$z^2 - C^2y^2 = C^2a^2$

$\frac{z^2}{C^2} - y^2 = a^2$

$\frac{z^2}{a^2C^2} - \frac{y^2}{a^2} = 1$ hyperbola

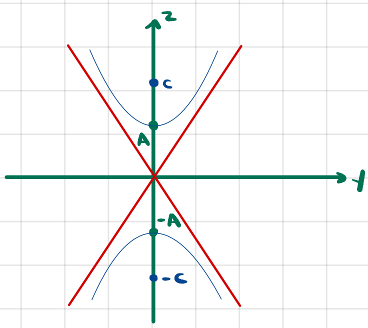
* $A^2 = a^2C^2 \Rightarrow A = \pm aC$

a new C, not the slope C

$B^2 = a^2 = C^2 - A^2 = C^2 - a^2C^2$

$\Rightarrow C^2 = a^2(1+C^2)$

$\Rightarrow C = a\sqrt{1+C^2}$

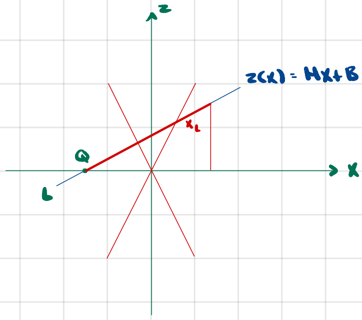


Case 2: non-vertical plane

$z(x,y) = Mx + B$ is a plane in 3D

The intersection w/ the cone is

$Mx + B = \pm C\sqrt{x^2+y^2}$



choose coordinate axes in plane P

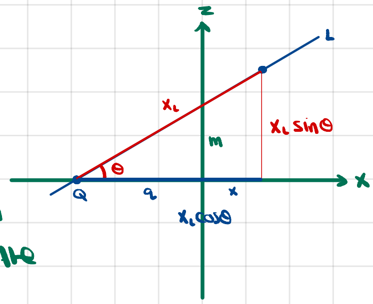
L is the first axis; the second axis passes through Q out of the screen

we consider coordinates in this new coord. system and in the old one

x_L means coordinate x_L along L axis

$x = \alpha x_L + \beta$

$y = y_Q$, where y_Q is the coordinate on the axis passing through Q out the page.



$\frac{m}{x_L\sin\theta} = \frac{q}{x_L\cos\theta}$

The expression $Mx + B = \pm C\sqrt{x^2+y^2}$ becomes, in the plane coords:

$x = x_L\cos\theta - q$
$= \alpha x_L + \beta$

$M(\alpha x_L+\beta) + B = \pm C\sqrt{(\alpha x_L+\beta)^2 + y_Q^2}$

Square both sides

$M^2(\alpha x_L+\beta)^2 + 2M(\alpha x_L+\beta)B + B^2 = C^2[(\alpha x_L+\beta)^2 + y_Q^2]$

$C^2y_Q^2 + (\alpha^2x_L^2 + 2\alpha\beta x_L + \beta^2)(C^2-M^2) - 2M\alpha B x_L - 2M\beta B - B^2 = 0$

$C^2y_Q^2 + x_L^2\alpha^2(C^2-M^2) + x_L[(C^2-M^2)2\alpha\beta - 2M\alpha B] - 2M\beta B - B^2 + \beta^2(C^2-M^2)$

$= C^2y_Q^2 - \alpha^2(M^2-C^2)x_L^2 + Ex_L + F = 0$

$= Ax^2 + Bx + Cy^2 + E = 0$

From problem 4-16 this is either parabola, ellipse, hyperbola.

$M = \pm C \Rightarrow C^2y_Q^2 + Ex_L + F = 0$ parabola, ie intersecting plane has same slope as the sides of the cone.

$C^2 > M^2 \Rightarrow C^2y_Q^2 + \alpha(C^2-M^2)x_L^2 + Ex_L + F = 0$

$$\frac{y_Q^2}{\alpha(C^2-M^2)} + \frac{x_L^2}{C^2} = \frac{-Ex_L - F}{C^2\alpha(C^2-M^2)}$$

$$C^2/Q^2 - \alpha^2 (N^2 - C^2) x_L^2 + E x_L + F = 0$$

$$\alpha^2 (C^2 - N^2) x_L^2 + E x_L$$

$$C^2/Q^2 + x_L^2 \alpha^2 (C^2 - N^2) + x_L \left[(C^2 - N^2) 2\alpha\beta - 2N\alpha B \right] - 2N\beta B - B^2 + \beta^2 (C^2 - N^2)$$