Formal Semantics

Principles of Programming Languages
Lecture 15

CAS CS 320

Practice Problem

```
<s> ::= A <s> B | <a>
<a> ::= A <a> | A <b>
<b> ::= B <b> | B
```

Show that the above grammar is ambiguous.

Answer

```
<s> ::= A <s> B | <a>
<a> ::= A <a> | A <b>
<b> ::= B <b> | B
```

<s>

A <s> B

A <a> B

A A <b> B

A A B B

<s>

<a>

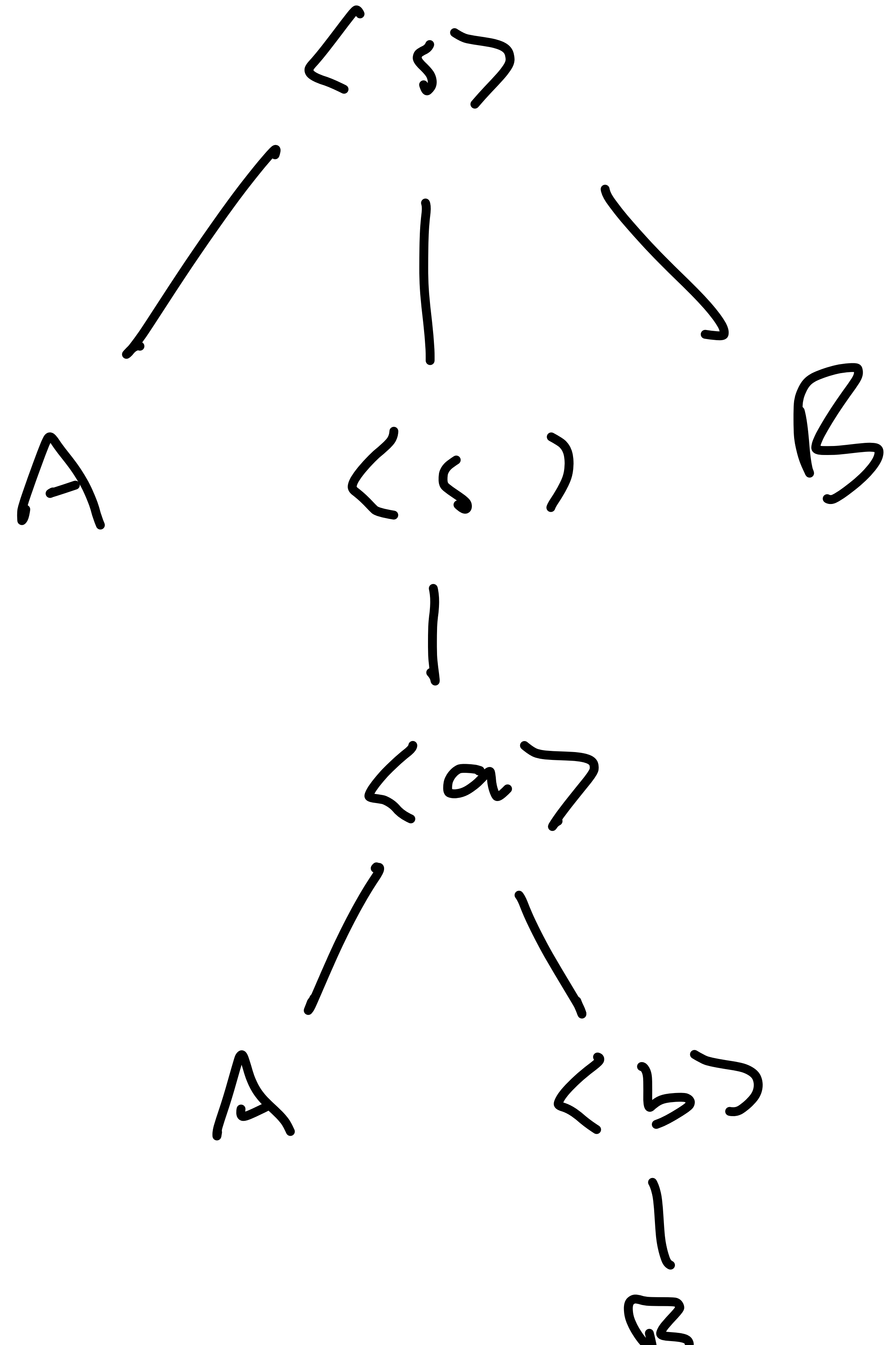
A <a>

A A <b>

A A B <b>

A A B B

A A B B



Outline

Discuss formal semantics in general

Look at small-step and big-step semantics with some examples

Learning Objectives

- Determine the value of the expression e according to a given operational semantics
- Describe the difference between big step and small step semantics
- Derive $e \longrightarrow^{\star} e'$ or $e \Downarrow v$
- Determine the order of evaluation given by a set of semantics rules (when possible)
- What does this program print?

Introduction

A Thought Experiment

```bash
x=3
function f () {
    x=2
}
f
echo $x
```

Bash

```python
x = 3
def f():
    x = 2
f()
print(x)
```

Python

```ocaml
let x = 3
let f () =
  let x = 2 in
  ()
let _ = f ()
let _ = print_int x
```

OCaml

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Question. *How do we know what will happen when a program executes?*

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Usually we build intuitions by writing programs and reading manuals

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OCaml

Question. *How do we know what will happen when a program executes?*

Usually we build intuitions by writing programs and reading manuals

But many decisions about what it means to execute a program are arbitrary (or based on concerns like efficiency)

Meaning

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Syntax is interested in the *form* of a program

```
let x = 2 in
let y = 3 in
x + y
```

Meaning

Syntax is interested in the *form* of a program

Semantics is interested in the *meaning* of a program

```
let x = 2 in
let y = 3 in
x + y
```

```
let y = 3 in
2 + y
```

```
2 + 3
```

```
5
```

Meaning

Syntax is interested in the *form* of a program

Semantics is interested in the *meaning* of a program

What is the meaning of meaning?

```
let x = 2 in
let y = 3 in
x + y
```

↓

```
let y = 3 in
2 + y
```

↓

```
2 + 3
```

↓

```
5
```

Meaning

Syntax is interested in the *form* of a program

Semantics is interested in the *meaning* of a program

What is the meaning of meaning?

Formal semantics is the mathematical study of meaning

```
let x = 2 in
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↓

```
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Aside: Denotational vs. Operational Semantics

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$$1 + 2 * 3 + 4 = 11$$
$$1 + 12 - 2 = 11$$

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Operational semantics is interested in how a programming language "operates" i.e. how a program *behaves* during execution

$$\begin{aligned} 1 + 2 * 3 + 4 &\longrightarrow 1 + 6 + 4 \\ &\longrightarrow 7 + 4 \\ &\longrightarrow 11 \end{aligned}$$

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This course

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Small-Step vs. Big-Step Semantics

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Small-step operational semantics is interested in *program transformation*, i.e., how a program transforms "one step at a time"

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let x = 2 in
let y = 3 in
x + y
```

```
let y = 3 in
2 + y
```

```
2 + 3
```

```
5
```

Small-Step vs. Big-Step Semantics

Small-step operational semantics is interested in *program transformation*, i.e., how a program transforms "one step at a time"

Big-step operational semantics is interested in *evaluation*, i.e., what is the value of the program once a program has finished evaluating

$$\frac{2 \Downarrow 2 \qquad \dfrac{3 \Downarrow 3 \qquad \dfrac{2 \Downarrow 2 \qquad 3 \Downarrow 3}{2 + 3 \Downarrow 5}}{\texttt{let y = 3 in 2 + y} \Downarrow 5}}{\texttt{let x = 2 in let y = 3 in x + y} \Downarrow 5}$$

Small-Step vs. Big-Step Semantics

Small-step operational semantics is interested in *program transformation*, i.e., how a program transforms "one step at a time"

Mini-projects

Big-step operational semantics is interested in *evaluation*, i.e., what is the value of the program once a program has finished evaluating

$$\dfrac{2 \Downarrow 2 \qquad \dfrac{3 \Downarrow 3 \qquad \dfrac{2 \Downarrow 2 \qquad 3 \Downarrow 3}{\texttt{2 + 3} \Downarrow 5}}{\texttt{let y = 3 in 2 + y} \Downarrow 5}}{\texttt{let x = 2 in let y = 3 in x + y} \Downarrow 5}$$

Static vs. Dynamic Semantics

Static vs. Dynamic Semantics

Static semantics refers to the meaning given to a program *before* it is evaluated

```
% ocaml silly.py
File "./silly.py", line 1, characters 8-9:
1 | let x = 2 +. 3.
            ^
Error: This expression has type int but an expression was
       expected of type
         float
  Hint: Did you mean '2.'?
```

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Dynamic semantics refers to the behavior of a program *during* evaluation

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utop # let x = 2 + 3;;
val x : int = 5
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Static vs. Dynamic Semantics

Type checking

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Evaluation

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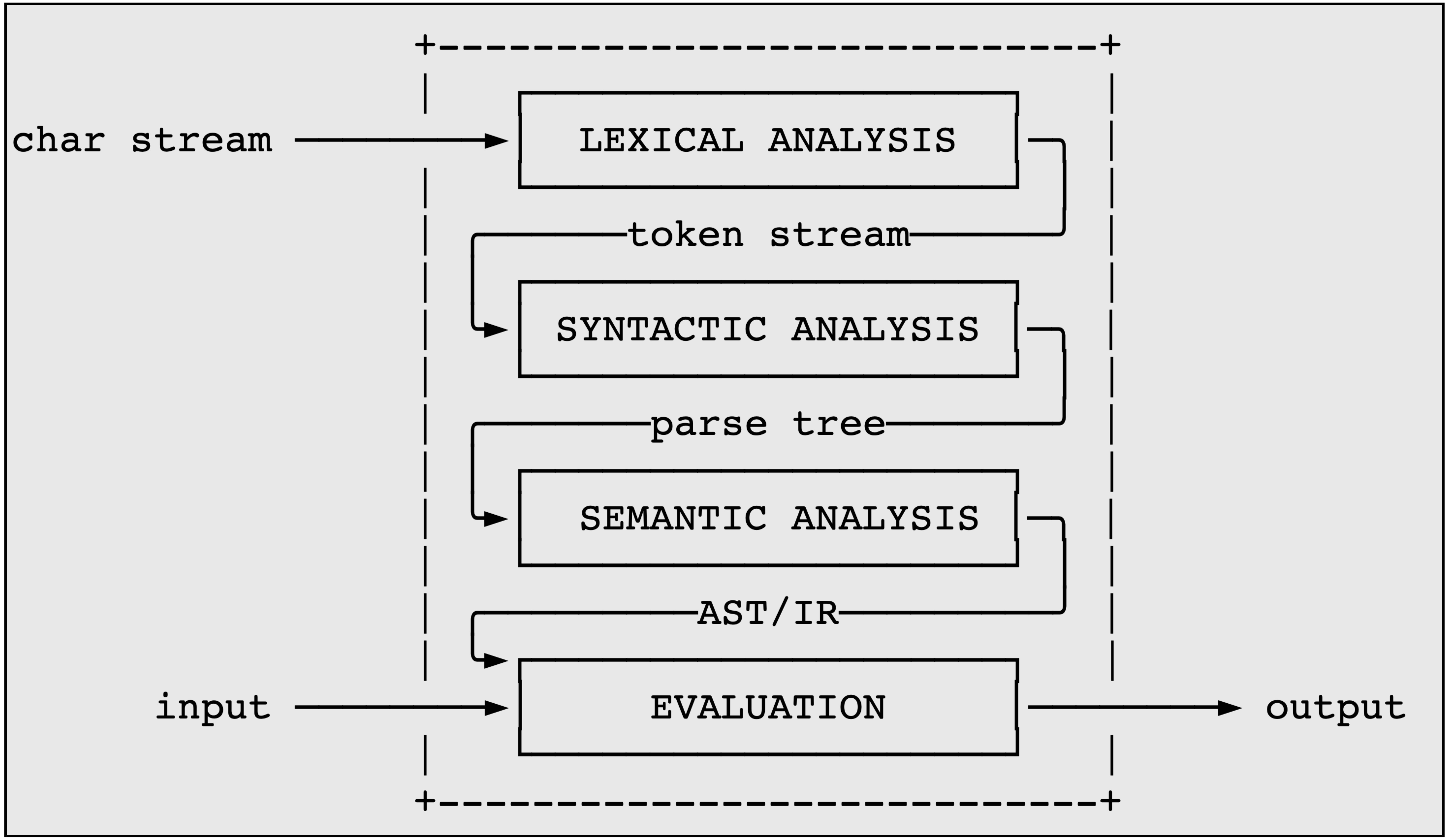
Recall: The Picture



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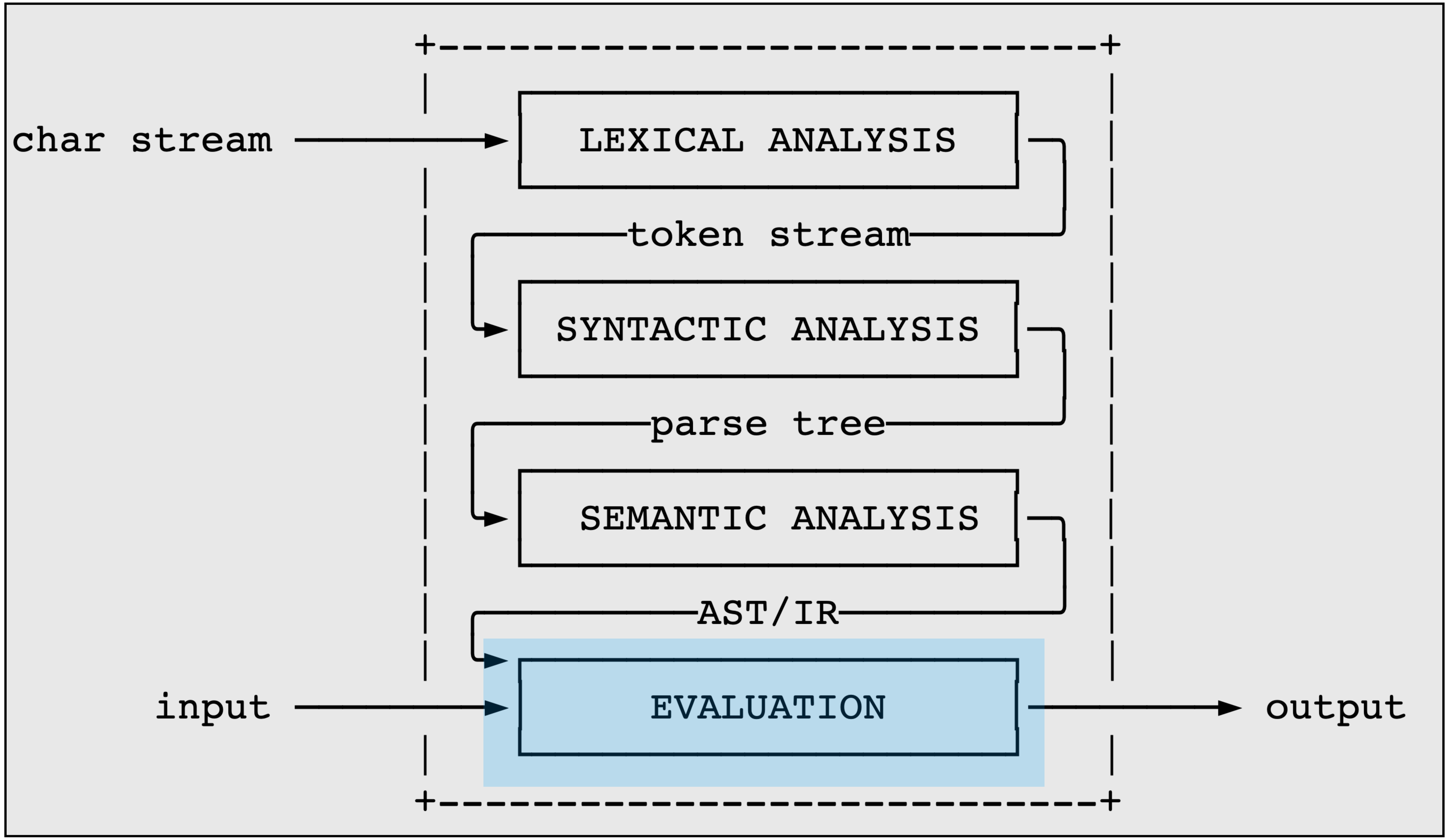

dynamic semantics (this week + next week)

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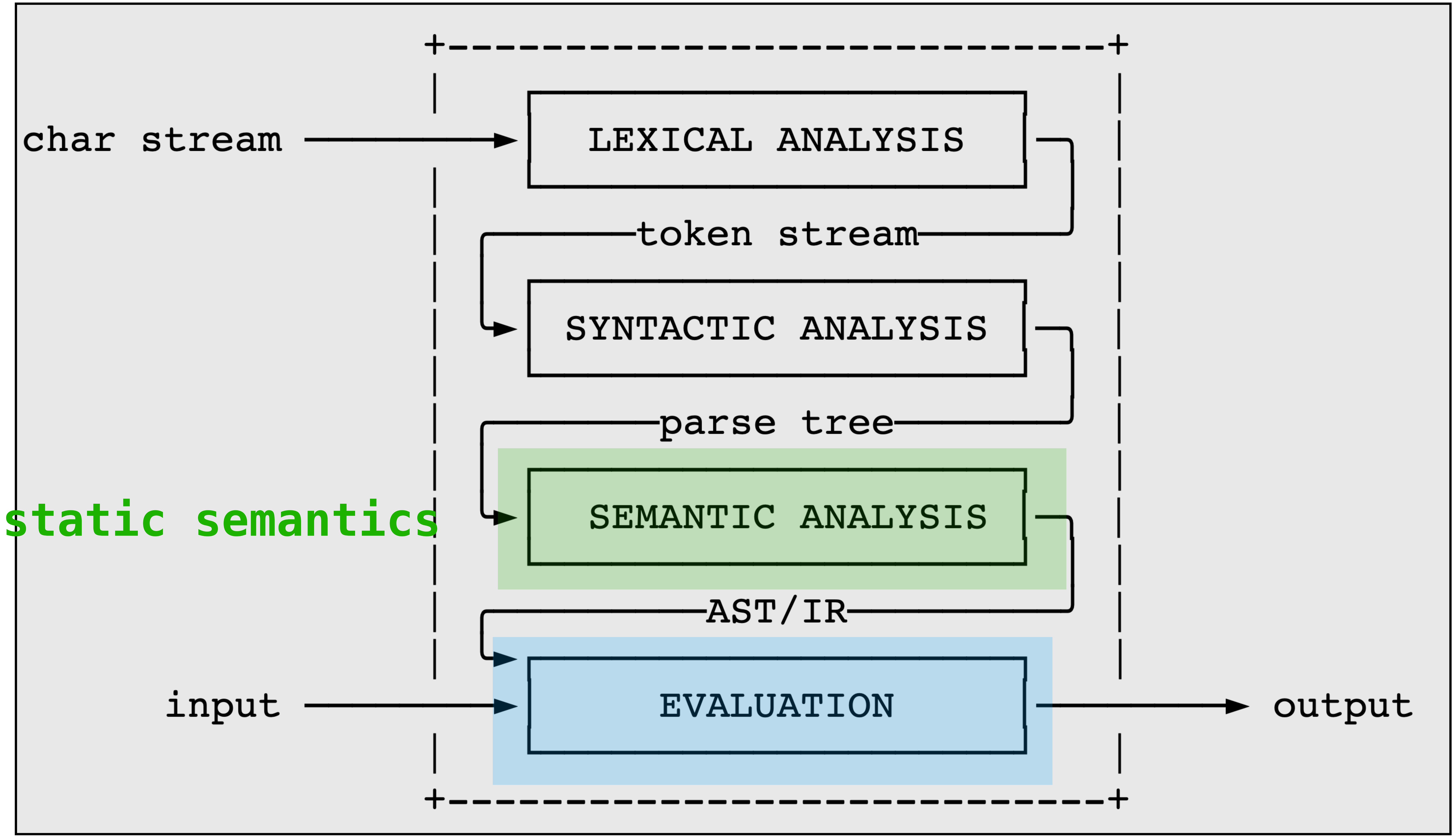

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Operational Semantics

Inference Rules

$$\dfrac{\overset{\texttt{premise}}{P_1} \qquad \overset{\texttt{premise}}{P_2} \qquad \cdots \qquad \overset{\texttt{premise}}{P_k}}{\underset{\texttt{conclusion}}{C}} \;\; \overset{\texttt{name}}{\texttt{red-rule}}$$

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Then general form of a reduction rule has a collection of **premises** and a **conclusion**

There may be no premises, this is called an **axiom**

Premises which are not of the same form as the conclusion are called **side-conditions**

Example

premise

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

conclusion

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub  | eq
<bool> ::= true | false
<int>  ::= ...
```

```
Example Programs:
(add 2 3)
(add (add 2 3) 5)
(eq (add 2 3) (sub 7 2))
(add true 2)
```

Example

premise

$$\frac{e_1 \longrightarrow e_1'}{(\mathsf{add}\ e_1\ e_2) \longrightarrow (\mathsf{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

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Example Programs:
(add 2 3)
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```

If e_1 *reduces to* e_1' *in one step, then* add e_1 e_2 *reduces to* add e_1' e_2 *in one step*

Another Example

$$\frac{n_1 \texttt{ is a number} \qquad n_2 \texttt{ is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

If n_1 *and* n_2 *are numbers then* $(\texttt{add}\ n_1\ n_2)$ *reduces in one step to the number* $n_1 + n_2$

In this case, the premises are side-conditions

We'll come back to these examples...

Small-Step Semantics

$$(S, p) \longrightarrow (S', p')$$

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In general, we define small-step semantics on a **configuration,** which is a program together with some stateful information

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Example: Arithmetic Expressions

$$\underbrace{(\varnothing}_{\text{state}}\;,\;\overbrace{10\times(2+3)}^{\text{program}})\longrightarrow(\varnothing\;,\;10\times5)\longrightarrow(\varnothing\;,\;50)$$

<u>State:</u> none

<u>Program:</u> arithmetic expression

Example: (Fragment of) OCaml

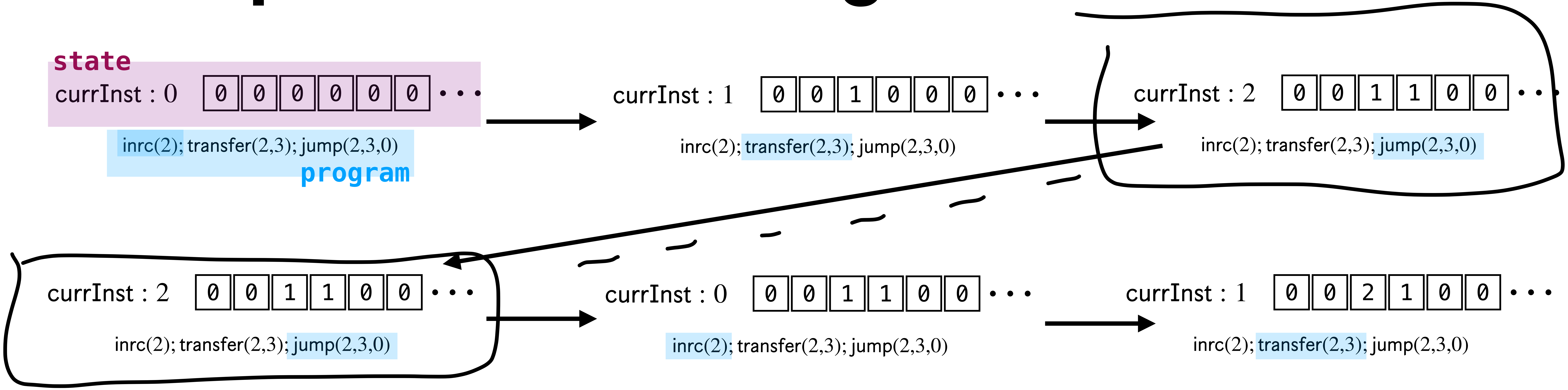
$(\varnothing$, `let x = 3 in if x > 10 then 4 else 5`$)$ $\longrightarrow$ $(\varnothing$, `if 3 > 10 then 4 else 5`$)$
state program
$\longrightarrow$ $(\varnothing$, `if false then 4 else 5`$)$
$\longrightarrow$ $(\varnothing$, `5`$)$

State: none

Program: OCaml expression

For purely functional languages
there is no state

Example: Unlimited Register Machines



<u>State:</u> (current instruction pointer) + (collection of number registers)

<u>Program:</u> sequence of commands for updating registers values and current instruction

Example: Stack-Oriented Language

$$\underset{\text{state}}{(\varnothing}\;,\;\underset{\text{program}}{\texttt{push 2; push 3; add}})\longrightarrow$$

$$(2 :: \varnothing,\ \texttt{push 3; add}) \longrightarrow$$

$$(3 :: 2 :: \varnothing,\ \texttt{add}) \longrightarrow$$

$$(5 :: \varnothing,\ \epsilon)$$

State: stack (i.e., list) of values

Program: sequence of commands for manipulating the stack

High-Level

$$\underbrace{(\overset{}{S}}_{\text{state}},\overset{\text{program}}{p}) \longrightarrow (\underbrace{S'}_{\text{transformed state}},\overset{\text{transformed program}}{p'})$$

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When we define the small-step semantics of a programming language, we need to define two things:

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» What kind of **state** are we manipulating?

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» What kind of **state** are we manipulating?

» What **rules** describe how to transform configurations?

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When we define the small-step semantics of a programming language, we need to define two things:

» What kind of **state** are we manipulating?

» What **rules** describe how to transform configurations?

(we'll elide the state for most of this course)

(add 2 3)

Example

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

$$\frac{e_2 \longrightarrow e_2'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 \boxed{+} n_2}\ \texttt{add-ok}$$

~~(add 2 5) → 7~~

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1'\ e_2)}\ \texttt{sub-left}$$

$$\frac{e_2 \longrightarrow e_2'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1\ e_2')}\ \texttt{sub-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{sub}\ n_1\ n_2) \longrightarrow n_1 - n_2}\ \texttt{sub-ok}$$

e' $\qquad$ e_1'

$$\frac{(\text{add } 2\ 3) \longrightarrow 5}{\text{add } (\text{add } 2\ 3)\ 5 \longrightarrow \text{add } 5\ 5}$$

e' $\qquad$ e_1'

Reduction is a Relation

$$\begin{tikzcd}
& (1+2)+(2+3) \arrow[dl] \arrow[dr] & \\
(1+2)+5 \arrow[dr] & & 3+(2+3) \arrow[dl] \\
& 8 &
\end{tikzcd}$$

It's important to recognize that **reduction is a *relation***

This means there may be multiple choices of reductions

When possible, we try do design our rules to avoid this

Reduction is a Relation

$$\frac{\texttt{add}\ 1\ 2 \longrightarrow 3}{(\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))}\ \texttt{add-left}$$

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There are two reductions from $(\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3))$ in our current rule set.

Reduction is a Relation

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There are two reductions from $(\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3))$ in our current rule set.

We can avoid this by *breaking symmetry*. We will enforce that the right argument can reduced only when the left argument is completely reduced.

Example: Addition

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

$$\frac{v\ \texttt{is a number} \qquad e_2 \longrightarrow e_2'}{(\texttt{add}\ v\ e_2) \longrightarrow (\texttt{add}\ v\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

Enforcing an Evaluation Order

$$\frac{\mathsf{add}\ 1\ 2 \longrightarrow 3}{(\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3)) \longrightarrow (\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3))}\ \texttt{add-left}$$

$$\frac{\mathsf{add}\ 2\ 3 \longrightarrow 5}{(\mathsf{add}\ \boxed{(\mathsf{add}\ 1\ 2)}\ (\mathsf{add}\ 2\ 3)) \longrightarrow (\mathsf{add}\ \boxed{(\mathsf{add}\ 1\ 2)}\ 5)}\ \texttt{add-right}$$

not a number

The new rule enforces that arguments of **add** are evaluated from left to right.

Practice Problem

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

*Write down the reduction rules for **eq** (to the best of your ability) so that the left argument is evaluated before the right argument.*

Answer

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{eq}\ e_1\ e_2) \longrightarrow (\texttt{eq}\ e_1'\ e_2)}$$

$$\frac{v\ \texttt{is a num or bool} \qquad e_2 \longrightarrow e_2'}{(\texttt{eq}\ v\ e_2) \longrightarrow (\texttt{eq}\ v\ e_2')}$$

$$\frac{b_1\ \texttt{is a bool} \qquad b_2\ \texttt{is a bool}}{(\texttt{eq}\ b_1\ b_2) \longrightarrow b_1 = b_2}$$

$$\frac{n_1\ \texttt{is a num} \qquad n_2\ \texttt{is a num}}{(\texttt{eq}\ n_1\ n_2) \longrightarrow n_1 = n_2}$$

Answer

```
<expr> ::= ( <op> <expr> <expr> )
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$$\frac{n_1 \ \texttt{is a num} \qquad n_2 \ \texttt{is a num}}{(\texttt{eq}\ n_1\ n_2) \longrightarrow n_1 = n_2}$$

Looks a lot like pattern matching.

Two Questions

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- Show that $C \longrightarrow C'$.
- Given C, determine a configuration C' such that $C \longrightarrow C'$ (and show that it holds).

Two Questions

Once we have an operational semantics, there are two questions we can ask (as PL designers and on the final exam):

- Show that $C \longrightarrow C'$.
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Derivations

$$\dfrac{10 \texttt{ is a number} \qquad \dfrac{\dfrac{1 \texttt{ is a number} \qquad 2 \texttt{ is a number}}{(\texttt{add } 1\ 2) \longrightarrow 3}\ \texttt{add-ok}}{(\texttt{add } (\texttt{add } 1\ 2)\ (\texttt{add } 2\ 3)) \longrightarrow (\texttt{add } 3\ (\texttt{add } 2\ 3))}\ \texttt{add-left}}{(\texttt{sub } 10\ (\texttt{add } (\texttt{add } 1\ 2)\ (\texttt{add } 2\ 3))) \longrightarrow (\texttt{sub } 10\ (\texttt{add } 3\ (\texttt{add } 2\ 3)))}\ \texttt{sub-right}$$

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Definition (Informal): A **derivation** is a tree of reductions, gotten by applying reduction rules. The leaves are trivial premises.

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Definition (Informal): A **derivation** is a tree of reductions, gotten by applying reduction rules. The leaves are trivial premises.

A derivation is a proof that the reduction step is valid in the operational semantics.

How To: Building Derivations

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow \texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))$$

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`We can build derivations from the ground up, applying rules in reverse.`

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If the reduction is valid, then at each step we should be able to find a rule to apply.

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$$
\dfrac{\texttt{10 is a number} \qquad \dfrac{(\texttt{add}\ 1\ 2) \longrightarrow 3}{(\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))}\ \texttt{add-left}}{\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow \texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))}\ \texttt{sub-right}
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Two Questions

Once we have a small-step semantics, there are two questions we can ask (as PL designers and on the final exam):

- Show that $C \longrightarrow C'$.
- Given C, determine a configuration C' such that $C \longrightarrow C'$ (and show that it holds).

Single-Step Evaluation

$$(\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3))) \longrightarrow ???$$

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The more "realistic" situation is to be given a program and then try to figure out what it evaluates to in a single step.

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The more "realistic" situation is to be given a program and then try to figure out what it evaluates to in a single step.

This is why we want to be careful about how we design our rules: *we don't want to get too caught up on which rule to apply.*

How To: Performing Single-Step Evaluation

$$(\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3))) \longrightarrow ??$$

We can perform a single evaluation step by again, build derivations from the ground up.

If we've designed our rules well (e.g., by enforcing evaluation order) there should always be a rule to use.

How To: Performing Single-Step Evaluation

$$\overbrace{(\mathsf{sub}\ 10\ (\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3)))}^{\mathsf{sub}\ n\ e} \longrightarrow ??$$

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$$\frac{10 \ \texttt{is a number} \qquad (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow ??}{(\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3))) \longrightarrow (\texttt{sub}\ 10\ ??)}\ \texttt{sub-right}$$

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Practice Problem

$$(\texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))) \longrightarrow (\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5))$$

Give a derivation of the above reduction

Answer

$$
\dfrac{10 \in \mathbb{Z} \qquad \dfrac{3 \in \mathbb{Z} \qquad \dfrac{2 \in \mathbb{Z} \qquad 3 \in \mathbb{Z} \qquad 2+3=5}{(\mathsf{add}\ 2\ 3) \longrightarrow 5}}{\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3) \longrightarrow (\mathsf{add}\ 3\ 5)}}{(\mathsf{sub}\ 10\ (\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3))) \longrightarrow (\mathsf{sub}\ 10\ (\mathsf{add}\ 3\ 5))}
$$

Multi-Step Reduction Relation

$C \longrightarrow C' \longrightarrow C'' \longrightarrow C''' \longrightarrow \cdots \longrightarrow D$

$$\frac{}{C \longrightarrow^\star C}\ \texttt{refl} \qquad \frac{C \longrightarrow C' \qquad C' \longrightarrow^\star D}{C \longrightarrow^\star D}\ \texttt{trans}$$

Given any single-step reduction relation, we can derive the **multi-step reduction relation:**

» Every configuration reduces to itself **(reflexivity)**

» Every $\longrightarrow^\star$ reduction can be extended by a single step **(transitivity)**

Two Questions (Again)

Once we have an operational semantics, there are two questions we can ask (as PL designers and on the final exam):

» Show that $C \longrightarrow^{\star} C'$.

» Given C, determine a configuration C' such that $C \longrightarrow^{\star} C'$ **and** C' **cannot be reduced.**

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Once we have an operational semantics, there are two questions we can ask (as PL designers and on the final exam):

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How To: Derivations of Multi-Step Reductions

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2$$

want to show

» Derive all necessary single-step evaluations.

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$\texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3)) \longrightarrow \texttt{sub}\ 10\ (\texttt{add}\ 3\ 5)$ (you did this)

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$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow \texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))$ `(we did this)`

$\texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3)) \longrightarrow \texttt{sub}\ 10\ (\texttt{add}\ 3\ 5)$ `(you did this)`

$\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5) \longrightarrow \texttt{sub}\ 10\ 8$ `(exercise)`

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$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^\star 2$

`want to show`

$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow \texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))$ `(we did this)`

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$\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5) \longrightarrow \texttt{sub}\ 10\ 8$ `(exercise)`

$\texttt{sub}\ 10\ 8 \longrightarrow 2$ `(easy)`

» Derive all necessary single-step evaluations.

How To: Derivations of Multi-Step Reductions

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2$$

- » Derive all necessary single-step evaluations
- » Combine them with the **transitivity rule.**

How To: Derivations of Multi-Step Reductions

$$\dfrac{\begin{array}{c}\texttt{(we did this)}\\ \vdots \\ \texttt{s}\ 10\ (\texttt{a}\ (\texttt{a}\ 1\ 2)\ (\texttt{a}\ 2\ 3)) \longrightarrow \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3))\end{array} \qquad \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3)) \longrightarrow^{\star} 2}{\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2}\ \texttt{trans}$$

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    \qquad
    \dfrac{
      \begin{array}{c}\texttt{(an exercise)}\\ \vdots \\ \texttt{s}\ 10\ (\texttt{a}\ 3\ 5) \longrightarrow \texttt{s}\ 10\ 8\end{array}
      \qquad
      \texttt{s}\ 10\ 8 \longrightarrow^{\star} 2
    }{
      \texttt{s}\ 10\ (\texttt{a}\ 3\ 5) \longrightarrow^{\star} 2
    }\ \texttt{trans}
  }{
    \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3)) \longrightarrow^{\star} 2
  }\ \texttt{trans}
}{
  \texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2
}\ \texttt{trans}
$$

- Derive all necessary single-step evaluations
- Combine them with the **transitivity rule.**

How To: Derivations of Multi-Step Reductions

(easy)

$$\dfrac{\dfrac{\vdots}{\mathsf{s}\ 10\ 8 \longrightarrow 2} \qquad 2 \longrightarrow^{\star} 2}{\mathsf{s}\ 10\ 8 \longrightarrow^{\star} 2}\ \texttt{trans}$$

(an exercise)

$$\dfrac{\dfrac{\vdots}{\mathsf{s}\ 10\ (\mathsf{a}\ 3\ 5) \longrightarrow \mathsf{s}\ 10\ 8} \qquad \mathsf{s}\ 10\ 8 \longrightarrow^{\star} 2}{\mathsf{s}\ 10\ (\mathsf{a}\ 3\ 5) \longrightarrow^{\star} 2}\ \texttt{trans}$$

(you did this)

$$\dfrac{\dfrac{\vdots}{\mathsf{s}\ 10\ (\mathsf{a}\ 3\ (\mathsf{a}\ 2\ 3)) \longrightarrow \mathsf{s}\ 10\ (\mathsf{a}\ 3\ 5)} \qquad \mathsf{s}\ 10\ (\mathsf{a}\ 3\ 5) \longrightarrow^{\star} 2}{\mathsf{s}\ 10\ (\mathsf{a}\ 3\ (\mathsf{a}\ 2\ 3)) \longrightarrow^{\star} 2}\ \texttt{trans}$$

(we did this)

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        \qquad
        \dfrac{}{2 \longrightarrow^{\star} 2}\,\texttt{refl}
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  }{\mathtt{s}\ 10\ (\mathtt{a}\ 3\ (\mathtt{a}\ 2\ 3)) \longrightarrow^{\star} 2}\,\texttt{trans}
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$$

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Two Questions (Again)

Once we have an operational semantics, there are two questions we can ask (as PL designers and on the final exam):

- Show that $C \longrightarrow C'$.
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How To: Evaluation

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star}\ ??$$

want to show

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow\ ??$$

If our rules are well defined, then should be easy:

Solve this single-step evaluation problem until you reach a configuration that cannot be further reduced

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want to show

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$$\texttt{sub}\ 10\ 8 \longrightarrow 2$$

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Solve this single-step evaluation problem until you reach a configuration that cannot be further reduced

When are we done?

When evaluating, there are **three** cases "end" cases:

» **value:** we reach the end of our computation and the value of our program

(add 2 3) → $\boxed{5}$ ✓

» **stuck:** we reach an expression that cannot be reduced, but that is not a value there are no stuck terms after type checking

(add true 3) $\not\to$

ex. divByZero

» **diverge:** the computation never reaches a point where the expression is not reducible

$e_1 \to e_2 \to e_3 \to e_1 \to e_2 \dots$

moving onto big-step...

Big-Step Semantics

$$(\mathsf{sub}\ 10\ (\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3))) \Downarrow 2$$

Big-Step Semantics

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This is what we've been doing in this course so far

Example

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

$$\frac{n \ \texttt{is a number}}{n \Downarrow n}\ \texttt{numEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \ \texttt{is a number} \qquad v_2 \ \texttt{is a number}}{(\texttt{add}\ e_1\ e_2) \Downarrow v_1 + v_2}\ \texttt{addEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \ \texttt{is a number} \qquad v_2 \ \texttt{is a number}}{(\texttt{sub}\ e_1\ e_2) \Downarrow v_1 - v_2}\ \texttt{subEval}$$

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we'll remove these side conditions once we have type-checking

Practice Problem

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

Write the rule for ***eq***

Answer

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

Relation to Small-Step

$$e \longrightarrow^{\star} v \qquad \approx \qquad e \Downarrow v$$

`The big-step relation` **`"cuts out the middle steps"`** `of a small-step relation`

`This means fewer and clearer rules, but less fine-grain control of the evaluation sequence`

`Note:` `We can't always have both small-step and big-step!`

Order of Evaluation

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$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{add } e_1\ e_2) \Downarrow v_1 + v_2}\ \texttt{addEval}$$

With small-step semantics, we can choose the order of evaluations based on the rules

With big-step semantics, we can't because our relation only deals with the *final* value, nothing intermediate

We will take the order of operations to be from left to right

What is a value?

$$e \Downarrow \underbrace{\boxed{v}}_{}$$

value

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`Anything we want it to be`

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But we get to **choose** what our values are (we will usually define them separately as an ADT)

This will turn out to be very useful for mini-project 2

Taking Stock

big-step	single-step	multi-step
$e \Downarrow v$	$e \longrightarrow e'$	$e \longrightarrow^\star e'$
e evaluates to v	*e reduces to e' in a single step*	*e reduces to e' in many steps*

demo

(how does this look in code)