Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 06 - Lax Richtmyer, Interpolation

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well posed problems : $f : \mathcal{X} \longrightarrow \mathcal{Y}$

1) + 2) $\forall x \in \mathcal{X} \quad \exists ! \; y \in \mathcal{Y}$ s.t. $f(x) = y$

3) $\exists \; K_{abs}$ s.t. $\forall \delta x, \forall x \quad f(x+\delta x) = y + \delta y$

$$\|\delta y\|_{\mathcal{Y}} \leq K_{abs} \|\delta x\|_{\mathcal{X}}$$

4) $\exists \; K_{rel}$ s.t.

$$\frac{\|\delta y\|_{\mathcal{Y}}}{\|y\|_{\mathcal{Y}}} \leq K_{rel} \frac{\|\delta x\|_{\mathcal{X}}}{\|x\|_{\mathcal{X}}}$$

Numerical approximation of f is a "sequence" of problems $\boxed{f_n}_{\mathcal{X}_n, \mathcal{Y}_n}$ $\boxed{(\text{or } f_h)}$ that "approximate" f.
depending on approximation parameter n or h

1) Approximation is consistent if "$f_n \to f$"
assuming that the domain of f_n is $\mathcal{X}$

$$\|f_n(x) - f(x)\|_{\mathcal{Y}} \to 0 \quad \text{as} \quad n \to \infty$$

is exactly the same as the definition of convergence

Example

$\mathcal{X} : C^1([x_0, x_0+1])$

$\mathcal{Y} : \mathbb{R}$

$$f(u) := u'(x_0)$$

$$f_n(u) := \frac{u\left(x_0+\frac{1}{n}\right) - u(x_0)}{1/n}$$

$$\|u\|_{\mathcal{X}} := \|u\|_{L^\infty([x_0,x_0+1])} + \|u'\|_{L^\infty([x_0,x_0+1])}$$

$$|f(u)| = |u'(x_0)| \leqslant \|u\|_{\mathcal{X}}$$

$$\leq n\left[\left|u\left(x_0+\tfrac{1}{n}\right)\right| + |u(x_0)|\right]$$

$$|f_n(u)| = n\left|u\left(x_0+\frac{1}{n}\right) - u(x_0)\right| \leqslant 2n\|u\|_{L^\infty} \leq 2n\|u\|_{\mathcal{X}}$$

$$y = f(u) = u'$$

$$f(\delta u + u) = \delta u' + u'$$

$$f(u+\delta u) = f(u) + f(\delta u)$$

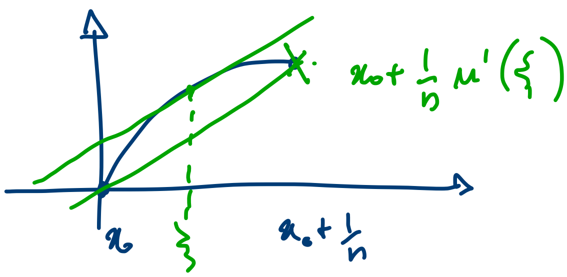
$$\delta y + y = f(\delta u + u) = \delta u' + u'$$

$$\Rightarrow \quad \delta y = \delta u'$$

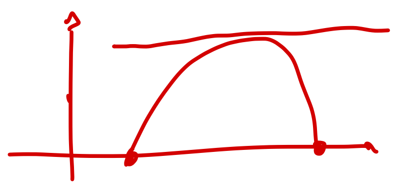
$$|\delta y_n| = n\left|\delta u\left(x_0+\frac{1}{n}\right) - \delta u(x_0)\right|$$

Alternative for f_n: Taylor expansion theorem

observe that $u\left(x_0+\frac{1}{n}\right) = u(x_0) + \frac{1}{n}u'(\xi)$ for $\xi \in \left(x_0, x_0+\frac{1}{n}\right)$



$\Rightarrow$



By definition $\exists\, \xi \in \left(x_0, x_0 + \frac{1}{n}\right)$

$$f_n(u) = u'(\xi)$$

$$|f_n(u)| = |u'(\xi)| \leq \|u'\|_{L^\infty}$$

$$\|f(\delta u)\| \leq 1 \cdot \|\delta u'\|_{L^\infty}$$

$$\|f_n(\delta u)\| \leq 1 \cdot \|\delta u'\|_{L^\infty}$$

Polynomial interpolation $\quad f, \mathcal{X}, \mathcal{Y}$

Given a set of $(n+1)$ interpolation points $\{x_i\}_{i=0}^{n}$,

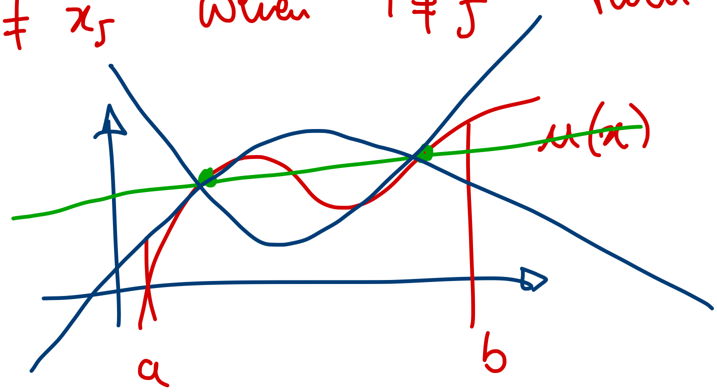
$$\mathcal{X} : C^0([a,b]), \quad \|u\|_{\mathcal{X}} := \max_{x\in[a,b]} |u(x)| =: \|u\|_\infty$$

$$\mathcal{Y} : \mathbb{P}^n([a,b]) \subset C^\infty([a,b]), \quad \|y\|_{\mathcal{Y}} := \|y\|_\infty$$

$$f : \mathcal{X} \longrightarrow \mathcal{Y}$$

$$u \longrightarrow p \quad \text{s.t.} \quad u(x_i) = p(x_i) \quad i = 0, \dots, n$$

if $x_i \neq x_j$ when $i \neq j$, then $\exists!\, p \in \mathbb{P}^n$ s.t. $u(x_i) = p(x_i)$ for $i = 0, \dots, n$



How do we solve it ?

1) choose a basis $\{v_i\}_{i=0}^{n} \in \mathbb{P}^n$ s.t. $\underline{\mathbb{P}^n = \text{span}\{v_i\}_{i=0}^{n}}$

$\Longrightarrow \forall p \in \mathbb{P}^n \quad \exists! \{p^i\}_{i=0}^{n}$ s.t.

$$p(x) = \sum_{i=0}^{n} p^i v_i(x)$$

2) Solve the system

$$p(x_i) = u(x_i) \qquad \text{for } i = 0 \ldots n$$

$$\underline{\sum_{J=0}^{n} p^J v_J(x_i) = u(x_i)} \iff \underline{\underline{V}}\, \underline{p} = \underline{u}$$

$$\underline{\underline{V}} \in \mathbb{R}^{n+1 \times n+1} \qquad \underline{p}, \underline{u} \in \mathbb{R}^{n+1}$$

$$V_{iJ}\, p^J = u_i \qquad p^J = V^{Ji} u_i$$

$$V_{iJ} V^{Jk} = \delta_i^{\ k} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

$$V_{iJ} := v_J(x_i)$$

Observe that :

$$\| p \|_\infty := \| p^J v_J \|_\infty = \max_{x \in [a,b]} \left| \sum_{J=1}^{n} p^J v_J(x) \right|$$

$$\leq \max_{x \in [a,b]} \left| \sum_{J=1}^{n} |p^J| \, |v_J(x)| \right| \leq \| \underline{p} \|_{\ell^\infty} \max_{x \in [a,b]} \sum_{J=0}^{n} |v_J(x)|$$

$u \in C^0([a,b])$

$f(u) = p \in \mathbb{P}^n([a,b])$

$\| f(u) \| \leq \overset{?}{K} \|u\|_\infty$

$u \in \mathcal{X}$

$f(u) \in \mathbb{P}^n$

$$\boxed{\text{Lebesgue function} \quad \Lambda(x) := \sum_{j=0}^{n} |v_j(x)|}$$

$\|f(u)\|_\infty \leq \|\underline{p}\|_{\ell^\infty} \|\Lambda\|_{L^\infty}$

$\max_{i \in [0,\dots,n]} \{|p_i|\}$ $\qquad$ $\max_{x \in [a,b]} |\Lambda(x)|$

can I control $\|p\|$ with $\|u\|$?

$\underline{\underline{V}}\, \underline{p} = \underline{u}$ $\qquad$ $\underline{u} := \{u(x_i)\}_{i=0}^{n}$

$\underline{p} = \underline{\underline{V}}^{-1} \underline{u}$ $\qquad$ $\|\underline{p}\|_{\ell^\infty} \leq \|V^{-1}\|_{\ell^\infty} \|\underline{u}\|_{\ell^\infty}$

$\leq \|V^{-1}\|_{\ell^\infty} \|u\|_\infty$

$\|f(u)\|_\infty \leq \|V^{-1}\|_{\ell^\infty} \|\Lambda\|_{L^\infty} \|u\|_{L^\infty}$

- Improve $\|V^{-1}\|_{\ell^\infty} \Rightarrow V^{-1} \equiv \mathrm{Id}$

$$v_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\Rightarrow$ Lagrange polynomial basis : $\ell_i := \prod_{\substack{i \neq j \\ j=0}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$

- Improve $\|\Lambda\|_{L^\infty} \Rightarrow$ choose chebyshev points.