Weierstrass approximation theorem

$\forall f \in C^0([a,b])$, $\forall \varepsilon > 0$

$\exists n \in \mathbb{N}$, $p \in \mathbb{P}^n$ s.t. $\|f - p\|_{L^\infty([a,b])} \leq \varepsilon$

Proof:

Let $B_n : C^0([a,b]) \longrightarrow \mathbb{P}^n$ be s.t.

1) B_n is $\underbrace{\text{linear}}_{1}$ and $\underbrace{\text{positive}}_{2}$

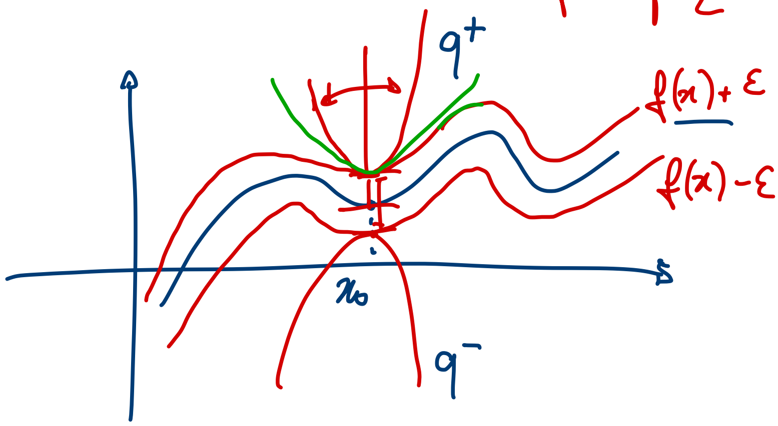
1.1: $B_n(\alpha u + \beta v) = \alpha B_n u + \beta B_n v$ $\quad \forall \alpha, \beta \in \mathbb{R}$, $\forall u, v \in C^0([a,b])$

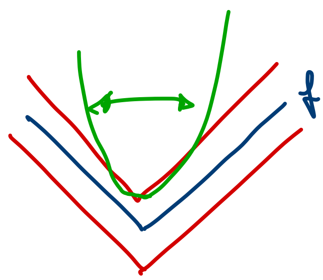
1.2: $f \geq 0 \implies B_n f \geq 0$

2) $\|B_n p - p\|_{L^\infty} \to 0 \quad \forall p \in \mathbb{P}^2$

Then $\|B_n f - f\|_{L^\infty} \to 0 \quad \forall f \in C^0([a,b])$

$\forall \varepsilon > 0 \ \exists n$ s.t. $\|B_n f - f\|_{L^\infty} \leq \varepsilon$





$\forall f \in C^0([a,b])$, $\forall x_0 \in [a,b]$ construct $q^\pm \in \mathbb{P}^2$ s.t.

- $q^-(x; x_0) \leq f(x) \leq q^+(x; x_0) \quad \forall x \in [a,b]$

. use $B_n q^+ \longrightarrow q^+$, $B_n q^- \longrightarrow q^-$ use (2)

. use $B_n(q^+ - f)(x) \geq 0$ $B_n(f - q^-)(x) \geq 0$ use (1.2)

$\forall f \in C^0([a,b])$, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon$ s.t.

by continuity on $[a,b]$ $\boxed{|x_1 - x_2| \leq \delta_\varepsilon \Rightarrow |f(x_1) - f(x_2)| \leq \frac{\varepsilon}{2}}$

$$q^\pm := f(x_0) \pm \left(\frac{\varepsilon}{2} + \frac{2\|f\|_{L^\infty([a,b])}}{\delta_\varepsilon^2} (x - x_0)^2 \right)$$

$$|f(x_1) - f(x_2)| \leq |f(x_1)| + |f(x_2)| \leq 2\|f\|_{L^\infty([a,b])}$$

$x - x_0 \longrightarrow q^\pm(x_1) - q^\pm(x_2)$ explicit computation

For varying x_0 $q^\pm = a^\pm(x_0)\, x^2 + b^\pm(x_0)\, x + c^\pm(x_0)$

$\boxed{a^\pm, b^\pm, c^\pm \text{ depend on } x_0, \|f\|_{L^\infty}, \delta_\varepsilon, \varepsilon}$

$$M := \max_{x_0 \in [a,b]} \left(|a^+|, |a^-|, |b^+|, |b^-|, |c^+|, |c^-| \right)$$

$\boxed{M \text{ will depend on } \delta_\varepsilon, \varepsilon, \|f\|_{L^\infty} \text{ but not on } x_0}$

Choose N s.t.

$$\|B_n x^i - x^i\|_{L^\infty} \leq \frac{\varepsilon}{6M} \quad \forall n \geq N \quad \text{for } i = 0,1,2$$

In x_0 : $\underbrace{\boxed{f(x_0) - \varepsilon} \leq q^-(x_0) - \frac{\varepsilon}{2}}_{\text{definition of } q^-} \leq \underbrace{B_n q^-(x_0) \leq B_n f(x_0)}_{\text{positivity}}$

$$\underbrace{\|B_n q^{\pm} - q^{\pm}\|_{L^\infty} \leq \frac{\varepsilon}{2}}$$

$$q^{\pm} = a^{\pm} x^2 + b^{\pm} x + c^{\pm}$$

$$\|B_n a^{\pm} x^2 - a^{\pm} x^2\|_{L^\infty} = \|a^{\pm}(B_n x^2 - x^2)\|_{L^\infty} \leq \left\| a^{\pm} \frac{\varepsilon}{6M} \right\|$$

$$\leq \frac{\varepsilon}{6}$$

$$\|B_n q^{\pm} - q^{\pm}\|_{L^\infty} \leq 3\left(\frac{\varepsilon}{6}\right) = \frac{\varepsilon}{2}$$

$$\underbrace{\boxed{B_n f(x_0)} \leq B_n q^{+}(x_0)}_{\text{positivity}} \leq \underbrace{q^{+}(x_0) + \frac{\varepsilon}{2} \leq \boxed{f(x_0) + \varepsilon}}_{\text{definition of } q^{+}}$$

$$\boxed{\|B_n f - f\|_{L^\infty([a,b])} \leq \varepsilon}$$

Construction of B_n is given by the Bernstein polynomials.

Set $[a,b] = [0,1]$

$$1 = 1^n = \boxed{((1-x)+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i}}$$

$$= \sum_{i=0}^{n} b_i^n(x)$$

$$b_i^n(x) := \binom{n}{i} x^i (1-x)^{n-i}$$

ith Bernstein polynomial of order n

$$B_n u := \sum_{i=0}^{n} b_i^n(x)\, u\left(\frac{i}{n}\right)$$

$$\sum_{i=0}^{n} b_i^n(x) = 1 \quad \Longrightarrow \quad \boxed{B_n 1 = 1}$$

$$\sum_{i=0}^{n} b_i^n(x)\left(\frac{i}{n}\right) = x \quad \Longrightarrow \quad \boxed{B_n x = x}$$

$$\sum_{i=0}^{n} b_i^n(x)\left(\frac{i}{n}\right)^2 = \left(\frac{n-1}{n}\right)x^2 + \frac{1}{n}x \qquad \boxed{B_n x^2 = \left(\frac{n-1}{n}\right)x^2 + \frac{1}{n}x}$$