Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 03 - LH - Basic concepts - Condition numbers

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- Vector Spaces (we will work mostly with Banach and Hilbert Spaces)
- Norms
- Subspaces, linear combinations, spans, ...

$\boxed{\text{• Vector Space}}$ Real Numbers and a set V of "vectors"

$\forall\, \underline{u}, \underline{v} \in V$, $a, b \in \mathbb{R}$, it "makes sense to write"

- $\underline{u} + \underline{v} = \underline{w} \in V$
- $a\underline{u} = \underline{w} \in V$
- $a\underline{u} + b\underline{v} = \underline{w} \in V$
- $(a+b)\underline{u} = a\underline{u} + b\underline{u} \in V$
- $(ab)\underline{u} = a(b\underline{u}) \in V$
- $\exists\, \underline{0} \in V$ s.t. $\underline{u} + \underline{0} = \underline{u} \quad \forall \underline{u} \in V$
- $\exists\, 1 \in \mathbb{R}$: $1\underline{v} = \underline{v} \quad \forall \underline{v} \in V$

$V = \mathbb{R}^n \quad (n=2)$

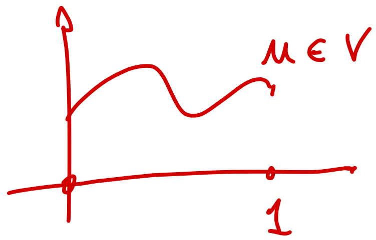
$u = (u_1, u_2)$

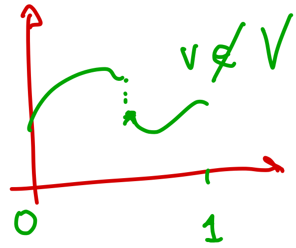
$v = (v_1, v_2)$

$a u := (a u_1, a u_2)$

$u + v := (u_1 + v_1, u_2 + v_2)$

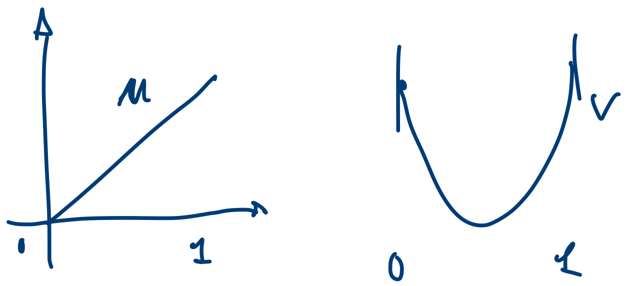
$V = C^0([0,1])$ all continuous functions in $[0,1]$



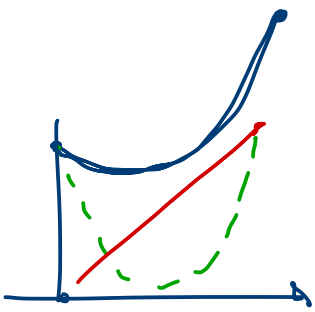


$u, v \in C^0([0,1])$

$w := u + v \qquad w(x) = u(x) + v(x) \ \forall x \in [0,1]$



$w = u + v$



· (Semi-) norm on vector space V

$$|\cdot| : V \longrightarrow \mathbb{R}_0^+$$

$$u \longrightarrow |u|$$

1) $|a u| = |a|\,|u| \qquad \forall a \in \mathbb{R}, \forall u \in V$

2) $|u + v| \leq |u| + |v|$

It is a norm, and we indicate it with $\|\cdot\|$ if

3) $\|u\| = 0 \iff \underline{u} = \underline{0}$

ℓ_p norm in $\mathbb{R}^n$

$$\|u\|_p := \left(\sum_{i=0}^{n-1} |u_i|^p \right)^{\frac{1}{p}}$$

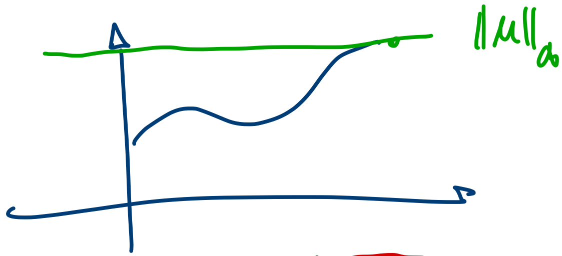
$$p = \infty \Rightarrow \|u\|_\infty := \max_i |u_i|$$

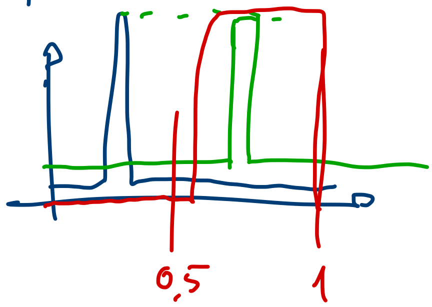
L_p norm in $C^0([0,1])$

??

$$\|u\|_p := \left(\int_0^{0.5} |u|^p \right)^{\frac{1}{p}}$$

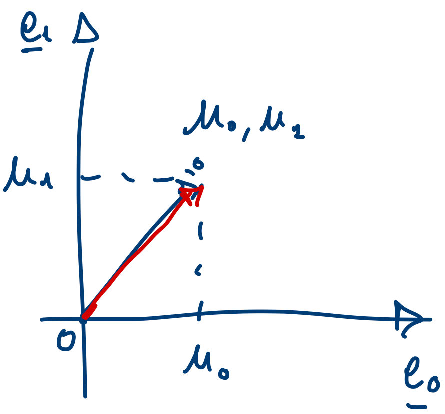
$$\|u\|_\infty := \max_{x \in [0,1]} |u(x)|$$





$\mathbb{R}^2$: euclidean space of dimension 2

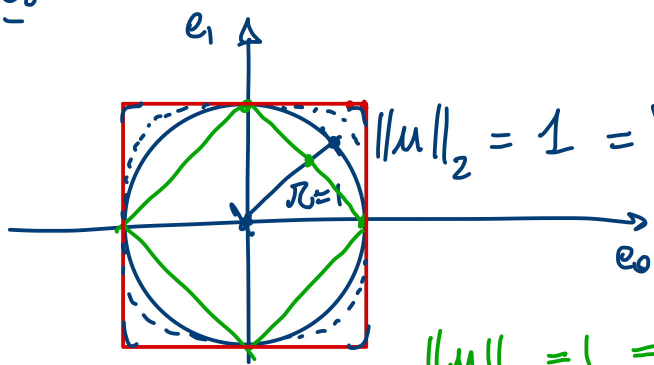
$u = (u_0, u_1) \qquad v = (v_0, v_1)$



$$\|u\|_2 := \sqrt{u_0^2 + u_1^2}$$

$$\|u\|_1 := |u_0| + |u_1|$$

$$\|u\|_\infty := \max(|u_0|, |u_1|)$$

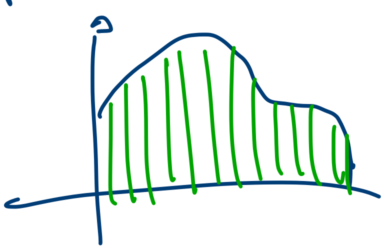


$$\|u\|_2 = 1 = \sqrt{u_0^2 + u_1^2}$$

$$\|u\|_1 = 1 = |u_0| + |u_1|$$

$$\|u\|_\infty = 1 = \max(u_0, u_1)$$

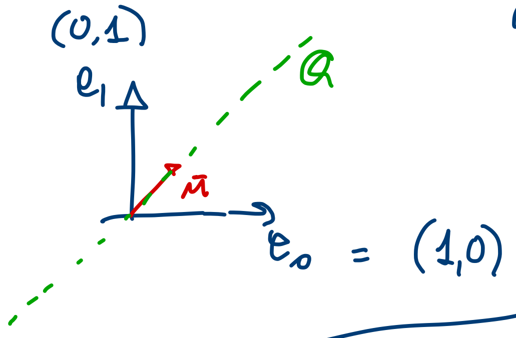
$f \in C^0([0,1])$



$$\|f\|_1 := \int_0^1 |f|$$

A Subspace $Q \subseteq V$ is a subset of V which is closed w.r.t. addition in V:

$$u, v \in Q \Rightarrow au + bv = w \in Q \quad \forall a, b \in \mathbb{R}$$



given $\bar{u} \in V$,
$Q = \{\alpha \bar{u}, \text{ with } \alpha \in \mathbb{R}\}$

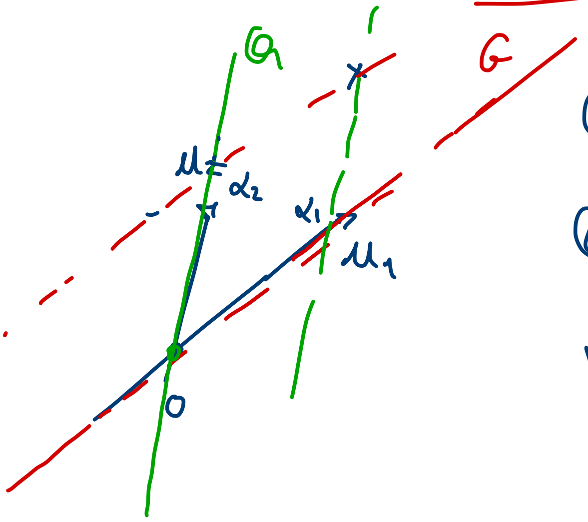
is a subset of $\mathbb{R}^2$

Span $\{u_i \in V\}_{i=1}^{n}$ is defined as the subspace of V s.t.

$$Q = \text{Span}\{u_i \in V\}_{i=1}^{n} := \left\{ \sum_{i=1}^{n} \alpha_i u_i, \text{ with } \alpha_i \in \mathbb{R} \right\}$$

Let Q, G be subspaces of V

Q and G are linearly independent if $Q \cap G = \underline{0}$



example:

$G = \text{Span}\{u_1\}$

$Q = \text{span}\{u_2\}$

$V = \text{Span}\{u_1, u_2\} = \mathbb{R}^2$

Dimension of a vector space V?

Minimum number of vectors (basis) I need to represent $\underline{V}$ $\quad \dim(V)$

Examples:

$C^0([0,1])$, $\mathbb{P}^n([0,1])$ polynomials of order n

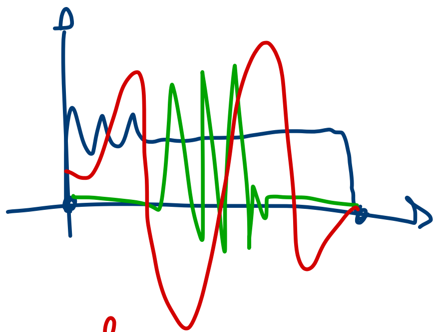
$\mathbb{P}^n([0,1])$ is a subspace of $C^0([0,1])$ of dimension $(n+1)$

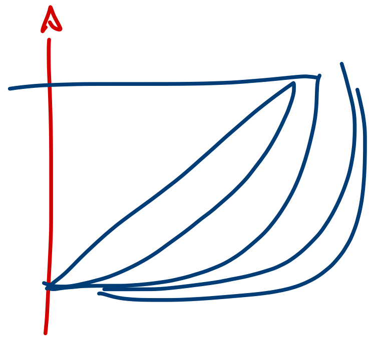
An example basis: monomials : $\{x^i\}_{i=0}^n$

$$\underline{\mathbb{P}^n([0,1]) := \text{Span}\{x^i\}_{i=0}^n}$$

$$\Longleftrightarrow$$

$$\forall p \in \mathbb{P}^n([0,1]), \quad \exists \{p_i\}_{i=0}^n \text{ s.t. } p = \sum_{i=0}^n p_i (x)^i$$





Example

Every polynomial of order 2, can be written as

$$p(x) = \overset{p_2}{a} x^2 + \overset{p_1}{b} x + \overset{p_0}{c}$$

and it is homeomorphic to $\mathbb{R}^3$

$$\underline{\mathbb{P}^n \text{ is homeomorphic to } \mathbb{R}^{n+1}}$$