Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 11 - L2 projection and polynomial integration

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Polynomial Interpolation

Bernstein Approximation

L^2 projection

V : Vector space + norm (we concentrate on Hilbert Space):

V is Hilbert if $\quad \|u\|_V := \sqrt{(u,u)} \quad \forall u \in V$

Q a subspace of V, we call $p \in Q$ the best approximation in Q of a function $u \in V$ iif:

① $\|p-u\|^2 \leq \|q-u\|^2 \quad \forall q \in Q$

for Hilbert spaces ① $\Leftrightarrow$ ②

② $(p,q) = (u,q) \quad \forall q \in Q$

$(p-u, q) = 0 \quad \forall q \in Q$

Assume that $\underline{Q = \text{span}\{v_i\}_{i=0}^{n}}$ finite dimensional of dimension $\underline{\underline{n+1}}$

then $\forall p \in Q \quad \exists! \ \{p^j\}_{j=0}^{n}$ s.t.

$$p(x) = \sum_{i=0}^{n} p^j v_j(x)$$

Asking (2) means :

$$\forall q \in Q \Rightarrow \exists! \ \{q^i\}_{i=0}^{n} \quad \text{s.t.} \quad q(x) = \sum_{i=0}^{n} q^i v_i(x)$$

$$(p - u, q) = 0 \iff \sum_j \sum_i \left(\underline{\underline{p^j v_j}} - u, \ \underline{\underline{q^i v_i}} \right) = 0 \quad \forall q^i \in \mathbb{R}$$

$$\iff [M][p] - [u] = 0 \qquad [p] = [M]^{-1}[u]$$

$$[M]_{ij} := (v_j, v_i) \qquad [u]_i = (u, v_i)$$

$$p^i = M^{ij} u_j \qquad M^{ij} M_{jk} = \delta^i_k$$

Example : $(u, v) := \int_a^b u v \, dx$ L^2 scalar product

$$\|u\| := \sqrt{(u, u)} = \left(\int_a^b u^2 \, dx \right)^{\frac{1}{2}}$$

$$Q := \mathbb{P}^n([a, b])$$ polynomials of order n

Monomials: $Q = \text{span}\{v_i\}_{i=0}^{n} = \text{span}\{x^i\}_{i=0}^{n}$

$$M_{ij} := \int_a^b x^i x^j \, dx \overset{\text{assume } a=0,\, b=1}{=} \ = \int_0^1 x^{(i+j)}$$

$$\left. \frac{x^{i+j+1}}{i+j+1} \right|_a^b = \frac{1}{i+j+1} \Rightarrow \text{Hilbert matrix}$$

Legendre basis functions: $\quad ([a,b] = [0,1])$

$$V_0 = 1$$

$$(v_i, v_j) = \delta_{ij} \quad \forall i,j \qquad v_i \in \mathbb{P}^i$$

Graham Schmidt procedure:

$$V_0 = 1 \qquad \psi_k := x V_{k-1} - \sum_{j=0}^{k-1} (x V_{k-1}, V_j) V_j$$

$$V_k := \frac{\psi_k}{\|\psi_k\|}$$

$$\underbrace{}_{} \delta_{jm} \quad \forall m < k$$

$$(\psi_k, V_m) = (x V_{k-1}, V_m) - \sum_{j=0}^{k-1} (x V_{k-1}, V_j)(V_j, V_m)$$

$$= (x V_{k-1}, V_m) - (x V_{k-1}, V_m) = 0$$

$$\forall i, \quad (p, v_i) = 0 \quad \forall\ p \in \mathbb{P}^k \text{ with } k < i$$

$$(u - p, q) = 0 \iff \|u-p\| \leq \|u-q\| \quad \forall q \in Q$$

"$\Leftarrow$" $\quad p$ is b.a. $\Rightarrow (u-p, q) = 0 \ \forall q \in Q$

$$\|u-p\|^2 \leq \|u-p+tq\|^2 \qquad \forall t \in \mathbb{R}, \ \forall q \in Q$$

$$\|u-p\|^2 \leq (u-p+tq, u-p+tq) = \|u-p\|^2 + 2(u-p, tq) + t^2\|q\|^2$$

$$0 \leq 2t(u-p, q) + t^2\|q\|^2$$

$$0 \leq 2(u-p,q) + t\|q\|^2 \qquad t>0$$

$$0 \geq 2(u-p,q) + t\|q\|^2 \qquad t<0$$

$$-|t|\|q\|^2 \leq 2(u-p,q) \leq |t|\|q\|^2 \qquad \forall t, \forall q$$

$$\Leftrightarrow \qquad (u-p,q) = 0 \quad \forall q$$

" $\Rightarrow$ "

$(u-p,q)=0 \quad \Rightarrow \quad p$ is b.a.

$$\|u-q\|^2 = \|u-p+p-q\|^2 = \|u-p\|^2 + \|p-q\|^2 + \cancel{2(u-p,p-q)}^{\,0}$$

$$\|u-q\|^2 = \|u-p\|^2 + \underbrace{\|p-q\|^2}_{\geq 0} \Rightarrow \|u-p\| \leq \|u-q\| \quad \forall q$$

For polynomials:

$$M_{ij} := (v_j, v_i) \qquad u_i := (u, v_i) := \int_a^b u\, v_i$$

We need a way to compute $\int_a^b f\,dx$ with $f = u v_i$

We use "Interpolatory quadrature rules".

Given $\{q_i\}_{i=0}^n$ $n+1$ quadrature points

- Construct I^n : polynomial interpolation

• Integrate $I^n u$ instead of u

We say that a quadrature formula is exact of order k if it integrates exactly polynomials of order k.

$$Int(u) := \int_a^b u \qquad \forall u \in C^0([a,b]) \cap L^2([a,b])$$

$$Int^n(u) := \int_a^b I^n u = \sum_{i=0}^{n} \overbrace{\int_a^b \ell_i}^{w_i} u(q_i)$$

$$= \sum_{i=0}^{n} w_i \, u(q_i)$$

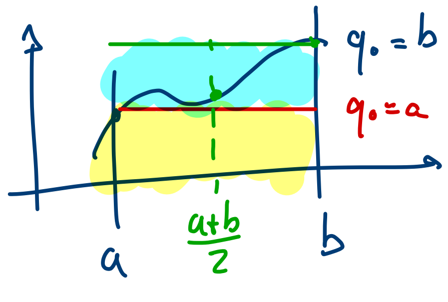
• $\{w_i\}_{i=0}^n$ are the "weights" of the quadrature formula

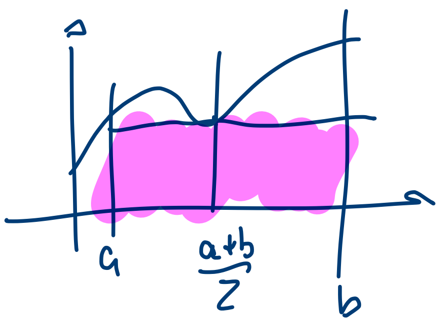
• ℓ_i are the Lagrange basis for $\{q_i\}_{i=0}^n$

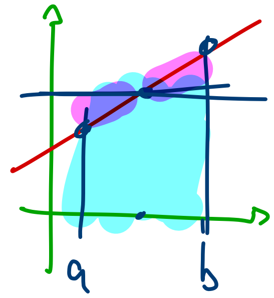
• $\{q_i\}_{i=0}^n$ are the "quadrature points"

Examples: $n=1$, $q_0 := \{a, b, \frac{a+b}{2}\}$ $\quad w_0 = (b-a)$

1):







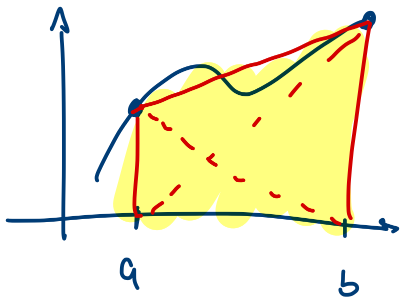
$\ell_0 = 1$

• Forward Euler: $q_0 = a$

• Backward Euler: $q_0 = b$

• Mid point $\quad q_0 = \frac{a+b}{2}$

2) $n = 2$ $\quad q_0 = a \;,\; q_1 = b$ $\quad$ Trapezoidal

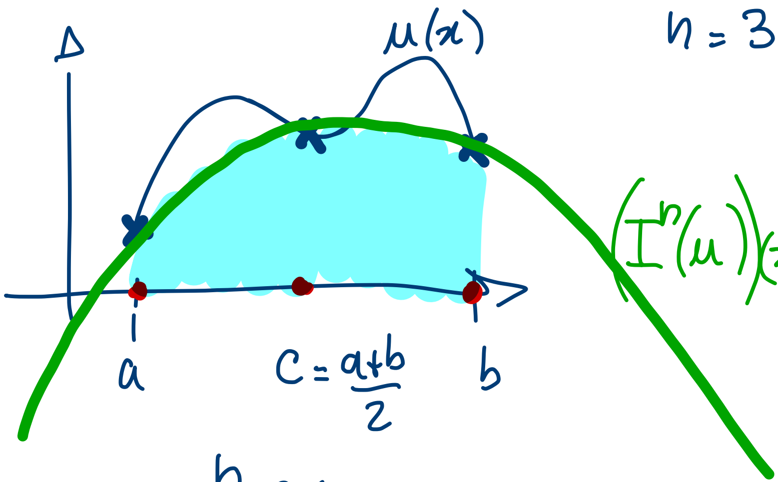


$$\ell_0 := \frac{(x-a)}{(b-a)} \qquad w_0 := \int_a^b \ell_0 \, dx = \frac{1}{2}(b-a)$$

$$\ell_1 := \frac{(x-b)}{(a-b)} \qquad w_1 := \int_a^b \ell_1 \, dx = \frac{1}{2}(b-a)$$

$$\mathrm{Int}^2(u) = (b-a)\left[\frac{u(a)}{2} + \frac{u(b)}{2}\right] = \boxed{\sum_{i=0}^{n} w_i\, u(q_i)}$$

3) $n = 3$ $\quad q_0 = a \quad q_1 = \frac{a+b}{2} \quad q_2 = b$



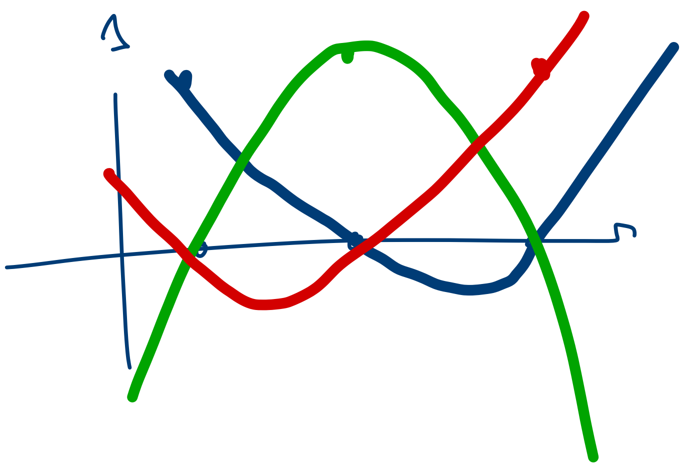
$\big(I^n(u)\big)(x) \in \mathbb{P}^2$

$$\mathrm{Int}^n(u) = \sum_{i=0}^{n} w_i\, u(q_i)$$

$$w_i := \int_a^b \ell_i(x)\, dx$$

$$\ell_0 := \frac{(x-c)(x-b)}{(a-c)(a-b)} \qquad \ell_1 := \frac{(x-a)(x-b)}{(c-a)(c-b)}$$

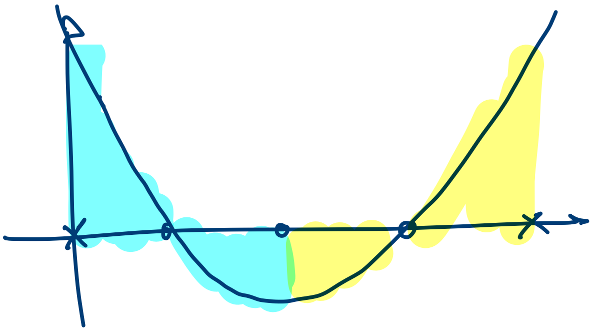
$$\ell_2 := \frac{(x-a)(x-c)}{(b-a)(b-c)}$$



$$w_0 = w_2 \neq w_1$$

$$w_1 > w_0$$

The order of accuracy of a quadrature rule with $n+1$ points is at least $\boxed{n}$. au $< 2(n+1)$

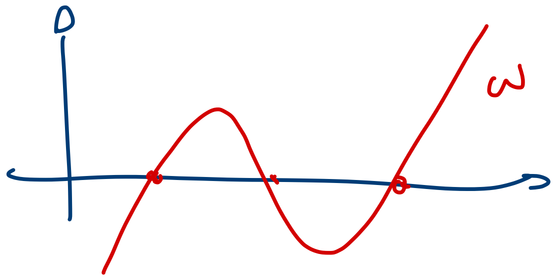


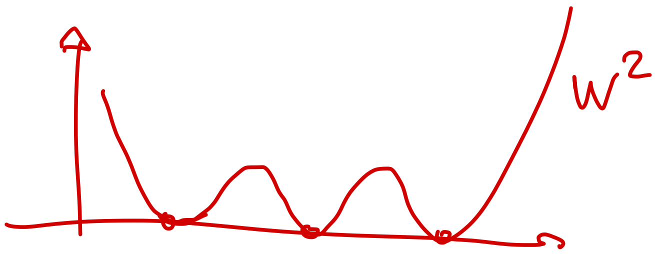
$$\omega := \prod_{i=0}^{n} (x - q_i)$$

$$\omega^2 \in \mathbb{P}^{2(n+1)}$$

$\underline{\underline{I^n \omega = 0}}$

$$\mathrm{Int}^n(\omega^2) \stackrel{?}{=} 0 \neq \mathrm{Int}(\omega^2)$$





<u>Theo</u>: Given $n+1$ quadrature points $\{q_i\}_{i=0}^{n}$, the quadrature rule will have order $n+1+m$ for $0 \leq m \leq n$ iif the following condition is satisfied:

$$\omega := \prod_{i=0}^{n} (x - q_i) \qquad \int_a^b \omega q = 0 \quad \forall q \in \mathbb{P}^m$$

$$\forall p \in \mathbb{P}^{n+1+m} \qquad \exists q \in \mathbb{P}^m, \; r \in \mathbb{P}^n \text{ s.t.}$$

$$p = \omega q + r$$ Ruffini's theorem

$$\mathrm{Int}^n(p) = \mathrm{Int}^n(\omega q) + \mathrm{Int}^n(r)$$

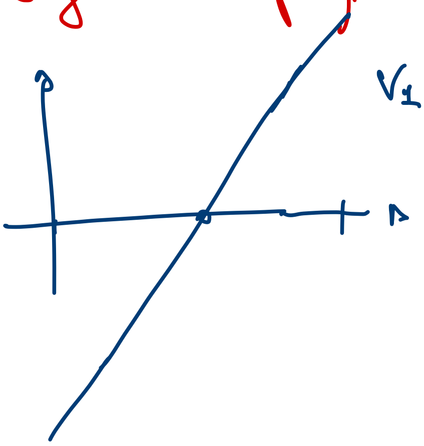
$$\mathrm{Int}(p) = \mathrm{Int}(\omega q) + \mathrm{Int}(r)$$

$$\mathrm{Int}^n(r) \equiv \mathrm{Int}(r)$$

$$Int(p) = Int^n(p) \iff Int^n(wq) = 0 = \int_a^b wq$$

$\Rightarrow \{q_i\}_{i=0}^{n}$ must contain the roots of the Legendre polynomial of order $\boxed{n+1}$

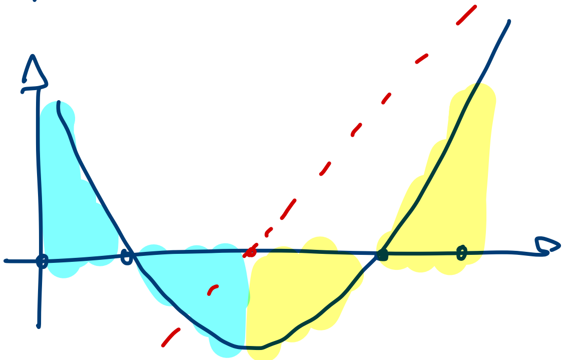
Legendre polynomials:



$V_1 := r(x-c) \qquad c = \frac{b-a}{2}$

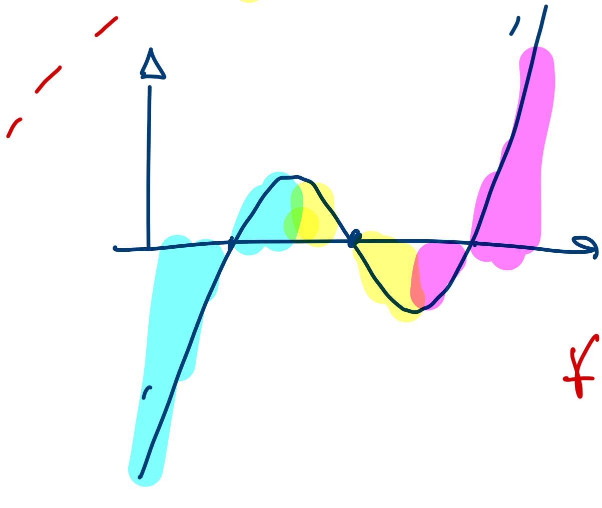
$\forall p \in \mathbb{P}^1 \ \exists \ p^0, p^1$ s.t.

$$p = p^1 V_1 + p^0 V_0$$



$\begin{cases} V_2 \perp V_0 & \int V_2 = 0 \quad (1) \\ V_2 \perp V_1 & \int V_2 V_1 = 0 \quad (2) \end{cases}$

(1)+(2) $\Rightarrow V_2 \perp q \ \forall q \in \mathbb{P}^1$



$V_2 \cdot V_1$

$\forall p \in \mathbb{P}^2, \ \exists \ p^2, p^1, p^0$

$$p = p^2 V_2 + p^1 V_1 + p^0 V_0$$

$\forall q \in \mathbb{P}^1 \qquad pq = p^2 V_2 q + p^1 V_1 q + p^0 V_0 q$

We reach $2n+1$ order of accuracy with Gauss-Legendre quadrature rules: $\{q_i\}_{i=0}^{n}$ are the roots of V_{n+1} Legendre basis of order $n+1$