Master in **High Performance Computing**

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 08 - Properties of polynomial interpolation

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Mathematical Analysis, Modeling, and Applications (math.sissa.it)
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SISSA mathLab (mathlab.sissa.it)

mathLab innovating with mathematics

ICTP The Abdus Salam International Centre for Theoretical Physics

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Given $\{x_i\}_{i=0}^n$ $(n+1)$ points, we seek $p \in \mathbb{P}^n([a,b])$ s.t. $p(x_i) = u(x_i)$ and we call p the polynomial interpolation of u in $\{x_i\}_{i=0}^n$ (or in $\mathbb{P}^n$)

$$(\mathcal{L}^n u)(x) = p(x) \qquad \mathcal{L}^n : C^0([a,b]) \longrightarrow \mathbb{P}^n([a,b]) \subset C^0([a,b])$$

- $\mathcal{L}^n p = p \quad \forall p \in \mathbb{P}^n$ ($\mathcal{L}^n$ is a projection)
  $(\mathcal{L}^n)^2 = \mathcal{L}^n$

Given $\{v_i\}_{i=0}^n$ s.t. $\mathbb{P}^n = \mathrm{span}\{v_i\}_{i=0}^n$

Define $V_{ij} := v_j(x_i)$, then

$\forall p \in \mathbb{P}^n$, $\exists! \{p^i\}_{i=0}^n$ s.t. $\underline{\underline{p(x) = \sum_{j=0}^n p^j v_j(x)}}$ $\rightarrow$ $p(x_i) = \sum_j p^j v_j(x_i)$
$= \sum_j V_{ij} p^j$

$$p^J := \sum_i (V^{-1})^{Ji} \, u(x_i)$$

$V^{Ji} := (V^{-1})^{Ji}$ notation $\quad V_{iJ} V^{Jk} = \delta_i^k$

$$V_{iJ} \, p^J = u_i \iff p^J = V^{Ji} u_i$$

$$\underline{\underline{V}}\,\underline{p} = \underline{u} \qquad \underline{p} = \underline{\underline{V}}^{-1} \underline{u}$$

$$(\mathcal{L}^n u)(x) := \sum_i \sum_J \underline{V^{Ji} u(x_i) \, V_J(x)}$$

$$:= \sum_J \underbrace{\left(\underline{\underline{V}}^{-1} \underline{u}\right)^J}\, V_J(x)$$

$$(\mathcal{L}^n u)(x) = p^J v_J(x) = p(x)$$

K_{abs} of $\mathcal{L}^n$ is $\quad \|\mathcal{L}^n\|_* := \sup_{\substack{u \neq 0 \\ u \in C^0([a,b])}} \dfrac{\|\mathcal{L}^n u\|_{L^\infty}}{\|u\|_{L^\infty}}$

By definition:

$$\|\mathcal{L}^n u\|_{L^\infty} \leq \|\mathcal{L}^n\|_* \, \|u\|_{L^\infty} \quad \forall u \in C^0([a,b])$$

$$\|\mathcal{L}^n u\|_{L^\infty} := \left\| \sum (V^{-1} \underline{u})^J v_J \right\|_{L^\infty}$$

$$\leq \|V^{-1}\|_{\ell^\infty} \, \Big\| \underbrace{\sum_J |v_J|}_{\Lambda} \Big\|_{L^\infty} \, \|u\|_{L^\infty}$$

$$\|\mathcal{L}^n\|_* \leq \|V^{-1}\|_{\ell^\infty} \, \|\Lambda\|_{L^\infty}$$

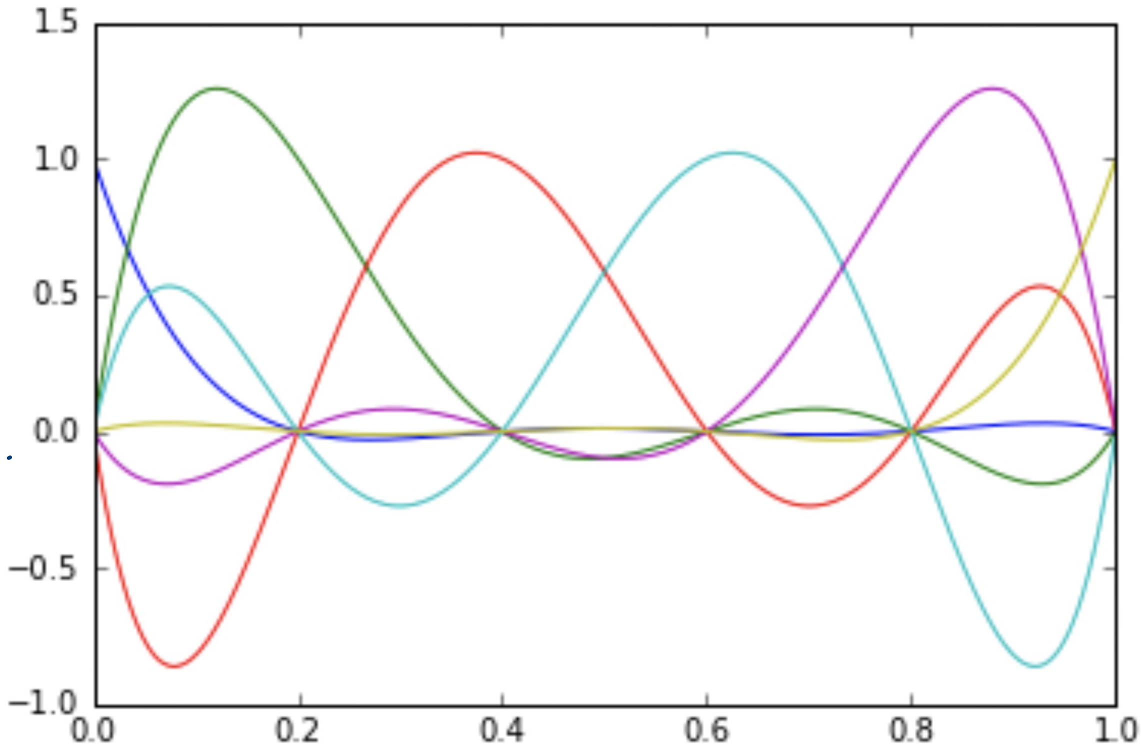
1) ∘ Minimize $\|V^{-1}\|_{\ell^\infty}$ $\Rightarrow$ choose v_i s.t. $v_i(x_j) = \delta_{ij}$

∘ Minimize $\|\Lambda\|_{L^\infty}$ $\Rightarrow$ choose x_i "appropriately"

1) $\Rightarrow$ choose $\{v_i\}_{i=0}^n$ as Lagrange basis:

$$\{\ell_i\}_{i=0}^n \qquad \ell_i := \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

$$= \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$



For $v_i \equiv e_i$ we have:

$$\left(\mathcal{L}^n u\right)(x) := \sum_i e_i(x) \; u(x_i)$$

$$\left(\mathcal{L}^n u\right)(x_j) := \sum_i e_i(x_j)\, u(x_i) = \sum_i \delta_{ij}\, u(x_i) = u(x_j)$$

How do we estimate

$$\| u - \mathcal{L}^n u \|_{L^\infty} \quad ?$$

$$\| u - \mathcal{L}^n u \|_{L^\infty} \leq \left(1 + \|\Lambda\|_{L^\infty}\right) \|u\|_{L^\infty}$$

How do we stand w.r.t. the "best possible choice"?

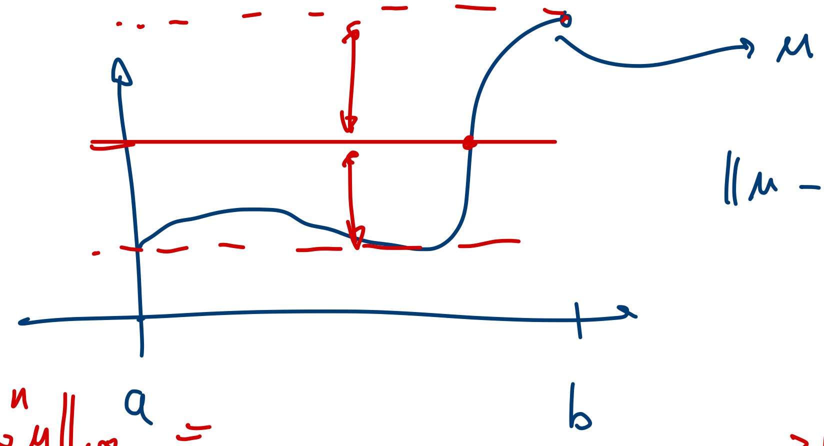
Given $\tilde{p}$ s.t.

$$\| u - \tilde{p} \|_{L^\infty} \leq \| u - q \|_{L^\infty} \quad \forall\, q \in \mathbb{P}^n$$

(p is often called "best approximation" of u in $\mathbb{P}^n$)

Is $\mathcal{L}^n u = p$ good or bad w.r.t. $\tilde{p}$?

$$\| u - \tilde{p} + \tilde{p} - \mathcal{L}^n u \|_{L^\infty}$$

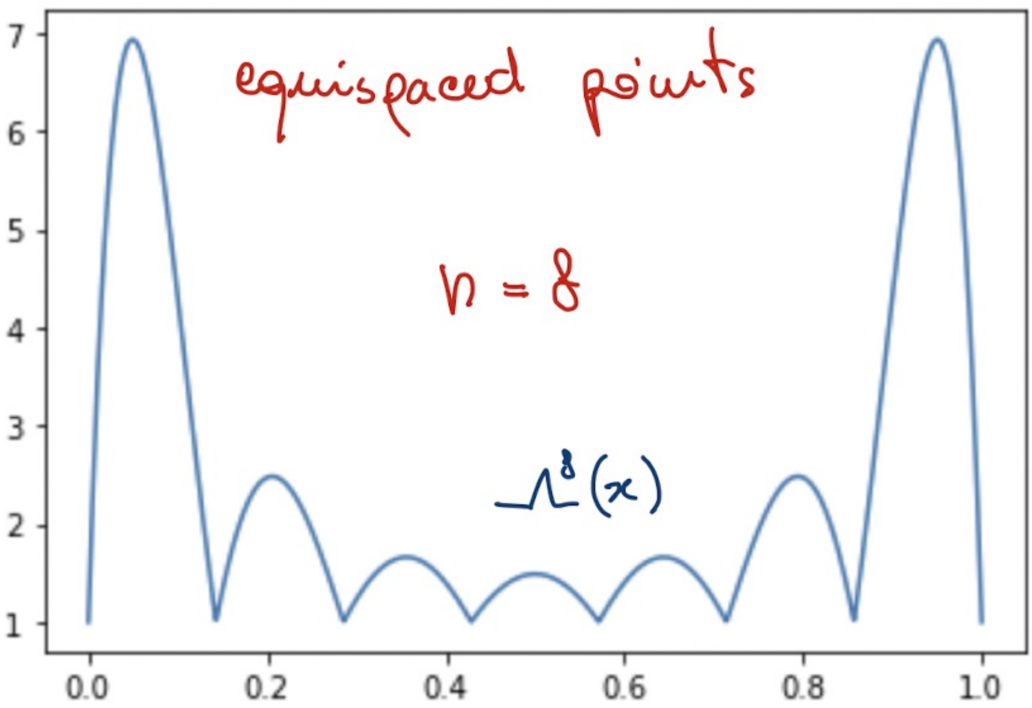


$$\|u-\tilde{p}\|_{L^\infty} \le \|u-q\|_{L^\infty} \quad \forall q \in P^{0}$$

$$\|u-\mathcal{L}^n u\|_{L^\infty} =$$

$$= \|u-\tilde{p} + \mathcal{L}^n(\tilde{p}-u)\|_{L^\infty} \le \Big(1+\overbrace{\|\Lambda\|_{L^\infty}}^{\ge 1}\Big)\|\tilde{p}-u\|$$

$$\Lambda := \sum_J |e_J| \qquad \|\Lambda\|_{L^\infty} \geq 1$$

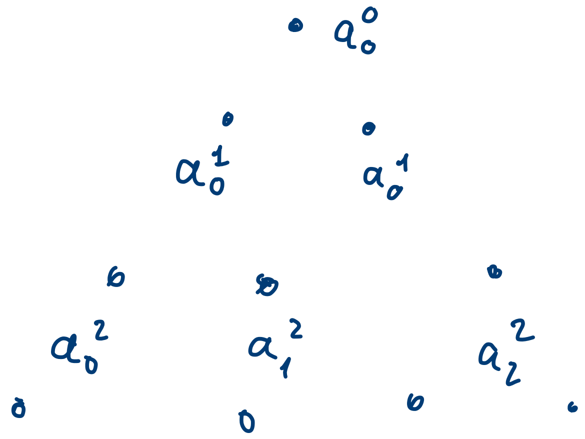


Theo (Erdos)

For any collection of points $\mathcal{X}_n \in \mathbb{R}^n$, $\mathcal{X}_{ni} = a_i^n$,
$\exists\, c > 0$ s.t.

$$\boxed{\|\Lambda^n\|_{L^\infty} \geq \frac{2}{\pi}\log(n-1) - c}$$

$\mathcal{X}_n$



$a_i^n \neq a_j^n \quad \forall i \neq j$

$\{a_i^n\}_{i=0}^n$ is a given collection of n interpolation points

<u>Theo</u> (Faber)

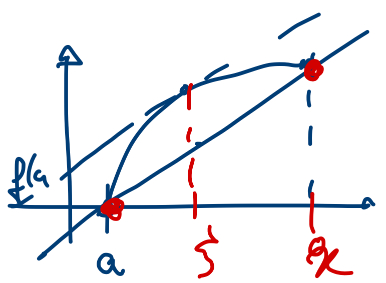
$\forall \mathcal{X}^n \in \mathbb{R}^n \quad \exists f \in C^0([a,b])$ s.t.

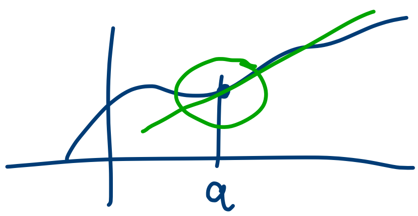
$$\boxed{\lim_{n\to\infty} \| f - \mathcal{L}^n f \|_{L^\infty} \longrightarrow \infty}$$

<u>Can we use it?</u> (yes, in some cases ....)

<u>Theo</u>: Taylor expansion. if f $C^{k+1}([0,1])$ and a is a point in $([0,1])$ $\forall x \in [0,1]$, $\exists\, \xi \in (a,x)$ s.t.

$$\boxed{f(x) = \underbrace{\sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i} + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1}}$$





Theo: if $f \in C^{n+1}([0,1])$, $a_i \in (0,1)$ $\forall a_i$ (interpolation points)

$\forall x \in [0,1]$, $\exists \xi \in (0,1)$

$$\left(f - \mathcal{L}^n f\right)(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\, \omega(x)$$

$$\omega(x) := \prod_{i=0}^{n} (x - a_i) \in \mathbb{P}^{n+1} \qquad p = \mathcal{L}^n f$$

$$G(t) = \left(f(t) - p(t)\right)\omega(x) - \left(f(x) - p(x)\right)\omega(t)$$

$f(t) = p(t)$ when $t = a_i$

$\omega(t) = 0$ when $t = a_i$ $\qquad$ $n+2$ zeros

$G(t) = 0$ when $t = x$

how many zeroes do I have in $G^{(n)}(t)$? 2!!

$\exists \xi$ s.t. $G^{(n+1)}(\xi) = 0$

$$\frac{\partial^{n+1}}{\partial t^{n+1}} \omega(x)$$

$$0 = \frac{d^{n+1} G(t)}{dt^{n+1}} := \omega(x) f^{(n+1)}(t) - \left(f(x) - p(x)\right)\underline{\underline{(n+1)!}}$$

if $t = \xi$

$$\boxed{f(x) - \mathcal{L}^n f(x) = \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi)}$$

$$\| f - \mathcal{L}^n f \|_{L^\infty} \leq \frac{\|\omega\|_{L^\infty}}{(n+1)!} \| f^{(n+1)} \|_{L^\infty}$$