Master in **High Performance Computing**

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 09 - Errors in polynomial interpolation

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Given $[a,b]$, $\{a_i\}_{i=0}^n \in [a,b]$, $I^n : C^0([a,b]) \longrightarrow \mathbb{P}^n([a,b])$

$$(I^n u)(x) := \sum_{i=0}^n u(a_i)\, \ell_i(x)$$

$$\boxed{\ell_i(x) = \prod_{\substack{i \neq j \\ j=0}}^{n} \frac{(x-x_j)}{(x_i-x_j)}}$$

Lagrange basis

Theo

If $u \in C^{n+1}([a,b])$, $\forall x \in [a,b]$, $\exists \xi \in (a,b)$ s.t.

$$(I^n u - u)(x) = \frac{\omega(x)}{(n+1)!} u^{n+1}(\xi)$$

$$\omega(x) := \prod_{i=0}^{n} (x - x_i)$$

$\Downarrow$

characteristic polynomial

$$\Rightarrow \|I^n u - u\|_{L^\infty} \leq \frac{\|\omega\|_{L^\infty}}{(n+1)!} \|u^{n+1}\|_{L^\infty}$$

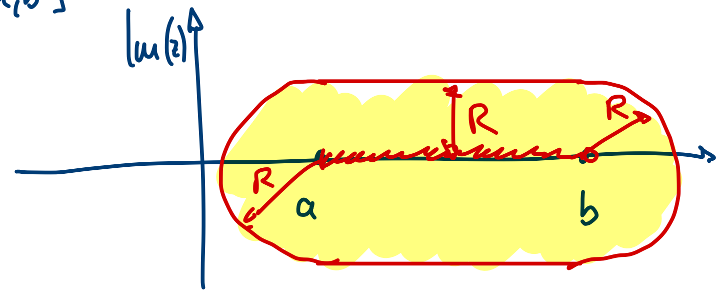
Can we say something about $\|u^{n+1}\|_{L^\infty}$? Yes.

Theorem Runge: $u \in C^0([a,b])$

If u is analytically extendible in an oval of radius R

$$\| u^{n+1} \|_{L^\infty} \leq \frac{(n+1)!}{R^{n+1}} \| \tilde{u} \|_{L^\infty(O(a,b,R))}$$

$$\tilde{u} \Big|_{[a,b]} \equiv u \qquad \tilde{u} : \mathbb{C} \longrightarrow \mathbb{R}$$



$$O(a,b,R) := \{ z \in \mathbb{C} \text{ s.t. } \operatorname{dist}(z, [a,b]) \leq R \}$$

$\tilde{u}$ is $C^\infty(O(a,b,R))$

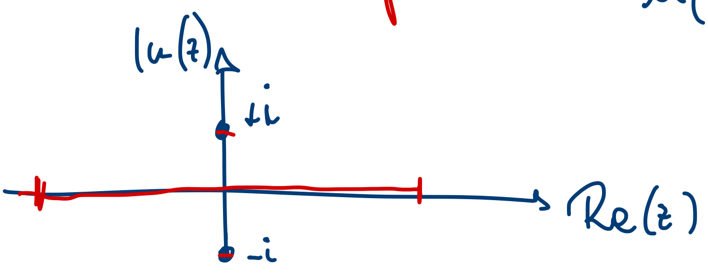
——— . ——— if u is A.E. on $O(a,b,R)$ then

$$\| I^n u - u \|_{L^\infty} \leq \frac{\| \omega \|_{L^\infty}}{(n+1)!} \| u^{n+1} \|_{L^\infty} \leq \frac{\| \omega \|_{L^\infty}}{\cancel{(n+1)!}} \frac{\cancel{(n+1)!}}{R^{n+1}} \| \tilde{u} \|_{L^\infty(O(a,b,R))}$$

$$\| \omega \|_{L^\infty} \leq (b-a)^{n+1}$$

$$\| I^n u - u \|_{L^\infty} \leq \left(\frac{(b-a)}{R} \right)^{n+1} \| \tilde{u} \|_{L^\infty(O(a,b,R))}$$

Runge counter example: $u(x) := \dfrac{1}{1+x^2}$



We also know :

$$\| I^n u - I^n p + p - u \|_{L^\infty} \le \Big(\overbrace{1 + \|\Lambda\|_{L^\infty}}^{\|I^n\|_{*} \sim ?} \Big) \|u - p\|$$

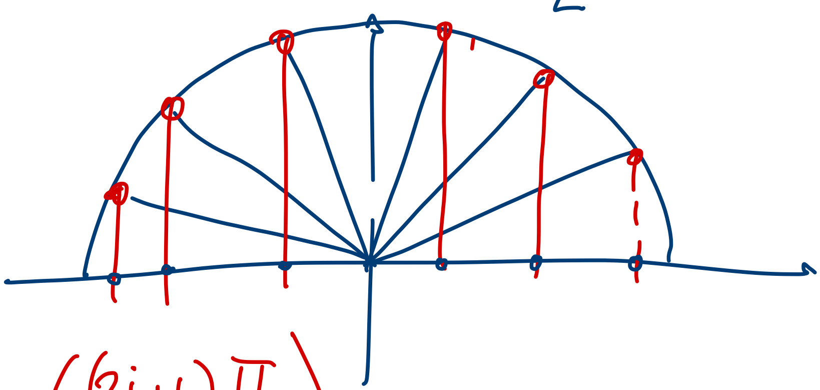
$$\ln(n+1) \lesssim \|\Lambda\|_{L^\infty} \lesssim \underbrace{\ln(n+1)}_{\text{true for Chebyshev points}}$$

$$\boxed{\{a_i\}_{i=0}^{n} := \arg\min \Big\| \underbrace{\sum_{i=0}^{n} |\ell_i|}_{\Lambda} \Big\|_{L^\infty}}$$

Def. of Chebishev points.

C.B. are those for which $\|\Lambda\|_{L^\infty}$ is minimum.

$[-1, 1]$



$$a_i := \cos\left(\frac{(2i+1)\pi}{2n+2}\right)$$