MATHEMATICS

Grade 10

Part - I

Educational Publications Department

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Contents

Message of the Board of Compilers

This textbook has been compiled in accordance with the new syllabus to be implemented from 2015.

Textbooks are compiled for students. Therefore, we have made an attempt to compile this textbook in a simple and detailed manner making it possible for you to read and understand it on your own.

We have included descriptions, activities and examples to introduce the subject concepts in an attractive manner and to establish them. Moreover, activities are organized from simple to complex to develop an interest to do them.

We have used the terms related to mathematical concepts in accordance with the glossary of technical terms of mathematics compiled by the Department of Official Languages.

Some subject matter learnt during the earlier grades is necessary to learn the subject content in the grade 10 syllabus. Thus, review exercises are included at the beginning of each chapter to revise previous knowledge. You will be prepared by them for the subject content of grade 10.

You will gain maximum benefit from this textbook by reading the chapters and doing the review exercises of each chapter even before your teacher teaches them in the classroom.

We hope that studying mathematics will be an interesting, joyful and productive experience.

Board of Compilers

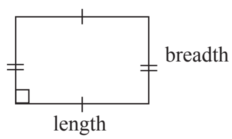
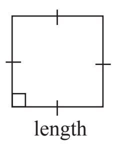
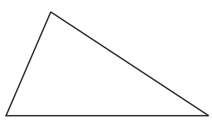
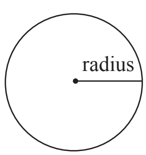
1 Perimeter

By studying this lesson you will be able to

- find the perimeter of a sector of a circle and
- solve problems related to the perimeter of plane figures containing sectors of circles.

Perimeter of Plane Figures

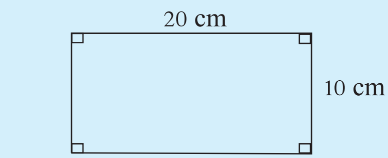
In previous grades you have learnt how to find the perimeter of plane figures such as a rectangle, a square, a triangle and a circle. Facts relating to these can be summarized as follows.

Plane Figure		Perimeter
Rectangle		2 (length + breadth)
Square		4 × length of a side
Triangle		Sum of the lengths of the three sides
Circle		2π × radius

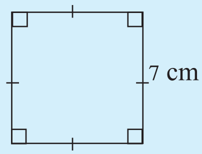
Review Exercise

1. Find the perimeter of each of the following plane figures.

(i)



(ii)



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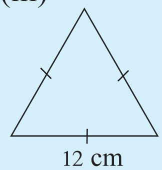

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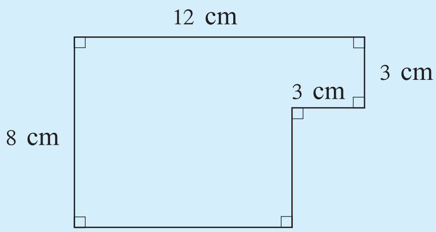
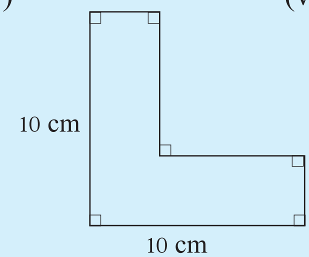
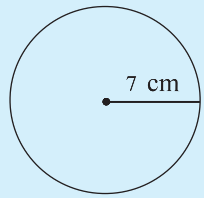


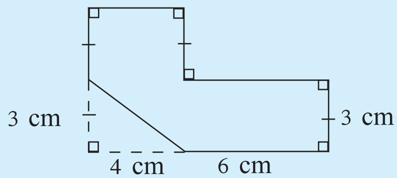
(v)



(vi)

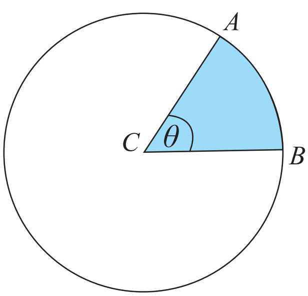


2. Find the perimeter of the following figure.



While doing the above exercise you would have recalled facts on finding the perimeter of some basic plane figures as well as of compound figures.
Now let us consider the perimeter of sectors of circles.

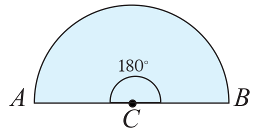
Sector of a circle



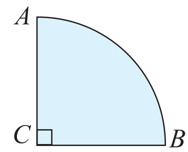
The region which is shaded in this figure is a portion of a circle with centre C which is bounded by two radii and a part of the circumference. Such a portion is called a **sector of a circle**. The angle θ ($A\hat{C}B$) which is the angle between the two radii is called the **angle at the centre**.

The angle at the centre can take any value from 0° to 360°.

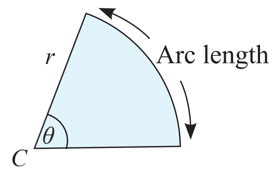
- The sector that is obtained when the angle at the centre is 180º is a semi-circle.



- The sector that is obtained when the angle at the centre is 90º is a quarter of the circle.



1.1 Finding the arc length of a sector of a circle



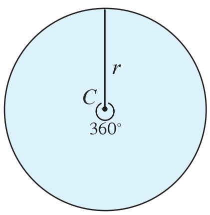
The figure illustrates a sector of a circle of radius r. Such a sector is called a sector of radius r and angle at the centre θ. Let us now consider how the arc length of such a sector of a circle is found.

Let us first find the arc length of a semi-circle of radius r.
We know that the circumference of a circle of radius r is $2\pi r$.

Therefore, due to symmetry, the arc length of a semi-circle of radius r is given by

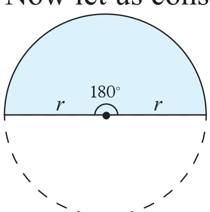
$$\frac{2\pi r}{2} = \pi r$$

Here, the reason for taking the arc length of the semi-circle to be πr, that is, the value of $2\pi r$ divided by 2, is the symmetry of the circle. The expression πr for the arc length of a semi-circle of radius r can be obtained by reasoning out in the following manner too.
Let us consider a circle and a semi-circle, both of radius r.



The angle at the centre of the circle is 360º. The arc length corresponding to this angle is the circumference of the circle which is $2\pi r$.

Now let us consider the semi-circle.



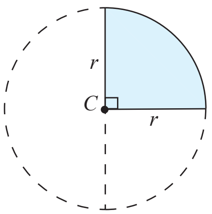
The angle at the centre of the semi-circle is 180º, which is $\frac{1}{2}$ of 360º. Therefore, the arc length of the semi-circle should be $\frac{1}{2}$ of the arc length of the circle.

That is, the arc length of the semi-circle is $\frac{1}{2} \times 2\pi r = \pi r$

Writing this in more detail,

$$\text{the arc length of the semi-circle} = \frac{180}{360} \times 2\pi r = \frac{1}{2} \times 2\pi r$$
$$= \underline{\underline{\pi r}}$$

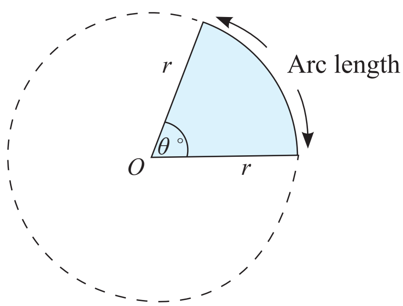
In the same manner, since the angle at the centre is 90° for a sector which is $\frac{1}{4}$ of the circle,



arc length of a sector which is $\frac{1}{4}$ of the circle $= \frac{90}{360} \times 2\pi r$

$$= \frac{1}{4} \times 2\pi r$$
$$= \underline{\underline{\frac{\pi r}{2}}}$$

Reasoning out in this manner, an expression can easily be obtained for the arc length of a sector of a circle of radius r with angle at the centre θ°.

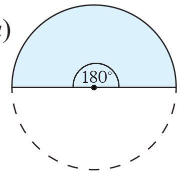
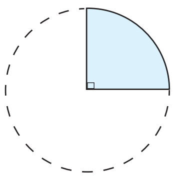
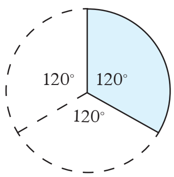
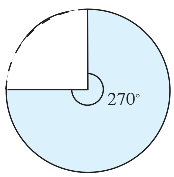
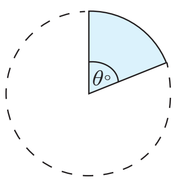


The circumference of the circle $= 2\pi r$

Arc length $= \frac{\theta}{360}$ of the circumference.

$$\boxed{\therefore \text{ Arc length} = \frac{\theta}{360} \times 2\pi r}$$

Study the following table to understand further about finding the arc length of a sector of a circle.

Sector of a circle	Length of the arc as a fraction of the circumference (According to the figure)	Angle at the centre	Angle at the centre as a fraction of the total angle at the centre
(*a*) 	$\frac{1}{2}$	180°	$\frac{180}{360} = \frac{1}{2}$
(*b*) 	$\frac{1}{4}$	90°	$\frac{90}{360} = \frac{1}{4}$
(*c*) 	$\frac{1}{3}$	120°	$\frac{120}{360} = \frac{1}{3}$
(*d*) 	$\frac{3}{4}$	270°	$\frac{270}{360} = \frac{3}{4}$
(*e*) 	$\frac{\theta}{360}$	θ°	$\frac{\theta}{360}$

Observe the 1st and 2nd columns of the table. If the length of the arc of the circle can be identified from the figure as a fraction of the circumference, then the length of the arc can easily be found. When the angle at the centre is in degrees, this fraction is

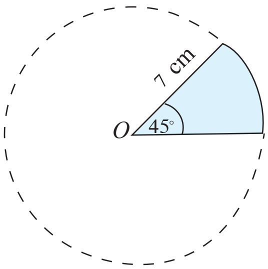
$$\frac{\text{angle at the centre}}{360}, \text{ as can be observed from column 4.}$$

Accordingly, it should be clearer to you now that the length of an arc with angle at the centre θ° and radius r is $\frac{\theta}{360} \times 2\pi r$.

Now let us consider several examples.

In the following examples and exercises it is assumed that the value of π is $\frac{22}{7}$.

Example 1



(i) What fraction of the circumference of the circle in the figure is the arc length of the shaded sector?

(ii) Find this arc length.

(i) $\frac{45}{360} = \underline{\underline{\frac{1}{8}}}$

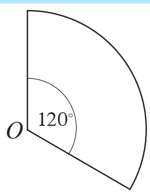
(ii) Arc length $= \frac{1}{8} \times 2\pi r$

$= \frac{1}{\cancel{8}_{\cancel{4}_{2}}} \times \cancel{2}^{1} \times \frac{\cancel{22}^{11}}{\cancel{7}_{1}} \times \cancel{7}_{1}$

$= 5.5$

$\therefore$ Arc length is 5.5 cm.

Example 2



The arc length of the sector in the figure is 44 cm. Find the radius of the corresponding circle. (i.e., the radius of the sector)

Let the radius be r cm.

Arc length $= \frac{120}{360}$ of $2\pi r$

$\therefore \quad 44 = \frac{120}{360} \times 2\pi r$

$44 = \frac{\cancel{120}^{1}}{\cancel{360}_{3}} \times 2 \times \frac{22}{7} \times r$

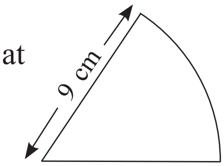
$\therefore \quad r = \frac{\cancel{44}^{\cancel{2}^{1}} \times 3 \times 7}{\cancel{2}_{1} \times \cancel{22}_{1}}$

$r = 21$

$\therefore$ The radius of the circle is 21 cm.

Example 3

The arc length of the sector in the figure is 11 cm. Find the angle at the centre of this sector.



Let θ° be the angle at the centre.

Then,

$$\text{arc length} = \frac{\theta}{360} \times 2\pi r$$

$$\therefore \quad 11 = \frac{\theta}{360} \times 2 \times \frac{22}{7} \times 9$$

$$\theta = \frac{\cancel{11}^{\,1} \times \cancel{360}^{\,\cancel{40}\,\cancel{20}\,10} \times 7}{\cancel{2}_{\,1} \times \cancel{22}_{\,\cancel{2}\,1} \times \cancel{9}_{\,1}}$$

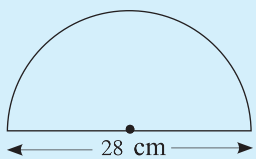
$$\theta = 70$$

Therefore, the angle at the centre is 70°.

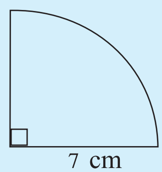
Exercise 1.1

1. Find the arc length of each of the sectors given below.

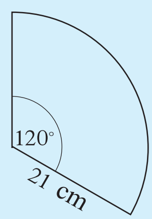
(i)



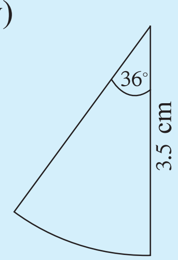
(ii)



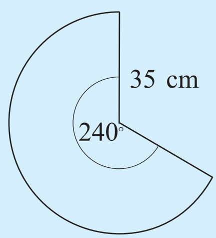
(iii)



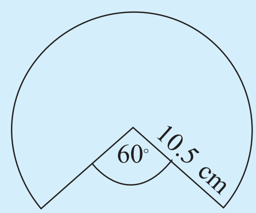
(iv)



(v)



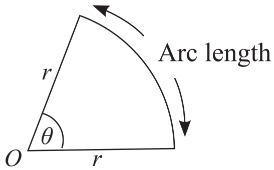
(vi)



1.2 Finding the perimeter of a sector of a circle

To obtain the perimeter of a sector of a circle, the arc length and the lengths of the two radii by which the sector is bounded should be added together.

That is,

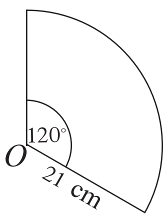


perimeter of the sector = arc length + radius + radius
= arc length + 2 × radius

Therefore,
the perimeter of a sector of a circle of radius r with angle at the centre θ°

$$\text{is } \boldsymbol{\frac{\theta}{360} \times 2\pi r + 2r}$$

Example 1



The figure denotes a sector of a circle of radius 21 cm with angle at the centre 120°. Find its perimeter.

$$\text{Arc length } = \frac{120}{360} \times 2\pi r$$

$$= \frac{\cancel{120}^{1}}{\cancel{360}_{3_1}} \times 2 \times \frac{22}{\cancel{7}_1} \times \cancel{21}^{\cancel{3}^1}$$

$$= 44$$

i.e., arc length is 44 cm.

$\therefore$ The perimeter of the sector $= 44 + 2 \times 21$
$= \underline{\underline{86 \text{ cm}}}$

Example 2

The perimeter of a sector which is $\frac{2}{3}$ of a circle is 260 cm. Find its radius.

Let us assume that the radius is r cm.

$$\text{Arc length} = 2\pi r \times \frac{2}{3}$$

$$= 2 \times \frac{22}{7} \times r \times \frac{2}{3}$$

$$= \frac{88r}{21}$$

The perimeter of the sector $= \frac{88r}{21} + 2r$

$$\therefore \frac{88r}{21} + 2r = 260$$

$$\therefore 88r + 42r = 260 \times 21$$

$$\therefore 130r = 260 \times 21$$

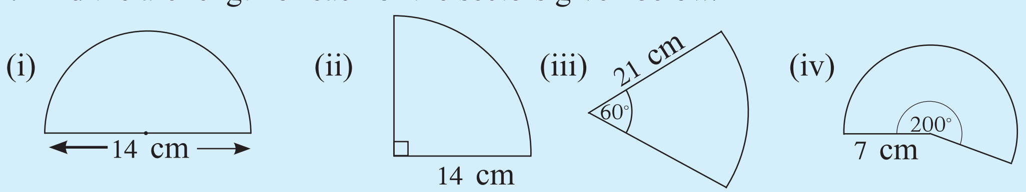
$$r = \frac{\cancel{260}^{2} \times 21}{\cancel{130}_{1}}$$

$$= 42$$

$\therefore$ The radius of the circle $= \underline{\underline{42 \text{ cm}}}$

Exercise 1.2

1. Find the arc length of each of the sectors given below.



2. Find the radius of the sector

(i) when the angle at the centre is 180° and the perimeter is 180 cm.

(ii) when the angle at the centre is 120° and the perimeter is 43 cm.

3. Find the angle at the centre of the sector,

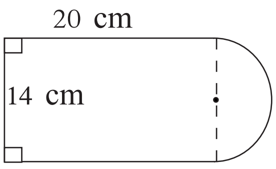
(i) when the perimeter is 64 cm and radius is 21 cm.

(ii) when the perimeter is 53 cm and the radius is 21 cm.

1.3 The perimeter of plane figures containing sectors of circles

Let us see how the perimeter of plane figures containing sectors of circles is found by considering several examples.

Example 1



The figure illustrates how a semi-circle has been joined to a rectangle of length 20 cm and breadth 14 cm such that the breadth of the rectangle is the diameter of the semi-circle. Find the perimeter of this figure.

Since the arc length of a semi-circle of radius r is $\frac{1}{2}\times 2\pi r$,

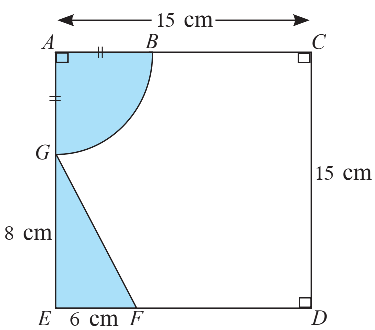
the arc length of the semi-circle of radius 7 cm $= \frac{1}{2}\times 2\times\frac{22}{7}\times 7$

$= 22$ cm

$\therefore$ The perimeter of the figure $= 20 + 20 + 14 + 22$

$= 76$ cm

Example 2



The figure represents a square lamina of side length 15 cm. If the shaded sector (AGB) and the triangular portion (GEF) are cut out, find the perimeter of the portion ($BCDFG$) that is left over.

Perimeter of $BCDFG$ is $BC + CD + DF + FG +$ arc length GB

First let us find the length of FG.

Let us apply Pythagoras' theorem to the right angled triangle GEF.

$$FG^2 = 8^2 + 6^2$$
$$= 64 + 36$$
$$= 100$$
$$\therefore\ FG = \sqrt{100}$$
$$= 10 \text{ cm}$$

Next let us find the arc length *GB*.
Since the angle at the centre of the sector *ABG* is 90º,

$$GB = \frac{\cancel{90}^{\,1}}{\cancel{360}_{\,\cancel{4}\,\cancel{2}\,1}} \times \cancel{2}^{\,1} \times \frac{\cancel{22}^{\,11}}{\cancel{7}_{\,1}} \times \cancel{7}^{\,1}$$

$$GB = 11\,\text{cm}$$

Finally let us find the length of *BC* and *DF*.

$$\begin{aligned} BC &= 15 - 7 \text{ cm} \\ &= 8 \text{ cm} \\ DF &= 15 - 6 \text{ cm} \\ &= 9 \text{ cm} \end{aligned}$$

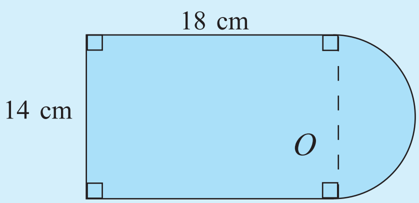
$$\begin{aligned} \text{Perimeter of } BCDFG &= BC + CD + DF + FG + \text{arc length } GB \\ &= 8 + 15 + 9 + 10 + 11 \\ &= 53 \text{ cm} \end{aligned}$$

$\therefore$ The perimeter of the remaining portion of the lamina is 53 cm

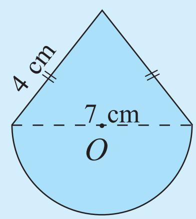
Exercise 1.3

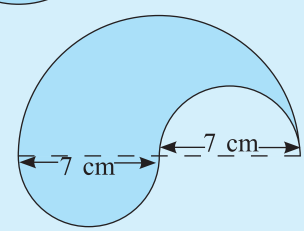
1. Find the perimeter of each of the following plane figures. *O* denotes the centre of the circle of the sector in the figure.

(i)



(ii)





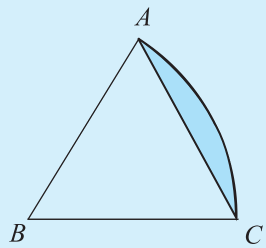
2. A semi-circular portion of diameter 7 cm has been cut out from a semi-circular lamina of radius 7 cm and welded to the lamina again as shown in the figure.

(i) Find the arc length of the sector of the circle of radius 7 cm.

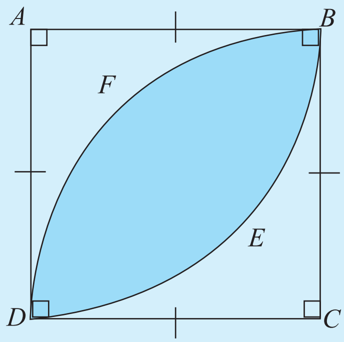
(ii) Find the arc length of the sector of the circle of diameter 7 cm

(iii) Find the perimeter of the shaded region.

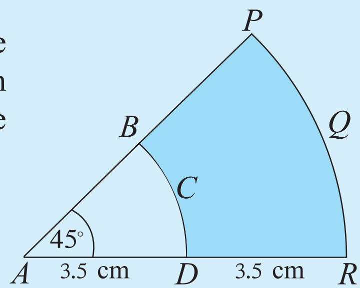
3. The figure illustrates how an equilateral triangle of side length 7 cm is drawn within a sector of a circle of radius the length of a side of the triangle.
(i) Find the arc length of the sector of the circle.
(ii) Find the perimeter of the shaded region.



4. The figure shows two sectors of circles *ABED* and *CDFB*. If $AB = 10.5$ cm, find the perimeter of the shaded region using the given data.

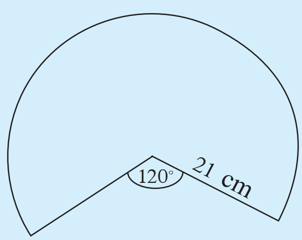


5. The figure shows two sectors of circles of centre *A* and radius *AD* and *AR* respectively. How much greater is the perimeter of the sector *APQR* than the perimeter of the shaded region?



Miscellaneous Exercise

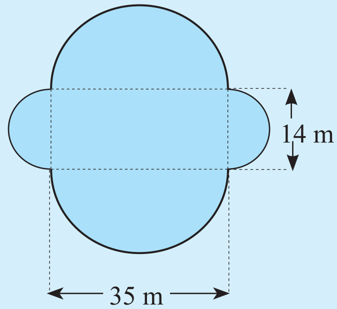
1. A sector of a circle as shown in the figure, with angle at the centre equal to 120° has been cut out from a circular lamina of radius 21 cm. Show that the perimeter of the remaining portion of the lamina is 130 cm.



2. The figure depicts a pond having semi-circular boundaries. A protective fence along the boundary of the pond has been planned.

(i) Find the perimeter of the pond.

(ii) It has been estimated that the cost of constructing 1 m of the fence is Rs. 5000. How much will it cost to complete the whole fence?

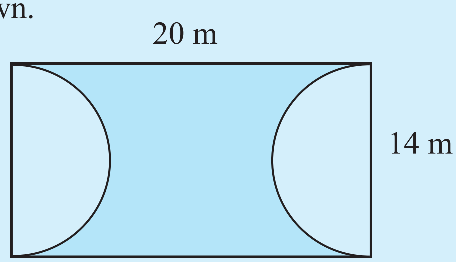


3. The figure depicts a rectangular plot of land with two semi-circular flower beds at the two ends. The shaded region is a lawn.

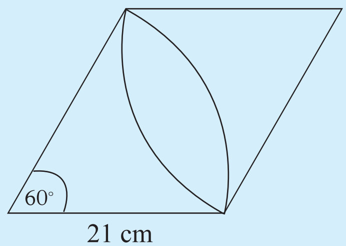
(i) Find the perimeter of the lawn.

It has been decided to lay tiles around the lawn. Each tile is of length 25 cm.

(ii) Find the minimum number of tiles that are required.



4. A portion of a grill to be fixed to a window has been made by combining two equal sectors of circles as shown in the figure. The person making the grill states that based on the given data, a wire of length 128 cm is required for it. Show with reasons that his statement is correct.



Summary

- The arc length of a sector of a circle of radius r with angle at the centre $\theta°$ is given by $\frac{\theta}{360} \times 2\pi r$
- The perimeter of the sector is given by $\frac{\theta}{360} \times 2\pi r + 2r$

2

Square Root

By studying this lesson you will be able to
- approximate the square root of a positive number which is not a perfect square
- use the division method to find an approximate value for the square root of a positive number which is not a perfect square.

2.1 The square root of a positive number

You have earlier learnt some facts about the square of a number, and the square root of a positive number. Let us briefly recall what you have learnt.
The value of 3×3 is 9. We denote 3×3 in short by 3^2. This is read as 'three squared'. The '2' in 3^2 denotes the fact that two threes are multiplied 'twice' over. Accordingly, three squared is 9 and this can be written as $3^2 = 9$.

The squares of several numbers are given in the following table.

Number	How the square of the number is obtained	How the square of the number is denoted	Square of the number
1	1×1	1^2	1
2	2×2	2^2	4
3	3×3	3^2	9
4	4×4	4^2	16
5	5×5	5^2	25

Numbers such as 1, 4, 9, 16 are perfect squares.
Finding the square root is the inverse of squaring. For example, since $3^2 = 9$, we say that the square root of 9 is 3. It will be clear to you, according to the first and last columns of the above table that

the square root of 1 is 1,
the square root of 4 is 2,
the square root of 9 is 3,
the square root of 16 is 4
and the square root of 25 is 5.

The symbol $\sqrt{\ }$ is used to denote the square root. Accordingly we can write
$\sqrt{1}=1$, $\sqrt{4}=2$, $\sqrt{9}=3$, $\sqrt{16}=4$, $\sqrt{25}=5$ etc.

It is clear that every number has a square. However, does every positive number have a square root? Let us investigate this.

According to the above table, the square root of 4 is 2 and the square root of 9 is 3. The square root of a number between 4 and 9 is a value between 2 and 3. Accordingly, it is clear that the square root of a number between 4 and 9 is not an integer. It is a decimal number. In this lesson we will consider how an approximate value is found for this. We call such a value an **approximation**.

Let us for example consider how an approximate value is obtained for the square root of 5.

Consider the following table.

Number	**How the square of the number is found**	**How the square of the number is written**	**The square of the number**
2	2×2	2^2	4
2.1	2.1×2.1	2.1^2	4.41
2.2	2.2×2.2	2.2^2	4.84
2.3	2.3×2.3	2.3^2	5.29
2.4	2.4×2.4	2.4^2	5.76
2.5	2.5×2.5	2.5^2	6.25
2.6	2.6×2.6	2.6^2	6.76
2.7	2.7×2.7	2.7^2	7.29

From the values in the right hand side column, the two values that are closest to 5 are 4.84 and 5.29. They are the squares of 2.2 and 2.3 respectively. According to the above table, the square roots of 4.84 and 5.29 are respectively 2.2 and 2.3. This can be written symbolically as $\sqrt{4.84}=2.2$ and $\sqrt{5.29}=2.3$.

Now let us examine which value from 4.84 and 5.29 is closer to 5.

The difference between 4.84 and 5 = 5 – 4.84 = 0.16

The difference between 5.29 and 5 = 5.29 – 5 = 0.29

Accordingly, the value that is closer to 5 is 4.84. Therefore, 2.2 can be taken as an approximate value for the square root of 5. The value that is obtained for the square root of a positive integer which is correct to the first decimal place is called the “**approximation to the first decimal place**” of the square root of the given number (or more simply the “**first approximation**”)

Accordingly, the approximation of square root of 5 to the first decimal place is 2.2. When an approximate value is given, the symbol $\approx$ is used. Accordingly, we can write $\sqrt{5} \approx 2.2$.

By providing reasons in a similar manner, we can conclude that the approximation of square root of 6 to the first decimal place is 2.4 and the approximation of square root of 7 to the first decimal place is 2.6

That is, $\sqrt{6} \approx 2.4$

$$\sqrt{7} \approx 2.6$$

By considering the following examples let us now learn a specific method of finding the first approximation of the square root of a positive number which is not a perfect square.

Example 1

Approximate $\sqrt{17}$ to the first decimal place.

- From the perfect square numbers which are less than 17, the one which is closest to it is 16, and from the perfect square numbers which are greater than 17, the one which is closest to it is 25.

  Accordingly, let us write

$$16 < 17 < 25$$

- When we write the square root of each of these numbers we obtain.

$$\sqrt{16} < \sqrt{17} < \sqrt{25}$$
$$\therefore \ 4 < \sqrt{17} < 5$$

Accordingly, the square root of 17 is greater than 4 which is the square root of 16, and less than 5 which is the square root of 25.

i.e., $\sqrt{17}$ lies between 4 and 5.

- To find an approximate value close to $\sqrt{17}$, let us check whether 17 is closer to 16 or to 25.

The difference between 16 and 17 is 1.

The difference between 17 and 25 is 8.

$\therefore$ 17 is closer to 16 than to 25.

$\therefore \sqrt{17}$ is a value close to 4.

i.e., one of the values 4.1, 4.2, 4.3 and 4.4 can be the first approximation of $\sqrt{17}$

- Let us now multiply each of these numbers by itself to identify the number which has a product which is closest to 17.

When the first two values are squared we obtain

$$4.1 \times 4.1 = 16.81$$
$$4.2 \times 4.2 = 17.64$$

Since the value of 4.2^2 exceeds 17, it is unnecessary to find 4.3^2 and 4.4^2.
16.81 is the closer value to 17 from these two.

$\therefore$ First approximation of $\sqrt{17}$ is 4.1

Example 2

Find the first approximation of $\sqrt{245}$.
Since $15^2 = 225$ and $16^2 = 256$,

write
$$225 < 245 < 256$$

Accordingly, $15 < \sqrt{245} < 16$

$\therefore \sqrt{245}$ is a value between 15 and 16.

Since 245 is closer to 256 than to 225, $\sqrt{245}$ is closer to 16 than to 15.
$\therefore$ The first approximation of $\sqrt{245}$ can be one of 15.5, 15.6, 15.7, 15.8 and 15.9.
Let us now determine this value.

$15.9 \times 15.9 = 252.81$
$15.8 \times 15.8 = 249.64$
$15.7 \times 15.7 = 246.49$
$15.6 \times 15.6 = 243.36$
From the above values, 246.49 is closest to 245.

$\therefore$ First approximation of $\sqrt{245}$ is 15.7.

Exercise 2.1

Find the first approximation of each of the following numbers.

(i) $\sqrt{5}$ (ii) $\sqrt{20}$ (iii) $\sqrt{67}$ (iv) $\sqrt{115}$ (v) $\sqrt{1070}$

2.2 The Division Method

Let us now consider a method of finding the square root of any positive number. This method is called the division method. Let us study this method by considering several examples.

Example 1 Find the square root of 1764.

Step 1

Separate 1764 as shown below, by grouping the digits of 1764 in pairs, starting from the units position and proceeding towards the left.

17 64

Step 2

Find the perfect square number which is closest to the leftmost digit or pair of digits of the separated number, and as indicated below, write its square root above and to the left of the drawn lines.

$$\begin{array}{r|l} & 4 \\ \hline 4 & 17\ \ 64 \end{array}$$

Step 3

Write down the product 4 x 4 of the number above and to the left of the lines, below the number 17 as indicated, and subtract it from 17.

$$\begin{array}{r|l} & \ \ 4 \\ \hline 4 & 17\ 64 \\ & \underline{16} \\ & \ \ 1 \end{array}$$

Step 4

Now carry down the next two digits 64, as indicated below.

$$\begin{array}{r|l} & \ \ 4 \\ \hline 4 & 17\ 64 \\ & \underline{16} \\ & \ \ 1\ 64 \end{array}$$

Step 5

Next, write on the left as shown below, the digit 8, which is two times the number above the line, leaving space for another number to be written. (i.e., leave space for the digit in the units position)

$$\begin{array}{r|l} & \ \ 4 \\ \cline{2-2} 4 & 17\ 64 \\ & 16 \\ \cline{2-2} 4\times 2 = 8 \rightarrow 8\ \square & \ \ 1\ 64 \end{array}$$

Step 6

The same digit should be written above the line to the right of 4 and in the space left in the units position on the left. This digit should be selected so that the product of this digit and the number obtained on the left when this digit is written in the units position (in this case 82), is equal to 164, or is the closest number less than 164 that can be obtained in this manner.

$$\begin{array}{r|r} & 4\ \boxed{2} \\ \cline{2-2} 4 & 17\ 64 \\ & 16\ \ \ \ \\ \cline{2-2} 8\ \boxed{2} & 1\ 64 \\ & 1\ 64 \\ \cline{2-2} & 0 \end{array}$$

Then $\sqrt{1764} = \underline{\underline{42}}$

When finding the square root of a decimal number, separate the digits in pairs on both sides of the decimal point, starting at the decimal point as shown below.

$3.61 \longrightarrow 3.\ 61$

$12.321 \longrightarrow 12.\ 32\ 10$

$143.456 \longrightarrow 1\ 43.\ 45\ 60$

Example 2

Find the value of $\sqrt{3.61}$.

$$\begin{array}{r|l} & 1.\ \boxed{9} \\ \cline{2-2} 1 & 3.\ 61 \\ & 1 \\ \cline{2-2} \boxed{1\times 2 = 2} \rightarrow 2\ \boxed{9} & 2\ \ 61 \\ & 2\ \ 61 \\ \cline{2-2} & \ \ \ \ 00 \end{array}$$

$\therefore \sqrt{3.61} = \underline{\underline{1.9}}$

Example 3

Find the value of $\sqrt{2737}$ accurate to two decimal places.

We must find the value to three decimal places and round off to two decimal places. To find to three decimal places we must write three pairs of zeros after the decimal point.

$$
\begin{array}{rr|l}
 & & \;5\;\boxed{2}.\boxed{3}\;\boxed{1}\;\boxed{6} \\
\hline
 & 5 & 27\;37.\;00\;00\;00 \\
 & & 25 \\
\cline{3-3}
\boxed{5 \times 2 = 10} \rightarrow & 10\boxed{2} & \;\;2\;37 \\
 & & \;\;2\;04 \\
\cline{3-3}
\boxed{52 \times 2 = 104} \rightarrow & 104\boxed{3} & \;\;\;\;33\;00 \\
 & & \;\;\;\;31\;29 \\
\cline{3-3}
\boxed{523 \times 2 = 1046} \rightarrow & 1046\boxed{1} & \;\;\;\;\;\;1\;71\;00 \\
 & & \;\;\;\;\;\;1\;04\;61 \\
\cline{3-3}
\boxed{5231 \times 2 = 10462} \rightarrow & 10462\boxed{6} & \;\;\;\;\;\;\;\;66\;39\;00 \\
 & & \;\;\;\;\;\;\;\;62\;77\;56 \\
\cline{3-3}
 & & \;\;\;\;\;\;\;\;\;\;3\;61\;44
\end{array}
$$

$\therefore \sqrt{2737} \approx \underline{\underline{52.32}}$

Example 4 Find the value of $\sqrt{3.421}$ accurate to two decimal places.

As above let us find the value to three decimal places and round off to two decimal places. For this there must be three pairs of decimal places after the decimal point.

$$
\begin{array}{r|l}
 & 1.\;\boxed{8}\;\;\boxed{4}\;\;\boxed{9} \\
\hline
1 & 3.\;\;42\;\;10\;\;00 \\
 & 1 \\
\cline{2-2}
2\boxed{8} & 2\;\;42 \\
 & 2\;\;24 \\
\cline{2-2}
36\boxed{4} & \;\;\;18\;\;10 \\
 & \;\;\;14\;\;56 \\
\cline{2-2}
368\boxed{9} & \;\;\;\;\;3\;\;54\;\;00 \\
 & \;\;\;\;\;3\;\;32\;\;01 \\
\cline{2-2}
 & \;\;\;\;\;\;\;\;\;21\;\;99
\end{array}
$$

$\therefore \sqrt{3.421} \approx \underline{\underline{1.85}}$

Exercise 2.2

1. Find the square root of each of the following numbers.

(i) 676 (ii) 1024 (iii) 2209 (iv) 2809 (v) 3721

2. Find the value accurate to one decimal place.

(a)

(i) $\sqrt{8}$ (ii) $\sqrt{19}$ (iii) $\sqrt{26}$

(iv) $\sqrt{263}$ (v) $\sqrt{2745}$ (vi) $\sqrt{3630}$

(b)

(i) $\sqrt{5.4}$ (ii) $\sqrt{3.45}$ (iii) $\sqrt{15.3}$ (iv) $\sqrt{243.2}$

(v) $\sqrt{4061.3}$ (vi) $\sqrt{85.124}$ (vii) $\sqrt{0.0064}$ (iv) $\sqrt{0.000144}$

2.3 Using the square roots of numbers to solve problems

Example 1

Find the length of a side of a square of area 441 cm^2.

Area of the square $=$ (side length)^2

$\therefore$ Length of a side of the square $= \sqrt{\text{area of the square}}$

Area of the square $= 441 \text{ cm}^2$

$\therefore$ Length of a side of the square $= \sqrt{441}$ cm

$= \underline{\underline{21 \text{ cm}}}$

$$
\begin{array}{r|l}
 & \;2\;\;\;1 \\
\hline
2 & 4\;41 \\
 & \underline{4} \\
41 & 0\;41 \\
 & \;\;\underline{41} \\
 & \;\;00
\end{array}
$$

Example 2

324 square shaped garden tiles of area 900 cm^2 each have been placed in a square shaped courtyard so that the courtyard is completely covered with the tiles. Find the length of a side of the courtyard.

The number of garden tiles in one row $= \sqrt{324}$

$= 18$

The length of a garden tile $= \sqrt{900}$ cm

$= 30$ cm

The length of a side of the courtyard $= 18 \times 30$ cm

$= 540$ cm

$= \underline{\underline{5.4 \text{ m}}}$

Exercise 2.3

1. What is the length of a side of a square shaped piece of cardboard of area 1225 cm^2?

2. What is the length of a side of a square of area the same as that of a rectangle of length 27 cm and breadth 12 cm?

3. 196 children participating in a drill display have been placed such that they form an equal number of rows and columns. How many children are there in a row?

4. The surface area of a cube is 1350 cm^2. Find the length of a side of the cube.

5. A rectangular walkway has been made by placing 10 flat square shaped concrete slabs along 200 rows. If the area of the flat surface of a concrete slab is 231.04 cm^2, what is the length and the breadth of the walkway?

Miscellaneous Exercise

1. Find the value accurate to the second decimal place.

(i) $\sqrt{3669}$ (ii) $\sqrt{4302}$ (iii) $\sqrt{22.79}$ (iv) $\sqrt{0.1296}$ (v) $\sqrt{5.344}$

2. The length and breadth of a rectangular shaped plot of land are respectively 25 m and 12 m. Find to the nearest metre, the least distance that a child standing at one corner of the plot should travel to reach the diagonally opposite corner of the plot.

3. If the length of the hypotenuse of an isosceles right angled triangle is 12 cm, find the length of a remaining side. (Give the answer to two decimal places).

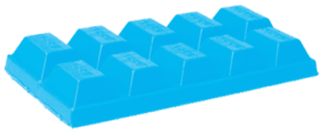
4. 9, 16, 25, … is a number pattern. Which term of the pattern is the number 729?

3 Fractions

By studying this lesson you will be able to

- identify situations in which fractions are applied
- solve problems involving fractions.

Fractions



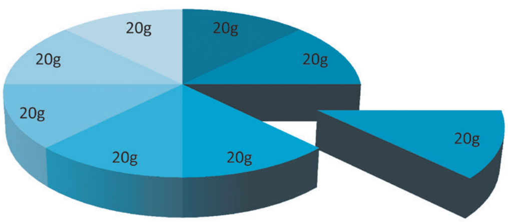
The figure depicts a slab of a certain type of chocolate. It has been divided into ten equal sized pieces such that the pieces can easily be broken off.

When the whole slab of chocolate is considered as one unit,

- one piece of chocolate that is broken off is considered as $\frac{1}{10}$ of the slab,
- two pieces of chocolate that are broken off is considered as $\frac{2}{10}$ of the slab,
- three pieces of chocolate that are broken off is considered as $\frac{3}{10}$ of the slab.

$\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$ etc., which are used to represent portions of a whole unit, are examples of fractions.

Now let us consider another example.



The above figure depicts how pieces of cheese have been arranged in a container. There are 8 equal pieces. One of the pieces has been removed from the container. This piece is $\frac{1}{8}$ of the amount of cheese in the container. If the total mass of all the pieces of cheese in the container is 160 g, then the piece that was removed which is $\frac{1}{8}$ of the total quantity of cheese in the container, is of mass 20 g.

The unit that has been considered here is the total mass of the cheese in the container which is 160 g.

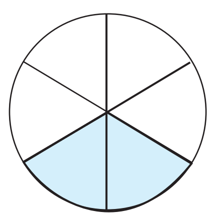
When talking about fractions, it is important to be aware of the unit from which the factions are obtained.

For example, in the statement "$\frac{2}{3}$ of all the students in the class are girls", the fraction $\frac{2}{3}$ is applied by taking "the total number of students in the class" as the unit.

In the following table, the units relevant to several statements involving fractions are given.

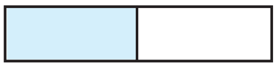
Situation	Total Unit
(*a*) $\frac{1}{5}$ of the volume of the atmosphere consists of oxygen.	The volume of the atmosphere
(*b*) $\frac{1}{4}$ of the 50 litres of water has been used up.	50 litres of water
(*c*) $\frac{2}{3}$ of a 200 m^2 plot of land has been used to cultivate vegetables.	200 m^2 plot of land
(*d*) $\frac{1}{4}$ of the harvest was kept for consumption.	The total harvest
(*e*) $\frac{3}{4}$ of a piece of wire of length 5 m was cut.	The piece of wire of length 5 m
(*f*) $\frac{1}{5}$ of the fruits in a collection of 25 oranges were ripe.	25 oranges
(*g*) A father wrote half ($^1/_2$) of his land in his son's name.	Area of the whole land

The circular shape illustrated in the following figure has been divided into 6 equal parts. We know that the fraction that has been shaded is $\frac{2}{6}$ of the shape.

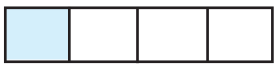


The denominator of $\frac{2}{6}$ is 6 and the numerator is 2. The total number of parts into which the unit has been divided is the denominator, and the number of parts being considered is the numerator. Here, the numerator is smaller than the denominator. Such fractions are called **proper fractions**. Fractions such as $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{1}{7}$ with numerator equal to 1 are called **unit fractions**.

In the following figures, quantities equal to $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ of the same unit have been shaded. Here the area of the rectangle has been taken as the unit.

$\frac{1}{2}$ 

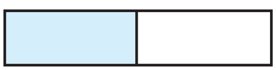
$\frac{1}{3}$ 

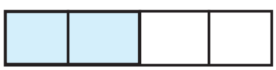
$\frac{1}{4}$ 

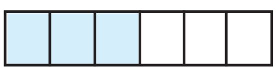
It is clear from the above figures that $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$.

Further, considering fractions such as $\frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6}$, where the numerators are equal and the denominators are not, as the denominator increases, the value represented by the fraction decreases. That is, $\frac{2}{3} > \frac{2}{4} > \frac{2}{5} > \frac{2}{6}$

The following figure illustrates the three fractions $\frac{1}{2}, \frac{2}{4}$ and $\frac{3}{6}$ obtained from the same unit.

$\frac{1}{2}$ 

$\frac{2}{4}$ 

$\frac{3}{6}$ 

According to the figure, the quantities represented by the three fractions are equal. That is, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$.

Such fractions which are equal to each other are called **equivalent fractions**. Observe that fractions which are equivalent to a given fraction are obtained when

both the numerator and the denominator are multiplied by the same number.
For example,

$$\frac{1}{2}=\frac{1\times 2}{2\times 2}=\frac{2}{4}, \qquad \frac{1}{2}=\frac{1\times 3}{2\times 3}=\frac{3}{6}, \qquad \frac{1}{2}=\frac{1\times 5}{2\times 5}=\frac{5}{10}$$

Equivalent fractions are also obtained when the numerator and the denominator of the given fraction are divided by the same number.
For example,

$$\frac{5}{10}=\frac{5\div 5}{10\div 5}=\frac{1}{2}, \qquad \frac{3}{6}=\frac{3\div 3}{6\div 3}=\frac{1}{2}, \qquad \frac{8}{16}=\frac{8\div 8}{16\div 8}=\frac{1}{2}$$

Now let us consider the two fractions $\frac{2}{3}$ and $\frac{3}{4}$ obtained from the same unit, which are unequal to each other.

First let us write down several fractions which are equivalent to each of the fractions $\frac{2}{3}$ and $\frac{3}{4}$.

$$\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}=\frac{12}{18}=\frac{14}{21}=\frac{16}{24}=\frac{18}{27}=\ldots..$$

$$\frac{3}{4}=\frac{6}{8}=\frac{9}{12}=\frac{12}{16}=\frac{15}{20}=\frac{18}{24}=\frac{21}{28}=\ldots..$$

It can be observed that among the fractions which are equivalent to $\frac{2}{3}$ and $\frac{3}{4}$, there are some which have the same denominator. $\frac{8}{12}$ and $\frac{9}{12}$ are two such fractions. $\frac{16}{24}$ and $\frac{18}{24}$ are two more such fractions.
For convenience, let us select $\frac{8}{12}$ and $\frac{9}{12}$ which have the smallest common denominator.

Comparing $\frac{8}{12}$ and $\frac{9}{12}$ we obtain $\frac{9}{12}>\frac{8}{12}$.

However, since $\frac{2}{3}=\frac{8}{12}$ and $\frac{3}{4}=\frac{9}{12}$, we can conclude that $\frac{3}{4}>\frac{2}{3}$.

Let us explain the comparison of $\frac{3}{4}$ and $\frac{2}{3}$ done above with equivalent fractions, using figures.

$\frac{2}{3}$ 

$\frac{3}{4}$  It is clear from the figure too that $\frac{3}{4} > \frac{2}{3}$.

Accordingly, it is clear that it is suitable to write equivalent fractions having a common denominator when comparing fractions.

Now let us consider the addition and subtraction of fractions.

You have learnt in earlier grades to add fractions with equal denominators, as for example

$\frac{2}{10} + \frac{3}{10} = \frac{5}{10}$.

Similarly, you know that fractions with equal denominators can be subtracted as for example,

$\frac{6}{7} - \frac{2}{7} = \frac{4}{7}$.

When adding and subtracting fractions of unequal denominators, the relevant fractions can first be converted into equivalent fractions of equal denominators.

For example,

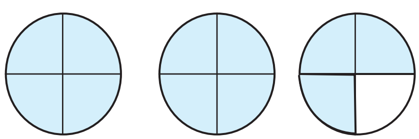
$$\begin{aligned}\frac{2}{3} + \frac{1}{4} &= \frac{2 \times 4}{3 \times 4} + \frac{1 \times 3}{4 \times 3}\\ &= \frac{8}{12} + \frac{3}{12}\\ &= \underline{\underline{\frac{11}{12}}}\end{aligned}$$

$$\begin{aligned}\frac{3}{5} - \frac{1}{3} &= \frac{3 \times 3}{5 \times 3} - \frac{1 \times 5}{3 \times 5}\\ &= \frac{9}{15} - \frac{5}{15}\\ &= \underline{\underline{\frac{4}{15}}}\end{aligned}$$

Fractions can also be used to represent quantities which are greater than a unit.

For example, let us see what quantity $\frac{3}{2}$ of a loaf of bread is. This is the amount that is obtained when a loaf of bread is divided into two equal parts and three such parts are considered. This quantity is one and a half loaves of bread. That is, $1 + \frac{1}{2}$ loaves of bread, which in short, is a quantity of $1\frac{1}{2}$.

As another example, let us illustrate $2\frac{3}{4}$ of a circle by a figure.



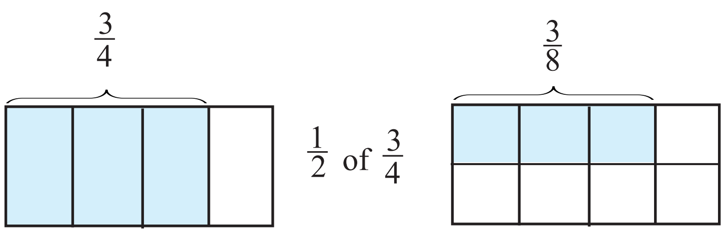
If these three figures are considered together as one unit instead of separately, then the shaded region represents $\frac{11}{12}$ of the unit. However, if any one of these circles is taken as the unit, then the shaded region represents $\frac{11}{4}$ of the unit. It is clear therefore that the mixed number $2\frac{3}{4}$ can also be written as the fraction $\frac{11}{4}$.

Another way in which the equality $2\frac{3}{4} = \frac{11}{4}$ can be obtained is given below.

$$\begin{aligned} 2\frac{3}{4} &= 1+1+\frac{3}{4} \\ &= \frac{4}{4}+\frac{4}{4}+\frac{3}{4} \\ &= \frac{11}{4} \end{aligned}$$

Accordingly, it is clear that $2\frac{3}{4} = \frac{11}{4}$. When this amount is denoted by $2\frac{3}{4}$, it is called a **mixed number**. When it is written as $\frac{11}{4}$, it is called an **improper fraction**. You have learnt in previous grades how a mixed number is converted into an improper fraction and how an improper fraction is converted into a mixed number.

Now let us recall what has been learnt about multiplying fractions. To do this, let us draw figures in the following manner to find out how much $\frac{1}{2}$ of $\frac{3}{4}$ is.



According to the figure, it is clear that $\frac{1}{2}$ of $\frac{3}{4}$ is equal to $\frac{3}{8}$

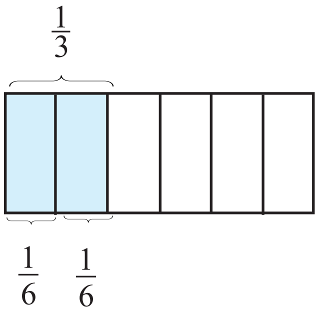
The above answer can be obtained by simplifying this in the following manner too.

$$\begin{aligned} \frac{1}{2} \text{ of } \frac{3}{4} &= \frac{1}{2} \times \frac{3}{4} \\ &= \frac{3}{8} \end{aligned}$$

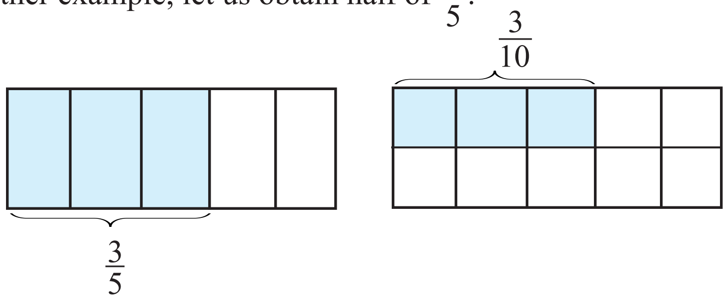
It is clear that 'of' in $\frac{1}{2}$ of $\frac{3}{4}$ denotes the mathematical operation multiplication and that the numerator can be taken as 1×3 and the denominator as 2×4.

Now let us consider the situation of dividing fractions.

Let us find how many $\frac{1}{6}$ there are in $\frac{1}{3}$. This is denoted by $\frac{1}{3} \div \frac{1}{6}$. It is clear from the following figure that this value is 2.



As another example, let us obtain half of $\frac{3}{5}$.



According to the figure, half of $\frac{3}{5}$ is equal to $\frac{3}{10}$.

However, since half of any quantity, is the amount that is obtained when the quantity is divided by 2,

$$\frac{3}{5} \div 2 = \frac{3}{10}$$

It is not convenient to draw figures every time we divide fractions. We need to identify a different method to do this. The above division of fractions which was done using figures, can also be presented in the following manner.

$$\frac{3}{5} \div 2 = \frac{3}{5} \div \frac{2}{1} \qquad \left(\text{Since } 2 = \frac{2}{1}\right)$$

$$= \frac{3}{5} \times \frac{1}{2} \qquad \left(\text{Multiplying by } \frac{1}{2} \text{ which is the same as dividing by 2}\right)$$

$$= \underline{\underline{\frac{3}{10}}}$$

That is, the same answer that was obtained with the figures is obtained using this method too.

Let us see whether this method works for $\frac{1}{3} \div \frac{1}{6}$.

$$\frac{1}{3} \div \frac{1}{6} = \frac{1}{\cancel{3}_1} \times \frac{\cancel{6}^2}{1}$$

$$= \underline{\underline{2}}$$

That is, the same answer that was obtained with the figures is obtained here.

When the numerator and the denominator of the fraction $\frac{1}{6}$ are interchanged we obtain $\frac{6}{1}$. We say that the reciprocal of $\frac{1}{6}$ is $\frac{6}{1}$; that is 6.

The reciprocal of a general term $\frac{a}{b}$ is $\frac{b}{a}$.

In general, when a fraction needs to be divided by another fraction, the first fraction can be multiplied by the reciprocal of the second fraction.

Let us recall all the facts that have been learnt so far regarding fractions by working out the following example.

$$\left(2\frac{2}{3} - 1\frac{1}{2} + \frac{5}{6}\right) \div \left(\frac{4}{5} \text{ of } 1\frac{2}{3}\right)$$

$$= \left(\frac{8}{3} - \frac{3}{2} + \frac{5}{6}\right) \div \left(\frac{4}{\cancel{5}_1} \times \frac{\cancel{5}^1}{3}\right)$$

$$\left(\frac{16 - 9 + 5}{6}\right) \div \frac{4}{3}$$

$$= \frac{12}{6} \div \frac{4}{3}$$

$$= 2 \div \frac{4}{3}$$

$$= \frac{{}^1\cancel{2}}{1} \times \frac{3}{\cancel{4}_2}$$

$$= \frac{3}{2}$$

$$= \underline{\underline{1\frac{1}{2}}}$$

The order in which the mathematical operations are manipulated when simplifying fractions, is as follows:

B – Brackets
O – Of
D – Division
M – Multiplication
A – Addition
S – Subtraction

Do the following exercise to further recall the facts that have been learnt about fractions.

Review Exercise

1. Complete the second table using the fractions in the first table.

$\frac{4}{5}, \frac{1}{7}, \frac{5}{7}, \frac{4}{9}, \frac{9}{4}, \frac{19}{15}, \frac{7}{12}, \frac{1}{15}, \frac{7}{8}, \frac{11}{9}, \frac{23}{50}, \frac{22}{7}, \frac{1}{3}, \frac{8}{7}, \frac{6}{5}$

Unit Fractions	
Proper Fractions	
Improper Fractions	

2. Fill in the blanks in the following table.

Mixed number	$2\frac{1}{2}$	$1\frac{3}{5}$	$3\frac{5}{6}$	......	....	......
Improper fraction	......	......	......	$\frac{7}{2}$	$\frac{16}{3}$	$\frac{22}{5}$

3. Fill in the blanks.

a. $\frac{1}{4} = \frac{1 \times \dots}{4 \times 3} = \frac{..}{12}$ **b.** $\frac{2}{3} = \frac{\dots}{12}$ **c.** $\frac{2}{7} = \frac{\dots}{14}$ **d.** $\frac{4}{16} = \frac{\dots}{\dots}$

e. $\frac{8}{20} = \frac{\dots \div \dots}{\dots \div \dots} = \frac{\dots}{5}$ **f.** $\frac{10}{12} = \frac{5}{\dots}$ **g.** $\frac{21}{30} = \frac{7}{\dots}$ **h.** $\frac{75}{100} = \frac{\dots}{\dots}$

4. Write the fractions in each of the following parts in ascending order.

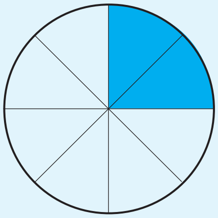
(i) $\frac{1}{7}, \frac{1}{3}, \frac{1}{10}, \frac{1}{2}$ (ii) $\frac{2}{5}, \frac{2}{9}, \frac{2}{11}, \frac{2}{3}$

(iii) $\frac{2}{3}, \frac{5}{6}, \frac{1}{2}, \frac{3}{4}$ (iv) $\frac{4}{5}, \frac{5}{8}, \frac{3}{4}, \frac{1}{2}$

5. If $\frac{3}{4}$ of a tank which was completely filled with water was used by a household for its needs on a certain day, what fraction of the tank still held water by the end of the day?

6. If the two pieces of wire A and B are of unequal length, is $\frac{1}{3}$ of the length of A equal to $\frac{1}{3}$ of the length of B, or are they unequal? Give reasons for your answer.

7. If the two shaded regions of the circular lamina that has been divided into eight equal parts as shown in the figure, are cut out and discarded,



(i) what fraction of the total lamina is remaining?

(ii) what fraction of the total lamina is exactly half the remaining portion equal to?

8. Simplify:

(i) $\frac{2}{5}+\frac{3}{5}+\frac{4}{5}$ (ii) $\frac{1}{2}+\frac{2}{3}+\frac{4}{5}$ (iii) $1\frac{1}{2}+2\frac{1}{4}-1\frac{2}{3}$

(iv) $\frac{1}{2}$ of $\left(\frac{4}{5}+\frac{2}{3}\right)$ (v) $\left(4\frac{1}{2}-\frac{3}{5}\right)\times 1\frac{2}{13}$ (vi) $\left(1\frac{2}{5}\times\frac{5}{7}\right)+\left(\frac{3}{4}\div\frac{1}{2}\right)$

(vii) $\frac{4}{5}$ of $2\frac{2}{5}\div 1\frac{1}{2}$ (viii) $2\frac{2}{5}\div 1\frac{1}{2}\times\frac{4}{5}$

9. Mother, who went to the market with Rs 500 in hand, spent Rs 300 to buy vegetables and Rs 150 to buy fruits.

(i) What fraction of the money in hand did she spend on vegetables?

(ii) What fraction of the money in hand did she spend on fruits?

(iii) If she had planned to save $\frac{1}{4}$ of the money she had taken to the market, was she able to fulfill her plan? Give reasons for your answer.

10. Sathis, who left home on a journey, travelled $\frac{1}{4}$ of the journey by bicycle, $\frac{2}{3}$ by bus and the remaining portion by a three wheeler.

(i) What fraction of the total journey did he travel by both bicycle and bus?

(ii) What fraction of the whole journey remained for him to travel by three wheeler?

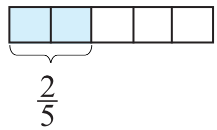
3.1 Applications of fractions

Fractions are used in many calculations that come up in day to day life. These calculations can easily be carried out with a correct understanding of fractions. Given below are several examples of such situations.

Example 1

$\frac{2}{5}$ of a mixture of flour prepared to make a certain type of food consists of kurakkan flour, while the remaining portion consists of wheat flour. A cook wishes to prepare 50 kg of this mixture to make the food item, to cater to an order. Find the amount of kurakkan flour and wheat flour required for this mixture.

Fraction of kurakkan flour in the mixture $= \frac{2}{5}$



The quantity of kurakkan flour in the mixture $= \frac{2}{5}$ of 50 kg

$= 50 \text{ kg} \times \frac{2}{5}$

Quantity of kurakkan flour in the mixture $= \underline{\underline{20 \text{ kg}}}$

The quantity of wheat flour in the mixture $= 50 - 20 \text{ kg}$

$= \underline{\underline{30 \text{ kg}}}$

Example 2

It takes 12 minutes to fill $\frac{1}{4}$ of a certain tank using a pipe through which water flows at a uniform rate. Find the time it takes to fill the whole tank with water using this pipe.

Time taken to fill $\frac{1}{4}$ of the tank $= 12$ minutes

$\therefore$ Time taken to fill $\frac{4}{4}$ of the tank (that is, the whole tank) $= 12 \times 4$ minutes

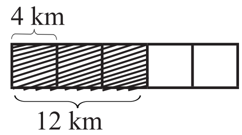
$= \underline{\underline{48 \text{ minutes}}}$

Example 3

Selva can travel $\frac{3}{5}$ of the distance from his home to the school by bus. This distance is 12 km. Find the distance from his home to the school.

$\frac{3}{5}$ of the distance from Selva's home to the school = 12 km

$\frac{1}{5}$ of the distance from Selva's home to the school $= 12 \text{ km} \div 3$



$= 4$ km

$\therefore$ Total distance to the school (that is, $\frac{5}{5}$ of the distance to the school) $= 4 \text{ km} \times 5$

$= \underline{\underline{20 \text{ km}}}$

Example 4

A tank was filled with water up to a $\frac{4}{5}$ of its capacity. After consuming 350 l of this amount, the quantity of water remaining filled $\frac{1}{3}$ of the tank.

(i) What fraction of the whole tank was used?

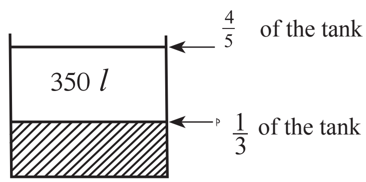
(ii) Find the capacity of the tank.

(i) The quantity of water used as a fraction of the whole tank $\Big\}$ $= \frac{4}{5} - \frac{1}{3}$

$= \frac{12-5}{15}$

$= \underline{\underline{\frac{7}{15}}}$

(ii) $\frac{7}{15}$ of the whole tank $= 350\ l$



$\therefore$ $\frac{1}{15}$ of the whole tank $= \frac{350\ l}{7}$

$\therefore$ Capacity of the tank $= \frac{\cancel{350}^{50}}{\cancel{7}_1} \times 15\ l$

$= \underline{\underline{750\ l}}$

Do the following exercise on problems involving fractions.

Exercise 3.1

1. Calculate the following quantities.

(i) $\frac{1}{2}$ of Rs 5 000 (iii) $\frac{3}{4}$ of 200 m (v) $\frac{2}{3}$ of 2.4 l

(ii) $\frac{1}{4}$ of 2000 ml (iv) $\frac{3}{5}$ of 250 kg (vi) $\frac{3}{4}$ of 4.8 km

2. Mr. Upul received Rs 24 000 as his salary last month. He spent $\frac{3}{8}$ of it for travel expenses. Find the amount he spent for travel expenses.

3. A household water tank was filled with water. When $\frac{3}{4}$ of its volume was used up, there was 200 l of water remaining in the tank.

(i) What fraction of the whole tank was the remaining volume of water equal to?

(ii) Find the capacity of the tank.

4. $\frac{3}{7}$ of a plot of land belonged to Pradeep. He bought $\frac{1}{4}$ of the portion of land which did not belong to him, and annexed it to his plot.

(i) What fraction of the total plot of land did Pradeep buy?

(ii) Show that Pradeep now owns more than half the plot of land.

(iii) If the area of the portion remaining after Pradeep had bought a part of it is 240 m^2, what is the area of the land owned by Pradeep in square metres?

5. Vishwa, who is saving money to buy a bicycle, has been able to save $\frac{5}{8}$ of the required amount. He requires Rs 2 700 more to buy the bicycle.

(i) What fraction of the value of the bicycle does he now need to save?

(ii) Find the value of the bicycle.

6. Mohamed wrote exactly half of his land in his daughter's name and $\frac{1}{3}$ of his land in his son's name. He donated the remaining 10 acres to a charity.

(i) What fraction of the total plot of land did Mohamed donate to charity?

(ii) How many acres is the whole plot of land?

(iii) Since the land that was donated to charity was not sufficient, his daughter was willing to give a portion of what she received, so that the land that the charity received would be double the initial amount. Show that when this is done, the plots that the daughter and the son have are equal in area.

7. Black pepper and cloves have been cultivated in $\frac{7}{8}$ of a plot of land. Black pepper has been cultivated in 450 m^2 while cloves have been cultivated in $\frac{1}{4}$ of the land. Find,

(i) the fraction of land on which black pepper has been cultivated.
(ii) the area of the whole land.
(iii) the area in which cloves have been cultivated.

8. A piece of wire was cut into three equal parts. One of the three parts was cut again into four equal pieces.

(i) What fraction of the length of the whole piece of wire is the length of one small piece?
(ii) Illustrate the above division of the wire by a figure and compare the answer you obtain with the answer in (i) above.
(iii) If a small piece of wire is 70 cm in length, find the length of the whole wire.

3.2 Further applications of fractions

After a portion of a unit has been separated, dividing the remaining portion repeatedly occurs in certain instances as applications of fractions. Such an instance is given in the following example.

Example 1

From the money that Raj received from his father, he spent $\frac{2}{3}$ on books and $\frac{1}{4}$ of the remaining amount on travel expenses. After that he had a balance of Rs 500.

(i) What fraction of the amount Raj received from his father was remaining after buying the books?
(ii) What fraction of the amount the father gave did he spend on travelling?
(iii) Find the amount he obtained from his father.

(i) The fraction that was spent to buy books $= \frac{2}{3}$

Fraction that remained after buying the books $= 1 - \frac{2}{3}$

$= \underline{\underline{\frac{1}{3}}}$

(ii) Fraction spent on travelling $= \frac{1}{4}$ of the remaining portion

$$= \frac{1}{4} \text{ of } \frac{1}{3}$$

$$= \frac{1}{4} \times \frac{1}{3}$$

$$= \underline{\underline{\frac{1}{12}}}$$

(iii) Fraction spent on buying books and on travelling

$$= \frac{2}{3} + \frac{1}{12}$$

$$= \frac{8+1}{12}$$

$$= \frac{9}{12}$$

$$= \underline{\underline{\frac{3}{4}}}$$

Fraction remaining after both the above expenses were met $= 1 - \frac{3}{4}$

$$= \frac{1}{4}$$

It is given that the balance is Rs 500.

$\frac{1}{4}$ of what the father gave = Rs 500

$\therefore$ Amount the father gave = Rs 500×4 = $\underline{\underline{\text{Rs } 2\,000}}$

Exercise 3.2

1. Mr. Austin who works in an office in the city, spends $\frac{2}{5}$ of his salary on food and sends $\frac{2}{3}$ of the remaining amount to his wife.

(i) What fraction of his salary remains after he has spent for his food?

(ii) What fraction of his salary does he send his wife?

(iii) What fraction of his salary is left over after these expenses are met?

2. From a certain amount of money, $\frac{1}{2}$ was given to A, $\frac{1}{3}$ of the remaining amount was given to B, and the rest was given to C.

(i) Find the fraction that C received from the amount that was distributed.

(ii) If instead of dividing the amount in the above manner, it was divided equally between the three, show that the amount that B would receive is twice the amount that he received initially.

(iii) If C received Rs 1000 through the initial distribution, find the total amount that was distributed among the three.

3. It has been decided to allocate $\frac{2}{3}$ of the floor area of a certain hall for classrooms, $\frac{2}{3}$ of the remaining portion for an office, and the rest of the floor area, which is 200 m^2 for the library.

(i) What fraction of the whole area has been allocated for the office?

(ii) What fraction of the whole area has been allocated for the library?

(iii) Find the floor area of the hall.

(iv) Find separately the floor area that is allocated for the classrooms and the office.

4. Of the amount that Anil spent on a trip, $\frac{4}{7}$ was spent on food and $\frac{2}{3}$ of the remaining amount on travelling. If Rs 800 was spent on other expenses, find how much Anil spent in total on the trip.

5. Saroja read $\frac{1}{3}$ of a book she borrowed from the library on the first day. On the second day she was only able to read $\frac{1}{2}$ of the remaining portion. On the third day, she finished the book by reading the remaining 75 pages. How many pages were there in total in the book?

Miscellaneous Exercise

1. Find the fraction suitable for the blank space: $3\frac{1}{2}+(1\frac{1}{2}\times.....)=4\frac{1}{2}$

2. Simplify: $\frac{4}{5}$ of $\dfrac{2\frac{1}{2}+1\frac{2}{3}}{1\frac{1}{5}\div\frac{4}{15}+\frac{1}{2}}$

3. *A, B* and *C* are the three owners of a business. The profit from the business was divided between the three of them based on the amount they each invested. *A* received $\frac{2}{7}$ of the profit, *B* received twice the amount *A* received, and *C* received the remaining portion. If the total amount that *A* and *B* received together is Rs 72 000, find the profit from the business.

4. An election was held between two persons to select a representative to a certain institution. All the voters who were registered cast their votes. The winner received $\frac{7}{12}$ of the votes and won by a majority of 120 votes.

(i) What fraction of the total votes did the person who lost receive?
(ii) How many registered voters were there?
(iii) Find the number of votes the winner received.

4 Binomial Expressions

By studying this lesson you will be able to

- multiply two binomial expressions
- expand the square of a binomial expression

Do the following exercise to recall the facts you have learnt earlier about simplifying algebraic expressions.

Review Exercise

1. Simplify the following algebraic expressions.

a. $2 \times 3a$ **b.** $4 \times (-2x)$ **c.** $(-3) \times 2x$
d. $2x \times 3y$ **e.** $3a \times (-5b)$ **f.** $(-2m) \times 4n$
g. $(-4p) \times (-2q)$ **h.** $3x \times 5x$ **i.** $(-5a) \times 3a$

2. Expand the following algebraic expressions.

a. $2(x+1)$ **b.** $3(b+3)$ **c.** $4(y-2)$
d. $-3(a+2)$ **e.** $-2(x-2)$ **f.** $x(2x+3)$
g. $2y(y+1)$ **h.** $-2x(4x+1)$ **i.** $-3b(a-b)$
j. $2(a-b-3c)$

3. Expand and simplify the following algebraic expressions.

(a) (i) $x(x+2)+2(x+2)$ **(ii)** $y(y+3)+3(y-2)$
(iii) $x(x+1)-3(x-1)$ **(iv)** $m(m-3n)-n(m-3n)$

(b) (i) $(x+5)(x+8)$ **(ii)** $(7+a)(3+a)$ **(iii)** $(x-5)(x+8)$
(iv) $(x+5)(x-8)$ **(v)** $(2+m)(3-m)$ **(vi)** $(x-5)(x-8)$

4.1 Product of two binomial expressions

What you simplified in 3(b) above is the product of two binomial expressions of the form $x + a$. In this lesson, we will learn about expanding the product of two binomial expressions of the form $ax + by$. Here, ax and by are the two terms of the binomial expression $ax + by$.

Example 1 Expand and simplify $(3x + 2)(2x + 3)$.

$(3x+2)(2x+3)$

$= 3x(2x+3)+2(2x+3)$

$= 6x^2+9x+4x+6$

$= \underline{\underline{6x^2+13x+6}}$

or

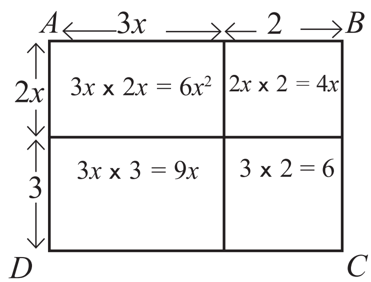
$(3x+2)(2x+3)$

$= (3x+2)\times 2x+(3x+2)\times 3$

$= 6x^2+4x+9x+6$

$= \underline{\underline{6x^2+13x+6}}$

The result that was obtained above can also be illustrated using the areas of rectangles. (All the measurements below are assumed to have been given in the same units).



For the rectangle $ABCD$,

length of $AB = 3x + 2$

length of $AD = 2x + 3$

Area $= (3x + 2)(2x + 3)$ ——— ①

According to the figure,

Area of the rectangle $ABCD$ = The sum of the areas of the four smaller rectangles

$= 6x^2 + 9x + 4x + 6$

$= 6x^2 + 13x + 6$ ——— ②

It is clear according to (1) and (2) that,

$(3x + 2)(2x + 3) = 6x^2 + 13x + 6$

Now study how the binomial expressions of different types given in the following examples have been expanded and simplified.

Example 2

$(3x-2)(2x+5)$

$(3x-2)(2x+5)$

$= 3x(2x+5)-2(2x+5)$

$= 6x^2+15x-4x-10$

$= \underline{\underline{6x^2+11x-10}}$

Example 3

$(2x + y)(x + 3y)$

$(2x + y)(x + 3y)$

$= 2x(x + 3y) + y(x + 3y)$

$= 2x^2 + 6xy + xy + 3y^2$

$= \underline{\underline{2x^2 + 7xy + 3y^2}}$

Example 4

$(3x+2y)(3x-2y)$

$(3x+2y)(3x-2y)$

$= 3x(3x-2y)+2y(3x-2y)$

$= 9x^2-6xy+6xy-4y^2$

$= \underline{\underline{9x^2-4y^2}}$

Example 5

$(5a-2b)\,(2a-3b)$

$$(5a-2b)\,(2a-3b)$$
$$=5a(2a-3b)-2b(2a-3b)$$
$$=10a^2-15ab-4ab+6b^2$$
$$=\underline{\underline{10a^2-19ab+6b^2}}$$

Example 6

$(a+b)\,(\frac{1}{3}a-\frac{1}{4}b)$

$$(a+b)\,(\frac{1}{3}a-\frac{1}{4}b)$$
$$=a(\frac{1}{3}a-\frac{1}{4}b)+b(\frac{1}{3}a-\frac{1}{4}b)$$
$$=\frac{1}{3}a^2-\frac{1}{4}ab+\frac{1}{3}ab-\frac{1}{4}b^2$$
$$=\underline{\underline{\frac{1}{3}a^2+\frac{1}{12}ab-\frac{1}{4}b^2}}$$

Exercise 4.1

1. Expand and simplify the following binomial expressions.

a. $(x+2)\,(x+2)$ **b.** $(x-3)\,(x-3)$ **c.** $(2x+3)\,(x+2)$
d. $(2p-5)\,(p-3)$ **e.** $(3x-1)\,(3x+1)$ **f.** $(-3x+2)\,(2x-3y)$
g. $(2a+b)\,(3a+2b)$ **h.** $(3x-5y)\,(4x+3y)$ **i.** $(-3p+4q)\,(3p-2q)$
j. $(-7k-5l)\,(3k+4l)$ **k.** $(4m-3n)\,(4m-3n)$ **l.** $(5x-2y)\,(5x-2y)$
m. $\left(\frac{1}{2}x+y\right)(2x+3y)$ **n.** $\left(\frac{1}{3}p+\frac{1}{2}q\right)\left(\frac{2}{3}p-\frac{3}{4}q\right)$ **o.** $(3x+4y)\,(5a+3b)$

2. If the length of a rectangular shaped field is $(2a+7)$ m and the breadth is $(2a-3)$ m, determine the area of the field in terms of a.

3. Piyumi made a square - shaped flower bed. Her sister made a rectangular flower bed. The length of the sister's bed was 3 metres more than of Piyumi's and the width was 2 metres less than of Piyumi's. Taking the length of one side of Piyumi's bed as x, write the length and width of the sister's bed in terms of x and then express its area in the form Ax^2+Bx+C.

4. A child bought a mandarins which were priced at x rupees each. He then decides to buy three times that number of apples. The price of an apple is twice the price of a mandarin.

(i) Write an expression in terms of a and x for the cost of the apples.

The fruit seller states that if the number of apples that are bought is increased by 5 fruits, he can reduce the price of each apple that is bought by 3 rupees. Accordingly, the child decides to buy 5 more apples.

(ii) If he does this, write down the number of apples he buys in terms of a.

(iii) Write down the price of an apple in terms of x.
(iv) Write down the cost of the apples in terms of a and x.
(v) Expand and simplify the binomial expression in (iv) above.

4.2 Squares of binomial expressions

Let us turn our attention again to the following expressions which you expanded in the above exercise.
Do you notice that the two binomial expressions to be multiplied are equal?

$$(x+2)(x+2),\ (x-3)(x-3),\ (5x-2y)(5x-2y)$$

In the same manner that we write $x \times x = x^2$, we can write $(x+2)(x+2) = (x+2)^2$.

Similarly,

$$(x-3)(x-3) = (x-3)^2$$
$$(5x-2y)(5x-2y) = (5x-2y)^2$$

Each of these expressions written in the form $(x+2)^2$, $(x-3)^2$ and $(5x-2y)^2$ is called the square of the respective binomial expression.

The same method that was used above to simplify the product of two binomial expressions can be used to expand such squares of binomial expressions too.

Example 1

Write $(x+2)^2$ as a product of two binomial expressions and expand it.

$$\begin{aligned}(x+2)^2 &= (x+2)(x+2)\\ &= x(x+2)+2(x+2)\\ &= x^2+2x+2x+4\\ &= \underline{\underline{x^2+4x+4}}\end{aligned}$$

Simplifying the square of a binomial expression can be done using another method too.

Now let us consider how the squares of binomial expressions of the form $(a+b)$ and $(a-b)$ are expanded.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+ab+ab+b^2\\ &= \underline{\underline{a^2+2ab+b^2}}\end{aligned}$$

It is important to remember this as a formula.

$$(a+b)^2 = \underbrace{a^2}_{\text{Square of the first term}} + \underbrace{2ab}_{\text{Two times the product of the first and second terms}} + \underbrace{b^2}_{\text{Square of the second term}}$$

Now let us consider the expansion of $(a-b)^2$

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - ab - ab + b^2\\ &= \underline{\underline{a^2 - 2ab + b^2}}\end{aligned}$$

That is, $\boxed{(a-b)^2 = a^2 - 2ab + b^2}$

Note : The expansion of $(a-b)^2$ can also be obtained by substituting $-b$ for b in the expansion of $(a+b)^2$

i.e., $(a+(-b))^2 = a^2 + 2(a)(-b) + (-b)^2 = a^2 - 2ab + b^2$

Also,

$$(-a+b)^2 = (-a)^2 + 2(-a)b + b^2 = a^2 - 2ab + b^2$$

$$(-a-b)^2 = (-a)^2 + 2(-a)(-b) + (-b)^2 = a^2 + 2ab + b^2$$

You may have observed that the expansions of $(a+b)^2$ and $(-a-b)^2$ are equal to each other and that the expansions of $(a-b)^2$ and $(-a+b)^2$ too are equal to each other.

Now study how the following squares of binomial expressions have been expanded.

Example 2

$$\begin{aligned}(x+3)^2 &= x^2 + 2\times x\times 3 + 3^2\\ &= \underline{\underline{x^2 + 6x + 9}}\end{aligned}$$

Example 3

$$\begin{aligned}(y-2)^2 &= y^2 - 2\times y\times 2 + 2^2\\ &= \underline{\underline{y^2 - 4y + 4}}\end{aligned}$$

Example 4

$$\begin{aligned}(3x+5y)^2 &= (3x)^2 + 2\times 3x\times 5y + (5y)^2\\ &= \underline{\underline{9x^2 + 30xy + 25y^2}}\end{aligned}$$

Example 5

$$\begin{aligned}(3a-2b)^2 &= (3a)^2 - 2\times 3a\times (2b) + (2b)^2\\ &= \underline{\underline{9a^2 - 12ab + 4b^2}}\end{aligned}$$

Example 6

$$(-y+5)^2 = (-y)^2 - 2\times(y)\times 5 + 5^2$$
$$= \underline{\underline{y^2 - 10y + 25}}$$

Example 7

$$(-2x-3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2$$
$$= \underline{\underline{4x^2 + 12xy + 9y^2}}$$

Let us consider how this method is used to find numerical values.

Example 8

Find the value of 105^2.

$105^2 = (100 + 5)^2$
$= 100^2 + 2 \times 100 \times 5 + 5^2$
$= 10000 + 1000 + 25$
$= \underline{\underline{11025}}$

Example 9

Find the value of 99^2

$99^2 = (100 - 1)^2$
$= 100^2 - 2 \times 100 \times 1 + 1^2$
$= 10000 - 200 + 1$
$= \underline{\underline{9801}}$

Example 10

Verify $(x + y)^2 = x^2 + 2xy + y^2$ for the values $x = 5$ and $y = 2$.

l.h.s

$(x+y)^2$
$= (5+2)^2$
$= 7^2$
$= \underline{\underline{49}}$

r.h.s.

$x^2 + 2xy + y^2$
$= 5^2 + 2\times 5\times 2 + 2^2$
$= 25 + 20 + 4$
$= \underline{\underline{49}}$

$\therefore$ l.h.s = r.h.s.

Exercise 4.2

1. For each expression in column *A* which is a square of a binomial expression, select and join the corresponding expansion from Column *B*.

Column *A*	**Column *B***
a. $(x + 5)^2$	$4x^2 + 4xy + y^2$
b. $(x - 5)^2$	$4y^2 + 4xy + x^2$
c. $(2x + 5)^2$	$x^2 - 10x + 25$
d. $(2x + y)^2$	$4x^2 - 4xy + y^2$
e. $(-2x + 5)^2$	$x^2 - 4xy + 4y^2$
f. $(x - 2y)^2$	$4x^2 - 12xy + 9y^2$
g. $(-2x + y)^2$	$4x^2 + 20x + 25$
h. $(2x + 3y)^2$	$4x^2 + 12xy + 9y^2$
i. $(2x - 3y)^2$	$x^2 + 10x + 25$
j. $(-2y - x)^2$	$4x^2 - 20x + 25$

2. Expand each of the following squares of binomial expressions.

a. $(x + 2)^2$ **b.** $(a + 3)^2$ **c.** $(p - 3)^2$ **d.** $(y - 1)^2$
e. $(2a + 3)^2$ **f.** $(3b + 2)^2$ **g.** $(3x - 1)^2$ **h.** $(4m - 5)^2$
i. $(3p + 4q)^2$ **j.** $(5m - 3n)^2$ **k.** $(-2y + 5)^2$ **l.** $(3a - 5b)^2$
m. $(-3m + n)^2$ **n.** $(-5m - 6n)^2$

3. Write down the term suitable for the blank space in each of the following expressions.

a. $(x + 3)^2 = x^2 + 6x +$ ___ **b.** $(y + 2)^2 = y^2 +$ ___ $+ 4$
c. $(m - 5)^2 = m^2 - 10m +$ ___ **d.** $(a +$ ___ $)^2 = a^2 + 8a + 16$
e. $($ ___ $+ b)^2 = 25 + 10b + b^2$ **f.** $($ ___ $- 7)^2 = x^2 - 14x + 49$
g. $(-3 +$ ___ $)^2 =$ ___ $- 6x + x^2$ **h.** $($ ___ $- x)^2 = +16 - 8x + x^2$

4. Write each of the following as a square of a binomial expression and find the value.

(i) 21^2 (ii) 102^2 (iii) 17^2 (iv) 98^2 (v) 9.9^2

5. If the length of a side of a square shaped room is given as $(2a + 3b)$ meters, write down an expression for the area of the room in terms of a and b and expand it.

6. Verify the following for the values $a = 2$ and $b = 3$.

(i) $(-a + b)^2 = a^2 - 2ab + b^2$
(ii) $(-a - b)^2 = a^2 + 2ab + b^2$

Miscellaneous Exercise

1. Verify that $(2x + 3y)(x + y) = 2x^2 + 5xy + 3y^2$ for each of the following cases.

(i) $x = 3, y = 2$ (ii) $x = 5, y = 0$
(iii) $x = 1, y = 1$ (iv) $x = -1, y = -2$

2. Write each of the following squares of binomial expressions with fractional coefficients as a product of two binomial expressions and simplify it.

(i) $(\frac{1}{2}x + y)^2$ (ii) $(\frac{1}{3}a - b)^2$ (iii) $(\frac{1}{4}m - \frac{2}{3}n)^2$

3. Fill in the blanks

(i) $(x + \_)^2 = x^2 + 6x + \_$ (ii) $(y + \_)^2 = y^2 + 8y + \_$
(iii) $(\_ + 5)^2 = x^2 + \_ + 25$ (iv) $(\_ + y)^2 = x^2 + \_ + y^2$

4. For each of the following expressions, determine the term that should be added so that it can be written as a square of a binomial expression, add the relevant term to the given expression and then write it as a square of a binomial expression.

(i) $x^2 + 6x$	(ii) $y^2 + 8y$	(iii) $m^2 + 10m$
(iv) $a^2 - 4a$	(v) $x^2 + 4xy$	(vi) $p^2 - 12pq$

5. Find the value of $x^2 + y^2$ when $x + y = 5$ and $xy = 6$.

6. Find the value of $a^2 + b^2$ when $a - b = 3$ and $ab = 28$.

7. Find the value of $x + y$ when $x^2 + y^2 = 25$ and $xy = 12$.

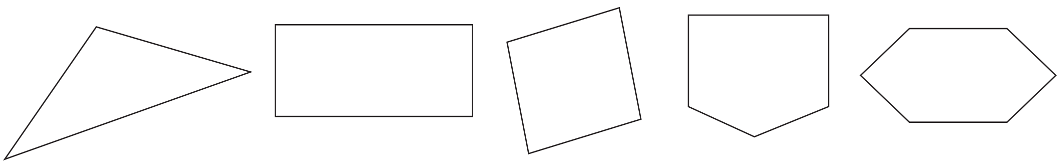
8. If $(x + k)^2 = x^2 + 6x + q$, find the values of k and q.

9. Find the value of $t^2 + \frac{1}{t^2}$ when $t + \frac{1}{t} = 2$.

5 Congruence of Triangles

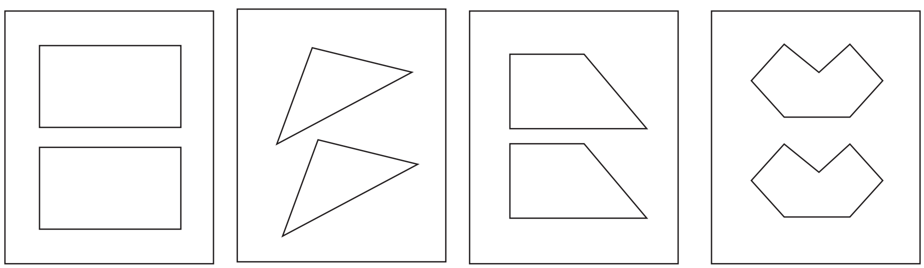
By studying this lesson you will be able to
- recognize the congruence of two plane figures
- identify the necessary conditions for two triangles to be congruent
- prove riders by using the congruence of triangles

Congruence of two plane figures



If we examine the above figures we see that they are all closed plane figures consisting of straight line segments. Such figures are called rectilinear plane figures. The sides and the angles are called the elements (parts) of these figures.

The pairs of rectilinear plane figures presented below in figures (i) to (iv), which are identical in shape and size can be placed on each other such that they coincide.



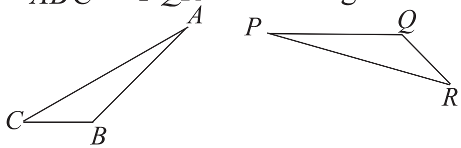
(i) (ii) (iii) (iv)

A pair of plane figures which may be made to coincide is called a pair of **congruent plane figures**. In this section we concentrate on the congruence of pairs of triangles.

5.1 Congruence of two triangles

A triangle has six elements. They are the three sides and the three angles of the triangle.

Let us assume that the two triangles *ABC* and *PQR* given below are congruent. Let us also assume that, when the two triangles are placed one on top of the other such that they coincide, then AB coincides with PQ, AC coincides with PR and BC coincides with QR. Then we say that, in the two triangles, the side corresponding to AB is PQ, the side corresponding to AC is PR and the side corresponding to BC is QR. Similarly, we say that the angle corresponding to $B\hat{A}C$ is $Q\hat{P}R$, the angle corresponding to $A\hat{B}C$ is $P\hat{Q}R$ and the angle corresponding to $A\hat{C}B$ is $P\hat{R}Q$.



Accordingly, in congruent triangles, the corresponding elements are equal to each other.

We indicate the fact that two triangles are congruent by using the symbol "$\equiv$". For example, if the two triangles ABC and PQR are congruent, we indicate this fact by writing $\Delta ABC \equiv \Delta PQR$

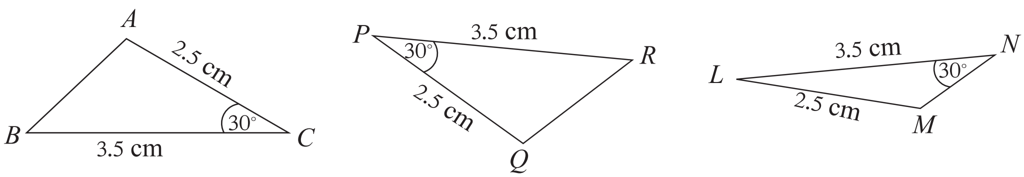
It is not necessary to show that the six elements of one triangle are equal to the six elements of the other triangle as indicated above to show that a pair of triangles is congruent. It is sufficient to show that three of the elements are equal. However, this does not mean that if any three elements of one triangle are equal to three elements of another triangle, then the two triangles are congruent. In certain cases, when three elements of one triangle are equal to three elements of another triangle, then the remaining elements too are equal and the triangles are congruent. There are four such cases. Let us now consider these four cases.

(*a*) First Case

The case in which two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle

Activity

Three triangles with two sides of length 2.5 cm and 3.5 cm and an angle of magnitude 30° are given below.



- Copy the triangle ABC onto a tissue paper and cut it out.
- Examine whether the cut out triangle can be made to coincide with the triangles PQR and LMN.
- Accordingly, select the triangle which is congruent to the triangle ABC.

It must be clear to you through the above activity, that only the triangle *PQR* is congruent to the triangle *ABC*. Both the triangles *PQR* and *LMN* have three elements which are equal to the given three elements of the triangle *ABC*. However, the triangle *ABC* is only congruent to the triangle *PQR*. It should be clear to you from this that, just because two triangles have three elements which are equal to each other, it does not mean that the triangles are congruent.

Let us consider another method by which we can identify that the triangle *ABC* is congruent to the triangle *PQR* but not to the triangle *LMN*. The 30^{o} angle of the triangle *ABC* is the included angle of the sides of length 2.5 cm and 3.5 cm. It is the same for the triangle *PQR*. However the 30^{o} angle is not the included angle of the sides of length 2.5 cm and 3.5 cm of the triangle *LMN*. Two sides and the included angle of triangle *ABC* are equal to two sides and the included angle of triangle *PQR*. But this cannot be said about the triangles *ABC* and *LMN*. $\therefore$ There is insufficient data to state that and ΔABC and $\Delta\ LMN$ are congruent.

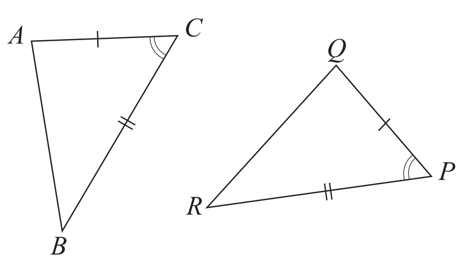
Note: Here the angle $A\hat{C}B$ which is 30^{o} is called the included angle of the sides *AC* and *CB*. Similarly, the angle $R\hat{P}Q$ is the included angle of the sides *PR* and *PQ* of the triangle *PQR*.

This result which you discovered through the above activity has been used as an axiom of geometry from ancient times.

If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, then the two triangles are congruent.

Showing that two triangles are congruent in the above manner is mentioned in short as being congruent under the case SAS.

The two triangles *ABC* and *PQR* given below can be shown to be congruent according to the above mentioned case using the given data as follows.



In the triangles *ABC* and *PQR*,

$$AC = PQ \quad \text{(Given)}$$
$$A\hat{C}B = R\hat{P}Q \quad \text{(Given)}$$
$$BC = PR \quad \text{(Given)}$$
$$\therefore \underline{\underline{\Delta ABC \equiv \Delta PQR}} \quad \text{(SAS)}$$

Since the above two triangles are congruent, the remaining corresponding elements are also equal.

That is,

the sides AB and QR which are opposite the equal angles $A\hat{C}B$ and $Q\hat{P}R$ are equal to each other, the angles $A\hat{B}C$ and $Q\hat{R}P$ which are opposite the equal sides AC and PQ are equal to each other,

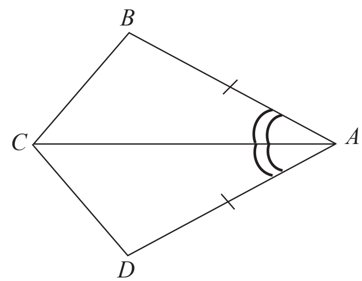
and the angles $C\hat{A}B$ and $P\hat{Q}R$ which are opposite the equal sides BC and PR are equal to each other.

Now let us consider an example.

Example 1

According to the data marked on the figure, prove that the triangles ABC and ADC are congruent, and write all the remaining equal corresponding elements.

Proof:



(i) In the triangles ABC and ACD,

$AB = AD$ (Given)

$B\hat{A}C = C\hat{A}D$ (Given)

AC is a common side

$\therefore \underline{\underline{\Delta ABC \equiv \Delta ADC}}$ (S A S)

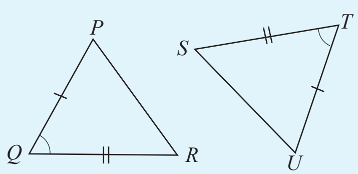
(ii) The corresponding elements of congruent triangles are equal.

$\therefore BC = CD,\ A\hat{B}C = A\hat{D}C$ and $A\hat{C}B = A\hat{C}D$.

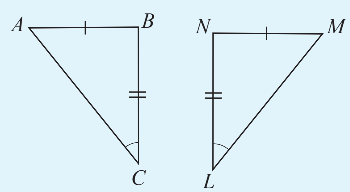
Exercise 5.1

1. Determine for which pairs of triangles the SAS case can be applied to prove congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

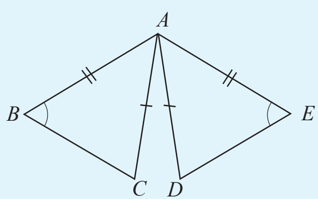
(i)



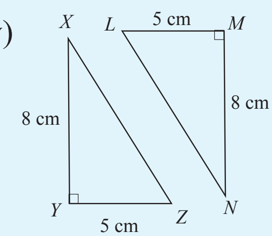
(ii)



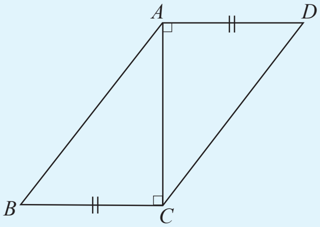
(iii)



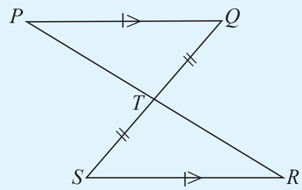
(iv)



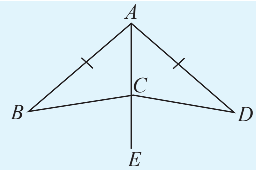
(v)



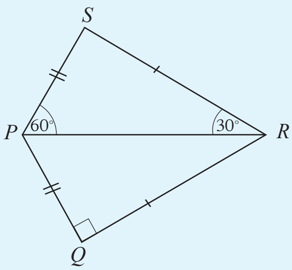
(vi)



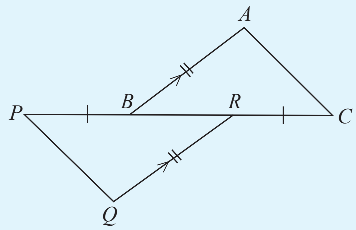
(vii)



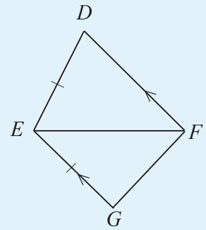
(viii)



(ix)



(x)



2. For each of the following parts, draw a sketch of the relevant pairs of triangles based on the information that is given. From these pairs of triangles, select the ones which are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

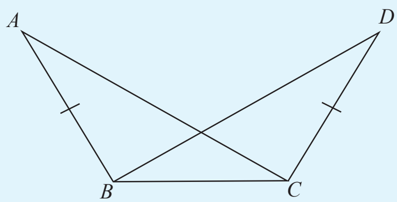
(i) In the triangles PQR and XYZ, $PQ = XZ, QR = XY, P\hat{Q}R = Y\hat{X}Z.$

(ii) In the triangles ABC and LMN, $AC = LN, BC = LM, A\hat{B}C = L\hat{M}N = 50°.$

(iii) In the triangles DEF and STU, $EF = TU, DF = SU, E\hat{F}D = T\hat{U}S.$

(iv) In the triangles ABC and PQR, $BC = PQ, C\hat{B}A = Q\hat{P}R, AC = PR.$

3.

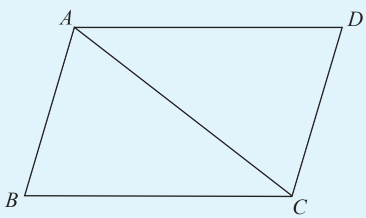


In the given figure, $AB = DC$ and $A\hat{B}C = B\hat{C}D$. Prove that,

(i) $\Delta ABC \equiv \Delta DCB$,

(ii) $AC = BD$.

4.



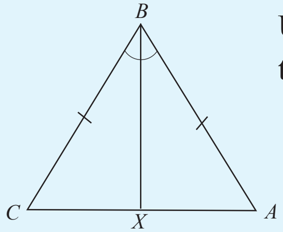
In the quadrilateral $ABCD$, the sides AD and BC are parallel and equal in length. Mark the given data in the diagram and prove that,

(i) $\Delta ABC \equiv \Delta ADC$,

(ii) $AB = DC$,

(iii) AB and DC are parallel to each other.

5. Using the information marked on the triangle ABC, prove that,



(i) $\Delta ABX \equiv \Delta CBX$

(ii) $A\hat{X}B = 90°$.

6. In the quadrilateral $ABCD$, the diagonals AC and BD bisect each other at O. Prove that,

(i) $\Delta AOD \equiv \Delta BOC$

(ii) the lines AD and BC are parallel to each other.

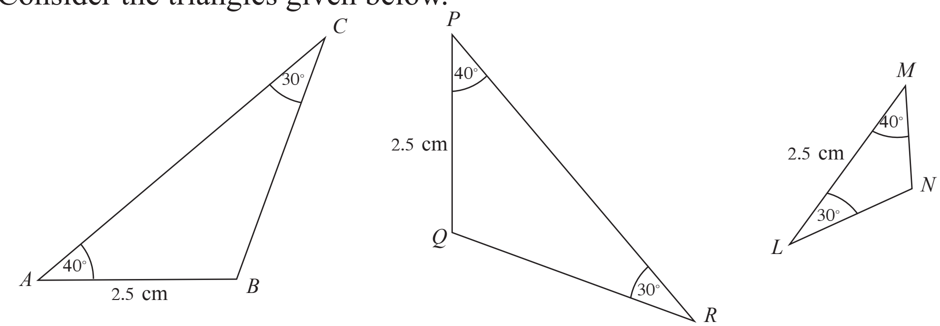
Now let us consider the second case by which the congruence of two triangles can be identified.

(*b*) Second Case

The case in which the magnitudes of two angles and the length of a side of a triangle are equal to the magnitudes of two angles and the length of a corresponding side of another triangle

Activity

Consider the triangles given below.



- Copy the triangle ABC onto a tissue paper and cut it out.
- Place it on the triangles PQR and LMN and examine which triangle it coincides with.
- Accordingly, which triangle is congruent to the triangle ABC?

It must be clear to you according to this activity that the trianlge ABC is congruent only to the triangle PQR.

In this case too, as in the case (*a*), the two triangles PQR and LMN have three elements which are equal to three elements of the triangle ABC. However, although the triangle ABC is congruent to the triangle PQR, it is not congruent to the triangle LMN. As before, it should be clear to you from this that, just because two triangles have three elements which are equal to each other, it does not mean that the triangles are congruent.

Therefore, let us consider another method by which we can identify that the triangle *ABC* is congruent to the triangle *PQR*. The given side of length 2.5 cm is opposite the given 30° angle of the triangle *ABC*. It is the same for the triangle *PQR*. However it is not the same in the triangle *LMN*. Accordingly, two angles of triangle *ABC* is equal to two angles of triangle *PQR*, and one side of triangle *ABC* is equal to the **corresponding** side of triangle *PQR*. However, the corresponding side of triangle *LMN* is not equal to that of triangle *ABC*.

Note: Here, corresponding sides are defined as those which are opposite equal angles of the two triangles.

If two angles and a side of one triangle are equal to two angles and the corresponding side of another triangle, then the two triangles are congruent.

Showing that two triangles are congruent under these conditions is stated concisely as being congruent according to the **AAS** case.

Using the given data, we can show as done below, that the triangles *STU* and *LMN* in the following figure are congruent according to the case mentioned above.

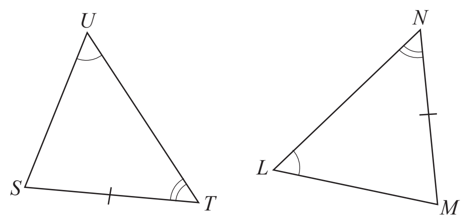
In the triangles *STU* and *LMN*

$S\hat{T}U = M\hat{N}L$ (Given)

$T\hat{U}S = N\hat{L}M$ (Given)

$ST = MN$ (Given)

$\therefore \Delta STU \equiv \Delta MNL$ (AAS)



Note: In the above two triangles *ST* and *MN* are a pair of corresponding sides which are also equal. Observe carefully that they are corresponding sides because they are opposite the angles $S\hat{U}T$ and $M\hat{L}N$ which are equal to each other.

Example 1

Based on the data marked on the figure, prove that, $\Delta ABX \equiv \Delta ACX$ and write all the remaining equal corresponding elements.

Proof:

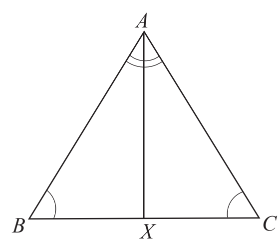
(i) In the triangles *ABX* and *ACX*,

$A\hat{B}X = A\hat{C}X$ (Given)

$B\hat{A}X = C\hat{A}X$ (Given)

AX is a common side

$\therefore \Delta ABX \equiv \Delta ACX$ (AAS)

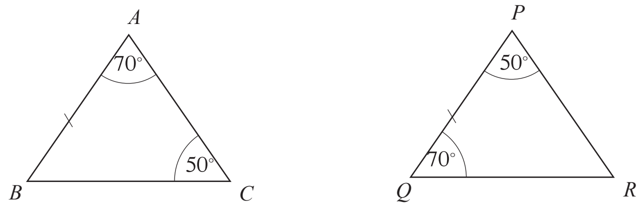


The corresponding elements of congruent triangles are equal.

Therefore $BX = CX,\ A\hat{X}B = A\hat{X}C,\ AB = AC$

Example 2

Determine whether the following pair of triangles is congruent under the case AAS.

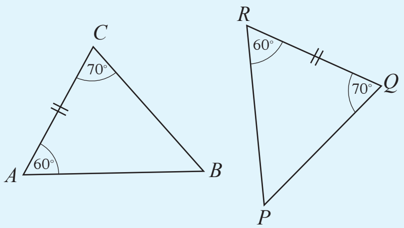


Two angles of the triangle ABC are equal to two angles of the triangle PQR. Also, $AB = PQ$. However, they are not corresponding sides. The reason for this is that the angles $A\hat{C}B$ and $P\hat{R}Q$ which are opposite these two sides are not equal to each other. ($A\hat{C}B = 50^\circ$, $P\hat{R}Q = 180^\circ - 50^\circ - 70^\circ = 60^\circ$) Therefore, there are insufficient reasons to say that the two triangles ABC and PQR are congruent to each other by the case AAS.

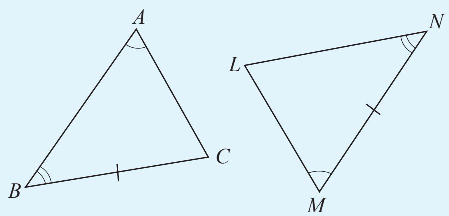
Exercise 5.2

Mention for which pairs of triangles the AAS conditions can be applied to prove congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the other pairs of corresponding elements which are equal to each other.

(i)



(ii)



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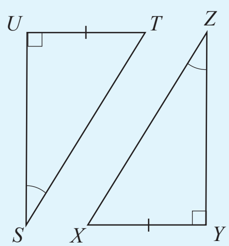
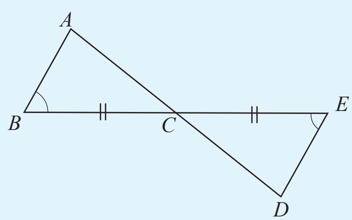


(iv)



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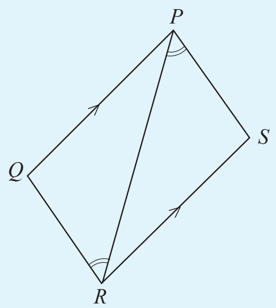

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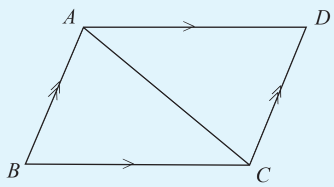

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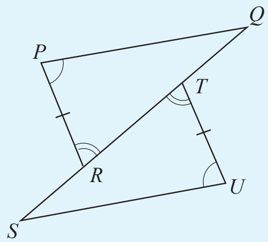

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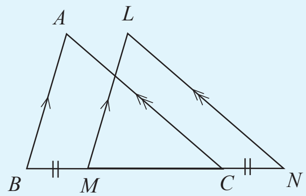


2. For each of the following parts, draw a sketch of the relevant pair of triangles based on the information that is given. From these pairs of triangles, select the ones that are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

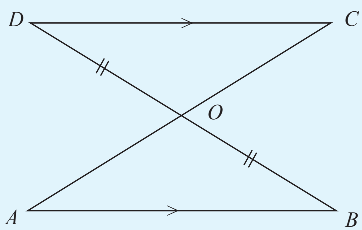
i. In the triangles ABC and PQR, $A\hat{B}C = P\hat{Q}R$, $A\hat{C}B = P\hat{R}Q$, $BC = QR$

ii. In the triangles XYZ and LMN, $X\hat{Y}Z = L\hat{M}N = 90^\circ$, $Y\hat{X}Z = 30^\circ$, $M\hat{N}L = 60^\circ$, $YZ = MN$

iii. In the triangles STU and PQR, $T\hat{S}U = Q\hat{R}P$, $TU = PR$, $T\hat{U}S = P\hat{Q}R$

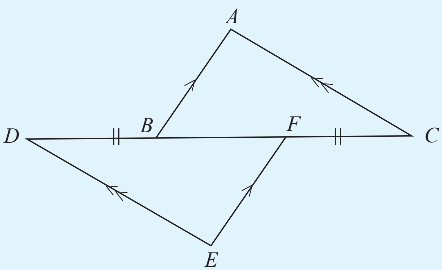
iv. In the triangles DEF and ABC, $E\hat{D}F = B\hat{A}C = 40^\circ$, $D\hat{F}E = A\hat{C}B = 60^\circ$, $DE = BA$

3.



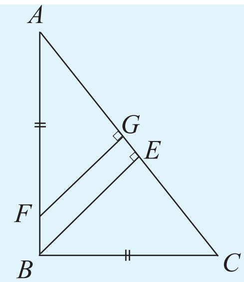
In the given figure, AB and CD are parallel to each other and $BO = OD$. Show that $\Delta AOB \equiv \Delta DOC$

4.



The pairs of sides AB and FE, and AC and DE are parallel to each other. Show that $\Delta ABC \equiv \Delta EFD$

5. In the triangle ABC, $A\hat{B}C = 90^\circ$. If $AF = BC$, prove that $\Delta AFG \equiv \Delta BCE$.



6. In the quadrilateral $ABCD$, $\hat{A} = \hat{C} = 90^\circ$. The angles $A\hat{D}C$ and $A\hat{B}C$ are bisected by BD. Prove that $\Delta ABD \equiv \Delta CBD$.

Let us consider the third case by which the congruence of two triangles can be identified.

(*c*) Third Case

The case of three sides of a triangle being equal to three sides of another triangle

Can a unique triangle be constructed when the lengths of the three sides of the triangle are given? To determine this, engage in the following activity.

Activity

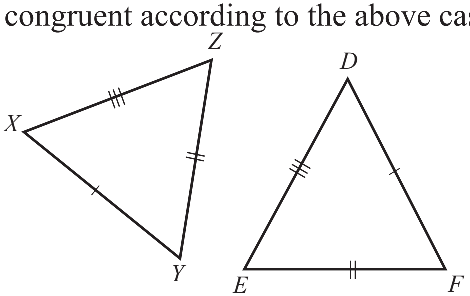
Break off pairs of ekel sticks of lengths 5 cm, 6 cm and 7 cm respectively. Make two triangles of side lengths 5 cm, 6 cm and 7 cm with these pieces. Do you see that the two triangles have to be congruent? By changing the positions of the pieces of ekel in one triangle, can you create a triangle which is not congruent to the other triangle? It must be clear to you that this is not possible.

This result which you established through the above activity can also be used as an axiom.

If the three sides of a triangle are equal to the three sides of another triangle, then the two triangles are congruent.

Showing that two triangles are congruent under these conditions is stated concisely as being congruent according to the **SSS** case.

We can show in the following manner that the pair of triangles XYZ and DEF are congruent according to the above case.



In the triangles XYZ and DEF,

$XY = DF$ (Given)

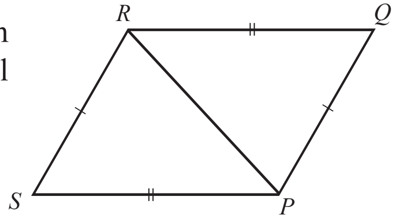
$YZ = EF$ (Given)

$ZX = DE$ (Given)

$\therefore \Delta XYZ \equiv \Delta DEF$ (SSS)

Example 1

Prove that $\Delta PQR \equiv \Delta PSR$ based on the information in the figure and write all the remaining equal corresponding elements.



Proof: In the triangles PQR and PSR,

$PQ = RS$ (Given)

$QR = PS$ (Given)

PR is a common side

$\therefore \Delta PQR \equiv \Delta RSP$ (SSS)

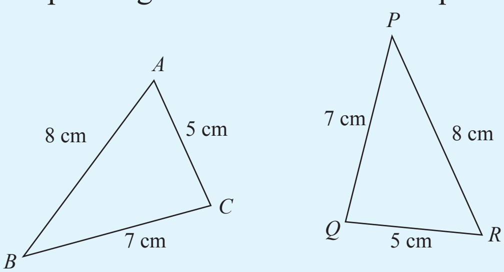
The corresponding elements of congruent triangles are equal

$\therefore \quad R\hat{S}P = P\hat{Q}R,\ S\hat{R}P = Q\hat{P}R,\ S\hat{P}R = Q\hat{R}P.$

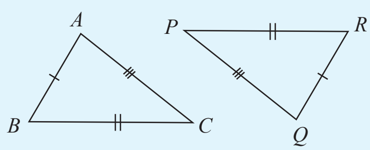
Exercise 5.3

1. Determine for which of the following pairs of the triangles the SSS conditions can be used to show congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

(i)



(ii)



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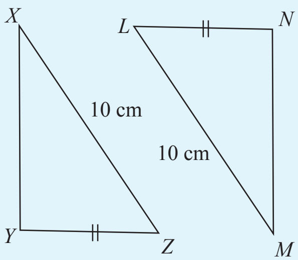

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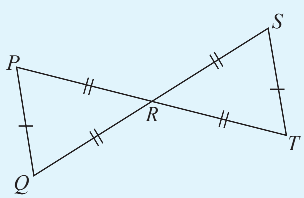

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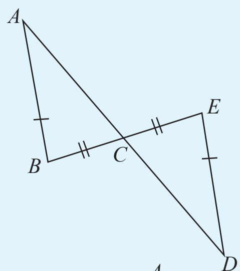

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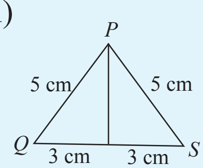

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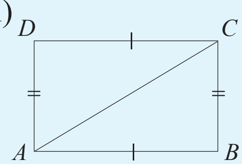

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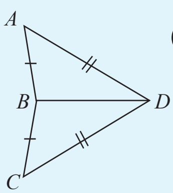
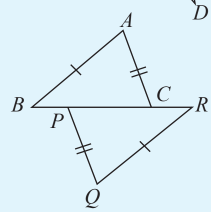
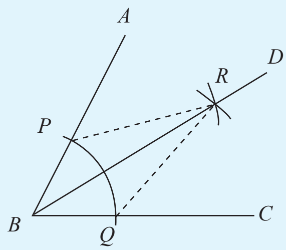


(ix)

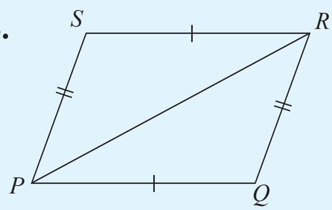


2. Draw a sketch of the following triangles based on the information that is given. Select the triangles (if there are any) which are congruent according to the SSS case and write down the remaining pairs of corresponding elements which are equal to each other.

In triangle PQR, $PQ = 4$ cm, $QR = 6$ cm, $RP = 5$ cm
In triangle XYZ, $XY = 6$ cm, $YZ = 8$ cm, $ZX = 10$ cm
In triangle LMN, $LM = 5$ cm, $NM = 4$ cm, $NL = 6$ cm
In triangle DEF, $DE = 8$ cm, $EF = 10$ cm, $FD = 6$ cm
In triangle ABC, $BC = 8$ cm, $CA = 7$ cm, $AB = 9$ cm
In triangle STU, $ST = 9$ cm, $TU = 7$ cm, $SU = 5$ cm

3. 

To bisect the angle $A\hat{B}C$, a student selects the point B as the centre and draws the arc PQ. The arc intersects AB and BC at the points P and Q respectively. Two equal arcs drawn from the points P and Q intersect at R. Prove that $P\hat{B}R = Q\hat{B}R$.

4. 

The opposite sides of the quadrilateral $PQRS$ are equal in length. Prove that,

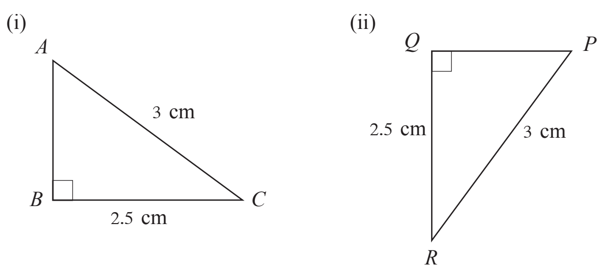
(i) $\Delta PSR \equiv \Delta PQR$
(ii) $P\hat{S}R = P\hat{Q}R$
(iii) the opposite sides of the quadrilateral are parallel.

5. Prove that the straight line joining one vertex of an equilateral triangle to the mid-point of the opposite side is perpendicular to that side.

(*d*) Fourth Case

The case of the hypotenuse and a side of a right - angled triangle being equal to the hypotenuse and a side of another right - angled triangle.

A pair of right-angled triangles drawn such that the hypotenuse is 3 cm and another side is 2.5 cm is shown below.



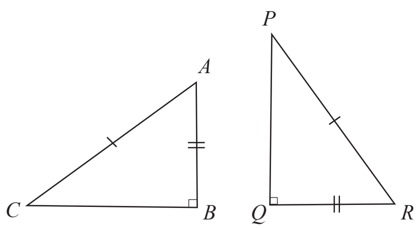
Activity

Copy the triangle in figure (i) onto a tissue paper and examine whether it can be made to coincide with the triangle in figure (ii).

Accordingly, the congruence of a pair of right-angled triangles can be expressed in terms of the equality of two elements as follows.

If the lengths of the hypotenuse and a side of a right-angled triangle are equal to the lengths of the hypotenuse and a side of another right-angled triangle, then the two triangles are congruent.
Showing that two triangles are congruent under these conditions is stated concisely as being congruent according to the **RHS** (right-angle-hypotenuse-side) case.

Let us prove that the two triangles given below are congruent based on the information that is given.



In the right-angled triangles ABC and PQR

$AC = PR$ (Given)

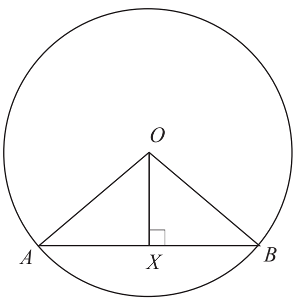
$AB = QR$ (Given)

$\therefore \; \Delta ABC \equiv \Delta PQR$ (RHS)

Since the above pair of triangles is congruent, the remaining pairs of corresponding elements are also equal.
That is, $BC = PQ,\ B\hat{A}C = P\hat{R}Q,\ A\hat{C}B = Q\hat{P}R$.

Example 1

Based on the information in the figure, prove that, $\Delta OXA \equiv \Delta OXB$ and write all the remaining equal corresponding elements.



Proof:
In the right-angled triangles OXA and OXB,

$OA = OB$ (radii of same circle)

OX is a common side

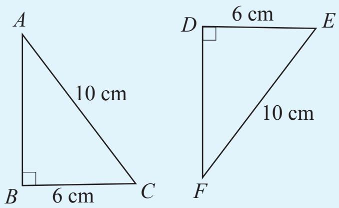
$\therefore \; \Delta OXA \equiv \Delta OXB$ (RHS)

Corresponding elements of congruent triangles are equal.

$\therefore \; O\hat{A}X = O\hat{B}X,\ AX = BX,\ A\hat{O}X = B\hat{O}X.$

Exercise 5.4

1. Determine for which of the following pairs of the triangles the RHS conditions can be used to show congruence based on the given information. For these pairs, prove that the two triangles are congruent and write down the remaining pairs of corresponding elements which are equal to each other.

(i)


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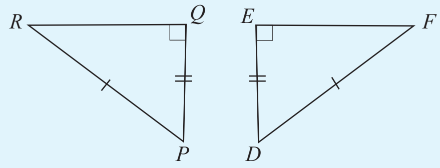

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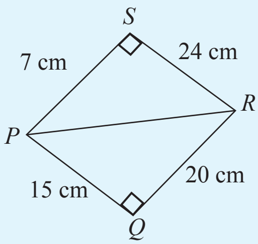

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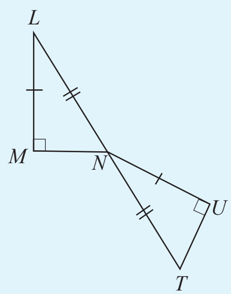

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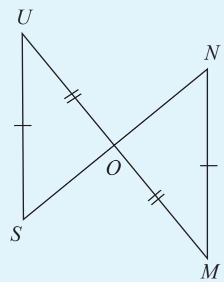

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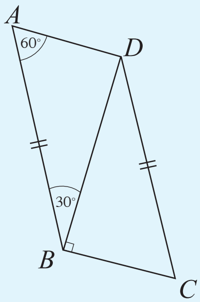

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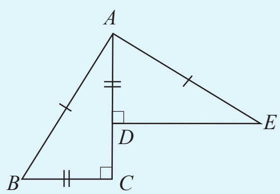

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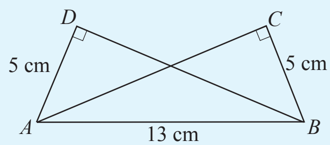

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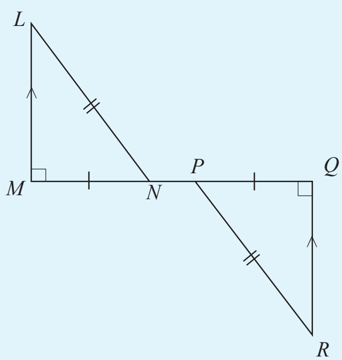
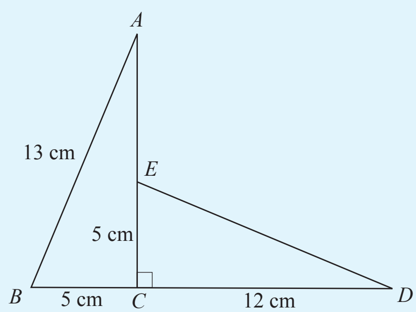


(x)


2. For each of the following parts, draw a sketch of the relevant pairs of triangles based on the information that is given. From these pairs, select the ones that are congruent (if there are any) and write down the remaining pairs of corresponding elements which are equal to each other.

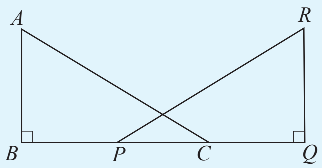
(i) In the triangles ABC and PQR, $A\hat{B}C = P\hat{Q}R = 90^\circ$, $AC = PR = 5$ cm, $BC = 3$ cm, $QP = 4$ cm.

(ii) In the triangles LMN and XYZ, $L\hat{M}N = X\hat{Y}Z = 90^\circ$, $LM = XY$, $MN = YZ$.

(iii) In the triangles DEF and PQR, $D\hat{E}F = P\hat{Q}R = 90^\circ$, $DF = PR$, $EF = PQ$.

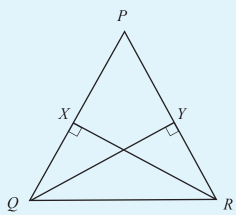
(iv) In the triangles ABD and ABC, $A\hat{D}B = A\hat{C}B = 90^\circ$, $AD = CB$

3.



If $AC = PR$ and $AB = RQ$ in the figure, show that $BP = CQ$.

4.



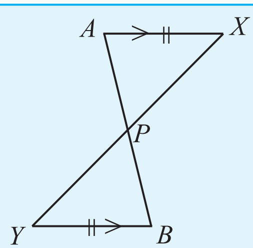
In the triangle PQR, the perpendiculars QY and RX are drawn from the points Q and R to the sides RP and QP respectively such that $QY=RX$. Prove that,

(i) $\Delta\, XQR \equiv \Delta\, YRQ$

(ii) $X\hat{R}Q = Y\hat{Q}R$

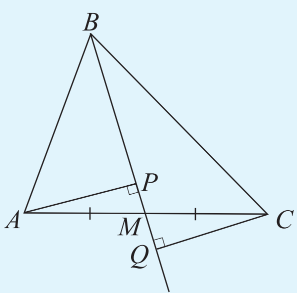
Miscellaneous Exercise

1.



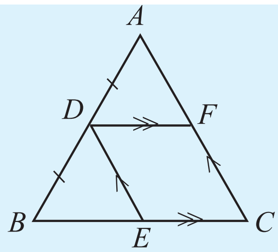
In the figure $AX//YB$ and $AX = YB$. Show that the straight lines AB and YX bisect each other at P.

2.



In the triangle ABC, the mid point of AC is M. The perpendiculars drawn from A and C meet BM and BM produced at P and Q respectively. Show that $\Delta\, AMP \equiv \Delta\, MQC$.

3. Using the information in the figure, show that $\Delta\, ADF \equiv \Delta\, DBE$.

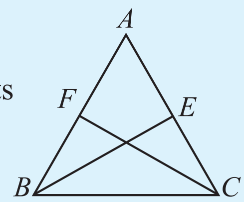


4. *ABC* in the figure is an equilateral triangle. The mid points of *AC* and *AB* are *E* and *F* respectively. Show that

(i) *AB* and *FC* are perpendicular

(ii) *AC* and *BE* are perpendicular

(iii) $CF = BE$.

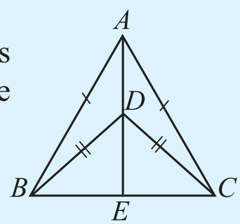


5. In the triangle *ABC* in the figure, $AB = AC$. The point *D* is such that $BD = DC$. *AD* produced meets *BC* at *E*. Prove that

(i) $\Delta\, ABD \equiv \Delta\, CAD$

(ii) $\Delta\, BAE \equiv \Delta\, CAE$

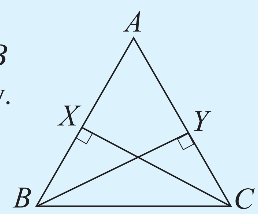
(iii) *AE* and *BC* are perpendicular to each other.



6. In the given triangle *ABC*, the perpendicular drawn from *B* and *C* to the sides *AC* and *AB* are *BY* and *CX* respectively. If $BY = CX$ then show that

(i) $AB = AC$

(ii) $X\hat{B}C = Y\hat{C}B$.



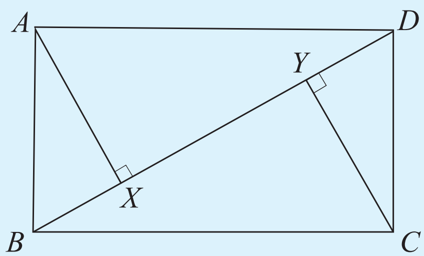
7. The perpendiculars drawn from *A* and *C* to the diagonal *BD* of the rectangle *ABCD* meet *BD* at *X* and *Y* respectively. Prove that

(i) $\Delta\, AXD \equiv \Delta\, BCY$

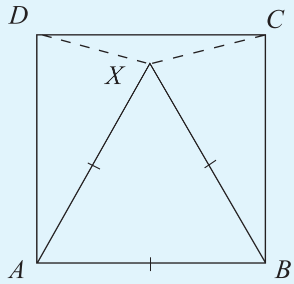
(ii) $AX = YC$

(iii) $BX = YD$

(iv) $\Delta\, YDC \equiv \Delta\, ABX$



8. The point X lies in the interior of the square $ABCD$ such that XAB is an equilateral triangle. Show that
   (i) $\Delta\, AXD \equiv \Delta\, CBX$
   (ii) DXC is an isosceles triangle.



9. The equilateral triangles BCF and DCE are drawn on the sides BC and DC of the square $ABCD$ such that the triangles lie outside the square.
   (i) Draw a rough sketch illustrating the above information.
   Show that,
   (ii) $\Delta\, EDA \equiv \Delta\, ABF$
   (iii) EAF is an equilateral triangle.

10. The perpendicular bisector of the side BC of the triangle ABC is AE. The point D lies on AE. Prove that,
   (i) $\Lambda\, ABE \equiv \Delta\, AEC$
   (ii) $\Delta\, BDE \equiv \Delta\, DEC$
   (iii) $\Delta\, ABD \equiv \Delta\, ACD$

11. $ABCDE$ is a regular pentagon.
   (i) Show that $\Delta\, ABC \equiv \Delta\, AED$.
   (ii) If the foot of the perpendicular drawn from A to the side CD is X, prove that $CX = XD$.

6

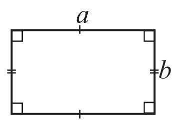
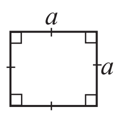
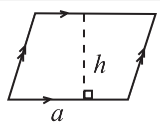
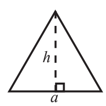
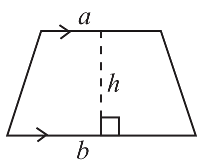
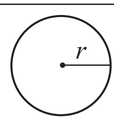
Area

By studying this lesson you will be able to

- find the areas of sectors of circles,
- solve problems related to the areas of plane figures containing sectors of circles.

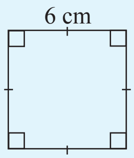
Areas of plane figures

Let us recall some facts you have learnt in previous grades under the topic **Area**.

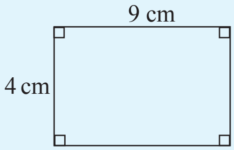
Name	**Plane Figure**	**How the area is calculated**	**Formula for the area (A)**
Rectangle		length × breadth	$A = a \times b$
Square		(length of a side)^2	$A = a^2$
Parallelogram		base × altitude	$A = a \times h$
Triangle		$\frac{1}{2}$ × base × altitude	$A = \frac{1}{2} \times a \times h$
Trapezium		$\frac{1}{2}$ × sum of the lengths of the parallel sides × altitude	$A = \frac{1}{2}(a+b) \times h$
Circle		π × (radius)^2	$A = \pi r^2$

Review Exercise

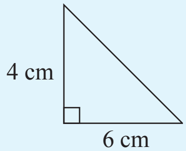
1. Find the area of each of the following plane figures.



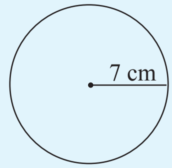
A



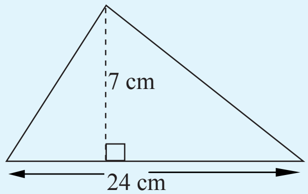
B



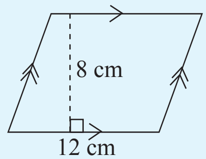
C



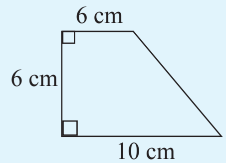
D



E



F



G

2. The rectangle in Figure C has been formed by joining together the trapezium in Figure A and the triangle in Figure B.

Figure A

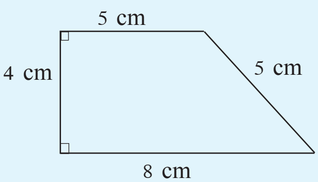


Figure B

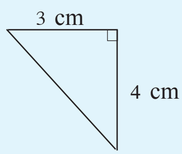
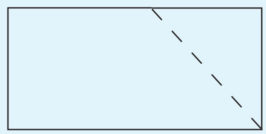


Figure C

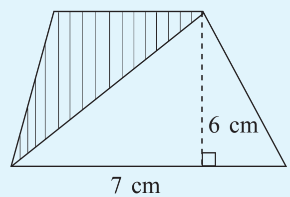


(i) Find the area of the trapezium in Figure A.

(ii) Find the area of the triangle in Figure B.

(iii) Find the area of the rectangle in Figure C in terms of the areas of Figure A and Figure B.

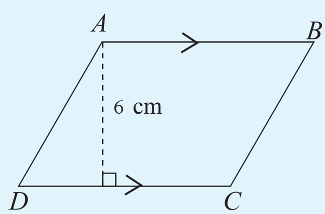
3. The figure denotes a trapezium of area 33 cm^2 that has been formed by joining two triangles together. Find the area of the triangle which is shaded.



4. The figure denotes a parallelogram of area 120 cm^2. Its perimeter is 64 cm. Determine the following based on the information that is given.

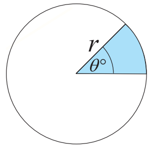
(i) The length of the side CD.

(ii) The length of the side BC.

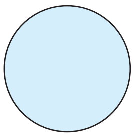
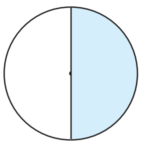
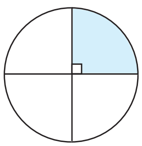
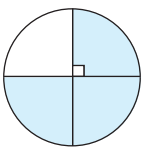
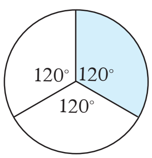
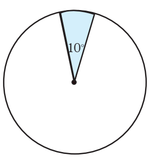
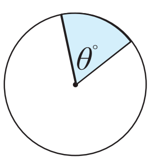


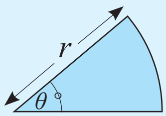
6.1 Area of a sector of a circle

We considered how the perimeter of a sector of a circle is found in the lesson on perimeters. Now let us consider how the area of a sector of a circle is found.



The following table shows how the area of a sector is found when the angle at the centre of the sector takes certain special values.

Sector	**Shaded Sector as a fraction of the circle**	**Area of the Sector**
	1	πr^2
	$\frac{1}{2}$	$\frac{1}{2} \times \pi r^2$
	$\frac{1}{4}$	$\frac{1}{4} \times \pi r^2$
	$\frac{3}{4}$	$\frac{3}{4} \times \pi r^2$
	$\frac{1}{3}$	$\frac{1}{3} \times \pi r^2$
	$\frac{10}{360}$	$\frac{10}{360} \times \pi r^2$
	$\frac{\theta}{360}$	$\frac{\theta}{360} \times \pi r^2$

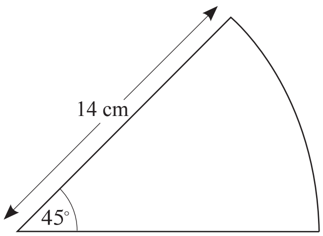


According to the pattern in the table, the area of the sector of radius r and angle at the centre $\theta°$ is $\frac{\theta}{360} \times \pi r^2$

Let us consider (through the following examples) how the area of a sector is found using this result. In the examples and exercises of this chapter, the value of π is taken as $\frac{22}{7}$.

Example 1

Find the area of the sector in the following figure.

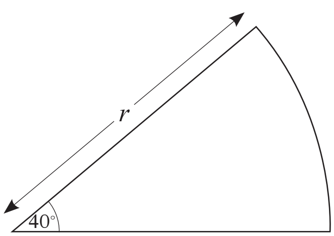


$$\text{Area} = \frac{45}{360} \times \pi r^2$$
$$= \frac{45}{360} \times \frac{22}{7} \times 14 \times 14$$
$$= 77$$
$\therefore$ Area is 77 cm^2.

Example 2

If the area of the sector in the figure is $17\frac{1}{9}$ cm^2, find the radius of the corresponding circle.

Let us take the radius as r cm.

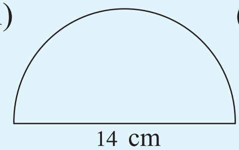


$$\text{Area} = \frac{40}{360} \times \pi r^2$$
$$17\frac{1}{9} = \frac{1}{9} \times \frac{22}{7} \times r^2$$
$$\frac{154}{9} = \frac{1}{9} \times \frac{22}{7} \times r^2$$
$$r^2 = \frac{154 \times 7}{22}$$
$$r^2 = 49$$
$$r = 7$$
$\therefore$ Radius is 7 cm.

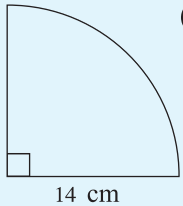
Exercise 6.1

1. Find the area of each sector.

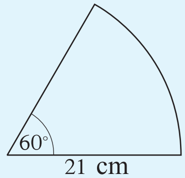
(i)



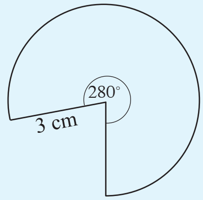
(ii)



(iii)

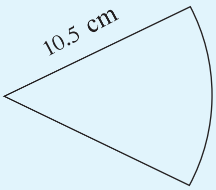


(iv)

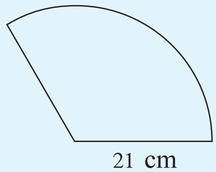


2. The areas of the two sectors of circles given below are 77 cm^2 and 462 cm^2 respectively. Find the angle at the centre of each sector.

(i)

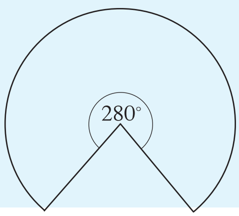


(ii)

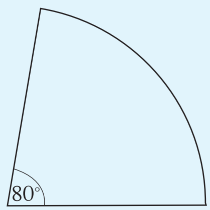


3. The areas of the two sectors of circles given below are 792 cm^2 and $6\frac{2}{7}$ cm^2 respectively. For each sector, find the radius of the corresponding circle.

(i)



(ii)

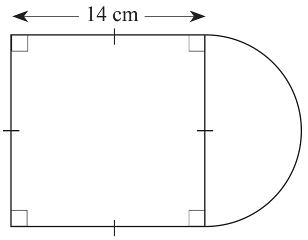


6.2 Plane figures containing sectors of circles

Let us consider the areas of plane figures formed by sectors of circles and other simple plane figures such as rectangles and triangles being joined together.

Example 1

The following denotes a plane figure consisting of a square and a semi-circle. Find its area.



Area of the square $= 14 \times 14$

$= 196 \text{ cm}^2$

Since the diameter of the semi-circle is equal to the length of a side of the square,

The radius of the circle is $14 \div 2 = 7$ cm.

Area of the semi-circle $= \dfrac{1}{2} \times \pi r^2$

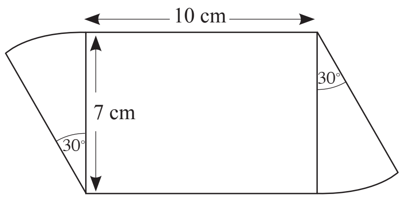
$= \dfrac{1}{\cancel{2}_1} \times \dfrac{\cancel{22}^{11}}{\cancel{7}_1} \times \cancel{7}^1 \times 7 = 77 \text{cm}^2$

Area of the compound plane figure $= 196 \text{ cm}^2 + 77 \text{ cm}^2$

$= \underline{\underline{273 \text{ cm}^2}}$

Example 2

The figure denotes a compound plane figure consisting of a rectangle and two sectors of a circle. Find its area.



Area of the rectangle $= 10 \times 7$

$= 70 \text{ cm}^2$

Area of a sector $= \dfrac{30}{360} \times \pi r^2$

$= \dfrac{30}{360} \times \dfrac{22}{7} \times 7 \times 7$

$= \dfrac{77}{6} \text{cm}^2$

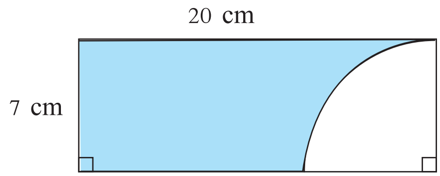
Area of both sectors $= \dfrac{77}{6} \times 2 = \dfrac{77}{3} = 25\dfrac{2}{3} \text{cm}^2$

Area of the compound plane figure $= 70 \text{cm}^2 + 25\dfrac{2}{3} \text{cm}^2$

$= \underline{\underline{95\dfrac{2}{3} \text{ cm}^2}}$

Example 3

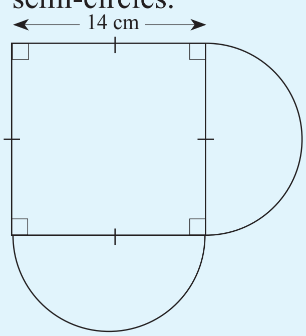
The shaded portion is obtained by cutting $\frac{1}{4}$ of a circle from a rectangular lamina. Find the area of the shaded portion using the given data.



$$\text{Area of the rectangle} = 20 \times 7$$
$$= 140\ \text{cm}^2$$
$$\text{Area of the sector} = \frac{90}{360} \times \pi r^2$$
$$= \frac{90}{360} \times \frac{22}{7} \times 7 \times 7$$
$$= 38.5\ \text{cm}^2$$
$$\therefore \quad \text{Area of the shaded portion} = 140 - 38.5$$
$$= \underline{\underline{101.5\ \text{cm}^2}}$$

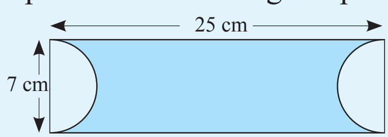
Exercise 6.2

1. The following is a compound plane figure consisting of a square and two semi-circles.



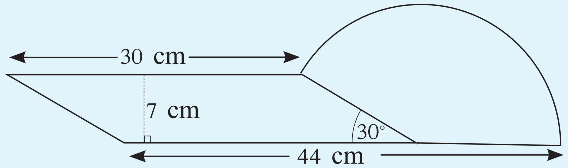
(i) Find the area of the square.
(ii) Find the radius of a semi-circular portion.
(iii) Find the total area of the two semi-circular portions.
(iv) Find the area of the compound plane figure.

2. The shaded portion in the figure was obtained by cutting out two semi-circular parts from a rectangular piece of paper.



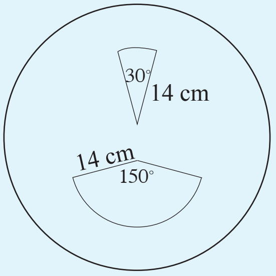
(i) Find the area of the rectangle.
(ii) Find the total area of the two semi-circular parts.
(iii) Find the area of the shaded portion.

3. The following is a compound plane figure consisting of a parallelogram and a sector of a circle.

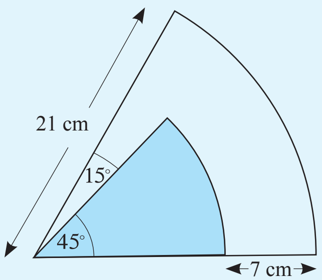


(i) Find the area of the parallelogram.
(ii) Find the area of the sector.
(iii) Find the area of the compound figure.

4. The figure denotes a circular lamina of radius 28 cm. The two sectors in the figure are to be cut out. Find the area of the remaining portion after the two sectors have been cut out.

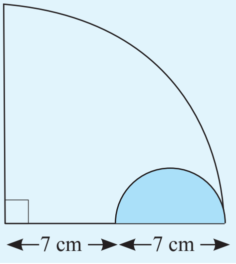


5. The following is a figure consisting of two sectors of circles.



Show that the ratio of the area of the smaller sector to the larger sector is 1 : 3.

6. Based on the measurements given in the figure, show that the area of the unshaded region of the larger sector is seven times the area of the shaded sector.

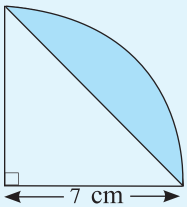


Summary
The area of a sector of a circle of radius r with angle at the centre θ° is $\frac{\theta}{360} \times \pi r^2$.

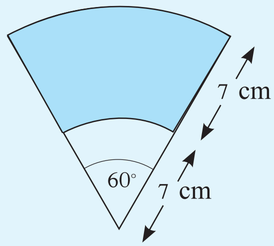
Miscellaneous Exercise

1. Find the area of the shaded portion in each of the figures given below which are formed by sectors of circles.

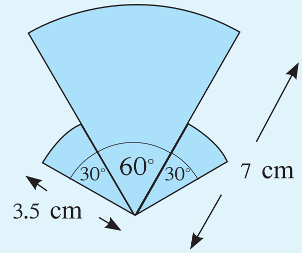
(i)



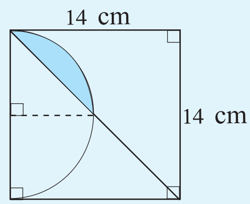
(ii)



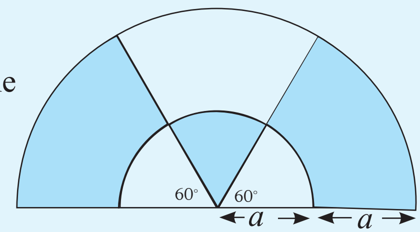
(iii)



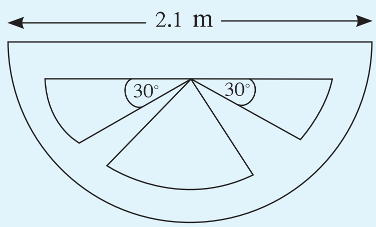
2. Find the area of the shaded region.
The curved line in the figure is the arc of a semi-circle.



3. The figure denotes two semi-circles.
Show that the ratio of the un-shaded region to the shaded region in the figure is 5 : 7.



5. A sketch of the area in front of a commemorative plaque is given in the figure. Grass has been grown in the three portions within the semi-circle which are in the shape of sectors of circles, while white sand has been spread in the other regions. The radius of each sector within the semi-circle is 84 cm.



(i) What is the radius of the semi-circle in centimetres?
(ii) Find the area of the semi-circular part in cm^2.
(iii) Find the area of one of the sectors with angle at the centre equal to 30^o.
(iv) Find the angle at the centre of the large sector if its area is 1848 square centimetres more than the sum of the areas of the other two sectors.

7 Factors of Quadratic Expressions

By studying this lesson you will be able to

- find the factors of trinomial quadratic expressions,
- find the factors of the difference of two squares.

Factors of algebraic expressions

We know that $2x + 6$ is a binomial algebraic expression. Since it can be expressed as $2(x + 3)$, we also know that 2 and $x + 3$ are its factors.

Similarly, since $4x^2 + 6x = 2x(2x + 3)$, we know that 2, x and $2x + 3$ are the factors of $4x^2 + 6x$.

Now let us find the factors of $a^2 - 2a + ab - 2b$

$$\begin{aligned} a^2 - 2a + ab - 2b &= a(a-2) + b(a-2) \\ &= (a-2)(a+b) \end{aligned}$$

Therefore, the factors of $a^2 - 2a + ab - 2b$ are $a - 2$ and $a + b$.

Do the following exercise to further recall what has been learnt earlier about factoring algebraic expressions.

Review Exercise

1. Write each of the following algebraic expressions as a product of its factors.

A. **a.** $3x+12$ **b.** p^2-p **c.** x^2+3xy

d. $2a-4a^2$ **e.** p^2q-pq **f.** $2pq-4p^2q$

g. $3m^2n+n^2$ **h.** $2a^2-4ab$ **i.** $2a^2-8ab-2b^2$

j. $5x^2-10x^2y^2-15x^2y$ **k.** $3x^2y-6x^2y^2+6xy^2$ **l.** $a^2bc+ab^2c-abc^2$

B. **a.** $x(a+b)+y(a+b)$ **b.** $2a(3x+y)-b(3x+y)$

c. $p(2a-3b)+q(2a-3b)$ **d.** $2(x-3)-xy+3y$

e. $3b+3+a(b+1)$ **f.** $x^2-xy+4x-4y$

g. $a^2-2ab-5a+10b$ **h.** $m-3mn-n+3n^2$

2. Fill in the blanks in (i) and (ii), and factor the expressions given below them accordingly.

(i) $a\,(2x - y) + b\,(y - 2x)$
$= a\,(2x - y) - b\,(........)$
$= \underline{\underline{(........)\,(........)}}$

(ii) $p\,(a - b) - q\,(b - a)$
$= p\,(a - b)\; ....\; q\,(a - b)$
$= \underline{\underline{(a - b)\,(........)}}$

a. $x\,(2p - q) - y\,(q - 2p)$
b. $3x\,(2a - b) + 2y\,(b - 2a)$
c. $m\,(l - 2n) - p\,(2n - l)$
d. $k\,(2x + y) - l\,(y + 2x)$
e. $a\,(x + 3y) - b\,(-x - 3y)$
f. $b\,(m - 2n) + d\,(2n - m)$

Defining trinomial quadratic expressions

Now let us consider factoring quadratic expressions such as $x^2 + 2x - 3$. This expression is of the form $ax^2 + bx + c$. Here a, b and c take integer values. We call an algebraic expression of the form $ax^2 + bx + c$, where a, b and c are non-zero, a trinomial quadratic expression in x. a is said to be the coefficient of x^2, b the coefficient of x and c the constant term. It is easy to find the factors of such an expression when the terms are written in this order. The coefficient of x^2 in $x^2 + 2x - 3$ is 1 while the coefficient of x is 2 and the constant term is -3. $4 + 2x - x^2$ is also a trinomial quadratic expression. To find its factors it can be written in the order $-x^2 + 2x + 4$.

When the trinomial quadratic expression $x^2 + 2xy - y^2$ is considered, it can be thought of as a quadratic expression in x or a quadratic expression in y. When it is considered as a quadratic expression in y, it is convenient to write it in the form $-y^2 + 2xy + x^2$.

For example, while $3x^2 - 2x - 5$, $a^2 + 2a + 8$, $y^2 + 2y - 5$ and $5 - 2x - 3x^2$ are trinomial quadratic expressions, the expressions $a + 2x + 3$ and $2x^3 + 3x^2 - 5x$ are not.

7.1 Factors of trinomial quadratic expressions

Let us recall how the product of two binomial expressions such as $x + 2$ and $x + 3$ is found.

$$\begin{aligned}(x+2)(x+3) &= x(x+3) + 2(x+3)\\ &= x^2 + \underbrace{3x + 2x} + 6\\ &= x^2 + 5x + 6\end{aligned}$$

Since $x^2 + 5x + 6$ is obtained as the product of $x + 2$ and $x + 3$, we obtain that $x + 2$ and $x + 3$ are factors of $x^2 + 5x + 6$. $x^2 + 5x + 6$ is a trinomial qudratic expression. How do we obtain $x + 2$ and $x + 3$ as its factors? Let us examine the steps that were performed to obtain the product of the above two expressions, in the reverse order.

- The middle term $5x$ of the trinomial quadratic expression $x^2 + 5x + 6$, has been expressed as a sum of two terms, as $3x + 2x$
- The product of the terms $3x$ and $2x = 3x \times 2x = 6x^2$
- The product of the initial and final terms of the trinomial quadratic expression $x^2 + 5x + 6$ is $x^2 \times 6 = 6x^2$

We can come to the following conclusions based on the above observations.
The middle term should be written as the sum of two terms. The product of these two terms should be equal to the product of the initial and final terms of the trinomial expression.

As an example, let us factor $x^2 + 7x + 10$ accordingly. Here, $7x$ is the middle term. This should be written as the sum of two terms. Also, the product of these two terms should be equal to $10x^2$.

The product of the initial and final terms = $x^2 \times 10 = 10x^2$

The middle term = $7x$

Let us find the pair of terms which has $10x^2$ as its product and $7x$ as its sum. Let us consider the following table to do this. The two terms in the first column have been selected such that their product is $10x^2$.

Pair of terms	Product	Sum
$x,\ 10x$	$x \times 10x = 10x^2$	$x + 10x = 11x$
$2x,\ 5x$	$2x \times 5x = 10x^2$	$2x + 5x = 7x$
$(-x),\ (-10x)$	$(-x) \times (-10x) = 10x^2$	$(-x) + (-10x) = -11x$
$(-2x),\ (-5x)$	$(-2x) \times (-5x) = 10x^2$	$(-2x) + (-5x) = -7x$

It is clear from the above table that the middle term $7x$ should be written as $2x + 5x$. Now, let us find the factors of the given trinomial quadratic expression.

$$\begin{aligned} x^2 + 7x + 10 &= x^2 + 2x + 5x + 10 \\ &= x(x+2) + 5(x+2) \\ &= (x+2)(x+5) \end{aligned}$$

Therefore, the factors of $x^2 + 7x + 10$ are $x + 2$ and $x + 5$.

Let us see whether the factors are different if we write the middle term of $x^2 + 7x + 10$ as the sum $5x + 2x$, instead of $2x + 5x$.

$$\begin{aligned} x^2 + 7x + 10 &= x^2 + 5x + 2x + 10 \\ &= x(x+5) + 2(x+5) \\ &= (x+5)(x+2) \end{aligned}$$

We observe that the same factors are obtained. Therefore, the order in which we write the two terms that were selected has no effect on the factors. Accordingly, the factors can be found by writing $7x$ as either $2x + 5x$ or as $5x + 2x$.

Example 1

Factor $a^2 - 8a + 12$.

The product of the initial and final terms $= a^2 \times 12 = 12a^2$

The middle term $= -8a$

Find the two terms which have $12a^2$ as their product and $-8a$ as their sum.
The following table contains pairs of terms of which the product is $12a^2$. The pair which when added equals $-8a$ has been shaded.

Pair of terms	Product	Sum
$a, 12a$	$a \times 12a = 12a^2$	$a + 12a = 13a$
$2a, 6a$	$2a \times 6a = 12a^2$	$2a + 6a = 8a$
$3a, 4a$	$3a \times 4a = 12a^2$	$3a + 4a = 7a$
$(-a), (-12a)$	$(-a) \times (-12a) = 12a^2$	$(-a) + (-12a) = -13a$
$(-2a), (-6a)$	$(-2a) \times (-6a) = 12a^2$	$(-2a) + (-6a) = -8a$
$(-3a), (-4a)$	$(-3a) \times (-4a) = 12a^2$	$(-3a) + (-4a) = -7a$

That is, $-8a$ can be written as $-2a - 6a$

$$\begin{aligned} a^2 - 8a + 12 &= a^2 - 2a - 6a + 12 \\ &= a(a-2) - 6(a-2) \\ &= \underline{\underline{(a-2)(a-6)}} \end{aligned}$$

Note: A table has been used here only as an illustration. The middle term can be obtained as a sum mentally too.

Example 2

Factor $x^2 - 7x - 8$.
The product of the initial and final terms $= x^2 \times (-8) = -8x^2$

The middle term $= -7x$

The two terms which have $-8x^2$ as their product and $-7x$ as their sum are $+x$ and $-8x$.

Accordingly,
$$\begin{aligned} x^2 - 7x - 8 &= x^2 + x - 8x - 8 \\ &= x(x+1) - 8(x+1) \\ &= \underline{\underline{(x+1)(x-8)}} \end{aligned}$$

Now let us consider how an expression such as $-x^2 - x + 6$ which has a negative quadratic term is factored. The factors can be found by re-writing the expression as $6 - x - x^2$, with the quadratic term at the end. By considering the example given below, let us recognize the fact that the factors can be found using either form.

Example 3

Factor $-x^2 - x + 6$.
The product of the initial and final terms $= -6x^2$
The middle term $= -x$

$\therefore$ $-x$ should be written as $2x - 3x$.

$$\begin{aligned} &-x^2 - x + 6 \\ &= -x^2 + 2x - 3x + 6 \\ &= x(-x+2) + 3(-x+2) \\ &= (-x+2)(x+3) \\ &= \underline{\underline{(2-x)(x+3)}} \end{aligned} \quad \text{or} \quad \begin{aligned} &6 - x - x^2 \\ &= 6 + 2x - 3x - x^2 \\ &= 2(3+x) - x(3+x) \\ &= (3+x)(2-x) \\ &= \underline{\underline{(2-x)(x+3)}} \end{aligned}$$

Example 4

Factor $a^2 - 4ab - 5b^2$.

We may consider this as a trinomial quadratic expression in a.
Then,
the product of the initial and final terms $= a^2 \times (-5b^2) = -5a^2b^2$
The middle term $= -4ab$

The two terms which have $-5a^2b^2$ as their product and $-4ab$ as their sum are ab and $-5ab$.

$$\begin{aligned} a^2 - 4ab - 5b^2 &= a^2 + ab - 5ab - 5b^2 \\ &= a(a+b) - 5b(a+b) \\ &= \underline{\underline{(a+b)(a-5b)}} \end{aligned}$$

Note: This can also be considered as a trinomial quadratic expression in b and factored. Then too the above answer is obtained.

Accuracy of the factors of a trinomial quadratic expression

The errors made in simplification can be minimized by examing the accuracy of the factors of a trinomial quadratic expression once they have been found. For example, let us factor $x^2 + 3x - 40$.

$$\begin{aligned} x^2 + 3x - 40 &= x^2 + 8x - 5x - 40 \\ &= x(x+8) - 5(x+8) \\ &= \underline{\underline{(x+8)(x-5)}} \end{aligned}$$

If the factors $x + 8$ and $x - 5$ are correct, by multiplying them together we should be able to get back our original expression. Let us find the product of $x + 8$ and $x - 5$

$$(x + 8)(x - 5) = x^2 - 5x + 8x - 40$$
$$= \underline{\underline{x^2 + 3x - 40}}$$

Since we have obtained our original expression $x^2 + 3x - 40$, the factors $x + 8$ and $x - 5$ are correct.

Exercise 7.1

1. Complete the following table.

Pair of algebraic terms	**Product**	**Sum**
$4x, x$	$4x^2$	$5x$
$2x, 7x$	....	....
$-5x, x$	....	....
$-3a, -7a$	....	....
$-p, -5p$	....	....
$2mn, -8mn$	....	....
....	$-4x^2$	$3x$
....	$-7x^2$	$6x$
....	$-10a^2$	$-3a$
....	$8p^2$	$6p$

2. Factor each of the following trinomial quadratic expressions.

A.

a. $x^2 + 6x + 8$ **b.** $a^2 - 8a + 15$ **c.** $p^2 + 8p + 12$

d. $x^2 - 10x + 21$ **e.** $m^2 + 11m + 24$ **f.** $y^2 - 11y + 18$

g. $n^2 + 15n + 14$ **h.** $x^2 - 17x + 30$ **i.** $a^2 + 14a + 49$

j. $p^2 - 12p + 35$ **k.** $p^2 + 8p - 20$ **l.** $x^2 - 3x - 10$

m. $p^2 + p - 20$ **n.** $n^2 - 4n - 21$ **o.** $a^2 + 3a - 28$

p. $y^2 - 4y - 12$ **q.** $m^2 - 40 + 6m$ **r.** $5p + p^2 - 24$

s. $45 + x^2 - 14x$ **t.** $n^2 - 28 - 12n$

B.

a. $10 - 3x - x^2$ **b.** $12 - p - p^2$ **c.** $12 - 4x - x^2$

d. $50 + 5x - x^2$ **e.** $18 + 7a - a^2$ **f.** $56 - y - y^2$

C. **a.** $a^2+7ab+10b^2$ **b.** $x^2+3xy+2y^2$

c. $p^2-7pq+12q^2$ **d.** $y^2+10ay+24a^2$

e. $a^2-10ab+21b^2$ **f.** $x^2-2xy-8y^2$

g. $p^2+pq-12q^2$ **h.** $y^2-3py-10p^2$

i. $a^2-ab-20b^2$ **j.** $x^2+6xy-40y^2$

3. A certain number is denoted by x. The product of the expression obtained by adding a certain number to x, and the expression obtained by subtracting a different number from x is given by the expression x^2+x-56.

(i) Find the factors of the given expression.

(ii) What is the number that has been added to the number denoted by x?

(iii) What is the number that has been subtracted from the number denoted by x?

7.2 The factors of trinomial quadratic expressions described further

So far we have considered only how the factors of trinomial quadratic expressions with the coefficient of x^2 equal to either 1 or -1 are found. Now let us consider how the factors are found when the coefficient of x^2 is some other integer. For example, let us consider the trinomial quadratic expression $3x^2+14x+15$. This is of the form ax^2+bx+c with $a=3$. In such cases too we can use the same method that we used above.

Example 1

Factor $3x^2+14x+15$.

The product of the initial and final terms $= 45x^2$

The middle term $14x$ needs to be written as $5x+9x$. (since $5x \times 9x = 45x^2$).

$$
\begin{aligned}
&3x^2+14x+15\\
&=3x^2+5x+9x+15\\
&=x(3x+5)+3(3x+5)\\
&=\underline{\underline{(3x+5)(x+3)}}
\end{aligned}
$$

Example 2

Factor $6x^2 + x - 15$.

$6x^2 + x - 15$

$= 6x^2 + 10x - 9x - 15$

$= 2x(3x+5) - 3(3x+5)$

$= \underline{\underline{(3x+5)(2x-3)}}$

Example 3

Factor $2a^2 + 13ab - 7b^2$.

$2a^2 + 13ab - 7b^2$

$= 2a^2 - ab + 14ab - 7b^2$

$= a(2a-b) + 7b(2a-b)$

$= \underline{\underline{(2a-b)(a+7b)}}$

Example 4

Factor $x^2 + \frac{5}{2}x + 1$.

First let us write the algebraic expression with a common denominator.

$$x^2 + \frac{5}{2}x + 1 = \frac{2x^2 + 5x + 2}{2}$$

$$= \frac{1}{2}(2x^2 + 5x + 2)$$

Now let us find the factors of the quadratic expression within brackets.

$$2x^2 + 5x + 2 = 2x^2 + x + 4x + 2$$

$$= x(2x+1) + 2(2x+1)$$

$$= (2x+1)(x+2)$$

Therefore, $x^2 + \frac{5}{2}x + 1 = \frac{1}{2}(2x+1)(x+2)$

Exercise 7.2

1. Factor each of the following trinomial quadratic expressions.

A.

a. $2x^2 + 3x + 1$	**b.** $5a^2 - 7a + 2$	**c.** $2x^2 - x - 1$
d. $4p^2 + 4p - 3$	**e.** $6x^2 + 3x - 3$	**f.** $2x^2 - 11xy + 15y^2$
g. $2y^2 - 5ya + 3a^2$	**h.** $2a^2 + 7ab + 6b^2$	**i.** $5p^2 - 9pq - 2q^2$
j. $2m^2 + 3mn - 2n^2$	**k.** $x^2y^2 + 10xy + 16$	**l.** $2x^3 - x^2y - 3xy^2$

2. Find the value of each of the following numerical expressions using the knowledge on the factors of trinomial quadratic expressions.

a. $8^2 + 7 \times 8 + 10$	**b.** $93^2 + 3 \times 93 - 28$
c. $27^2 - 4 \times 27 - 21$	**d.** $54^2 + 2 \times 54 - 24$

7.3 The factors of expressions which are in the form of the difference of two squares

Consider the product of the two binomial expressions $(x - y)$ and $(x + y)$

$$\begin{aligned}(x-y)(x+y) &= x^2 + xy - xy - y^2\\ &= x^2 - y^2\end{aligned}$$

We have obtained $x^2 - y^2$ which is a difference of two squares. We observe from the above expansion that the factors of $x^2 - y^2$ are $x - y$ and $x + y$. Further, $x^2 - y^2$ can be considered as a quadratic expression in x and factored. By taking its middle term to be 0, we can write it in the form of a trinomial quadratic expression as $x^2 + 0 - y^2$. Let us factor this expression.

Product of the initial and final terms $= - x^2y^2$

Middle term $= 0$

Accordingly, the pair of terms with product $- x^2y^2$ and sum 0 is $- xy$ and xy.

$$\begin{aligned}x^2 + 0 - y^2 &= x^2 - xy + xy - y^2\\ &= x(x-y) + y(x-y)\\ &= \underline{\underline{(x-y)(x+y)}}\end{aligned}$$

Therefore, by this too we obtain $x^2 - y^2 = (x - y)(x + y)$.

Let us consider the following examples of the factorization of expressions which are the difference of two squares.

Example 1

Factor (i) $x^2 - 4$ (ii) $4x^2 - 9$ (iii) $25a^2 - 16b^2$

(i)

$$\begin{aligned}&x^2 - 4\\ &= x^2 - 2^2\\ &= \underline{\underline{(x-2)(x+2)}}\end{aligned}$$

(ii)

$$\begin{aligned}&4x^2 - 9\\ &= (2x)^2 - 3^2\\ &= \underline{\underline{(2x-3)(2x+3)}}\end{aligned}$$

(iii)

$$\begin{aligned}&25a^2 - 16b^2\\ &= (5a)^2 - (4b)^2\\ &= \underline{\underline{(5a-4b)(5a+4b)}}\end{aligned}$$

Do the following exercise after studying the above examples.

Exercise 7.3

1. Fill in the blanks.

(i) $x^2 - 36$
$= x^2 - \ldots\ldots^2$
$= \underline{\underline{(x-6)(x+6)}}$

(ii) $9 - y^2$
$= \ldots\ldots - \ldots\ldots$
$= \underline{\underline{(\ldots\ldots)(\ldots\ldots)}}$

(iii) $25x^2 - 4y^2$
$= (\ldots\ldots)^2 - (\ldots\ldots)^2$
$= \underline{\underline{(\ldots\ldots)(\ldots\ldots)}}$

(iv) $2a^2 - 8b^2$
$= 2(\ldots\ldots\ldots\ldots)$
$= 2(a^2 - (\ldots\ldots)^2)$
$= \underline{\underline{2(\ldots\ldots)(\ldots\ldots)}}$

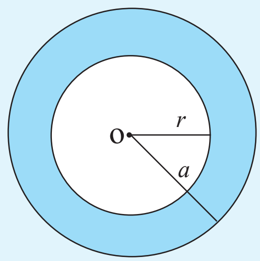
(v) $3p^2 - 27q^2$
$= 3(\ldots\ldots - \ldots\ldots)$
$= 3\left[(\ldots\ldots)^2 - (\ldots\ldots)^2\right]$
$= \underline{\underline{\ldots\ldots(\ldots\ldots)(\ldots\ldots)}}$

(vi) $a^2b^2 - 1$
$= (ab)^2 - 1$
$= \underline{\underline{(\ldots\ldots - \ldots\ldots)(\ldots\ldots + \ldots\ldots)}}$

2. Factor the following algebraic expressions.

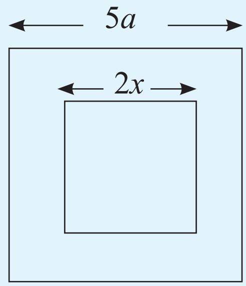
a. $y^2 - 81$ **b.** $16 - b^2$ **c.** $100 - n^2$ **d.** $m^2n^2 - 1$
e. $16a^2 - b^2$ **f.** $4x^2 - 25$ **g.** $9p^2 - 4q^2$ **h.** $400 - 4n^2$
i. $8x^2 - 2$ **j.** $4x^2y^2 - 9y^2$

3. The figure depicts two concentric circles of centre O. The radius of the smaller circle is r and the radius of the larger circle is a.



(i) Express the area of the smaller circle in terms of π and r.
(ii) Express the area of the larger circle in terms of π and a.
(iii) Write an expression for the area of the shaded region in terms of π, r and a and then represent it as a product of two factors.

4. The figure depicts two squares of side length $5a$ units and $2x$ units respectively.



(i) Express the area of the smaller square in terms of x.
(ii) Express the area of the larger square in terms of a.
(iii) Show that the area of the larger square is greater than the area of the smaller square by $(5a + 2x)(5a - 2x)$ square units.

7.4 Factors of the difference of two squares described further

There are many algebraic expressions that can be factored by considering them as the difference of two squares. Given below are two such expressions.

Example 1

Factor each of the following algebraic expressions.

(i) $(x + 2)^2 - y^2$ (ii) $(a - 2)^2 - (a + 5)^2$

(i) $(x+2)^2 - y^2$

$= [(x+2) - y]\,[(x+2) + y]$

$= \underline{\underline{(x+2-y)(x+2+y)}}$

(ii) $(a-2)^2 - (a+5)^2$

$= [(a-2) - (a+5)]\,[(a-2) + (a+5)]$

$= [a-2-a-5][a-2+a+5]$

$= \underline{\underline{-7(2a+3)}}$

Exercise 7.4

Factor the following expressions.

a. $(x+1)^2 - 4$ **b.** $(y-2)^2 - 9$ **c.** $(2a+3)^2 - 49$

d. $(4x-3y)^2 - 25$ **e.** $(2p+3)^2 - 4q^2$ **f.** $25 - (x+3)^2$

g. $4 - (a-2)^2$ **h.** $16 - (m+2)^2$ **i.** $(m+2)^2 - (m+1)^2$

j. $(2x+3)^2 - (x-2)^2$

Miscellaneous Exercise

1. Factor the following expressions

a. $(x-y)^2 - 4a^2b^2$ **b.** $x^2y^2 + 10xy + 16$

c. $p^2q^2 - pq - 20$ **d.** $2x^3 - x^2y - 3xy^2$

e. $6x^2 - 2x - 4$ **f.** $(x+1)^2 - (x-3)^2$

g. $x(x+5) - 14$ **h.** $(2x-1)^2 - 4$

2. Factor the following expressions. (Hint: Take $x^2 = y$)

a. $x^4 + 5x^2 + 6$ **b.** $x^4 - 16$

c. $2x^4 + 14x^2 + 24$ **d.** $1 - 81x^4$

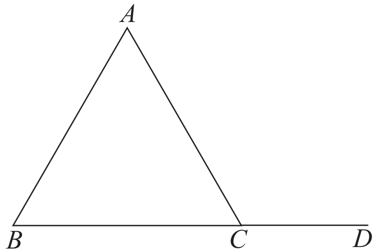
8 Triangles I

By studying this lesson you will be able to

- prove riders using the theorems related to the angles of a triangle.

8.1 The interior and exterior angles of a triangle

The angles $B\hat{A}C$, $A\hat{B}C$ and $A\hat{C}B$ which are within the triangle ABC are called the interior angles of the triangle ABC (in short, these are called the angles of the triangle ABC).



The side BC of the triangle ABC has been produced up to D as indicated in the figure. The angle $A\hat{C}D$ which is then formed is an **exterior angle** of the triangle. Since BCD is a straight line, $A\hat{C}B$ is an adjacent supplementary angle of $A\hat{C}D$. The angles $B\hat{A}C$ and $A\hat{B}C$ which are the interior angles of triangle ABC apart from $A\hat{C}B$ are called the **interior opposite angles** of the exterior angle $A\hat{C}D$. In the same manner, there are pairs of interior opposite angles corresponding to the exterior angles formed by producing the other two sides of the triangle too.

The following theorem specifies a relationship that exists between an exterior angle of a triangle and its interior opposite angles.

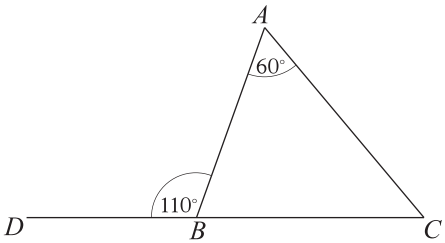
Theorem: The exterior angle formed when a side of a triangle is produced is equal to the sum of the two interior opposite angles.

Accordingly, for the above triangle ABC,

$$A\hat{C}D = A\hat{B}C + B\hat{A}C$$

Let us now consider how problems are solved using this theorem.

Example 1



Find the magnitude of $A\hat{C}B$ based on the information in the figure.

According to the theorem,

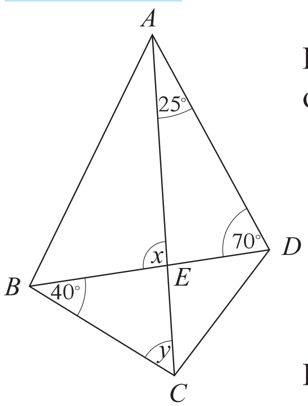
$B\hat{A}C + A\hat{C}B = A\hat{B}D$ (The exterior angle is equal to the sum of the interior opposite angles)

$\therefore\ 60^\circ + A\hat{C}B = 110^\circ$

$\therefore\ A\hat{C}B = 110^\circ - 60^\circ$

$\underline{\underline{A\hat{C}B = 50^\circ}}$

Example 2



Based on the information in the figure, find the magnitudes of $A\hat{E}B$ and $B\hat{C}E$.

Let us denote $A\hat{E}B$ by x and $B\hat{C}E$ by y.

It is clear that $A\hat{E}B$ is an exterior angle of the triangle AED

Accordingly, $x = 25^\circ + 70^\circ$ (The exterior angle is equal to the sum of the interior opposite angles)

$= 95^\circ$

Since $A\hat{E}B$ is an exterior angle of the triangle BCE

$y + 40^\circ = x$ (The exterior angle is equal to the sum of the interior opposite angles)

$\therefore\ y + 40^\circ = 95^\circ$

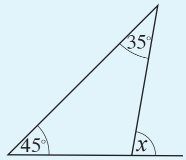
$\therefore\ y = 95^\circ - 40^\circ$

$\underline{\underline{y = 55^\circ}}$

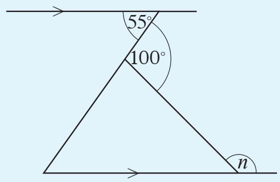
Exercise 8.1

1. Find the magnitudes of the angles represented by the letters in each of the figures.

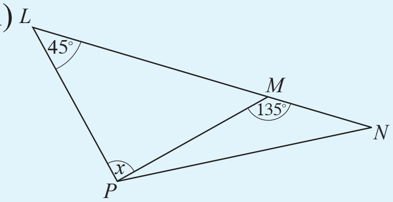
(i)



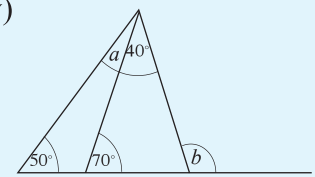
(ii)



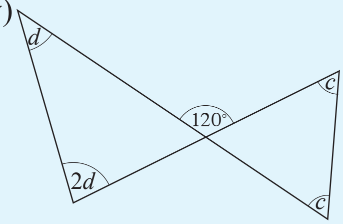
(iii)



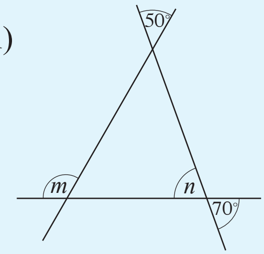
(iv)



(v)

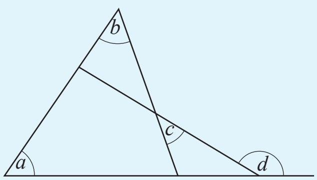


(vi)

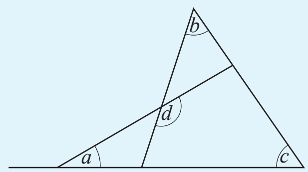


2. Express d in terms of a, b and c based on the information in each figure.

(i)

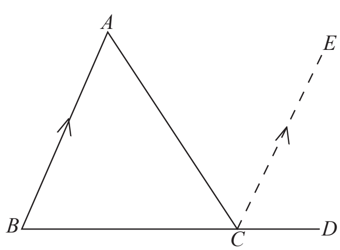


(ii)



8.2 The formal proof and applications of the theorem that the exterior angle formed when a side of a triangle is produced is equal to the sum of the two interior opposite angles

Formal Proof:



Data : The side BC of the triangle ABC has been produced up to D.

To be proved: $A\hat{C}D = A\hat{B}C + B\hat{A}C$

Construction: CE is drawn parallel to BA.

Proof:

$$E\hat{C}D = A\hat{B}C \quad (BA \,//\, CE \text{ since corresponding angles})$$

$$A\hat{C}E = B\hat{A}C \quad (BA \,//\, CE \text{ since alternate angles})$$

From ① and ②

$$E\hat{C}D + A\hat{C}E = A\hat{B}C + B\hat{A}C \text{ (Axiom)}$$

However according to the figure, the sum of the adjacent angles $E\hat{C}D$ and $A\hat{C}E$ is $A\hat{C}D$.

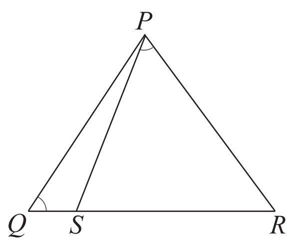
$$\therefore \underline{\underline{A\hat{C}D = A\hat{B}C + B\hat{A}C}}$$

Let us now prove several riders by using the formally proved exterior angle theorem and the other theorems you have learnt so far.

Example 1

The point S is located on the side QR of the triangle PQR such that $P\hat{Q}S = S\hat{P}R$. Prove that $Q\hat{P}R = P\hat{S}R$.

Let us first draw a sketch and include the given information in it.



Proof:

$P\hat{S}R$ is an exterior angle of the triangle PQS.

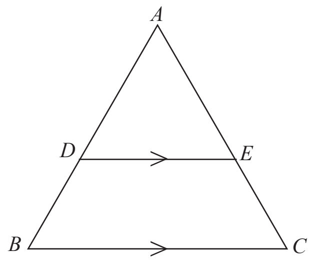
$$\therefore \; Q\hat{P}S + P\hat{Q}S = P\hat{S}R$$

$$\therefore Q\hat{P}S + S\hat{P}R = P\hat{S}R \quad (\text{Since } P\hat{Q}S = S\hat{P}R)$$

$$\text{But, } Q\hat{P}S + S\hat{P}R = Q\hat{P}R \quad (\text{Adjacent angles})$$

$$\therefore \underline{\underline{Q\hat{P}R = P\hat{S}R \quad (\text{Axiom})}}$$

Example 2



Based on the information in the figure, prove that $B\hat{A}C + A\hat{B}C = D\hat{E}C$

$C\hat{E}D$ is an exterior angle of the triangle ADE.

$\therefore\ D\hat{E}C = D\hat{A}E + A\hat{D}E$

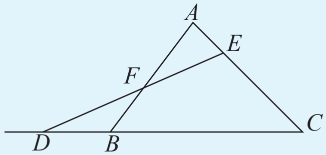
Also $D\hat{A}E$ and $B\hat{A}C$ are the same angle.

$\therefore\ A\hat{D}E = A\hat{B}C$ (Corresponding angles, $DE \,//\, BC$)

$\therefore\ \underline{\underline{D\hat{E}C = B\hat{A}C + A\hat{B}C}}$

Exercise 8.2

1. If $B\hat{D}F = E\hat{A}F$ in the figure given below, fill in the blanks in the following statements to prove that $F\hat{B}C = F\hat{E}C$.



Proof: Since $F\hat{B}C$ is an exterior angle of the triangle DBF,

$F\hat{B}C =$ ...... + ...

But $B\hat{F}D =$ ..... (Vertically opposite angles)

and $B\hat{D}F =$ ..... (.................)

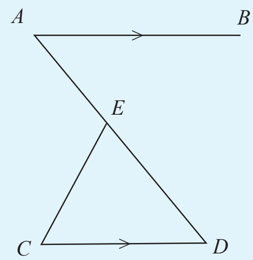
$\therefore\ F\hat{B}C =$ ..... + ...

But $F\hat{E}C$ is an exterior angle of the triangle AFE.

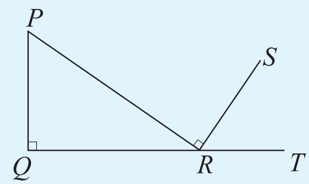
$\therefore\ F\hat{E}C =$ ..... + ... (.................)

$\therefore\ \underline{\underline{F\hat{B}C = F\hat{E}C}}$

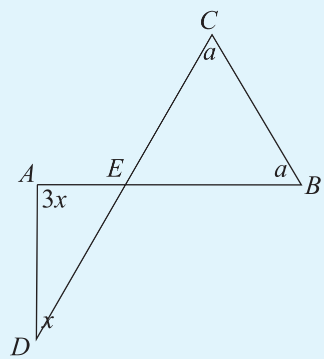
2. As indicated in the figure, AB and CD are parallel to each other. Prove that $A\hat{E}C = B\hat{A}D + E\hat{C}D$.



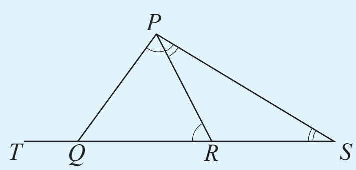
3. As indicated in the figure, $P\hat{Q}R$ and $P\hat{R}S$ are right angles. If QRT is a straight line, prove that $Q\hat{P}R = S\hat{R}T$.



4. The lines AB and CD intersect each other at point E as indicated in the figure. Based on the information in the figure, show that $a = 2x$.



5. In the given figure, $P\hat{R}Q = Q\hat{P}R$ and $R\hat{P}S = P\hat{S}R$. Based on the information in the figure, show that $P\hat{Q}T = 4\,P\hat{S}R$. (Hint: Take $P\hat{S}R = x$)



6. PQR is a triangle. RS is perpendicular to PQ and QT is perpendicular to PR. (S and T are on PQ and PR respectively). The lines SR and QT intersect at U. Prove that $S\hat{Q}U = T\hat{R}U$.

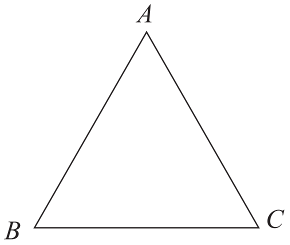
7. The side BC of the triangle ABC has been produced to E. The straight line AD has been drawn such that it meets the side CE at the point D and such that $B\hat{A}C = C\hat{A}D$. Also, $A\hat{B}C = B\hat{A}C$.

Prove that,

(i) $A\hat{C}D = 2\,A\hat{B}C$ and

(ii) $A\hat{D}E = 3\,A\hat{B}C$.

8.3 Theorem related to the interior angles of a triangle



The interior angles of the triangle ABC are $A\hat{B}C$, $B\hat{A}C$ and $A\hat{C}B$. We know that the sum of the magnitudes of these angles is equal to 180º.

This is expressed as a theorem as follows.

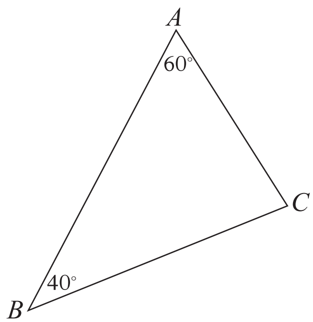
Theorem: The sum of the interior angles of a triangle is 180º

Accordingly, in relation to the above triangle, $A\hat{B}C + B\hat{A}C + A\hat{C}B = 180^\circ$

Let us now consider how problems are solved using this theorem.

Example 1

Find the magnitude of $A\hat{C}B$ based on the information in the figure.



$B\hat{A}C + A\hat{B}C + A\hat{C}B = 180^\circ$ (The sum of the interior angles)

$\therefore\ 60^\circ + 40^\circ + A\hat{C}B = 180^\circ$

$\therefore\ \underline{\underline{A\hat{C}B = 80^\circ}}$

Example 2

Find the value of x based on the information in the figure.

Since the sum of the interior angles of the triangle is 180º,

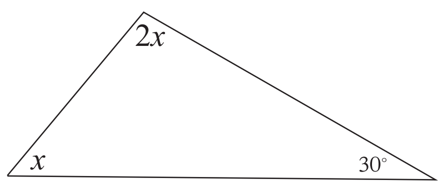
$\therefore x + 2x + 30^\circ = 180^\circ$

$\therefore\ 3x + 30^\circ = 180^\circ$

$\therefore\ 3x = 180^\circ - 30^\circ$

$3x = 150^\circ$

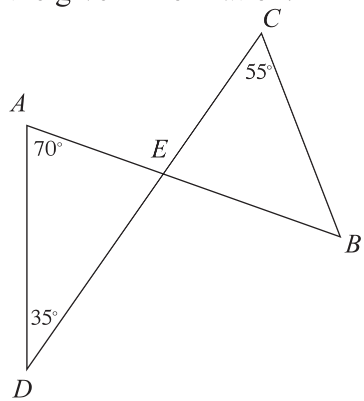
$\therefore\ \underline{\underline{x = 50^\circ}}$



Example 3

The straight lines AB and CD intersect at E. If $A\hat{D}E = 35^\circ$, $D\hat{A}E = 70^\circ$ and $E\hat{C}B = 55^\circ$ find the magnitude of $C\hat{B}E$.

First draw a figure with the given information.



According to the given information, considering the triangle ADE,

$A\hat{D}E + D\hat{A}E + A\hat{E}D = 180^\circ$ (Sum of the interior angles of a triangle)

$35^\circ + 70^\circ + A\hat{E}D = 180^\circ$

$\therefore \quad A\hat{E}D = 180^\circ - 105^\circ$

$= 75^\circ$

But, $A\hat{E}D = B\hat{E}C$ (Vertically opposite angles)

$B\hat{E}C = 75^\circ$

In the triangle BEC,

$B\hat{E}C + B\hat{C}E + C\hat{B}E = 180^\circ$ (Sum of the interior angles of a triangle)

$C\hat{B}E = 180^\circ - (75^\circ + 55^\circ)$

$= 180^\circ - 130^\circ$

$= \underline{\underline{50^\circ}}$

Example 4

Determine whether there can be a triangle with interior angles 55°, 60° and 75°.

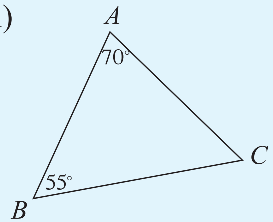
$55^\circ + 60^\circ + 75^\circ = 190^\circ$.

The sum of the interior angles of any triangle is 180°. Since the sum of the above three angles is not equal to 180°, there cannot be a triangle with the given angles as the interior angles.

Exercise 8.3

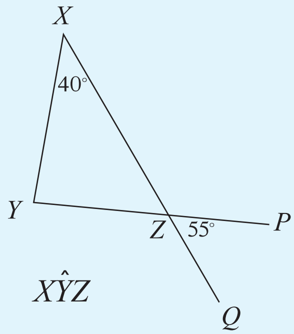
1. Using the information in each figure, determine the magnitude of the angle given below the figure.

(i)



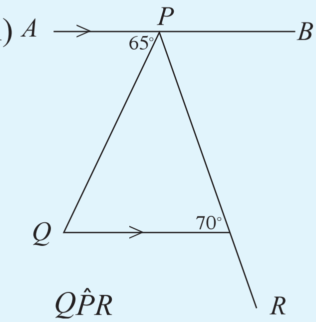
$A\hat{C}B$

(ii)



$X\hat{Y}Z$

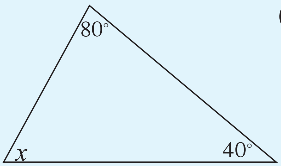
(iii)



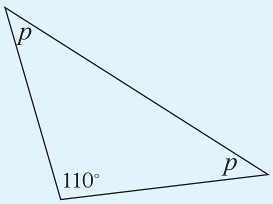
$Q\hat{P}R$

2. Using the given information, find the magnitude of each of the angles denoted by a letter in the following figures.

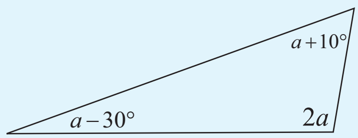
(i)



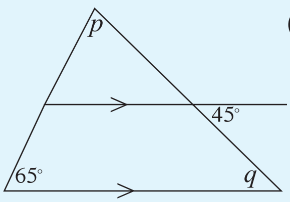
(ii)



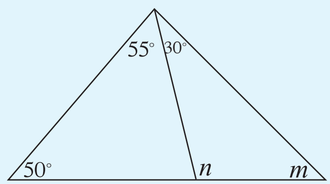
(iii)



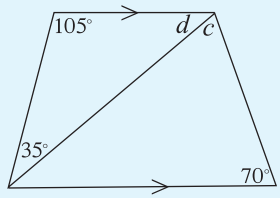
(iv)



(v)



(vi)



3. For each of the triplets of angles given below, determine whether they can be the interior angles of a triangle.

(i) $50^\circ, 40^\circ, 90^\circ$ (ii) $70^\circ, 30^\circ, 75^\circ$ (iii) $55^\circ, 72^\circ, 58^\circ$

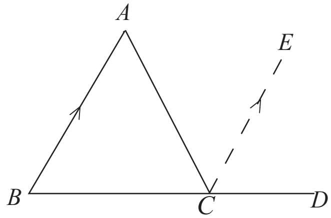
(iv) $60^\circ, 60^\circ, 60^\circ$ (v) $100^\circ, 20^\circ, 65^\circ$ (vi) $53^\circ, 49^\circ, 78^\circ$

4. The interior angles of a triangle are in the ratio 2 : 3 : 4. Find the magnitude of each of the angles.

5. The magnitude of the largest angle of a triangle is three times the magnitude of the smallest angle and the magnitude of the other angle is twice as large as the smallest angle. Find the magnitude of each of the angles.

8.4 Formal proof and applications of the theorem that the sum of the interior angles of a triangle is 180°.

The formal proof of the theorem "The sum of the interior angles of a triangle is 180°" is given below.



Data : ABC is a triangle

To be proved : $A\hat{B}C + B\hat{A}C + A\hat{C}B = 180^\circ$

Construction : Produce BC up to D and draw CE parallel to BA

Proof: $A\hat{B}C = E\hat{C}D$ (Corresponding angles, $BA \,//\, CE$) --------------- (1)

$B\hat{A}C = A\hat{C}E$ (Alternate angles, $BA \,//\, CE$) --------------- (2)

From (1) and (2),

$$A\hat{B}C + B\hat{A}C = E\hat{C}D + A\hat{C}E$$

By adding $A\hat{C}B$ to both sides,

$$A\hat{B}C + B\hat{A}C + A\hat{C}B = E\hat{C}D + A\hat{C}E + A\hat{C}B$$

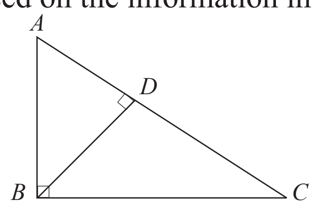
But,

$E\hat{C}D + A\hat{C}E + A\hat{C}B = 180^\circ$ (Angles on the straight line BCD)

$$\underline{\underline{A\hat{B}C + B\hat{A}C + A\hat{C}B = 180^\circ}}$$

Example 1

Prove that $A\hat{B}D = B\hat{C}D$ based on the information in the figure.



In the triangle BDC,

$B\hat{D}C = 90^\circ$ (Given)

Also $B\hat{D}C + D\hat{B}C + B\hat{C}D = 180^\circ$ (The sum of the interior angles of a triangle)

$90^\circ + D\hat{B}C + B\hat{C}D = 180^\circ$

$D\hat{B}C + B\hat{C}D = 180^\circ - 90^\circ$

$= 90^\circ$ —— ①

In the triangle ABC,

$$A\hat{B}C = 90^\circ \text{ (Given)}$$

$$\text{Since} \quad A\hat{B}C = A\hat{B}D + D\hat{B}C,$$

$$A\hat{B}D + D\hat{B}C = 90^\circ \text{ ——— } ②$$

By (1) and (2)

$$D\hat{B}C + B\hat{C}D = A\hat{B}D + D\hat{B}C$$

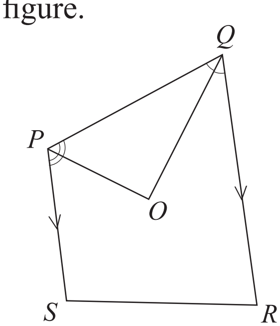
By subtracting $D\hat{B}C$ from both sides

$$\underline{\underline{B\hat{C}D = A\hat{B}D}}$$

Example 2

In the quadrilateral $PQRS$, the sides PS and QR are parallel to each other. The bisectors of the interior angles P and Q meet at O. Prove that $P\hat{O}Q$ is a right angle.

First let us draw the relevant figure.



Proof: Since $PS//QR$

$$S\hat{P}Q + P\hat{Q}R = 180^\circ \text{ (Allied angles)}$$

By dividing both sides by two

$$\frac{1}{2}SPQ + \frac{1}{2}PQR = \frac{180^\circ}{2} \quad \text{(Axiom)}$$

Since PO is the bisector of $S\hat{P}Q$ and QO is the bisector of $P\hat{Q}R$,

$$\frac{1}{2}S\hat{P}Q = Q\hat{P}O \quad \text{and}$$

$$\frac{1}{2}P\hat{Q}R = P\hat{Q}O$$

$$\therefore \quad Q\hat{P}O + P\hat{Q}O = 90^\circ$$

In the triangle POQ,

$$P\hat{O}Q + Q\hat{P}O + P\hat{Q}O = 180^\circ \text{ (Sum of the interior angles of a triangle)}$$

$$P\hat{O}Q + 90^\circ = 180^\circ$$

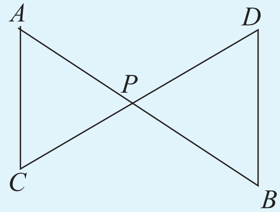
$$P\hat{O}Q = 90^\circ$$

$$\therefore \quad \underline{\underline{P\hat{O}Q \text{ is a right angle.}}}$$

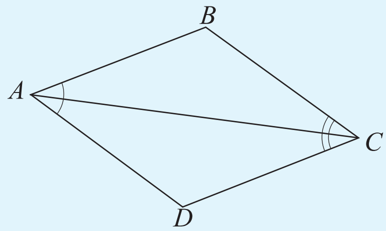
Now engage in the following exercise which contains problems involving proofs.

Exercise 8.4

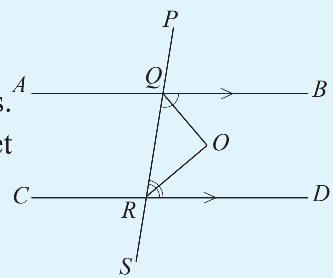
1. In the given figure, $A\hat{C}P = P\hat{B}D$. Prove that $C\hat{A}P = P\hat{D}B$.



2. In the given figure, the diagonal AC of the quadrilateral $ABCD$ bisects the angles $B\hat{A}D$ and $B\hat{C}D$. Prove that $A\hat{B}C = A\hat{D}C$.



3. In the given figure, AB and CD are parallel lines. The bisectors of the angles $B\hat{Q}R$ and $Q\hat{R}D$ meet at O.



(i) Find the value of $O\hat{Q}R + Q\hat{R}O$.

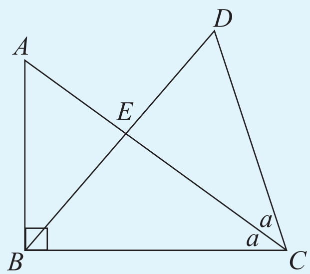
(ii) Prove that $Q\hat{O}R$ is a right angled triangle.

4. Using the information in the figure

(i) write down the magnitude of $B\hat{A}E$, in terms of a.

(ii) write down the magnitude of $B\hat{D}C + D\hat{B}C$ in terms of a.

(iii) Show that $B\hat{D}C + D\hat{B}C = 2\ B\hat{A}E$.



5 In the triangle ABC, $\hat{A} = \hat{B} = \hat{C}$. The bisector of $B\hat{A}C$ meets the side CB at D.

(i) Find the magnitude of $B\hat{A}C$.

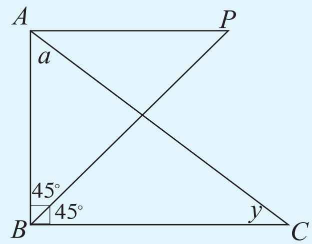
(ii) Prove that ABD is a right angled triangle.

Miscellaneous Exercise

1. If $\hat{A} + \hat{B} = 110^\circ$, and $\hat{B} + \hat{C} = 120^\circ$ in the triangle ABC, find the magnitude of each angle of the triangle.

2. The magnitude of a $B\hat{A}C$ of the triangle ABC is 100°. The bisectors of the interior angles $A\hat{B}C$ and $A\hat{C}B$ meet at O. Find the magnitude of $B\hat{O}C$.

3. In the figure, the straight line drawn from the point A perpendicular to the side BA of the triangle ABC meets the bisector of $A\hat{B}C$ at P. Prove that $B\hat{A}C + A\hat{C}B = 2\,A\hat{P}B$.



4. $A\hat{C}B = 3\,A\hat{B}C$ in the triangle ABC. The bisector of $B\hat{A}C$ meets BC at E. The point D lies on AE produced such that $AD \perp BD$. Prove that BC is the bisector of $A\hat{B}D$ (Hint: Take $A\hat{B}C = x$ and $B\hat{A}C = 2a$).

5. The line PQ has been drawn through the point A, parallel to the side BC of the triangle ABC. Prove that the sum of the interior angles of the triangle ABC is 180°.

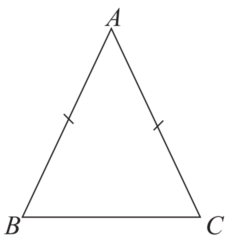
9 Triangles II

By studying this lesson you will be able to

solve problems by applying the theorem on isosceles triangles and its converse.

9.1 Isosceles triangles

If two sides of a triangle are equal, then it is called an isosceles triangle. The triangle ABC in the figure given below is an isosceles triangle. In this triangle, $AB = AC$. The angle in front of each side of a triangle is called the angle opposite that side.



The angle opposite the side AB is $A\hat{C}B$.
The angle opposite the side AC is $A\hat{B}C$.
The angle opposite the side BC is $B\hat{A}C$.
Further, A, which is the vertex at which the two equal sides meet is called the apex of the triangle.

A theorem related to isosceles triangles is given below.

Theorem: If two sides are equal in a triangle, the angles opposite the equal sides are equal.

According to the theorem, since $AB = AC$ in the above isosceles triangle ABC, $A\hat{B}C = A\hat{C}B$. Let us engage in the following activity to verify the truth of the above theorem.

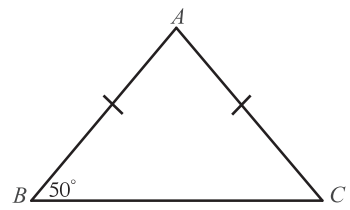
Activity

- Mark three points A, B and C (not collinear) such that $AB = AC = 5$cm.
- Complete the triangle ABC by joining the points A, B and C.
- Cut out the shape of the triangle ABC.
- Fold the triangular shaped piece of paper so that AB and AC coincide.
- Observe that $A\hat{B}C$ and $A\hat{C}B$ are equal.

Now let us consider several problems that can be solved by applying the above theorem together with theorems that have been learnt previously.

Example 1

In the triangle ABC, $AB = AC$ and $A\hat{B}C = 50^{\circ}$.
Find the magnitude of
(i) $A\hat{C}B$ (ii) $B\hat{A}C$



(i) $A\hat{C}B = A\hat{B}C$ ($AB = AC$, angles opposite equal sides in the isosceles triangle)
$\therefore A\hat{C}B = \underline{\underline{50^{\circ}}}$

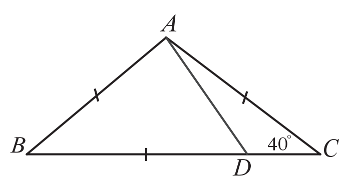
(ii) Since the sum of the interior angles of a triangle is 180°,

$$B\hat{A}C + A\hat{B}C + A\hat{C}B = 180^{\circ}$$
$$\therefore B\hat{A}C + 50^{\circ} + 50^{\circ} = 180^{\circ}$$
$$\therefore B\hat{A}C = 180^{\circ} - (50^{\circ} + 50^{\circ})$$
$$= 180^{\circ} - 100^{\circ}$$
$$= \underline{\underline{80^{\circ}}}$$

Example 2

In the triangle ABC, $AB = AC$ and $A\hat{C}B = 40^{\circ}$. The point D has been marked on the side BC such that $AB = BD$, and AD has been joined. Find separately the magnitude of each of the angles in the triangle ABD.

First, let us draw the figure with the given information.



According to the figure,
$A\hat{B}C = A\hat{C}B$ (Angles opposite equal sides in triangle ABC)
$\therefore A\hat{B}C = 40^{\circ}$
That is, $A\hat{B}D = 40^{\circ}$
In the triangle ABD,
$B\hat{A}D = B\hat{D}A$ ($AB = BD$)
$A\hat{B}D + B\hat{A}D + B\hat{D}A = 180^{\circ}$ (The sum of the interior angles of the triangle ABD is 180°)

$$40^{\circ} + 2B\hat{A}D = 180^{\circ} \text{ (Since } B\hat{A}D = B\hat{D}A)$$
$$2B\hat{A}D = 180^{\circ} - 40^{\circ}$$
$$2B\hat{A}D = 140^{\circ}$$
$$B\hat{A}D = 70^{\circ}$$
$$B\hat{D}A = 70^{\circ} \text{ (Since } B\hat{A}D = B\hat{D}A)$$

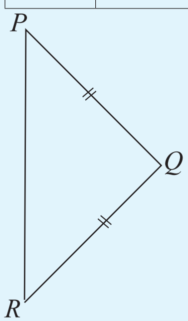
The magnitudes of the angles of the triangle ABD are 70º, 70º and 40º.

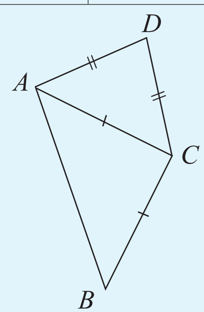
Do the following exercise by applying the theorem on isosceles triangles.

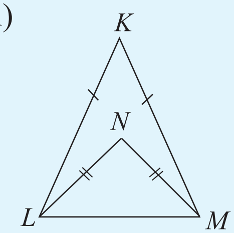
Exercise 9.1

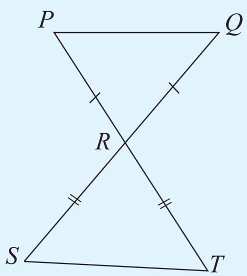
1. Complete the following table by identifying all the isosceles triangles in the figures in each of the parts given below.

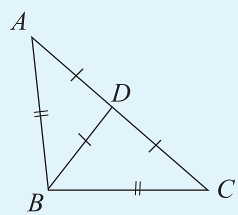
Figure	Triangle	Pair of equal sides	Pair of angles opposite equal sides
(i)	PQR	PQ, RQ	$Q\hat{P}R$, $Q\hat{R}P$
(ii)	ACD	AD, DC	$A\hat{C}D$, $D\hat{A}C$
	ABC		
(iii)	KLM		
	LMN		
(iv)	PQR		
	RST		
(v)	ABD		
	BCD		
	ABC		

(i)


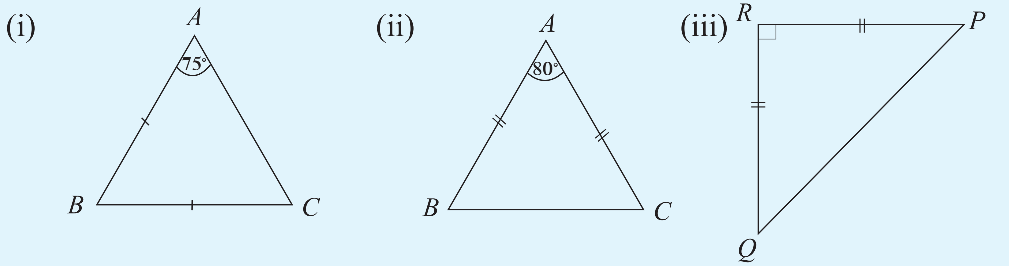
(ii)


(iii)


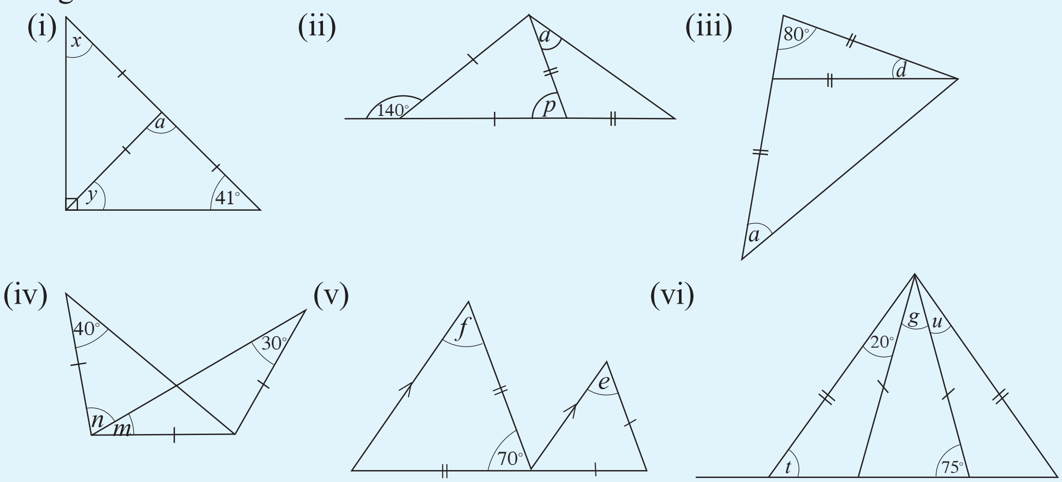
(iv)


(v)


2. In each of the following triangles, the magnitude of one of the angles is given. Separately find the magnitude of the other two angles.

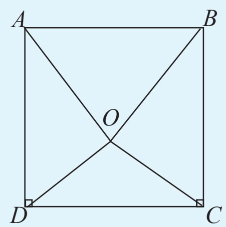


3. Find the value of each of the angles denoted by an unknown in the following figures.



4. In a certain isosceles triangle, the included angle of the two equal sides is 110°. Find the magnitudes of the other two angles.

5. The point O lies within the square $ABCD$ such that AOB is an equilateral triangle. Find the magnitude of $D\hat{O}C$.



6. In the triangle ABE, while $\hat{A}$ is an obtuse angle, $AB = AE$. The point C lies on BE such that $AC = BC$. The bisector of the interior angle $C\hat{A}E$ meets BE at D.

(i) Illustrate this information in a figure.

(ii) If $A\hat{B}C = 40°$, find the magnitude of $D\hat{A}E$.

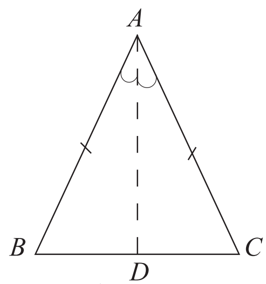
9.2 The formal proof of the theorem on isosceles triangles and applications of the theorem

Let us formally prove the theorem "In an isosceles triangle, the angles opposite equal sides are equal".

Data : $AB = AC$ in the triangle ABC

To prove that: $A\hat{B}C = A\hat{C}B$

Construction : Draw the bisector AD of the interior angle $B\hat{A}C$ such that it meets BC at D.



Proof: In the triangles ABD and ACD,

$AB = AC$ (Data)

$B\hat{A}D = D\hat{A}C$ (The bisector of $B\hat{A}C$ is AD)

AD is common to both the triangles

$\therefore \Delta ABD \equiv \Delta ACD$ (SAS)

Since the corresponding elements of congruent triangles are equal,

$A\hat{B}D = A\hat{C}D$

$\therefore A\hat{B}C = A\hat{C}B$

Now let us consider how several results on triangles are proved using the above theorem.

Example 1

$AB = AC$ in the triangle ABC in the figure.

Show that the following coincide.

(i) The perpendicular drawn from A to BC.

(ii) The bisector of the interior angle $B\hat{A}C$.

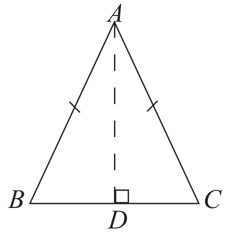
(iii) The straight line joining A and the midpoint of BC.

(iv) The perpendicular bisector of BC.

Let us first draw the perpendicular from the vertex A to the opposite side BC.

Construction : Draw the perpendicular from A to BC.

Proof : In the triangles ABD and ACD,



(i) $AB = AC$ (Data)

$A\hat{D}B = A\hat{D}C = 90^\circ$ (Construction)

AD is the common side

$\therefore \Delta ABD \equiv \Delta ACD$ (RHS)

Since the corresponding elements of congruent triangles are equal,

(ii) $B\hat{A}D = C\hat{A}D$

i.e., AD is the bisector of $B\hat{A}C$.

(iii) $BD = DC$ (corresponding sides of congruent triangles)

Therefore AD is the line joining A and the mid point of BC.

(iv) $A\hat{D}B = A\hat{D}C = 90^\circ$ (construction)

Also $BD = DC$

Therefore, AD is the perpendicular bisector of BC.

In an isosceles triangle,

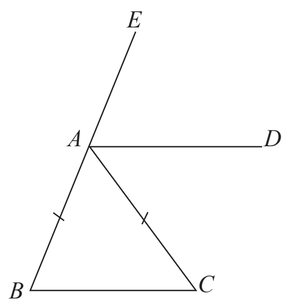
1. the perpendicular drawn from the apex to the opposite side
2. the bisector of the apex angle,
3. the straight line joining the apex to the midpoint of the opposite side and
4. the perpendicular bisector of the side opposite the apex,

coincide with each other.

In some instances, a result in geometry can be proved in several ways.
Now, let us consider such a result.

Example 2

$AB = AC$ in the triangle ABC. The side BA has been produced to E. The angle $C\hat{A}E$ is bisected by AD. Prove that AD and BC are parallel to each other.



To prove that $AD \parallel BC$, let us show that either a pair of alternate angles or a pair of corresponding angles are equal to each other.

Proof:

Method 1

In the triangle ABC, $A\hat{B}C = A\hat{C}B$ ($AB = AC$, angles opposite equal sides)

Since the side BA of the triangle ABC has been produced to E,

$$E\hat{A}C = A\hat{B}C + A\hat{C}B \text{ (Theorem on the exterior angle)}$$

$$E\hat{A}C = 2A\hat{C}B \text{ (Since } A\hat{B}C = A\hat{C}B) \text{ ——— (1)}$$

From the figure, $E\hat{A}C = E\hat{A}D + D\hat{A}C$

$$E\hat{A}C = 2D\hat{A}C \text{ (Since } AD \text{ is the bisector of } E\hat{A}C) \text{ ——— (2)}$$

$$2A\hat{C}B = 2D\hat{A}C$$

$$\therefore\ A\hat{C}B = D\hat{A}C$$

This is a pair of alternate angles. Since a pair of alternate angles are equal, AD is parallel to BC.

Method 2

According to the above figure, $A\hat{B}C$ and $E\hat{A}D$ are a pair of corresponding angles. By showing that these two angles are equal in the same manner as above, we can show that $BC \parallel AD$.

Method 3

The above proof could also be given as follows using algebraic symbols.

In the triangle ABC, let $A\hat{B}C = x$ ——— (1)

$$A\hat{B}C = A\hat{C}B \text{ (Since } AB = AC)$$

$$\therefore\ A\hat{C}B = x$$

Since the side BA of the triangle ABC has been produced to E,

$$E\hat{A}C = A\hat{B}C + A\hat{C}B \text{ (Theorem on the exterior angle)}$$

$$E\hat{A}C = x + x$$

$$= 2x$$

$$E\hat{A}D = x \text{ (Since } AD \text{ is the bisector of } E\hat{A}C) \text{ ——— (2)}$$

From (1) and (2), $E\hat{A}D = A\hat{B}C$

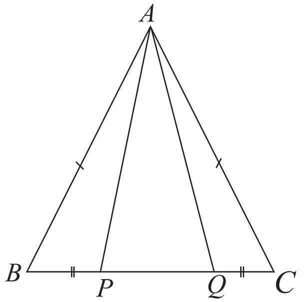
This is a pair of corresponding angles. Since a pair of corresponding angles are equal, $AD//BC$.

Example 3

In the triangle ABC, $AB = AC$. The points P and Q lie on the side BC such that $BP = CQ$. Prove that

(i) $\Delta APB \equiv \Delta AQC$

(ii) $A\hat{P}Q = A\hat{Q}P$



Proof : In the triangles APB and ACQ,

$AB = AC$ (Data)

$A\hat{B}P = A\hat{C}Q$ (Since $AB = AC$)

$BP = CQ$ (Data)

$\therefore \Delta ABP \equiv \Delta AQC$ (SAS)

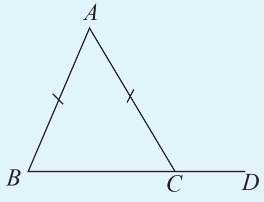
(ii) Since $\Delta ABP \equiv \Delta AQC$, $AP = AQ$ (Corresponding elements of congruent triangles)

In the triangle APQ,

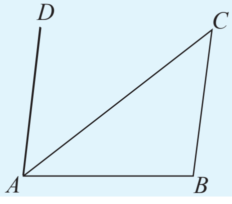
$A\hat{P}Q = A\hat{Q}P$ ($AP = AQ$, angles opposite equal sides)

Do the following exercise by applying the above theorem on isosceles triangles, and the other theorems learnt previously.

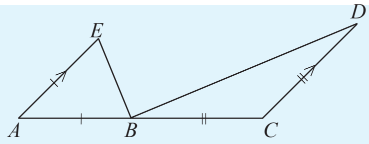
Exercise 9.2

1. 

Based on the information in the figure, prove that $A\hat{B}C + A\hat{C}D = 180^\circ$

2. 

In the figure, $AB = BC$ and $AD//BC$. Prove that the bisector of $D\hat{A}B$ is AC.

3. 

In the given figure, ABC is a straight line. Provide answers based on the information in the figure.

(i) Find the value of $B\hat{A}E + B\hat{C}D$

(ii) Show that $D\hat{B}E = 90^\circ$

4. D is the midpoint of the side BC of the triangle ABC. If $BD = DA$, prove that $B\hat{A}C$ is a right angle.

5. $AB = AC$ in the triangle ABC. The points P, Q and R lie on the sides AB, BC and AC respectively, such that $BP = CQ$ and $BQ = CR$.

(i) Draw a figure with this information included in it.

(ii) Prove that $\Delta PBQ \equiv \Delta QRC$.

(iii) Prove that $Q\hat{P}R = Q\hat{R}P$.

6. In the triangle ABC, $\hat{B}$ is a right angle. BD has been drawn perpendicular to AC. The point E lies on AC such that $CE = CB$.

(i) Draw a figure with this information included in it.

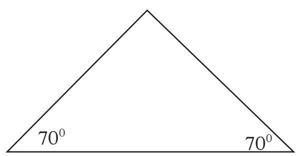
(ii) Prove that $A\hat{B}D$ is bisected by BE.

7. Prove that the angles in an equilateral triangle are 60°.

9.3 The converse of the theorem on isosceles triangles

Let us now examine whether the sides opposite equal angles are equal for a triangle.

Activity



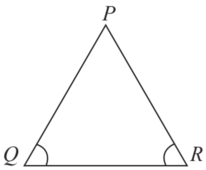
- Draw a line segment of length 5 cm and mark an angle of 70° at one end point using a protractor.
- Draw an angle of 70° at the other endpoint of the line segment too.
- Produce the two sides of the angles that were drawn, until they meet.
- Then a figure such as the above triangle is obtained.
- Cut out the triangle and fold it such that the equal angles coincide.
- Now identify the sides of the triangle which are equal.
- What is the special property that can be mentioned about the sides opposite the equal angles of the triangle?
- Draw several more triangles in the above manner, each time changing the magnitude of the equal angles, cut them out and see whether the above

- property is true for these triangles too.
- Observe that the sides opposite equal angles of a triangle are equal to each other.

The result that was obtained through the above activity is true in general and is given as a theorem below.

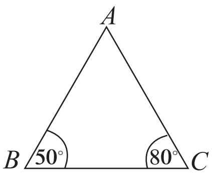
Theorem (Converse of the theorem on isosceles triangles):

The sides opposite equal angles of a triangle are equal.



According to the theorem, if $P\hat{Q}R = P\hat{R}Q$ in the triangle PQR, then $PR = PQ$.

Example 1



Find the equal sides of the triangle ABC in the figure.

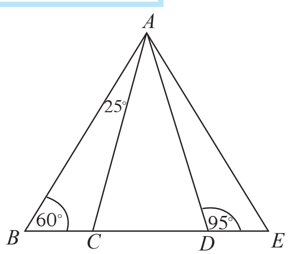
In the triangle ABC,

$$\hat{A} + \hat{B} + \hat{C} = 180^\circ \text{ (Sum of the interior angles of the triangle)}$$
$$\hat{A} + 50^\circ + 80^\circ = 180^\circ$$
$$\hat{A} = 180^\circ - (50^\circ + 80^\circ)$$
$$= 180^\circ - 130^\circ$$
$$= 50^\circ$$

$\hat{A} = \hat{B}$

$\therefore \underline{\underline{AC = BC}}$ (Sides opposite equal angles of a triangle)

Example 2



Based on the information in the figure, show that $AC = AD$.

Consider the triangle ABC.

$$A\hat{C}D = A\hat{B}C + B\hat{A}C \quad \text{(Exterior angle = sum of the interior opposite angles)}$$
$$= 60^\circ + 25^\circ$$
$$= 85^\circ$$

Since CDE is a straight line,

$A\hat{D}C + A\hat{D}E = 180^\circ$ (Adjacent angles on a straight line)

$A\hat{D}C = 180^\circ - 95^\circ$

$= 85^\circ$

In the triangle ACD,

$A\hat{C}D = 85^\circ$

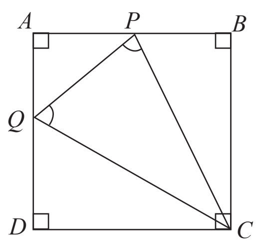
$A\hat{D}C = 85^\circ$

$\therefore \quad A\hat{D}C = A\hat{C}D$

$\therefore \quad AC = AD$ (Sides opposite equal angles)

Example 3

In the square $ABCD$, the points P and Q lie on the sides AB and AD respectively, such that $Q\hat{P}C = P\hat{Q}C$. Prove that $BP = QD$.



In the triangle PQC,

$Q\hat{P}C = P\hat{Q}C$ (Data)

$\therefore QC = PC$ (Sides opposite equal angles)

In the triangles PBC and DQC,

$P\hat{B}C = Q\hat{D}C = 90^\circ$ (Vertex angles of the square)

$BC = DC$ (Sides of the square)

$CP = CQ$ (Proved)

$\therefore \Delta PBC \equiv \Delta DQC$ (RHS)

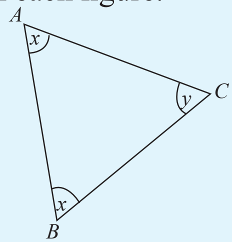
Since the corresponding elements of congruent triangles are equal,

$\underline{\underline{BP = QD}}$

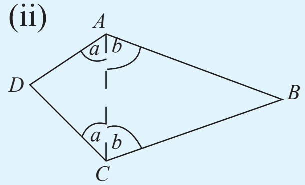
Exercise 9.3

1. Select the isosceles triangles from the following figures, based on the information in each figure.

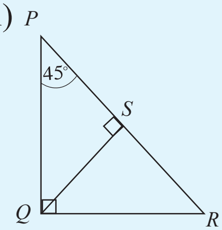
(i)



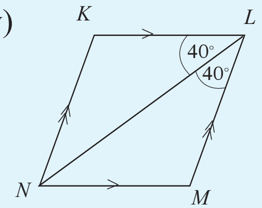
(ii)



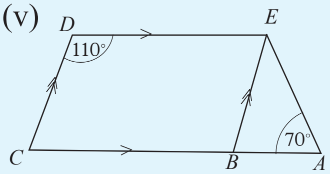
(iii)



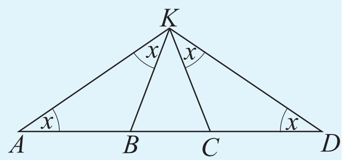
(iv)



(v)

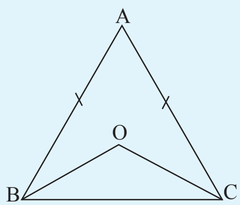


(vi)



2. If $A\hat{B}C = B\hat{C}A = B\hat{A}C$ in the triangle ABC, prove that it is an equilateral triangle.

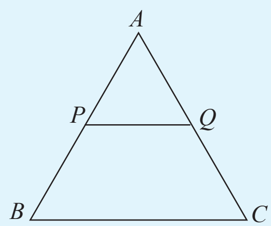
3.



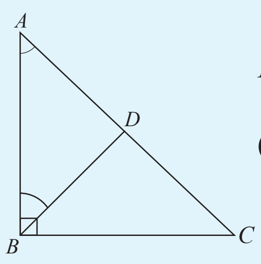
$AB = AC$ in the given triangle. The bisectors of the angles $A\hat{B}C$ and $A\hat{C}B$ meet at O. Prove that BOC is an isosceles triangle.

4. In the figure, $AB = AC$ and $BC//PQ$. Prove that

(i) $AP = AQ$
(ii) $BP = CQ$



5.



The point D lies on the side AC of the figure, such that $B\hat{A}D = D\hat{B}A$. Also $A\hat{B}C = 90^{\circ}$. Prove that,

(i) $D\hat{B}C = D\hat{C}B$
(ii) D is the midpoint of AC.

6. The bisectors of the angles $\hat{B}$ and $\hat{C}$ of the triangle ABC, meet at the point R. The points P and Q lie on AB and AC respectively such that PQ passes through R and is parallel to BC. Prove that,

(i) $PB = PR$

(ii) $PQ = PB + QC.$

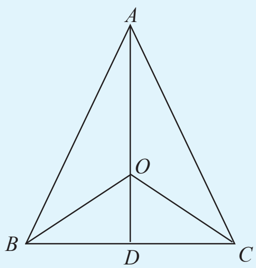
7. In the triangle ABC, the point P lies on AC such that $A\hat{C}B = A\hat{B}P$. The bisector of $P\hat{B}C$ meets the side AC at Q. Prove that $AB = AQ$.

8. In the quadrilateral $PQRS$, $PQ = SR$. The diagonals PR and QS which are equal in length intersect at T. Prove that,

(i) $\Delta PQR \equiv \Delta SQR$

(ii) $QT = RT.$

9.



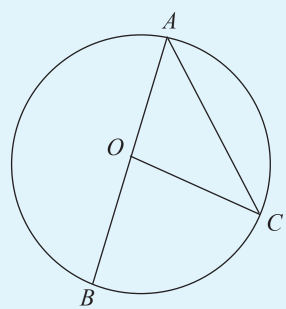
In the triangle $ABC, AB = AC$. The bisectors of $A\hat{B}C$ and $A\hat{C}B$ meet at O. AO produced meets BC at D. Prove that,

(i) BOC is an isosceles triangle.

(ii) $\Delta AOB \equiv \Delta AOC$

(iii) AD is perpendicular to BC.

10. A circle of centre O is given in the figure. Show that $B\hat{O}C = 2B\hat{A}C$



10 Inverse Proportions

By studying this lesson you will be able to

solve problems related to inverse proportions.

Ratios

Do the following exercise to recall the facts you have learnt earlier about ratios and direct proportions.

Review Exercise

1. Find the number suitable for the box, for each of the following to be a direct proportion.

(i) $5 : 2 = 20 : \boxed{}$ (iii) $4 : \boxed{} = 20 : 25$

(ii) $2 : 3 = \boxed{} : 15$ (iv) $\boxed{} : 4 = 60 : 80$

2. The daily income earned by a vehicle used in a taxi service is Rs 8000, while the daily cost incurred by it is Rs 4500. Write the ratio of the daily income to the daily cost in its simplest form.

3. In a certain scale diagram, an actual distance of 1000 m is represented by 2 cm. Express this scale as a ratio.

4. The gravitational force on earth is six times the gravitational force on the moon. Therefore, the ratio of the weight of an object on the moon to the weight of the object on earth is 1 : 6. How much would the weight of an astronaut be on the moon, if his weight on earth is 540 N.

5. To make a cement mixture, cement and sand are mixed together in the ratio 1 : 6.

(i) What would be the fraction of cement in such a mixture?

(ii) How many pans of cement are required for 18 pans of sand?

(iii) A certain bag of cement contains 5 pans of cement. If it is necessary to make a cement mixture using the whole bag, how many pans of sand are required?

(iv) Find separately the number of pans of cement and the number of pans of sand that are required to make 70 pans of the cement mixture.

10.1 Inverse Proportions

We know that, when we consider two quantities, if when one quantity increases according to a certain ratio, the other quantity also increases in the same ratio, or when one quantity decreases according to a certain ratio, the other quantity also decreases in the same ratio, then the two quantities are said to be **directly proportional** to each other.

In an **inverse proportion**, when one quantity increases according to a certain ratio, the other quantity decreases in the same ratio, or when one quantity decreases according to a certain ratio, the other quantity increases in the same ratio.

Let us establish this fact further by considering the following example.

In a certain hostel, there is sufficient food stored for 12 hostellers for 4 days. Keeping this amount of food in mind, let us consider the following questions.

(i) If the number of hostellers is 15, will the food be sufficient for four days?

(ii) If the number of hostellers is 6, for how many days will the food be sufficient?

(iii) When the number of hostellers decreases, does the number of days for which the food is sufficient decrease? Or does it increase?

(iv) For how many days will this food, which is sufficient for 12 hostellers for 4 days, be sufficient for one person?

It is clear that the food which is sufficient for 12 people for 4 days will be sufficient for 6 people for 8 days, and for one person for 48 days. The following relationships between the number of hostellers and the number of days for which the food is sufficient can easily be identified.

Number of hostellers	Number of days
12	4
8	6
6	8
4	12
2	24
1	48

Let us see how the two quantities, namely the number of hostellers and the number of days for which the food is sufficient change proportionally. According to the above chart, when the number of hostellers decreases from 8 to 2, the number of days for which the food is sufficient increases from 6 to 24.

In this case, the ratio of the number of hostellers = 8 : 2 = 4 : 1
The number of days for which the food is sufficient in this case has increased from 6 to 24.

The ratio of the number of days = 6 : 24 = 1 : 4

Although the ratio 1 : 4 is not equal to the ratio 4 : 1, the ratio that is obtained by interchanging the two numbers of one ratio is the same as the other ratio.

Then,

the ratio of the number of hostellers = 8 : 2 = 4 : 1
the ratio of the corresponding number of days interchanged = 24 : 6 = 4 : 1

A relationship such as the above one between the number of hostellers and the number of days for which the food is sufficient is called an **inverse proportion**.

Let us consider two more instances of the relationship between the number of hostellers and the number of days for which the food is sufficient.

Number of hostellers	**Number of days**
12	4
1	48

Ratio of the number of hostellers = 12 : 1
Ratio of the corresponding number of days interchanged = 48 : 4 = 12 : 1

The inverse proportion relationship should hold for any two cases that are considered, in the same manner that was observed in the above cases.

Two more examples of inverse proportions are given below.

(i) The number of people required to complete a particular task and the amount of time taken to complete the task.
(ii) When a vehicle travels a constant distance with uniform speed, the speed of the car and the time taken to complete the journey.

Now, let us consider the following example.

Example 1

It takes 5 men 8 days to complete a certain task. Find the number of days required by 10 men to complete the same task.

Let us look at two methods that can be used to solve this problem.
Note that this is a problem on inverse proportions.

Method 1

Let us take the number of days required by 10 men to complete the task as x. Then,

Number of men	**Number of days**
5	8
10	x

Since this is an inverse proportion,

$$
\begin{aligned}
5 : 10 &= x : 8 \\
\frac{5}{10} &= \frac{x}{8} \\
10x &= 8 \times 5 \\
&= 40 \\
\therefore x &= \frac{40}{10} \\
&= 4
\end{aligned}
$$

$\therefore$ The number of days required by 10 men to complete the task = $\underline{\underline{4}}$

Method 2

Time taken for 5 men to complete the task = 8 days

Time taken for one man to complete the task = 8 days $\times$ 5

= 40 days

$\therefore$ Time taken for 10 men to complete the task = 40 days $\div$ 10

= 4 days

Note: In the above example, the number of days taken by one man to complete the task, which is 40, can be taken as a measurement of the magnitude of the task. This value is called the **number of man days**.

Magnitude of the task = Amount of time required by one man to complete the task

= Number of men $\times$ Number of days

Accordingly, the magnitude of the above task can be considered as 40 man days.
The magnitude of a task can be measured in terms of man days as well as in terms of man hours.

Example 2

It takes 5 men 8 days to complete a certain task. How many men are required to complete the same task in 2 days?

Let us use method (ii) of example 1.

The number of days required by 5 men to complete the task = 8

$\therefore$ The number of days required by one man to complete the task = 8 $\times$ 5

$\therefore$ The magnitude of the task = 8 $\times$ 5 man days

= 40 man days

$\therefore$ The number of men required to complete the task in 2 days = $40 \div 2$

= $\underline{\underline{20}}$

Example 3

Food sufficient for 12 days for a group of 40 men employed at a worksite has been stored at the site. If 8 more men join the workforce after 6 days, for how many more days will the food that is remaining be sufficient?
Let us now consider how this problem is solved by two different methods.

Method 1

Amount of food sufficient for 40 men for 12 days $= 40 \times 12$
$= 480$

Amount of food sufficient for 40 men for 6 days $= 40 \times 6$
$= 240$

Amount of food remaining $= 480 - 240$
$= 240$

Number of days this food is sufficient for 48 men $= 240 \div 48$
$= 5$ days

Let us now see how this sum can be solved algebraically.

Method 2

The food which is sufficient for the group of 40 men for 12 days will be sufficient for the 40 of them for 6 days and for a few more days for the 48 men, which includes the 8 who joined later.

Let us take the number of days for which the food will be sufficient for the 48 men after the 6^{th} day as x. We can then equate the amount of food sufficient for the 40 men for 12 days to the sum of the amount of food sufficient for the 40 men for 6 days and the 48 men for x days.

$$
\begin{aligned}
\therefore\ 40 \times 12 &= (40 \times 6) + (48 \times x) \\
480 &= 240 + 48x \\
48x &= 480 - 240 \\
&= 240 \\
\therefore x &= \frac{240}{48} \\
&= 5
\end{aligned}
$$

$\therefore$ Therefore, the remaining food will be sufficient for 5 days.

Exercise 10.1

1. For each of the situations described below, select the suitable answer from (*a*), (*b*) and (*c*) and write it within the brackets next to it.

(*a*) Is not a proportion   (*b*) Is a direct proportion   (*c*) Is an inverse proportion

(i) The number of soldiers in a camp and the amount of food stored for their consumption (.......)
(ii) The radius of a circle and its area (.......)
(iii) The distance travelled by a vehicle travelling at a uniform speed and the time taken for the journey (.......)
(iv) The length and breadth of a rectangle of constant area (.......)
(v) The amount of sugar bought at a store by a person and the amount paid for it (.......)

2. It takes 8 men 9 days to complete a certain task.
(i) How many days will it take one man to complete the same task?
(ii) What is the magnitude of the task in man days?
(iii) If 12 men are assigned the task, how many days will it take them to complete the task?

3. A land owner estimates that it would take 10 men 8 days to clear his land. He hires 12 men for the initial two days.
(i) What is the magnitude of the task in man days?
(ii) How much of the task will be completed during the first two days?
(iii) If the landowner wishes to get the task done in 6 days, how many more men should he employ for the next four days?

4. In a certain farm there was sufficient food for 12 cattle for 10 days. After two days another four cattle were bought and brought to the farm.
(i) For how many days is the food in the farm sufficient for one of the cattle?
(ii) What is the reduction in the number of days for which the food is sufficient, due to the increase in the number of cattle?

5. Food sufficient for 24 trainees for 8 days was stored at a certain training camp. However, two days after the camp commenced, 6 of the trainees had to leave the camp due to illness. Show that the remaining food is sufficient for 2 extra days than was initially planned.

6. A particular tank of water can be emptied in four hours using three equal pumps. The three pumps were used to empty the tank. However one pump stopped working after an hour. Thereafter, the tank was emptied using the other two pumps. Find how much more time was required to empty the tank due to breakdown of one pump.

7. It takes half an hour for a certain vehicle travelling at a speed of 40 kmh^{-1} to complete a certain journey. Find the time in minutes that it would take the vehicle to complete the same journey, if it travels at a speed of 50 kmh^{-1}.

8. Four men who were given the responsibility of completing a task were able to finish only $\frac{2}{3}$ of it by working 6 hours a day for 3 days.

(Hint: Man hours = no.of men $\times$ no.of days $\times$ no.of working hours per day)

(i) What is the magnitude of the task in man hours?

(ii) The four men decide to complete the task on the next day. How many hours will they have to work to achieve this?

10.2 Representing inverse proportions algebraically

If it takes eight men one day to complete a certain task, then

- four men would require two days to complete the same task
- if only two men are engaged, four days would be required to complete the task
- it would take one man eight days to complete the task

Observe that in all four of the above situations, the product of the number of men and the number of days is a constant. That is,

$$\text{number of men} \times \text{number of days} = \text{a constant}$$

This constant value is the magnitude of the task. The unit used to measure the magnitude of the task can be taken as man days.

Accordingly, if x is the number of men and y is the number of days, then

$$xy = k \quad (k \text{ is a constant})$$

$$\therefore \quad x = \frac{k}{y} \quad \text{or } y = \frac{k}{x}$$

According to the definition of direct proportion, this can be expressed as $x \propto \frac{1}{y}$. That is, x and $\frac{1}{y}$ are directly proportional to each other. In this case we say that x and y are inversely proportional to each other.

Example 1

8 men can complete a certain task in 9 days. However, it was possible to employ only 6 men. How many days will it take these 6 men to complete the task?

Let us denote the number of men by x and the number of days by y. Then, by using the equation $xy = k$ and substituting the data, we obtain the equations,

$$8 \times 9 = k \text{ and}$$
$$6y = k$$

Note that we can use the same constant k as we refer to the same task.
Since it is the same task, the constant k is the same.
Substituting for k, we obtain the equation

$$8 \times 9 = 6y$$
$$\text{i.e., } y = \frac{8 \times 9}{6}$$
$$= 12$$

Therefore, it will take these 6 men 12 days to complete the task.

Example 2

A group of men, who completed a certain task in 9 days, recruited 3 more men to work on a similar task. If together they completed the task in 6 days, find how many men there were in the initial group.
Suppose the number of men who were in the first group is x.
Then, according to the data, we obtain the equations,

$$x \times 9 = k \text{ and}$$
$$(x+3) \times 6 = k$$

From these we obtain

$$9x = 6(x+3)$$
$$9x = 6x + 18$$
$$3x = 18$$
$$x = 6$$

Therefore, the number of men in the initial group was 6.

Solve the sums in the exercise given below algebraically.

Exercise 10.2

1. It took 5 men 4 days to complete a certain task. How many days will 4 men require to complete the task?

2. It was necessary to engage 9 men for 4 days, at 5 hours per day to clear a certain land. How many men working 6 hours per day can complete the same task in 10 days?

3. A certain task can be completed in 6 days by 18 men. It is expected to complete another task which is twice the size of the initial task in 9 days.
Find how many men are required to complete the second task in 9 days.

11 Data Representation

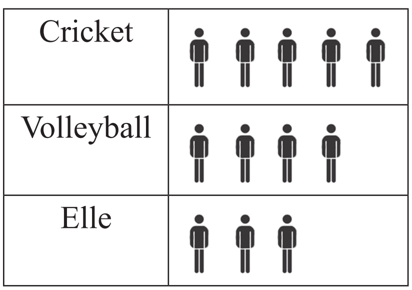
By studying this lesson you will be able to
- draw pie charts based on the given data
- extract information from a pie chart

11.1 Representing data in pie charts

The information gathered from the grade 10 students of a certain school regarding the sport they like the most, of the three sports, namely cricket, volleyball and elle is given below.

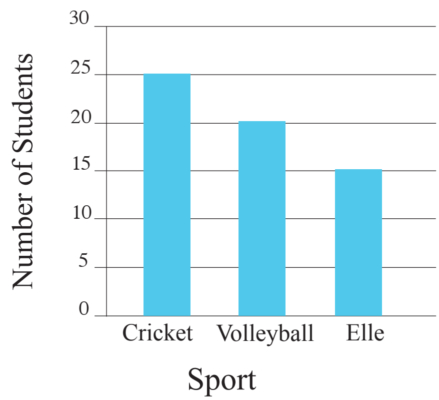
Sport	Number of students
Cricket	25
Volleyball	20
Elle	15

The above information can be represented in a picture graph and a column graph as follows.



Scale: 5 students are represented by 

Picture graph



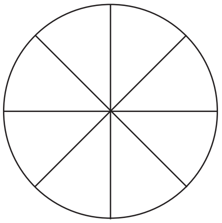
Column graph

The height of each column in the column graph denotes the number of students who like the relevant sport. In the picture graph this is denoted by the figures.

The pie chart is another method of representing data. Pie charts are also called pie graphs.

In a pie chart, the area of the whole circle represents the total number of data. The different frequencies are represented by suitable sectors.

Let us now consider how these sectors are found.



As an example, let us consider the above circle which has been divided into 8 equal sectors.

Since the circle has been divided into 8 equal sectors, the area of each sector is equal to $\frac{1}{8}$ of the area of the circle.

Also, the angle around the centre of this circle is divided into eight equal parts by the sectors.

Therefore, the angle at the centre of each sector is $\frac{1}{8}$ of the angle around the centre.

The angle around the center is 360°. Therefore the angle at the centre of each sector is $\frac{1}{8}$ of 360°.

That is, the angle at the centre of a sector which is $\frac{1}{8}$ of the circle $= \frac{1}{8} \times 360^\circ$

$= \underline{\underline{45^\circ}}$

Similarly, the angle at the centre of the sector which is $\frac{3}{8}$ of the circle $= \frac{3}{8} \times 360^\circ$

$= \underline{\underline{135^\circ}}$

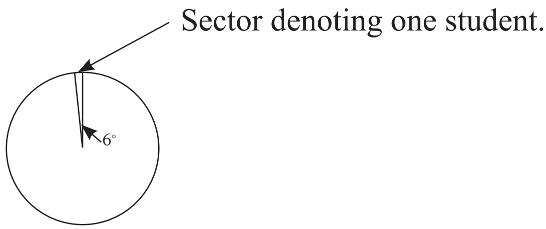
Now let us draw a suitable pie chart to represent the data in the above table.

First, let us draw a circle of suitable radius (about 3 cm.)

Let us represent the 60 students by the total area of the circle, which corresponds to the angle of 360° around the centre of the circle.

Then the angle at the centre of the sector which denotes one student. $= 360° \times \frac{1}{60}$

$= \underline{\underline{6^o}}$

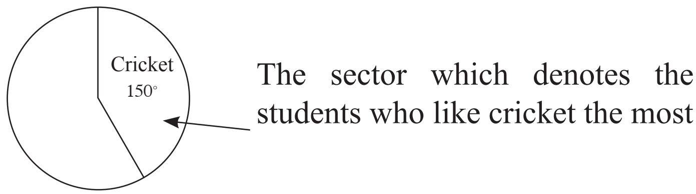


Accordingly, the angle at the centre of the sector which denotes the 25 students who like cricket the most $\Big\} = 360^o \times \frac{25}{60}$

$= 6^o \times 25$

$= \underline{\underline{150^o}}$

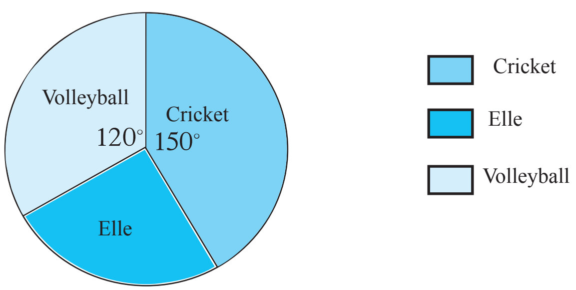
Now let us represent this within the circle as follows



Similarly, the angle at the centre of the sector which denotes the 20 students who like volleyball the most $\Big\} = 360^o \times \frac{20}{60}$

$= 120^o$

The remaining sector of the circle represents the students who like elle. The corresponding angle at the centre can be found using $360° \times \frac{15}{60}$. However this is not necessary, since this angle is equal to the angle obtained by subtracting the above two angles from 360º. All the above facts can be shown in a pie chart as follows.



Usually the angle at the centre is not denoted in a pie chart. Instead, the value represented by each sector is given as a percentage.

The comparison of data is facilitated by the sectors being shaded in different colours or covered with different patterns. Since all the data is represented by one circle comparisons such as 'more than' or 'less than' can easily be made.

Example 1

The information on the type of lunch packet preferred by 600 people who participated in a 'Shramadana' is given below.

Type of lunch packet	Number of people
Fish	250
Egg	150
Chicken	75
Vegetable	125
Total	600

Let us represent the above information in a pie chart.

The angle at the centre of the sector denoting the 250 people who prefer fish $\Big\} = 360^\circ \times \frac{250}{600}$

$= \underline{\underline{150^\circ}}$

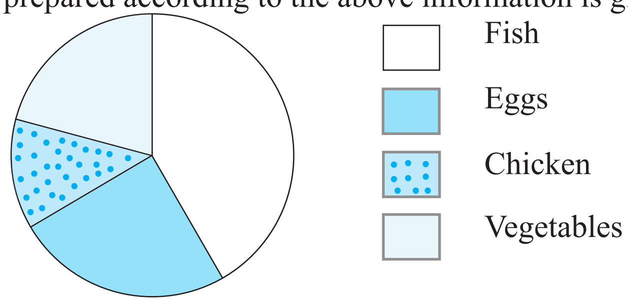
The angle at the centre of the sector denoting the 150 people who prefer eggs $\Big\} = 360^\circ \times \frac{150}{600}$

$= \underline{\underline{90^\circ}}$

The angle at the centre of the sector denoting the 75 people who prefer chicken $\Big\} = 360^\circ \times \frac{75}{600}$

$= \underline{\underline{45^\circ}}$

It is not neccessary to calculate the angle at the centre of the sector denoting those who prefer vegetables since the remaining portion of the circle denotes this.
The pie chart prepared according to the above information is given below.



Exercise 11.1

1. There are 40 children in a certain class. Each of these children has selected either dancing, music or art as their aesthetic subject. 20 of these children are studying art while 15 are studying music. The rest are studying dancing. Represent the above information in a pie chart.

2. The following table provides information on the subject streams followed by the students in the advanced level classes of a certain school.

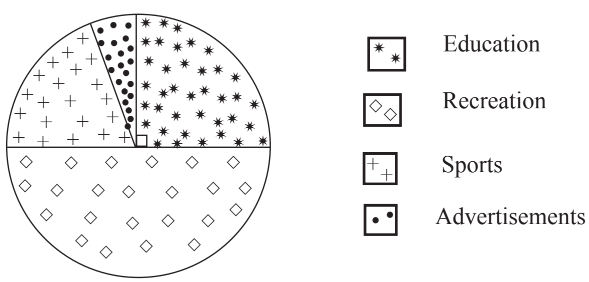
Subject Stream	Number of students
Arts	45
Science	20
Commerce	25
Technology	30

Draw a pie chart to represent the above information.

3. The number of newspapers that were sold at a newspaper stall on a certain day of the week was 540. The number of Sinhalese papers sold was 210 while the number of Tamil papers sold was 150. The rest of the papers that were sold were English papers. Represent this information in a pie chart.

11.2 Extracting information from a pie chart

Example 1



The above pie chart illustrates how a certain TV channel which telecasts programmes 18 hours a day, has divided its telecasting time between its programmes.

Let us answer the following questions by extracting the required information from the pie chart.

(i) For which type of programme has the most amount of time been allocated?

(ii) For which type of programme has the least amount of time been allocated?

(iii) (*a*) What is the magnitude of the angle at the centre of the sector which denotes the time allocated for educational programmes?

(*b*) Write the time allocated for educational programmes as a fraction of the total telecasting time.

(*c*) How much time has been allocated for educational programmes?

(*d*) Represent in the simplest form, the ratio of the time allocated for educational programmes to the time allocated for recreational programmes.

(iv) (*a*) If the time allocated for sports is 3 hours find the angle at the centre of that sector allocated for sports.

(*b*) Find the time allocated for advertisements.

(i) The largest sector of the pie chart represents the time allocated for recreational programmes.
Therefore, the most amount of time has been allocated for recreational programmes.

(ii) The smallest sector of the pie chart represents the time allocated for advertisements.
Therefore, the least amount of time has been allocated for advertisements.

(iii) *a*) 90°

b) Angle at the centre of the sector which denotes the time allocated for educational programmes $\Big\} = 90^\circ$

Angle at the centre representing the total telecasting time $= 360^\circ$

$\therefore$ The time allocated for educational programmes as a fraction of the total telecasting time $= \dfrac{90}{360}$

$= \underline{\underline{\dfrac{1}{4}}}$

(*c*) Time allocated for educational programmes $= 18 \times \dfrac{90}{360}$ hours

$= \underline{\underline{4\dfrac{1}{2}}}$ hours

(*d*) The angle at the centre of the sector which denotes the time allocated for educational programmes $\Big\} = 90^\circ$

The angle at the centre of the sector which denotes the time allocated for recreational programmes $\Big\} = 180^\circ$

$\therefore$ The ratio of the time allocated for educational programmes to the time allocated for recreational programmes $\Big\} = 90^\circ : 180^\circ$

$= \underline{\underline{1 : 2}}$

(iv) (*a*) Time allocated for sports as a fraction of the total time $= \frac{3}{18} = \frac{1}{6}$

The angle of the the sector which denotes sports $= 360^\circ \times \frac{1}{6}$

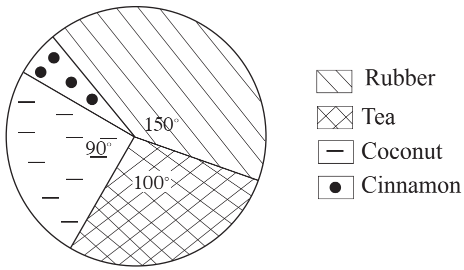
$= \underline{\underline{60^\circ}}$

(*b*) Angle at the centre of the sector which denotes advertisements $\Big\} = 360^\circ - 180^\circ - (90^\circ + 60^\circ) = 30^\circ$

Time allocated for advertisements $= \frac{30}{60} \times 3 = \underline{\underline{1\frac{1}{2} \text{ hours}}}$

Example 2

The following pie chart illustrates the information on the types of crops cultivated on a certain plot of land comprising of 720 hectare.



Answer the following questions using the pie chart.

(i) Which crop has been cultivated in the greatest extent of land?
(ii) Which crop has been cultivated in the smallest extent of land?
(iii) What is the extent of land on which tea has been cultivated?
(iv) What is the extent of land on which cinnamon has been cultivated?

Answers

(i) Rubber
(ii) Cinnamon
(iii) The angle at the centre of the sector denoting the extent of land on which tea has been cultivated $\Big\} = 100^\circ$

$\therefore$ The extent of land on which tea has been cultivated $= \frac{100}{360} \times 720$ hectare

$= \underline{\underline{200 \text{ hectare}}}$

(iv) The angle at the centre of the sector denoting the extent of land on which cinnamon has been cultivated $\Big\}$ $= 360^\circ - (100^\circ + 150^\circ + 90^\circ)$

$= 360^\circ - 340^\circ$

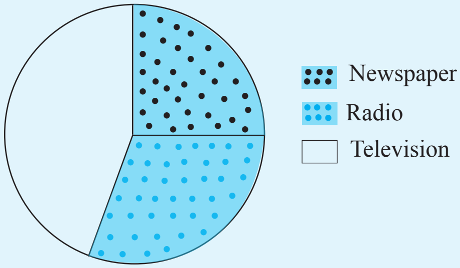
$= 20^\circ$

$\therefore$ The extent of land on which cinnamon has been cultivated $= \frac{20}{360} \times 720$ hectare

$= \underline{\underline{40 \text{ hectare}}}$

Exercise 11.2

1. The pie chart drawn with the information gathered from 40 grade 10 students of a certain school regarding the type of mass media they most prefer is given below.



Answer the following questions using the pie chart.

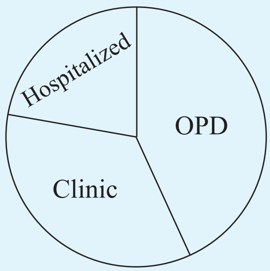
(i) Which type of mass media is preferred by the greatest number of children?

(ii) Which type of mass media is preferred by the least number of children?

(iii) If the angle at the centre of the sector which denotes the children who prefer television is 162°, find the number of children who prefer television.

(iv) If the angle at the centre of the sector which denotes the children who prefer newspapers is 90°; find the number of children who prefer newspapers.

2. The following pie chart illustrates the information on the number of patients who received treatment from the different units of a hospital on a certain day. The total number of patients who received treatment at the hospital on that day was 600.

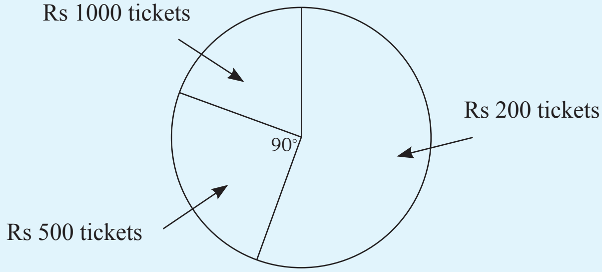


(i) From which unit did the most number of patients receive treatment on that day?

(ii) If the angle at the centre of the sector denoting this unit (the unit in (i)) is 150°, how many patients received treatment from that particular unit on that day?

(iii) If the number of patients who received treatment while being hospitalized is 130, find the angle at the centre of the sector which denotes these patients.

3. The tickets printed for a drama were valued at Rs 1000, Rs 500 and Rs 200. The following pie chart illustrates the number of tickets of each type that was sold.



(i) Of what value were the tickets that were sold the most?

(ii) What fraction of the total number of tickets sold is the number of Rs 500 tickets sold?

(iii) The number of Rs 1000 tickets that was sold was 140. If the angle at the centre of the sector that denotes the number of Rs 1000 tickets that was sold is 70°, find the number of Rs 200 tickets that was sold.

(iv) What was the total income that was received by selling the tickets?

Miscellaneous Exercise

1. A certain central school has classes from grade 1 to the advanced level. There are 600 students in grades 1 to 5 and 500 students in grades 6 to 11. The number of students in the advanced level classes is 340. Represent this information in a pie-chart.

2. The information gathered from the employees of a certain factory with the aim of providing transport is given below.

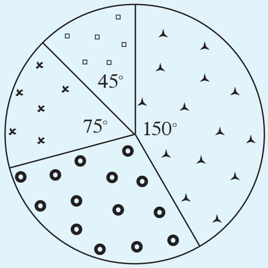
Mode of transport	Number of employees
Walking	110
Cycling	100
By bus	690

Represent this information in a pie chart.

3. The sum of the water, electricity and telephone bills of a certain household for the month of January was Rs 2700. The electricity bill was Rs 1440 and the water bill was Rs 750. Represent the above information in a pie chart.

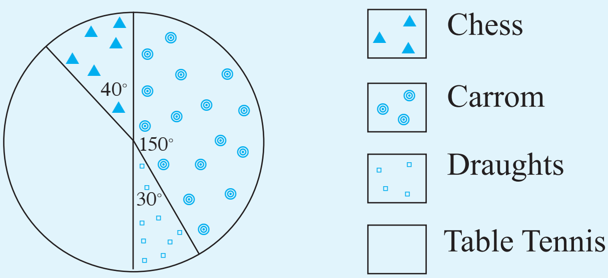
4. A certain welfare society decided to select one of the following three cities, Polonnaruwa, Anuradhapura, Kandy as the destination of their annual trip. $\frac{1}{4}$ of the members expressed their preference for Polonnaruwa. while 36 members preferred Kandy and the remaining 54 preferred Anuradhapura.
   (i) How many members did the welfare society have?
   (ii) Represent the above data in a pie chart.

5. The following pie chart illustrates the number of votes received by four parties at an election. The party which received the most number of votes received a total of 9300 votes.



   (i) How many votes in total did all four parties receive?
   (ii) How many votes in total did the party in third position receive?
   (iii) Express the number of votes that the party in fourth position received as a fraction of the total number of votes.
   (iv) Using the pie chart, write down the number of votes that the party in second position received as a percentage of the total number of votes.

6. The pie chart given below illustrates the information gathered from the members of a sports club regarding the indoor sport they like the most.



The number of members who like chess the most is 8.
According to the pie chart,
   (i) which indoor game is liked the most by the greatest number of members?
   (ii) how many members like carrom the most?
   (iii) how many members like table tennis the most?

12 Least Common Multiple of Algebraic Expressions

By studying this lesson you will be able to

find the least common multiple of several algebraic expressions.

Finding the least common multiple of numbers

The least common multiple (LCM) of several whole numbers is the smallest number which is divisible by all these numbers. You have learnt earlier how this is found. Let us recall what you have learnt.

Let us find the least common multiple of the numbers 6, 8 and 12 by writing these numbers as a product of their prime factors.

$$6 = 2\times 3 = 2^1\times 3^1$$
$$8 = 2\times 2\times 2 = 2^3$$
$$12 = 2\times 2\times 3 = 2^2\times 3$$

The distinct prime factors of the above three numbers are 2 and 3. When all three numbers are considered,

the greatest power of 2 = 2^3

the greatest power of 3 = 3^1

$\therefore$ The least common multiple $= 2^3\times 3$

$= \underline{\underline{24}}$

Accordingly, the least common multiple of several numbers can be found in the following manner.

1. Write each number as a product of its prime factors.
2. From the factors of all the numbers, select the greatest power of each prime number.
3. By multiplying all these powers together, obtain the required LCM.

Review Exercise

1. Find the least common multiple of each of the following triples, by writing each number as a product of its prime factors.

(i) 12, 18, 24 (ii) 6, 10, 15 (iii) 20, 30, 60

(iv) 8, 12, 24 (v) 24, 36, 48

2. An ice-cream producing company has three ice-cream vans. The three vans arrive at the housing complex "Isuruvimana" once every 3 days, once every 6 days and once every 8 days respectively. If all three vans arrived at "Isuruvimana" on a certain day, after how many days will they all arrive at this housing scheme again on the same day?

3. Mr. Gunatillake visits Galle Face grounds every Sunday evening to observe the setting of the sun. Mr. Mohamed and Mr. Perera too visit the same location once every 6 days and once every 8 days respectively for the same reason. If the three of them met each other for the first time in Galle Face grounds on Sunday the 8th of December 2013, after how many days did they all meet each other again at the same location? On which date did they meet again?

4. When a number is divided by each of the numbers 5, 6 and 7, a remainder of 1 is obtained. Find the smallest such number.

12.1 Finding the least common multiple of algebraic terms

Now let us consider what is meant by the least common multiple of several algebraic terms and how it is found.

Let us find the least common multiple of $4a^2$, $6ab$ and $8b$.

Let us write each term as a product of its factors.

$$4a^2 = 2\times 2\times a\times a = 2^2\times a^2$$

$$6ab = 2\times 3\times a\times b = 2^1\times 3^1\times a^1\times b^1$$

$$8b = 2\times 2\times 2\times b^1 = 2^3\times b^1$$

The distinct factors of these algebraic terms are 2, 3, a and b.

The largest power of 2 is 2^3
The largest power of 3 is 3^1
The largest power of a is a^2
The largest power of b is b^1

$$\therefore\ \text{LCM} = 2^3\times 3\times a^2\times b$$

$$= \underline{\underline{24a^2b}}$$

The least common multiple is obtained by taking the product of the largest powers of all the distinct factors of the given algebraic terms.

Exercise 12.1

1. For each of the following parts, find the LCM of the given terms.

(i) xy, xy^2 (ii) a^2b, ab^2
(iii) $6, 3a, 8b$ (iv) $24, 8x, 10x^2$
(v) $4m, 8mn, 12m^2$ (vi) $6p, 4pq, 12pq^2$
(vii) $4, 6x^2y, 8y$ (viii) m^2n, nm, nm^2
(ix) $ab, 4a^2b, 8a^2b^2$ (x) $5xy, 10x^2y, 2xy^2$

12.2 Finding the least common multiple of algebraic expressions which include binomial expressions

Let us find the least common multiple of $2x + 4$ and $3x - 9$.

To find the LCM of such expressions, their factors need to be found first.

$$2x + 4 = 2\,(x + 2)$$
$$3x - 9 = 3\,(x - 3)$$

The distinct factors are 2, 3, $(x + 2)$ and $(x - 3)$. The index of each of these factors is 1.

The product of the largest powers of these factors $= 2 \times 3 \times (x+2) \times (x-3)$

$$\therefore \text{ LCM} = \underline{\underline{6\,(x+2)\,(x-3)}}$$

Example 1

Find the least common multiple of $15x^2$, $20\,(x + 1)$, $10\,(x + 1)^2$

$$15x^2 = 3 \times 5 \times x^2$$
$$20\,(x + 1) = 2 \times 2 \times 5 \times (x+1) = 2^2 \times 5 \times (x + 1)$$
$$10\,(x + 1)^2 = 2 \times 5 \times (x+1)^2$$

The distinct factors are 2, 3, 5, x and $(x + 1)$.

$$\therefore \text{ LCM} = 2^2 \times 3 \times 5 \times x^2 (x+1)^2$$
$$= \underline{\underline{60x^2(x+1)^2}}$$

Example 2

Find the least common multiple of the algebraic expressions $(b - a)$, $2(a - b)$ and $4a^2\,(a - b)^2$

$$(b-a) = (-1) \times (a-b)$$
$$2(a-b) = 2 \times (a-b)$$
$$4a^2(a-b)^2 = 2 \times 2 \times a^2 \times (a-b)^2$$
$$= 2^2 \times a^2 \times (a-b)^2$$

> Since $(a - b)$ is a factor of two of these expressions, it is necessary to express $(b - a)$ as $-(a - b)$.

The distinct factors are 2, (-1), a and $(a-b)$.

$\therefore$ The product of the largest powers $= 2^2 \times (-1) \times a^2 \times (a-b)^2$

$\therefore$ LCM $= \underline{\underline{-4a^2(a-b)^2}}$

Note: Knowing that although $a-b=-(b-a)$, we have $(a-b)^2=(b-a)^2$, facilitates problem solving.

Exercise 12.2

1. Find the LCM of the algebraic expressions in each of the following parts.

a. $3x+6,\ 2x-4$
b. $2a+8,\ 3a+12$
c. $p-4,\ 8-2p$
d. $8(x+5),\ 20(x+5)^2$
e. $3x,\ 15(x+1),\ 9(x-1)$
f. $a^2,\ 2(a-b),\ (b-a)$
g. $3(x-2),\ 5(3-x),\ (x-2)(x-3)$
h. $3x,\ 15(x-3),\ 6(x-3)^2$
i. $(t-1),\ (1-t)^2$
j. $2a-4,\ 12(a-2)^2,\ 8(a+2)(2-a)^2$

12.3 Finding the least common multiple of algebraic expressions, described further

(a) When there is a difference of two squares

Example 1

Find the least common multiple of the algebraic expressions $2x-6,\ 4x(x-3)^2$ and $6(x^2-9)$

$2x-6 = 2(x-3)$

$4x(x-3)^2 = 2 \times 2 \times x \times (x-3)^2$

$6(x^2-9) = 2 \times 3 \times (x-3)(x+3)$

The distinct factors are $2,\ 3,\ x,\ (x-3)$ and $(x+3)$

$\therefore$ LCM $= 2^2 \times 3 \times x \times (x+3) \times (x-3)^2$

$= \underline{\underline{12x(x+3)(x-3)^2}}$

(b) When there are trinomial quadratic expressions

Example 2

Find the least common multiple of the algebraic expressions $3(x+2)^2,\ x^2+5x+6,$ and $2x^2+7x+3$

$3(x+2)^2 = 3 \times (x+2)^2$

$x^2+5x+6 = (x+2)(x+3)$

$2x^2+7x+3 = (x+3)(2x+1)$

The distinct factors are $3,\ (x+2),\ (x+3)$ and $(2x+1)$

$\therefore$ LCM $= \underline{\underline{3(x+3)(2x+1)(x+2)^2}}$

Exercise 12.3

1. Find the LCM of the following algebraic expressions.

a. $3(x-2), (x^2-4)$

b. $6(x-1), 2x(x^2-1)$

c. $3x-9, 4x(x-3), (x^2-9)$

d. $(a-b), (a^2-b^2)$

e. $p(p-q), pq(p^2-q^2)$

f. $x^2+2x+1, 2(x+1)$

g. $x^2-8x+15, 2x^2-x-15$

h. $x^2-4, 3x^2-5x-2, 3x^2-9x-12$

i. m^2-5m+6, m^2-2m-3

j. $x^2-a^2, x^2-ax, x^2-2ax+a^2$

13 Algebraic Fractions

By studying this lesson you will be able to

simplify algebraic fractions with unequal denominators.

Algebraic Fractions

Given below are several examples of algebraic fractions.

$$\frac{x}{4}, \quad \frac{2x+1}{x+3}, \quad \frac{3}{1+6y}, \quad \frac{x^2+x+1}{x^3-3x}$$

There are algebraic expressions in the numerator, the denominator or both the numerator and the denominator of each of these algebraic fractions.

Do the following exercise by applying what you have learnt earlier about adding and subtracting algebraic fractions.

Review Exercise

Simplify the following algebraic fractions.

(i) $\frac{x}{3} + \frac{x}{3}$ (ii) $\frac{x+1}{5} + \frac{2x+3}{3}$ (iii) $\frac{x}{3} + \frac{x}{2} + \frac{x}{4}$

(iv) $\frac{x+1}{3} + \frac{x+3}{6}$ (v) $\frac{2}{a} + \frac{3}{a} - \frac{1}{a}$ (vi) $\frac{5}{x+2} - \frac{3x+1}{x+2}$

13.1 Simplifying fractions with unequal algebraic terms in the denominator

Simplify

$$\frac{2}{x} + \frac{3}{2x}$$

x and $2x$ are the two terms in the denominators of $\frac{2}{x}$ and $\frac{3}{2x}$ respectively. Since they are unequal, the two fractions cannot be added directly. Therefore, let us first write equivalent fractions that have a common denominator and then do the simplification.

That is,

$$\frac{2}{x} + \frac{3}{2x} = \frac{2 \times 2}{x \times 2} + \frac{3}{2x}$$

$$= \frac{4}{2x} + \frac{3}{2x}$$

$$= \frac{7}{2x}$$

Here $2x$ is the denominator of the equivalent fractions. Observe that $2x$ is the least

common multiple of the denominators (x and $2x$) of the given two fractions.
Now consider the following algebraic fractions which have been simplified in a similar manner.

Example 1

$$\frac{5}{3a} - \frac{3}{4a}$$
$$= \frac{5\times4}{3a\times4} - \frac{3\times3}{4a\times3}$$
$$= \frac{20}{12a} - \frac{9}{12a}$$
$$= \underline{\underline{\frac{11}{12a}}}$$

Example 2

$$\frac{2}{3x} + \frac{5}{4y^2}$$
$$= \frac{2\times4y^2}{3x\times4y^2} + \frac{5\times3x}{4y^2\times3x}$$
$$= \frac{8y^2}{12xy^2} + \frac{15x}{12xy^2}$$
$$= \underline{\underline{\frac{8y^2+15x}{12xy^2}}}$$

Example 3

$$\frac{3b}{4a} + \frac{2a}{3b^2} + \frac{a}{2b}$$
$$= \frac{3b\times3b^2}{4a\times3b^2} + \frac{2a\times4a}{3b^2\times4a} + \frac{a\times6ab}{2b\times6ab}$$
$$= \frac{9b^3}{12ab^2} + \frac{8a^2}{12ab^2} + \frac{6a^2b}{12ab^2}$$
$$= \underline{\underline{\frac{9b^3+8a^2+6a^2b}{12ab^2}}}$$

Exercise 13.1

1. Simplify the following algebraic fractions.

a. $\frac{3}{x} + \frac{1}{3x}$ **b.** $\frac{7}{4a} - \frac{1}{2a}$ **c.** $\frac{3}{5m} + \frac{5}{4m^2}$ **d.** $\frac{1}{p} + \frac{1}{q}$

e. $\frac{7}{3x} - \frac{5}{4x}$ **f.** $\frac{3}{2a} + \frac{2}{a} - \frac{1}{3a}$ **g.** $\frac{3}{4x} - \frac{2}{3x} + \frac{4}{2x}$ **h.** $\frac{5}{m} + \frac{n}{3m}$

i. $\frac{a}{b} - \frac{b}{a}$ **j.** $\frac{1}{4a^2} + \frac{3}{5a}$ **k.** $\frac{3n}{m^2} - \frac{4}{5m}$ **l.** $\frac{3}{2a^2} - \frac{5}{4b} + \frac{4b}{3}$

13.2 Simplifying algebraic fractions with unequal binomial expressions in the denominators

As was done in 13.1 above, here too the LCM of the algebraic expressions in the denominators is first found and the simplification is done after the equivalent fractions with a common denominator are found.

Example 1

Simplify $\frac{1}{(p+1)} + \frac{1}{(p+5)}$

Since the LCM of $(p+1)$ and $(p+5)$ is $(p+1)\,(p+5)$,

$$\frac{1}{(p+1)} + \frac{1}{(p+5)} = \frac{(p+5)}{(p+1)(p+5)} + \frac{(p+1)}{(p+1)(p+5)}$$

$$= \frac{(p+5)+(p+1)}{(p+1)(p+5)}$$

$$= \frac{2p+6}{(p+1)(p+5)}$$

$$= \frac{2(p+3)}{(p+1)(p+5)}$$

Example 2

$$\frac{4}{x+3} - \frac{3}{x+4}$$

$$= \frac{4\,(x+4)}{(x+3)(x+4)} - \frac{3(x+3)}{(x+3)(x+4)}$$

Since the LCM of $(x+3)$ and $(x+4)$ is $(x+3)(x+4)$

$$= \frac{4\,(x+4)-3(x+3)}{(x+3)(x+4)}$$

$$= \frac{4x+16-3x-9}{(x+3)(x+4)}$$

$$= \frac{x+7}{(x+3)(x+4)}$$

When there are quadratic expressions in the denominators, they have to be first written in terms of their factors, and then the simplification needs to be done as above by finding the LCM of the denominators.

Example 3

$$\frac{1}{(x+2)} + \frac{1}{(x^2-3x-10)}$$

$$= \frac{1}{(x+2)} + \frac{1}{(x+2)(x-5)}$$

$$= \frac{(x-5)+1}{(x+2)(x-5)}$$

$$= \frac{(x-4)}{(x+2)(x-5)}$$

Example 4

$$\frac{1}{(x-1)} + \frac{3}{(x+1)} - \frac{2}{(x^2-1)}$$

$$= \frac{(x+1)}{(x-1)(x+1)} + \frac{3(x-1)}{(x-1)(x+1)} - \frac{2}{(x-1)(x+1)}$$

$$= \frac{x+1+3x-3-2}{(x-1)(x+1)}$$

$$= \frac{4x-4}{(x-1)(x+1)}$$

$$= \frac{4\cancel{(x-1)}}{\cancel{(x-1)}\,(x+1)}$$

$$= \frac{4}{(x+1)}$$

Exercise 13.2

Simplify the following algebraic fractions.

(*A*) **a.** $\frac{1}{a}+\frac{2}{a+2}$

b. $\frac{5}{x}+\frac{3}{x+1}$

c. $\frac{1}{x+1}+\frac{2}{x+3}$

d. $5+\frac{2}{x+3}$

e. $\frac{5}{4x+1}-\frac{1}{3(2x+1)}$

f. $\frac{8}{x+5}-\frac{3}{5-x}$

g. $\frac{2}{x+5}+\frac{3}{x-2}+\frac{1}{x}$

h. $\frac{2}{1-x}-\frac{3}{5-x}$

i. $\frac{3}{2(y-2)}+\frac{2}{3(y-2)}$

j. $\frac{1}{m-3}-\frac{2}{2m-1}$

k. $\frac{3}{x-6}-\frac{2}{2x-5}$

l. $\frac{4}{3(x+1)}-\frac{2}{5(x-1)}$

(*B*) **a.** $\frac{x+3}{x^2-1}+\frac{1}{x+1}$

b. $\frac{t-1}{t+1}+\frac{1}{t^2-1}$

c. $\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{1}{x^2-1}$

d. $\frac{1}{a-3}+\frac{1}{a^2-a-6}$

e. $\frac{1}{x+3}+\frac{1}{x^2+x-6}$

f. $\frac{3}{x^2+x-2}-\frac{1}{x^2-x-6}$

g. $\frac{4}{p^2+p-6}-\frac{2}{p^2+5p+6}$

h. $\frac{1}{x^2+4x+4}-\frac{1}{(x-2)(x+2)}$

i. $\frac{3}{a^2+5a+6}+\frac{1}{a^2+4a+3}$

j. $\frac{1}{2a+1}+\frac{1}{a^2+3a+2}$

14 Percentages

By studying this lesson you will be able to

- identify different types of taxes and solve related problems
- solve problems on simple interest.

Do the following exercise to recall what you have learnt so far about percentages.

Review Exercise

1. Complete the table by writing a suitable value in each empty box.

Fractional form	Decimal form	Percentage form
$\frac{1}{2}$	0.5	50%
$\frac{3}{5}$	0.6	
	0.8	80%
$\frac{1}{4}$		25%
	0.06	
		8%

2. Represent as a percentage.

(i) Rs. 50 as a percentage of Rs. 200 (ii) 25 cents as a percentage of one rupee
(iii) 8 cm as a percentage of 2 m (iv) 50 g as a percentage of 1 kg
(v) 300 m*l* as a percentage of 1 *l* (vi) 15 students as a percentage of 40 students

3. Find the quantity.

(i) 60% of Rs. 500 (ii) 20% of 250 km (iii) 25 % of 24 hours
(iv) 3% of 2 *l* (v) 15% of 1 kg (vi) 12% of 1.5 m

14.1 Taxes

The government of each country collects money from its citizens to meet recurrent expenses. This money is called taxes. Different types of taxes are collected in different ways and in varying amounts. The method that is used most to collect taxes is charging a percentage of the amount involved. Taxes that an individual has to pay directly to the government are called direct taxes. Several types of direct taxes are given below.

- Rates
- Customs duty
- Income tax

Taxes that are not charged directly are called indirect taxes. Value added tax (VAT) is one such tax that is charged at present.

Rates

Rates are taxes charged by urban councils, municipal councils, provincial councils etc., from households and businesses that come within their administrative domain. These taxes are used for various purposes such as the maintenance of roads and street lights, garbage collection and disposal etc. Properties owned by individuals and businesses are assessed annually by the government, and the relevant council charges a percentage of the assessed annual value of the property as rates. This is calculated annually and can be paid in total at the beginning of the year, or in 4 equal quarterly installments once every three months.

Example 1

A house of assessed annual value Rs. 36 000 is charged annual rates of 4%. Calculate the rates that have to be paid for a quarter.

$$\text{Annual rates} = \text{Rs. } 36\,000 \times \frac{4}{100}$$
$$= \text{Rs. } 1\,440$$

$\therefore$ The rates that have to be paid for a quarter $= \text{Rs. } 1\,440 \div 4$

$$= \underline{\underline{\text{Rs. } 360}}$$

Example 2

The assessed annual value of a shop within the administrative domain of a certain urban council is Rs. 24 000. The quarterly rates payable on this property is Rs. 300. Calculate the rates percentage charged by the council.

$$\begin{aligned}\text{Assessed annual value of the shop} &= \text{Rs. } 24\ 000\\ \text{Rates charged for a quarter} &= \text{Rs. } 300\\ \therefore \text{Rates charged for the year} &= \text{Rs. } 300 \times 4\\ &= \text{Rs. } 1\ 200\end{aligned}$$

$$\begin{aligned}\text{Accordingly, the rates percentage charged} &= \frac{1\ 200}{24\ 000} \times 100\%\\ &= \underline{\underline{5\%}}\end{aligned}$$

Customs Duty

When certain items are imported or exported, a certain percentage of the value of the item has to be paid as taxes to the government. Such taxes are called customs duty. This tax is charged by Sri Lanka Customs.

Individuals who have worked abroad are entitled to a duty free quota on their return, which may be used to purchase certain items without paying customs duty, from the duty free shops housed in the international airport.
Further, certain government servants are entitled to a reduction in customs duty when importing vehicles.

Example 1

If customs duty of 10% of the value of the item has to be paid when a certain type of clock is imported, how much duty has to be paid when a clock of this type of value Rs. 5 000 is imported?

$$\begin{aligned}\text{Customs duty that has to be paid} &= \text{Rs. } 5\ 000 \times \frac{10}{100}\\ &= \underline{\underline{\text{Rs. } 500}}\end{aligned}$$

Example 2

When a car is imported, 60% of its value has to be paid as customs duty. Find the total amount with the duty included, that it costs a person to import a car worth Rs. 2 000 000.

Method 1

Amount of customs duty that has to be paid $= \text{Rs. } 2\,000\,000 \times \frac{60}{100}$

$= \text{Rs. } 1\,200\,000$

Cost of the vehicle with the customs duty included $= \text{Rs. } 2\,000\,000 + 1\,200\,000$

$= \underline{\underline{\text{Rs. } 3\,200\,000}}$

Method 2

Cost of the vehicle with the customs duty included $= \text{Rs. } 2\,000\,000 \times \frac{160}{100}$

$= \underline{\underline{\text{Rs. } 3\,200\,000}}$

Example 3

If Rs. 18 000 was charged as customs duty when a stock of fruits worth Rs. 300 000 was exported to the Middle East, find the percentage of the value of the stock that the exporter had to pay as customs duty.

The value of the stock of fruits $= \text{Rs. } 300\,000$

Customs duty that was paid $= \text{Rs .} 18\,000$

Percentage charged as customs duty $= \frac{18\,000}{300\,000} \times 100\%$

$= \underline{\underline{6\%}}$

Example 4

The value of a television set with a 15% customs duty added to it is Rs. 28 750. What is the value of the television set without the duty?

The value of the television set without the customs duty $= \text{Rs. } 28\,750 \times \frac{100}{115}$

$= \underline{\underline{\text{Rs. } 25\,000}}$

Income Tax

The government imposes a tax on the annual income of an individual, which is earned as salary, from property or through a business, and which exceeds a certain limit. Such taxes are called income taxes.

The income tax for a particular year can either be paid annually or in quarterly installments. A table with the information on how the taxes are calculated, which was implemented in 2011 by the Inland Revenue Department is given below. (These values may change in the future)

Annual Income	Tax Percentage
Initial Rs. 500 000	Tax free
Next Rs. 500 000	4%
Next Rs. 500 000	8%
Next Rs. 500 000	12%
Next Rs. 500 000	16%
Next Rs. 500 000	20%
Balance taxable income	24%

(Table 13.1)

This table is used to perform calculations related to the examples and problems on income tax in this lesson.

Example 1

If the annual income of an individual is Rs. 575 000, calculate the income tax he has to pay for the year.

$$\begin{aligned}
\text{The annual income} &= \text{Rs. } 575\,000 \\
\text{Tax free income} &= \text{Rs. } 500\,000 \\
\text{Taxable income} &= \text{Rs. } 575\,000 - 500\,000 \\
&= \text{Rs. } 75\,000 \\
\text{Payable tax} &= \text{Rs. } 75\,000 \times \frac{4}{100} \\
&= \underline{\underline{\text{Rs. } 3\,000}}
\end{aligned}$$

Example 2

If the annual income of a certain businessman is Rs. 1 650 000, calculate the total annual income tax he has to pay.

First let us separate the annual income as shown below.

$$\underbrace{1,650\ 000}_{\text{Annual income}} = \underbrace{500\ 000}_{\text{Tax free income}} + \underbrace{500\ 000}_{4\%\ \text{tax}} + \underbrace{500\ 000}_{8\%\ \text{tax}} + \underbrace{150\ 000}_{12\%\ \text{tax}}$$

Tax free income = Rs. 500 000

Tax charged on the next Rs. 500 000 $= \text{Rs. } 500\ 000 \times \frac{4}{100}$

$= \text{Rs. } 20\ 000$

Tax charged on the next Rs. 500 000 $= \text{Rs. } 500\ 000 \times \frac{8}{100}$

$= \text{Rs. } 40\ 000$

Tax charged on the next Rs. 150 000 $= \text{Rs. } 150\ 000 \times \frac{12}{100}$

$= \text{Rs. } 18\ 000$

Total income tax that has to be paid = Rs. 20 000 + 40 000 + 18 000

= <u>Rs. 78 000</u>

Value Added Tax (VAT)

When a product or a service is bought, a percentage of its total value is charged as value added tax. The trader who sells the product and the service provider who are both bound to pay this tax, charge it from the consumer.

Example 1

A certain person's monthly telephone charges are Rs. 2 500. If a VAT of 15% is added to this charge, what is his monthly bill?

Method 1

$$\text{VAT} = \text{Rs. } 2\,500 \times \frac{15}{100}$$

$$= \text{Rs. } 375$$

$$\text{Total bill} = \text{Rs. } 2\,500 + 375$$

$$= \underline{\underline{\text{Rs. } 2\,875}}$$

Method 2

$$\text{Rs. } 2\,500 \times \frac{115}{100}$$

$$\text{Rs. } 2\,875$$

Example 2

The production cost of a pair of shoes produced by a certain organization is given below.

Raw materials	= Rs. 1 200
Labour	= Rs. 300
Other expenses	= Rs. 200
Total cost	= Rs. 1 700

The organization hopes to gain a profit of Rs. 250 from this product. A VAT of 15% has to be paid to the government on the price of the pair of shoes. Find the selling price of the pair of shoes with the VAT included.

$$\text{Price without VAT} = \text{Rs. } 1\,700 + 250 = \text{Rs. } 1\,950$$

$$\text{VAT that needs to be paid} = \text{Rs. } 1\,950 \times \frac{15}{100}$$

$$= \text{Rs. } 292.50$$

$$\text{Selling price of the pair of shoes} = \text{Rs. } 1950 + 292.50$$

$$= \underline{\underline{\text{Rs. } 2\,242.50}}$$

Method 2

$$\text{Rs. } 1\,950 \times \frac{115}{100}$$

$$\text{Rs. } 2\,242.50$$

Exercise 14.1

1. The assessed annual value of a certain house is Rs. 15 000. If the relevant provincial council institution charges 5% of the value of the house as rates, calculate the rates that have to be paid for a year.

2. If the rates that have to be paid by a shop of assessed annual value Rs. 18 000 is 6% of the value of the shop,
(i) how much has to be paid as rates for a year?
(ii) how much has to be paid as rates for a quarter?

3. If Rs. 270 has to be paid as quarterly rates for a house of assessed annual value Rs. 18 000, which lies within the limits of a certain urban council, calculate the percentage that the urban council charges as rates.

4. A canteen which lies within the limits of a certain municipal council which charges 8% of the assessed annual value of the property as rates, has to pay quarterly rates of Rs. 1 200. What is the assessed annual value of the canteen?

5. Mr. Silva who owns a house assessed to be of annual value Rs. 30 000, has rented it for a year to Mr. Perera. The monthly rent charged by Mr. Silva is Rs. 3 000. The relevant provincial council charges 4% of the assessed annual value of the house as rates. Mr. Silva has to spend 15% of the rent on maintenance. How much of the rent is Mr. Silva left with at the end of the year, after the relevant expenses are met?

6. The value of a refrigerator which is being imported is Rs. 40 000. If the customs duty that has to be paid is 20% of the value of the item, find the amount that has to be paid as duty.

7. If Rs. 3 000 had to be paid as customs duty when a camera worth Rs. 12 000 was imported, what is the percentage that was charged as customs duty?

8. The percentage charged as customs duty when a three wheeler is imported is 50% of its value. If the value of the three wheeler with the customs duty included is Rs. 450 000, what is the value of the three wheeler without the customs duty?

9. If customs duty of 12% of the value of the items is charged when a stock of ready-made garments worth Rs. 50 000 is exported, what is the value of the stock of garments after the customs duty has been paid?

10. When a certain type of motorcycle is imported, customs duty of 15% of its value has to be paid. The value of a motorcycle of this type which was imported is Rs. 175 000.
    (i) What is the value of the motorcycle after the customs duty is paid?
    (ii) At what price should the motorcycle be sold to make a profit of 10%?

11. The annual income of a certain individual is Rs. 550 000. How much income tax does he have to pay according to the income tax rates implemented in year 2011 (table 13.1)?

12. How much should a person whose annual income is Rs. 1 800 000 pay as income tax according to table 13.1?

13. If a businessman paid an annual income tax of Rs. 56 000 in 2012 according to the income tax rates implemented in 2011 (table 13.1), what is his annual income?

14. The monthly telephone charges of a certain household is Rs. 1 200. If the VAT charges are 10%, what is the total value of the bill?

15. The charges that had to be met by a person who imported a certain vehicle are as follows.

Import price of vehicle	= Rs. 600 000
Customs duty	= Rs. 400 000
Port charges and transportation cost	= Rs. 50 000

If VAT of 15% was charged on all these expenses, how much did he have to pay in total for the vehicle?

16. VAT of 15% is added to the monthly water bill of a household. If Prathapa pays a total of Rs. 1 725 as his monthly water bill, find the value of the bill without the VAT.

14.2 Interest

When a loan has been taken from an individual or an institution, the extra amount that has to be paid when settling the loan after a period of time is called the **interest**. Similarly, when money has been deposited in a bank or some other financial institution for a period of time, the extra amount that is received at the end of the period is also called the **interest**.
Usually, the interest that has to be paid for a year is a percentage of the loan amount (or deposit amount). This percentage is called the annual **interest rate**. The interest rate can also be given monthly or semi-annually.

Simple Interest

When calculating the interest for a specific period, if only the initial amount (principal) is taken into consideration, then such interest is called **simple interest**.

Example 1

It took a man 2 years to settle a loan of Rs. 5 000 borrowed at an annual simple interest rate of 10%. How much interest did he pay in total?

$$\begin{aligned}\text{Simple interest paid for a year} &= \text{Rs. } 5\,000 \times \frac{10}{100} \\ &= \text{Rs. } 500\end{aligned}$$

$$\begin{aligned}\text{Simple interest paid for two years} &= \text{Rs. } 500 \times 2 \\ &= \underline{\underline{\text{Rs. } 1\,000}}\end{aligned}$$

Example 2

Find the total amount that Senal has to pay to settle a loan in three months, if he borrowed Rs. 8 000 at a monthly simple interest rate of 2%.

Interest for a month $= \text{Rs. } 8\,000 \times \frac{2}{100}$
$= \text{Rs. } 160$

Interest for three months $= \text{Rs. } 160 \times 3$
$= \text{Rs. } 480$

Since total amount = Loan + Interest,
total amount to be paid in three months $= \text{Rs. } 8\,000 + 480$
$= \underline{\underline{\text{Rs. } 8\,480}}$

Example 3

How long will it take for a person who gave a loan of Rs 10 000 at an annual simple interest rate of 12%, to receive an interest of Rs. 3 600?

Simple interest received for a year $= \text{Rs. } 10\,000 \times \frac{12}{100}$
$= \text{Rs. } 1\,200$

Number of years it takes to receive Rs. 3 600 as interest $= \frac{3\,600}{1\,200}$

$= \underline{\underline{3 \text{ years}}}$

Example 4

If the simple interest that had to be paid for two years, on a loan of Rs.7 500 was Rs.1 200, find the annual simple interest rate that was charged.

Interest for two years = Rs.1 200
Interest for a year $= \text{Rs.}1\,200 \div 2$
= Rs.600

Annual simple interest rate $= \frac{600}{7\,500} \times 100\%$

$= \underline{\underline{8\%}}$

Example 5

How many years after taking a loan of Rs. 25 000 at an annual simple interest rate of 7.5%, does a person have to pay a total amount of Rs. 28 750 to settle the loan?

Interest for a year $= \text{Rs. } 25\,000 \times \frac{7.5}{100}$
$= \text{Rs. } 1\,875$
Loan = Rs. 25 000
Total amount to be paid = Rs. 28 750

$$\text{Total interest to be paid} = \text{Rs. } 28\ 750 - 25\ 000$$
$$= \text{Rs. } 3\ 750$$

$$\text{Time period for which interest has to be paid} = \frac{3\ 750}{1\ 875}$$
$$= \underline{\underline{2 \text{ years}}}$$

Example 6

If the total amount that had to be paid after 4 months to settle a loan borrowed at a monthly simple interest rate of 1.5% was Rs.5 300, find the loan amount.
The meaning of "A monthly simple interest rate of 1.5%" is that "Rs.1.50 has to be paid as interest for a month for a loan amount of Rs 100.

That is, interest paid for 1 month on a loan of Rs.100 $=$ Rs.1.50

$\therefore$ Interest paid for 4 months on a loan of Rs. 100 $=$ Rs.1.50 $\times$ 4 $=$ Rs.6

$\therefore$ Total amount paid in 4 months for a loan of Rs. 100 $=$ Rs.100 $+$ 6 $=$ Rs.106

$\therefore$ $\left.\begin{array}{r}\text{Loan amount for which the total amount of Rs.106} \\ \text{has to be paid after 4 months}\end{array}\right\} = \text{Rs.100}$

$\therefore$ $\left.\begin{array}{r}\text{Loan amount for which the total amount of Rs. 5300} \\ \text{has to be paid after 4 months}\end{array}\right\} = \dfrac{100}{106} \times 5300$

$$= \underline{\underline{\text{Rs.5000}}}$$

Exercise 14.2

1. How much interest is received in three years when a loan of Rs.5 000 is given at an annual simple interest rate of 12%?
2. A man deposits Rs.50 000 in a bank at a monthly interest rate of 1.5%. If he withdraws the interest he receives monthly, how much interest does he get in total during a period of 6 months?
3. Find the simple interest that has to be paid for 1 year and 5 months on a loan of Rs.2 500 borrowed at a monthly interest rate of 3%.
4. If a person who borrowed Rs. 500 settled the loan in one year by paying back Rs.560, find the annual interest rate that was charged.
5. If a person who gave a loan of Rs. 6 000 at an annual simple interest rate, received Rs.3 600 as interest for four years, what is the annual simple interest rate that was charged?
6. If a person who borrowed Rs.600 paid Rs.135 as interest for 1 year and 3 months, find the monthly interest rate that was charged.

7. If a person who gave a loan of Rs.8 000 received a total amount of Rs.9 680 at the end of two years, find the annual simple interest rate that was charged.

8. At what annual simple interest rate should a loan of Rs.5 000 be given to receive the same interest that is received when a loan of Rs.6 000 is given at an annual simple interest rate of 8%?

9. For what period of time should an interest of Rs.540 be paid on a loan of Rs.1 500 taken at an annual simple interest rate of 12%?

10. In how many months will an interest of Rs.420 be earned on a loan of Rs.2 000 given at a monthly simple interest rate of 3%?

11. After how long is a person who has taken a loan of Rs.6 000 at an annual simple interest rate of 18% able to settle the loan by paying back Rs.9 240?

12. After how long has a person to pay a total amount of Rs.5 000 to settle a loan of Rs.2500 taken at an annual simple interest rate of 10%?

13. A person takes a loan of Rs.5 000, promising to pay interest at a monthly simple interest rate of 5%. How much does he have to pay in total to settle the loan in 6 months?

14. What is the total amount that has to be paid in three years to settle a loan of Rs.8 000 taken at an annual simple interest rate of 15%?

15. Calculate the loan amount if a total amount of Rs.3 100 had to be paid in 8 months to settle the loan taken at a monthly simple interest rate of 3%.

16. A person takes a loan on annual simple interest, promising to settle the loan in two years by paying Rs.5 000. However he was only able to settle the loan in 5 years, at which time he had to pay Rs.6 500.

(i) Calculate the interest he has paid for a year.
(ii) What was the amount he borrowed?
(iii) What was the annual simple interest rate that was charged?

17. If a person took a loan of Rs. P for a period of T years at an annual simple interest rate of R.

(i) Write down an expression for the interest he has to pay for a month.
(ii) Write down an expression for the total interest I that he has to pay for a period of T years.
(iii) Calculate I using the expression in (ii) above when $P = 4000$, $R = 8$ % and $T = 5$.

15 Equations

By studying this lesson you will be able to

- construct and solve simple equations with algebraic fractions
- construct and solve simultaneous equations
- solve quadratic equations using factors

Solving simple equations

Let us work on the following exercise in order to revise the knowledge we have acquired on solving simple equations.

Review Exercise

1. Solve the following equations.

a. $2x+8 = x+12$ **b.** $2(x-3) = 4$ **c.** $5x-8 = 2(3-x)$

d. $2(y+3) = 3(y-1)$ **e.** $4-5(3-p) = 2(p-1)$ **f.** $\frac{x}{2}+1 = 3$

g. $5-\frac{x}{4} = 1$ **h.** $3-\frac{2x}{5} = 1$ **i.** $\frac{x}{3}+\frac{x}{4} = 7$

j. $\frac{5x-2}{4} = 2$ **k.** $\frac{(a-3)}{2}+1 = 4$ **l.** $\frac{(x+1)}{2}+\frac{(x-3)}{4} = \frac{1}{2}$

15.1 Solving simple equations described further:

Let us learn how to construct and solve an equation.

Did you observe that some of the equations in the above exercise contain fractions. The unknown term (x, y, p, a) was in the numerator of the fraction?. Now we are going to construct and solve equations with fractions of which the unknown term is in the denominator. Let us construct and solve such an equation.

Twelve is divided by two numbers of which the value of one number is double that of the other. The difference between the two quotients is 2. Find the two numbers.

Let us examine how this problem can be solved by the trial and error method.

Case I: Can the two numbers be 2 and 4?

$\frac{12}{2}=6$ and $\frac{12}{4}=3$, so the difference is $6 - 3 = 3$, and they do not work.

Case II: Can the two numbers be 6 and 12?

$\frac{12}{6}=2$ and $\frac{12}{12}=1$, so the difference is $2-1=1$, and they do not work.

Case III: Can the two numbers be 3 and 6?

$\frac{12}{3}=4$ and $\frac{12}{6}=2$, so the difference is $4-2=2$, and they work!

Thus, the problem can be solved using the method of trial and error. However, for some problems, the trial and error method is too long. Also, some problems cannot be solved by trial and error method. A suitable method of solving this type of problems is by using equations. Now, let us see how it can be solved by constructing an equation.

Suppose 12 is divided by a number x and its double $2x$.

Then the quotient of 12 divided by x is $\frac{12}{x}$.

The quotient of 12 divided by twice the x, i.e., $2x$, is $\frac{12}{2x}$.

Because the difference between the two quotients is 2, we get $\frac{12}{x}-\frac{12}{2x}=2$.

The value of x obtained by solving this equation gives us the required number. Let us solve this equation. This is an equation having fractions with algebraic terms in the denominator. The first fraction contains x in the denominator. The second fraction has $2x$ in the denominator. Let us make the denominators of both fractions equal. The easiest way to do this is by replacing $\frac{12}{x}$ by the equivalent fraction $\frac{12\times 2}{x\times 2}$, which is $\frac{24}{2x}$.

Then,

$$\frac{24}{2x}-\frac{12}{2x}=2$$

$$\therefore \frac{12}{2x}=2$$

Multiplying both sides by $2x$ $\qquad \frac{12}{2x}\times 2x = 2\times 2x$

$$\text{i.e., } 12 = 4x$$

Dividing both sides by 4

$$\frac{12}{4}=\frac{4x}{4}$$

$$\therefore 3 = x, \text{ i.e., } x=3$$

It follows that the two numbers are 3 and 6.

Note:This equation can be solved by writing the equation $\frac{12}{2x} = 2$ as $12 = 4x$ by cross multiplication too.

Example 1

Several friends shared 60 mangoes equally. One of them, Amal, sold three of his mangoes, and then he had only two mangoes left. How many friends shared the mangoes?

In reality, this problem can easily be solved mentally. Nevertheless, let us solve this equation as follows just to illustrate the construction and solving of equations.

Let x be the number of friends.
Then, the number of mangoes one person obtains $= \frac{60}{x}$

Number of mangoes Amal sold $= 3$

Then, the number of mangoes left $= \frac{60}{x} - 3$

Since he had only two left, we may write,

$$\frac{60}{x} - 3 = 2$$

Let us now solve this equation.
Let us add 3 to both sides.

$$\frac{60}{x} - 3 + 3 = 2 + 3$$

$$\therefore \frac{60}{x} = 5$$

Then, $5x = 60$
Hence, $x = 12$.

Therefore, 12 friends shared mangoes.

Observe how the following equations are solved.

Example 2

$$\frac{3}{a} + \frac{2}{a} = \frac{1}{2}$$

$$\frac{5}{a} = \frac{1}{2}$$

By cross multiplying,

$$\underline{\underline{a = 10}}$$

Example 3

$$\frac{3}{(x+2)} = \frac{1}{2}$$

By cross multiplying we obtain,

$$1 \times (x+2) = 2 \times 3$$

$$x + 2 = 6$$

$$\underline{\underline{x = 4}}$$

Example 4

$$\frac{2}{(x+5)} = \frac{3}{2(x-3)}$$
$$4(x-3) = 3(x+5)$$
$$4x-12 = 3x+15$$
$$4x-3x = 15+12$$
$$\underline{\underline{x = 27}}$$

Example 5

$$\frac{2}{(x-1)} - \frac{1}{2(x-1)} = \frac{3}{4}$$
$$\frac{4-1}{2(x-1)} = \frac{3}{4}$$
$$\frac{3}{2(x-1)} = \frac{3}{4}$$
$$3 \times 2\,(x-1) = 3 \times 4$$
$$\cancel{3}^{1} \times \cancel{2}^{1}(x-1) = \cancel{3}^{1} \times \cancel{4}^{2}$$
$$x-1 = 2$$
$$\underline{\underline{x = 3}}$$

Exercise 15.1

1. A father and his sons equally shared an amount of Rs 270. Then the amount each person had was Rs 45. Taking the number of sons as x, construct an equation. Solve this equation and hence find the number of sons the father has.

2. When the same number was added both to the numerator and the denominator of the fraction $\frac{3}{5}$, the resulting fraction was equal to $\frac{9}{10}$. What number was added?

3. Solve the following equations.

a. $\frac{5}{m} + \frac{2}{m} = \frac{1}{2}$ **b.** $\frac{3}{5x} + \frac{1}{x} = 2$ **c.** $\frac{5}{6x} - \frac{2}{3x} = \frac{1}{6}$

d. $\frac{4}{5x} - \frac{1}{3x} = \frac{7}{30}$ **e.** $\frac{21}{4m+1} = 3$ **f.** $\frac{3}{x+2} = \frac{3}{7}$

g. $\frac{10}{a-3} = \frac{5}{8}$ **h.** $\frac{4}{x+1} = \frac{3}{x-2}$ **i.** $\frac{2}{x-3} = \frac{3}{x+8}$

j. $\frac{1}{a+1} + \frac{3}{a+1} = \frac{2}{3}$ **k.** $\frac{5}{x-2} + \frac{3}{x-2} = 2$ **l.** $\frac{5}{2(p+1)} + \frac{1}{p+1} = \frac{7}{8}$

m. $\frac{3}{x+2} - \frac{1}{3(x+2)} = \frac{8}{15}$ **n.** $\frac{1}{2x-3} + \frac{4}{x+3} = 0$ **o.** $\frac{15}{2(p+1)} - \frac{3}{p+1} = 2$

p. $\frac{1}{a-1} + \frac{3}{4} = \frac{4}{a-1}$ **q.** $\frac{2x}{x+1} + \frac{2}{3} = 2$ **r.** $\frac{x+1}{x+3} = \frac{4}{5}$

15.2 Simultaneous Equations

Consider the following pair of simultaneous equations

$$2x + y = 5$$

$$2x + 3y = 8$$

The coefficient of the unknown x in each of the equations is 2. That means, they are equal. We have seen how to solve simultaneous equations of this nature, i.e., when the coefficients of one unknown are equal. Let us see how simultaneous equations are solved when the coefficients of both unknowns are un equal, i.e., the coefficients of each unknown are different.

Example 1:

Sajithi and Sanjana have certain amounts of money. When twice the amount of money Sanjana has is added to the amount of money Sajithi has, we get Rs. 110. When thrice the amount of money Sanjana has is added to twice the amount of money Sajithi has, the amount is Rs. 190. Find the amount of money each has.

Let us see how simultaneous equations can be used to solve this problem. Let the amount of money Sajithi has be Rs. x and the amount of money Sanjana has be Rs. y. Then, the sum of the amount of money Sajithi has and twice the amount of money Sanjana has is Rs. $x + 2y$. Because this amount is equal to Rs. 110, we get

$$x + 2y = 110 \text{ ——— } (1)$$

Also, the sum of twice the amount of money Sajithi has and thrice the amount of money Sanjana has is Rs. $2x + 3y$. Because this amount is equal to Rs 190, we get

$$2x + 3y = 190 \text{ ——— } (2)$$

The coefficients of neither x nor of y of the equations (1) and (2) are equal. Therefore, let us equate the coefficients of one of the unknowns. In order to make the coefficient of x in the first equation 2, let us multiply that equation by 2. Then we get

$$\therefore \ 2x + 4y = 220 \text{ ——— } (3)$$

Now, the coefficients of x in both equations are equal.
Therefore, from (3) and (2), we get,

$$2x + 4y - (2x + 3y) = 220 - 190$$
$$2x + 4y - 2x - 3y = 30$$
$$y = 30$$

Substituting the value of y in the first equation,

$$x + 2y = 110$$
$$x + 2 \times 30 = 110$$
$$x + 60 = 110$$
$$x = 110 - 60$$
$$x = 50$$

Therefore, the amount of money Sajithi has is Rs 50 and the amount of money Sanjana has is Rs 30.

Example 2

Solve: $2m + 3n = 13$
$3m + 5n = 21$

Let us denote the two equations as

$$2m + 3n = 13 \quad \text{—— (1)}$$
$$3m + 5n = 21 \quad \text{—— (2)}$$

From (1) $\times$ 3, we get, $6m + 9n = 39$ —— (3)
From (2) $\times$ 2, we get, $6m + 10n = 42$ —— (4)
Then, from (4) and (3), we get,

$$6m + 10n - (6m + 9n) = 42 - 39$$
$$6m + 10n - 6m - 9n = 3$$
$$n = 3$$

Substituting $n = 3$ in (1), we get

$$2m + 3n = 13$$
$$2m + 3 \times 3 = 13$$
$$2m = 13 - 9$$
$$2m = 4$$
$$m = 2$$

i.e., $m = 2$ and $n = 3$

Example 3

The price of two oranges and one king coconut is Rs. 80. Three king coconuts cost the same amount as two oranges. Find the price of an orange and a king coconut separately.

Let us construct two equations from the given information.

Let the price of one orange be Rs. x and that of a king coconut be Rs. y.

Then, the price of two oranges and a king coconut is $2x + y$. Since this is equal to Rs. 80, we get

$$2x + y = 80.$$

Because the price of two oranges is equal to that of three king coconut, we get,

$$2x = 3y.$$

Let us denote the equations as

$$2x + y = 80 \text{ ——— } (1)$$
$$2x = 3y \text{ ——— } (2)$$

These two equations can be solved just the way we did in the previous example, by writing the second equation as $2x - 3y = 0$. However, there is an easier way to do it in this case. That is, substituting from one equation to the other.

Substitute $3y$ from equation (2) for $2x$ in equation (1) to get

$$3y + y = 80$$
$$4y = 80$$
$$y = 20$$

Substitute $y = 20$ in (1) to get

$$2x + 20 = 80$$
$$2x = 60$$
$$x = 30$$

Therefore, the price of one orange is Rs. 30 and that of a king coconut is Rs. 20.

Example 4

Solve: $x = 3y$

$$2x + 3y = 18$$

Denote the two equations as

$$x = 3y \text{ ——— } (1)$$
$$2x + 3y = 18 \text{ ——— } (2)$$

Substitute for x in equation (2) from equation (1) to get

$$2 \times (3y) + 3y = 18$$
$$6y + 3y = 18$$
$$9y = 18$$
$$y = 2$$

Substitute $y = 2$ in (1) to get

$$x = 3y$$
$$x = 3 \times 2$$
$$x = 6$$

i.e., $x = 6$ and $y = 2$.

Exercise 15.2

1. Solve each of the following pair of simultaneous equations.

(a) $x + 2y = 10$
$2x - 5y = 2$

(b) $x = 3y$
$x + 3y = 12$

(c) $2m + n = 5$
$m + 2n = 4$

(d) $3x + y = 14$
$2x + 3y = 21$

(e) $2x + 5y = 9$
$3x + 2y = 8$

(f) $4m - 3n = 7$
$7m - 2n = 22$

(g) $8x - 3y = 1$
$3x + 2y = 16$

(h) $6x + 5y = 5$
$9x - 4y = 19$

(*i*) $3x + 4y = 9$
$2x - 5y + 17 = 0$

(*j*) $3x - 4y = 8\,(2 - y) + 1$
$2\,(2x + 3y\,) = 26 - y$

2. The price of two baby shirts and three pairs of baby shorts is Rs. 1150. Three baby shirts and a pairs of baby shorts cost Rs. 850. Taking the price of a baby shirt as Rs. x and that of a pairs of baby shorts as Rs. y, construct two simultaneous equations and find the price of a baby shirt and that of a pairs of baby shorts separately by solving the two equations.

3. Dinithi's father tells Dinithi, "My age is now four times your age. 8 years earlier, I was twelve times older than you." Find the present ages of the father and Dinithi separately.

15.3 Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation where $a \neq 0$. The terms b and c might be 0. Observe the following equations.

(i) $x^2 + 5x + 6 = 0$
(ii) $2x^2 - 5x = 0$
(iii) $x^2 - 9 = 0$

Each of the above equations is a quadratic equation since $a \neq 0$, whereas $c = 0$ in the second equation and $b = 0$ in the third equation.

Before solving the quadratic equations, let us consider the following:

- When any number is multiplied by zero, the product is zero.
- If the product of two numbers is zero, then at least one of the numbers is zero.

So, let us see for what values of x the expression $(x - 1)(x - 3)$ is zero. The expression is zero only when $x - 1 = 0$ or $x - 3 = 0$, that is, only when $x = 1$ or $x = 3$.

Now, consider the equation $(x - 1)(x - 3) = 0$. The values $x = 1$ and $x = 3$ satisfy this equation. Then, 1 and 3 are called the roots of this equation.

Now, consider the equation $x^2 + 5x + 6 = 0$.
Since $x^2 + 5x + 6 = (x + 3)(x + 2)$, the equation can be rewritten as $(x + 3)(x + 2) = 0$.
Then, $x + 3 = 0$ or $x + 2 = 0$.
There fore, $x = -3$ and $x = -2$ satisfy the equation. This can be verified as follows.
When $x = -3$,

$$\begin{aligned} x^2 + 5x + 6 &= (-3)^2 + 5(-3) + 6 \\ &= 9 + (-15) + 6 \\ &= 0 \end{aligned}$$

When $x = -2$,

$$\begin{aligned} x^2 + 5x + 6 &= (-2)^2 + 5(-2) + 6 \\ &= 4 + (-10) + 6 \\ &= 0 \end{aligned}$$

Thus, the roots of the equation $x^2 + 5x + 6 = 0$ are -3 and -2. In other words, the solutions of the equation are $x = -3$ and $x = -2$.

Example 1

Solve the equation $x^2 + 2x = 0$

$$\begin{aligned} x^2 + 2x &= 0 \\ x(x + 2) &= 0 \\ x = 0 \quad &\text{or} \quad x + 2 = 0 \\ x = 0 \quad &\text{or} \quad x = -2 \end{aligned}$$

Hence, $x = 0$ and $x = -2$ are the solutions of the given equation.

Example 2

Solve the equation $x^2 - 3x + 2 = 0$

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ (x - 1)(x - 2) &= 0 \\ x - 1 = 0 \quad &\text{or} \quad x - 2 = 0 \\ x = 1 \quad &\text{or} \quad x = 2 \end{aligned}$$

Hence, $x = 1$ and $x = 2$ are the solutions of the given equation.

Example 3

Solve the equation $x^2 - 4x - 21 = 0$

$$x^2 - 4x - 21 = 0$$

$$(x - 7)\ (x + 3) = 0$$

$$x - 7 = 0 \text{ or } x + 3 = 0$$

$$x = 7 \text{ or } x = -3$$

Hence, $x = 7$ and $x = -3$ are the solutions of the given equation.

Note: When there are two factors to a quadratic expression, there are two distinct roots to the that equation.

Exercise 15.3

Solve each of the following quadratic equations.

(*a*) $(x - 2)(x - 3) = 0$

(*b*) $(x + 2)(x - 5) = 0$

(*c*) $(x - 4)(x - 4) = 0$

(*d*) $(x - 1)(2x - 1) = 0$

(*e*) $x(x + 3) = 0$

(*f*) $y(2y - 3) = 0$

(*g*) $x^2 - 16 = 0$

(*h*) $4x^2 - 1 = 0$

(*i*) $9x^2 - 27x = 0$

(*j*) $x^2 + 15x + 36 = 0$

(*k*) $2x^2 - 5x + 2 = 0$

(*l*) $2x^2 - 5x = 0$

(*m*) $2x^2 = 6x$

(*n*) $x^2 = 25$

(*o*) $(x + 3)^2 = 16$

(*p*) $x^2 = 9x + 36$

(*q*) $(2x - 3)^2 = 0$

(*r*) $2x^2 - 5x = 0$

(*s*) $(x - 1)(x - 2) = 2x^2 - 3x - 2$

(*t*) $\dfrac{x + 3}{2} = \dfrac{3x + 2}{x}$

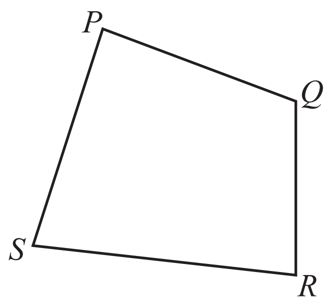
16 Parallelograms I

By studying this lesson you will be able to

solve problems and prove riders using the properties of parallelograms.

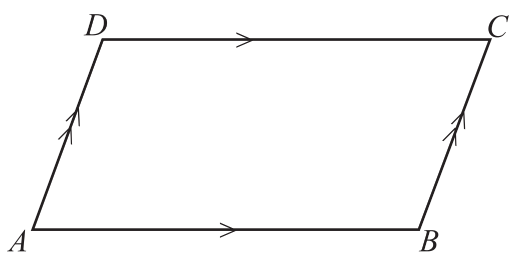
Parallelograms

A closed plane figure bounded by four straight line segments is called a quadrilateral. Let us consider the opposite sides and the opposite angles of a quadrilateral.



In the quadrilateral *PQRS*, *PQ* and *SR* are a pair of opposite sides, and *PS* and *QR* are the other pair of opposite sides. While $S\hat{P}Q$ and $S\hat{R}Q$ form a pair of opposite angles, the other pair of opposite angles is $P\hat{Q}R$ and $P\hat{S}R$.

A quadrilateral with both pairs of opposite sides parallel is defined as a **parallelogram.**



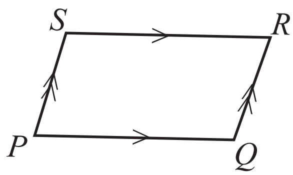
A pair of arrowheads have been used to indicate that the sides *AB* and *DC* of the above parallelogram are parallel to each other. Also, two arrowheads each have been used to indicate that the sides *BC* and *AD* are parallel.

16.1 Properties of Parallelograms

Do the following activities first to identify the properties of parallelograms.

Activity 1

Draw a parallelogram using a set square and a ruler. Name it $PQRS$ as indicated in the figure.



1. In the parallelogram $PQRS$ that you drew,

- measure the lengths of the sides PQ, QR, SR and PS.
- What can you say about the lengths of the pair of opposite sides PQ and SR, as well as the lengths of PS and QR?

It should be clear to you that $PQ = SR$ and $PS = QR$.

2. In the parallelogram that you drew above,

- measure the magnitudes of $P\hat{Q}R$, $Q\hat{P}S$, $P\hat{S}R$ and $Q\hat{R}S$.
- What can you say about the magnitudes of the pair of opposite angles $Q\hat{P}S$ and $Q\hat{R}S$, as well as the magnitudes of the pair $P\hat{S}R$ and $P\hat{Q}R$?

It should be clear to you that $Q\hat{P}S = Q\hat{R}S$ and that $P\hat{S}R = P\hat{Q}R$.

3. Now copy the parallelogram $PQRS$ onto a piece of tissue paper, and using it draw two copies of $PQRS$ and cut them out.

- In one parallelogram, draw the diagonal PR.
- Now cut this parallelogram along the diagonal and see whether the two triangles that you obtain coincide.

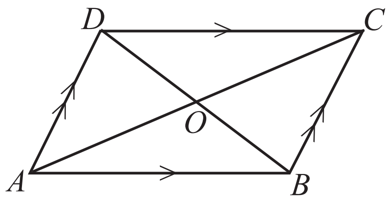
It should be clear to you that the two triangles coincide. Therefore, the areas of the two triangles are equal. Similarly, use the other parallelogram you cut out to verify that the areas of the two triangles obtained by cutting the parallelogram along the other diagonal are also equal.

According to the above activity,

The opposite sides of a parallelogram are equal, the opposite angles of a parallelogram are equal, and the area of the parallelogram is bisected by each diagonal.

Activity 2

As in activity 1, draw a parallelogram using a set square and a ruler. Name it *ABCD* as in the following figure.

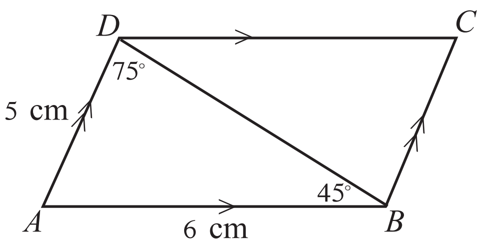


- Now draw the diagonals *AC* and *BD*. Name their intersection point as *O*.
- Measure the lengths of *OA, OB, OC* and *OD*.
- What can you say about the lengths of *OA* and *OC*?
- What can you say about the lengths of *OB* and *OD*?
- It should be clear to you that $OA = OC$ and that $OB = OD$.

Accordingly, it is clear that the diagonals of a parallelogram bisect each other.

Now let us consider how various elements of a parallelogram are found based on the data that is given.

Find the lengths of the sides and the magnitudes of the angles given below based on the data given in the parallelogram *ABCD*.



(i) Length of *BC*
(ii) Length of *DC*
(iii) Magnitude of $B\hat{A}D$
(iv) Magnitude of $B\hat{C}D$
(v) Magnitude of $A\hat{B}C$
(vi) Magnitude of $A\hat{D}C$

(i) Since the opposite sides of a parallelogram are equal, $AD = BC$ and $AB = CD$.

$\therefore BC = 5$ cm

(ii) $DC = 6$ cm

(iii) Since the sum of the interior angles of a triangle is 180°,

$$B\hat{A}D = 180° - 75° - 45°$$
$$= \underline{\underline{60°}}$$

(iv) Since the opposite angles of a parallelogram are equal,

$$B\hat{C}D = B\hat{A}D$$
$$\underline{\therefore B\hat{C}D = 60°}$$

(v) $A\hat{D}B = C\hat{B}D$ ($AD // BC$, Alternate angles)

$\therefore C\hat{B}D = 75°$

$A\hat{B}C = A\hat{B}D + C\hat{B}D$

$\therefore A\hat{B}C = 45° + 75°$

$\underline{\underline{= 120°}}$

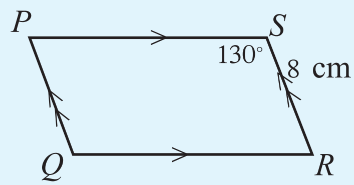
(vi) Since the opposite angles of a parallelogram are equal to each other,

$A\hat{B}C = A\hat{D}C$

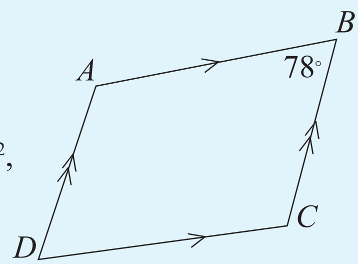
$\therefore \underline{\underline{A\hat{D}C = 120°}}$

Exercise 16.1

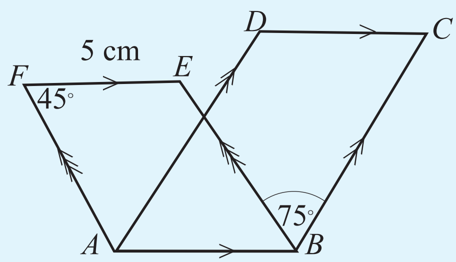
1. According to the information in the parallelogram $PQRS$,
  (i) find the length of the side PQ.
  (ii) find the magnitude of each of the angles $Q\hat{P}S$, $P\hat{Q}R$ and $Q\hat{R}S$.



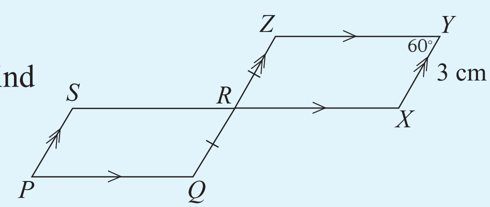
2. Based on the data in the figure,
  (i) find the magnitude of $B\hat{C}D$.
  (ii) If the area of the parallelogram $ABCD$ is 24 cm^2, what is the area of the triangle BCD?
  (iii) What is the area of the triangle ACD?



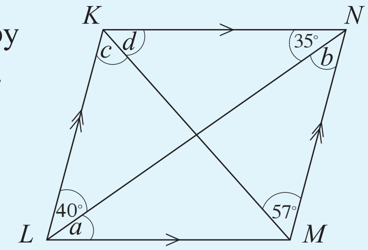
3. Based on the information in the figure,
  (i) find the length of DC.
  (ii) find the magnitude of $A\hat{B}E$.
  (iii) find the magnitude of $A\hat{D}C$.
  (iv) find the magnitude of $B\hat{C}D$.



4. Based on the information in the figure, find
  (i) the length of PS.
  (ii) the magnitude of $Q\hat{P}S$.
  (iii) the magnitude of $P\hat{Q}R$.



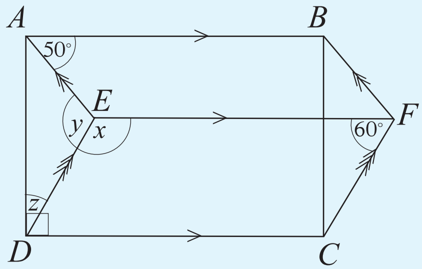
5. Find the magnitude of each of the angles denoted by *a, b, c* and *d*, based on the information in the figure.



6. Based on the information in the figure,

(i) write down two sides which are equal in length to *DC*.

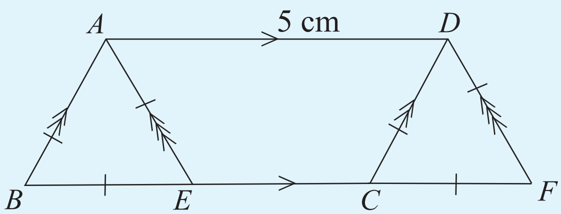
(ii) find the magnitudes of the angles denoted by *x, y* and *z*.



7. The figure depicts two parallelograms *ABCD* and *ADFE*. According to the information given in the figure,

(i) find the length of *BC*.

(ii) find the magnitude of each of the angles, $C\hat{F}D, A\hat{D}C$ and $E\hat{C}D$.



16.2 Properties of a parallelogram

Since the properties that we observed above in parallelograms are common to all parallelograms, we can present them in the form of a theorem as follows.

Theorem 1: In a parallelogram,

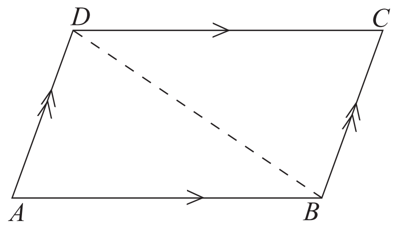
(i) opposite sides are equal.

(ii) opposite angles are equal.

(iii) the area of the parallelogram is bisected by each diagonal.

Theorem 2: In a parallelogram, the diagonals bisect each other.

Now let us consider how the first three parts of this theorem are proved formally.



Data: $ABCD$ is a parallelogram.

To be proved: (i) $AB = DC$ and $AD = BC$

(ii) $B\hat{A}D = B\hat{C}D$ and $A\hat{D}C = A\hat{B}C$

(iii) Area of $\Delta\, ABD$ = Area of $\Delta\, BCD$

Area of $\Delta\, ACD$ = Area of $\Delta\, ABC$

Construction: Join BD

We can obtain the three results by showing that the triangles ABD and BCD are congruent. Let us prove that the two triangles are congruent under the case AAS as follows.

Proof: In the triangles ABD and BCD,

$A\hat{D}B = C\hat{B}D$ (Alternate angles, $AD \,//\, BC$)

$A\hat{B}D = B\hat{D}C$ (Alternate angles, $AB \,//\, DC$)

BD is the common side.

$\therefore \Delta\, ABD \equiv \Delta\, BCD$ (AAS)

Since the corresponding elements of congruent triangles are equal,

$AB = DC$ and $AD = BC$.

Also $B\hat{A}D = B\hat{C}D$.

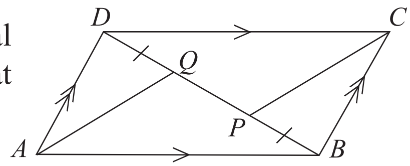
Area of $\Delta\, ABD$ = Area of $\Delta\, BCD$ (Since $\Delta\, ABD \equiv \Delta\, BCD$)

$\therefore$ The area of the parallelogram $ABCD$ is bisected by the diagonal BD.

The above facts can also be proved by using the diagonal AC.

Example 1

The points P and Q are marked on the diagonal BD of the parallelogram $ABCD$ such that $BP = DQ$. Prove that,



(i) $\Delta\, ADQ \equiv \Delta\, BPC$

(ii) $AQ // PC$

Proof: (i) In the triangles ADQ and $PBC,$

$DQ = BP$ (Given)

$AD = BC$ (The opposite sides of a parallelogram are equal)

$A\hat{D}Q = P\hat{B}C$ (Alternate angles, $AD \,//\, BC$)

$\therefore \underline{\underline{\Delta ADQ \equiv \Delta BPC}}$ (SAS)

(i) Since the triangles ADQ and PBC are congruent, the corresponding elements of the two triangles are equal.

$\therefore\ A\hat{Q}D = B\hat{P}C.$

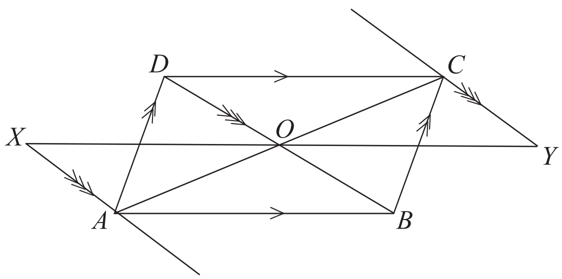
$\therefore A\hat{Q}P = Q\hat{P}C. \quad \left(A\hat{Q}D + A\hat{Q}P = B\hat{P}C + C\hat{P}Q = 180^0\right)$

But $A\hat{Q}P$ and $Q\hat{P}C$ are alternate angles.

Since a pair of alternate angles are equal, $AQ \,//\, PC.$

Example 2

According to the information given in the diagram, prove that O is the mid point of XY.



We need to prove that $XO = OY$. To do this, let us first show that the triangles AOX and COY are congruent.

Proof:

In the triangles AOX and COY,

$A\hat{X}O = C\hat{Y}O$ ($AX//CY$, Alternate angles)

$A\hat{O}X = C\hat{O}Y$ (Vertically opposite angles)

$AO = OC$ (Diagonals of a parallelogram bisect each other)

$\therefore\ \Delta AOX \equiv \Delta COY$ (A A S)

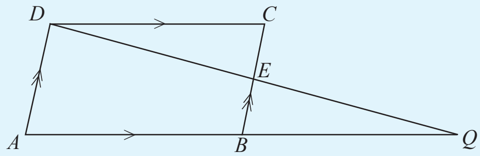
Corresponding elements of congruent triangles are equal.

$\therefore\ OX = OY$

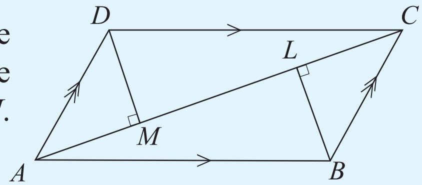
i.e., O is the mid point of XY.

Exercise 16.2

1. In the parallelogram $ABCD$, the midpoint of BC is E. DE and AB produced meet at Q. Prove that $AB = BQ$.



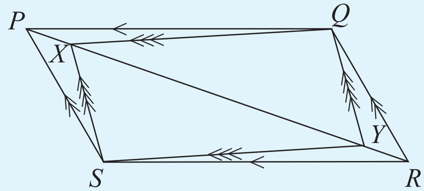
2. In the parallelogram $ABCD$ in the figure, the perpendiculars drawn from B and D to AC are BL and DM respectively. Show that $BL = DM$.



3. The figure illustrates two parallelograms $PQRS$ and $QYSX$. Prove that,

(i) $PX = RY$.

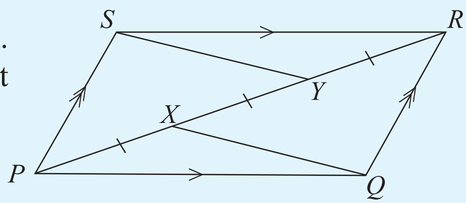
(ii) Area of $PSXQ$ = Area of $SRQY$



4. $PQRS$ in the figure is a parallelogram. The points X and Y lie on PR such that $PX = XY = YR$. Prove that,

(i) $QX = SY$,

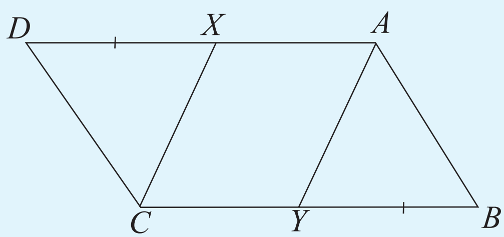
(ii) $QX // SY$.



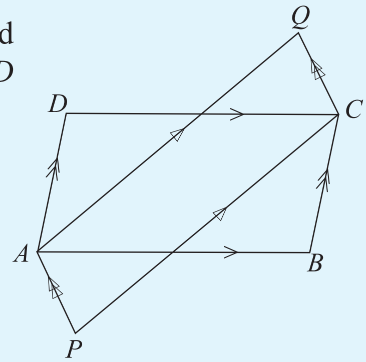
5. The figure illustrates a parallelogram $ABCD$. The points X and Y lie on the sides AD and BC respectively, such that $DX = BY$.

(i) Prove that $\Delta ABY \equiv \Delta DCX$

(ii) Show that $AY // XC$.



6. The figure depicts two parallelograms named *ABCD* and *APCQ*. Prove that, the lines *AC, BD* and *PQ* are concurrent.



7. In the parallelogram *PQRS,* the bisectors of $P\hat{S}R$ and $Q\hat{R}S$ meet at the point *X* on the side *PQ*.

(i) Draw a figure with the above information included in it.

(ii) Prove that $PX = PS$.

(iii) Prove that X is the mid-point of *PQ*.

(iv) Prove that $PQ = 2\,PS$.

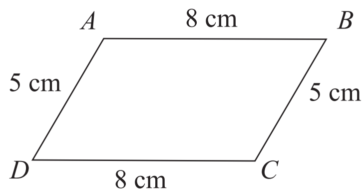
17 Parallelograms II

By studying this lesson you will be able to

identify the conditions that need to be satisfied for a quadrilateral to be a parallelogram.

Theorem: If the opposite sides of a quadrilateral are equal, then it is a parallelogram.

For example, in the given figure, $AB = DC$ and $AD = BC$.
Therefore, $ABCD$ is a parallelogram.

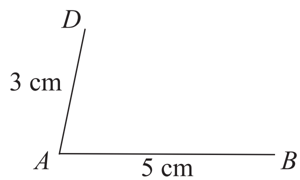


Let us engage in the following activity to establish the truth of the above theorem.

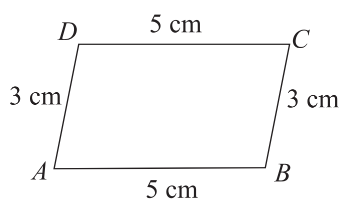
Activity 1

- Draw $D\hat{A}B$ as shown in the figure, such that the sides of the angle are of length 5 cm and 3 cm.

- As shown in the second figure, obtain the point C which is 3 cm from B and 5 cm from D.
- Now complete the quadrilateral $ABCD$.
- Then it can be seen that $AB = DC$ and $AD = BC$.



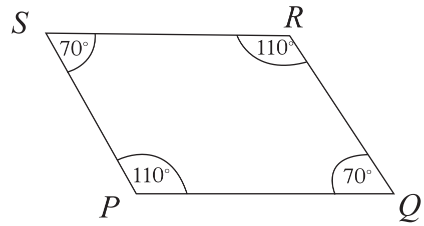
- By using a set square and a ruler or by measuring the angles and obtaining that the sum of a pair of allied angles is 180°, observe that the opposite sides of the quadrilateral are parallel. That is, obtain that $AB//DC$ and $AD//BC$.



It can be observed that in a quadrilateral with opposite side equal, the opposite sides are parallel too.

Theorem: If the opposite angles of a quadrilateral are equal, then it is a parallelogram.

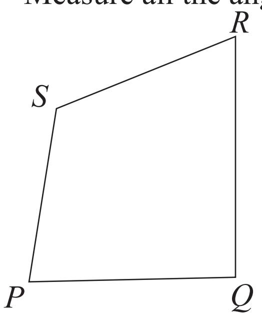
For example, in the given figure, since $P\hat{Q}R = P\hat{S}R$ and $Q\hat{R}S = Q\hat{P}S$, by the theorem, *PQRS* is a parallelogram.

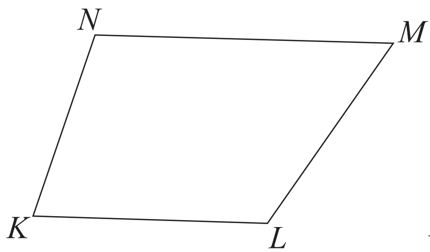


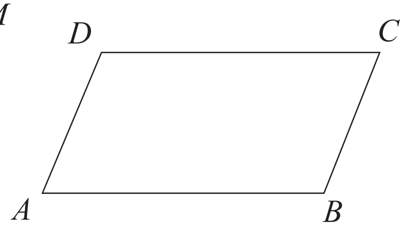
Let us engage in the following activity to establish the truth of the above theorem.

Activity 2

- Measure all the angles in each of the following quadrilaterals.





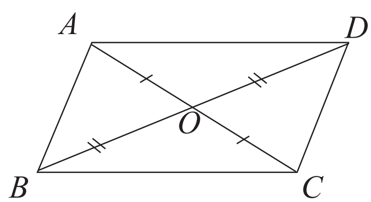


- Check whether the opposite pairs of angles of each quadrilateral are equal.
- Check whether the pairs of opposite sides are parallel in the quadrilateral with opposite angles equal. (See whether the sum of the allied angles is 180°.)

Accordingly, observe that in the quadrilateral with opposite angles equal, the opposite sides are parallel to each other.

Theorem: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

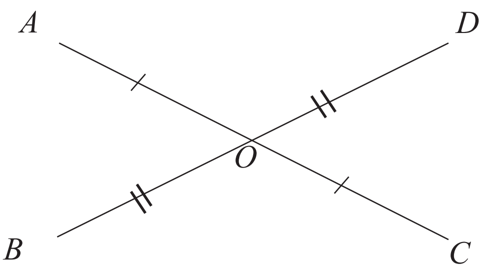
For example, in the quadrilateral *ABCD*, since $AO = OC$ and $BO = OD$, according to the theorem, *ABCD* is a parallelogram.

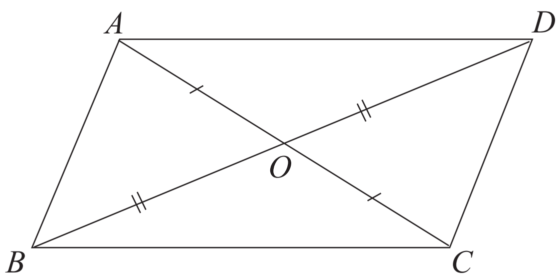


Engage in the following activity to establish the truth of the above theorem.

Activity 3

- To draw the quadrilateral $ABCD$ with diagonals AC and BD, first draw the diagonal AC and name its midpoint O.
- Now draw another straight line segment such that it intersects the diagonal AC at O. Mark the points B and D on this line segment such that $OB = OD$.



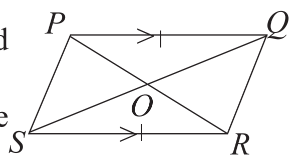


- Now complete the quadrilateral $ABCD$ as shown above.
- Check whether the sides AB and DC as well as the sides BC and AD in the quadrilateral $ABCD$ are parallel to each other by using a set square and a ruler or by measuring a pair of alternate angles.

Accordingly, it can be seen that if the diagonals of a quadrilateral bisect each other, then the opposite sides are parallel to each other.

Theorem: In a quadrilateral, if a pair of opposite sides is equal and parallel, then the quadrilateral is a parallelogram.

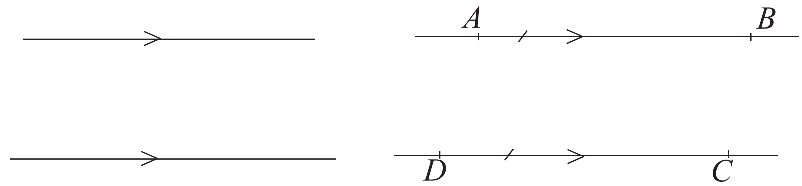
For example, in the quadrilateral $PQRS$, since $PQ = SR$ and $PQ//SR$, it is a parallelogram.



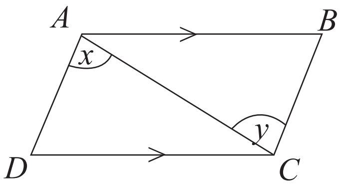
Engage in the following activity to establish the truth of the above theorem.

Activity 4

- Draw a pair of parallel lines using a set square and a ruler or by some other method.
- Mark two points A and B on one of these lines.
- Mark a length equal to AB on the other line as shown in the figure, and name it DC.



• Now complete the quadrilateral *ABCD* and draw the diagonal *AC* as shown in the figure.



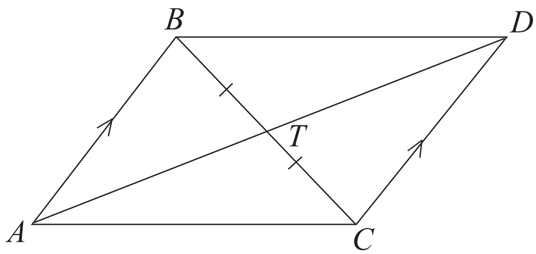
By measuring the pair of alternate angles x and y using a protractor, or by using a set square and ruler, observe that *AD* and *BC* are parallel to each other.

Accordingly, it can be seen that in a quadrilateral, if a pair of opposite sides is equal and parallel, then the quadrilateral is a parallelogram.

Now, by considering the following example, let us see how riders are proved using the above theorems.

Example 1

T is the midpoint of the side *BC* of the triangle *ABC*. The straight line drawn through *C*, parallel to *AB* meets *AT* produced at *D*. Prove that *ABDC* is a parallelogram.
First, let us draw the figure according to the given information.



We know that in a quadrilateral, if a pair of opposite sides is equal and parallel, then it is a parallelogram. Therefore, let us show that *ABDC* is a parallelogram by showing that a pair of opposite sides is equal and parallel. It is given that $AB//CD$. Let us show that $AB = CD$.

To obtain this, let us show that the triangles *ABT* and *CTD* are congruent.

In the triangles ABT and CTD

$BT = TC$ (Given)

$A\hat{T}B = C\hat{T}D$ (Vertically opposite angle)

$A\hat{B}T = T\hat{C}D$ $(AB // CD,$ Alternate angles$)$

$\therefore \Delta\, ABT \equiv \Delta\, CTD$ (A.A.S.)

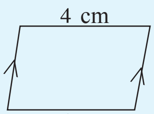
Since corresponding elements of congruent triangles are equal,

$AB = CD$.

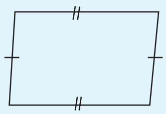
Since $AB = CD$ and $AB // CD$, we obtain that $ABDC$ is a parallelogram.

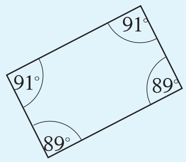
Exercise 17.1

1. From the following quadrilaterals, select the ones which can be concluded to be parallelograms, based on the given information.

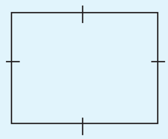


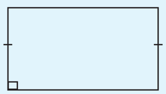
(*a*)

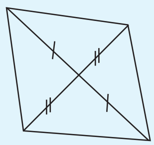

(*b*)

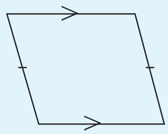


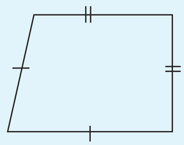
(*c*)

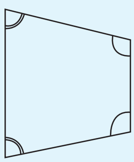

(*d*)

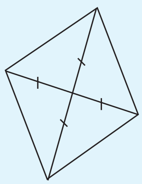

(*e*)

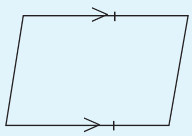

(*f*)

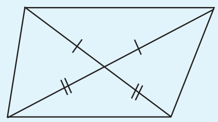

(*g*)

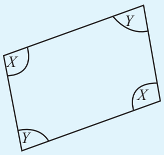

(*h*)


(*i*)

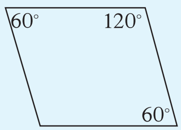

(*j*)


(*k*)

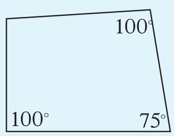

(*l*)



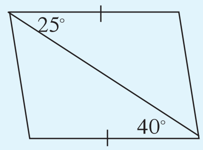
(*m*)



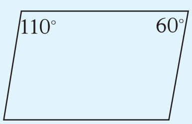
(*n*)



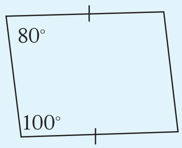
(*o*)



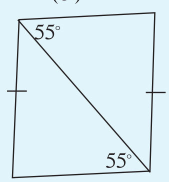
(*p*)



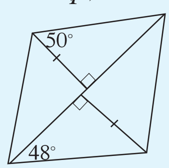
(*q*)



(*r*)



(*s*)

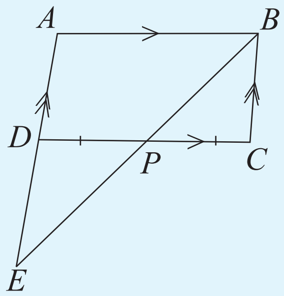


(*t*)

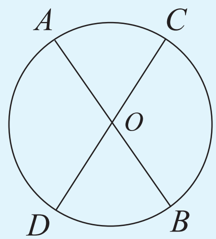
2. The midpoint of the side DC of the parallelogram $ABCD$ in the figure is P. AD and BP produced meet at E.

(i) Prove that $\Delta BCP \equiv \Delta DPE$

(ii) Prove that the quadrilateral $BCED$ is a parallelogram.



3. AB and CD are two diameters of the circle of centre O in the given figure. Prove that A, C, B and D are the vertices of a parallelogram.



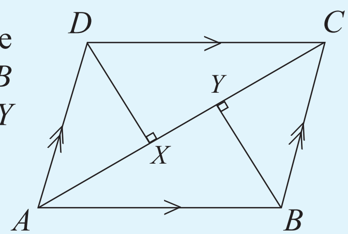
4. In the parallelogram $ABCD$ in the figure, the perpendiculars drawn from the points D and B to the diagonal AC meet AC at the points X and Y respectively.

Prove that,

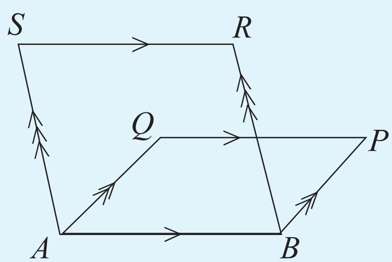
(i) $\Delta AXD \equiv \Delta BYC$,

(ii) $DX = BY$,

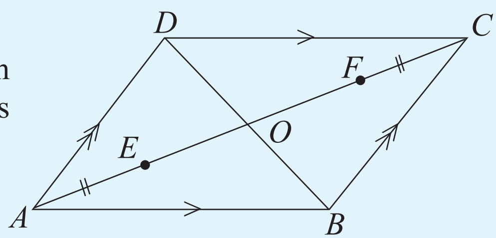
(iii) $BYDX$ is a parallelogram.



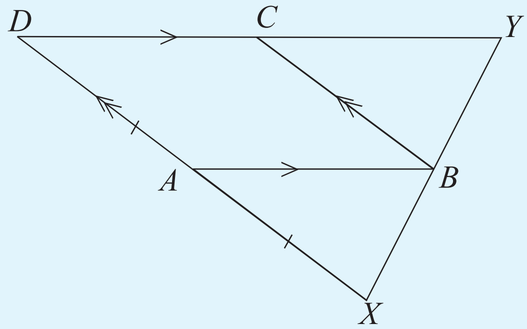
5. Two parallelograms $ABPQ$ and $ABRS$ are given in the figure. Prove that $QPRS$ is a parallelogram.



6. The figure illustrates a parallelogram $ABCD$. If $AE = FC$, prove that $EBFD$ is a parallelogram.



7. In the given figure, $ABCD$ is a parallelogram. The side DA has been produced up to X such that $DA = AX$. Also, DC produced and XB produced meet at Y. Prove that,
   (i) $AXBC$ is a parallelogram,
   (ii) $ABYC$ is a parallelogram,
   (iii) $DC = CY$.



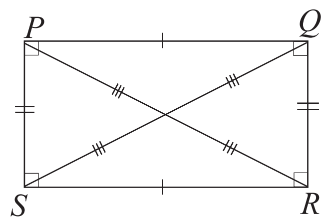
8. The diagonals of the parallelogram $PQRS$ intersect each other at O. The points M and T lie on PO and OR respectively and the points L and N lie on QO and OS respectively such that $PM = RT$ and $SN = QL$.
   Prove that,
   (i) $MO = OT$,
   (ii) $LMNT$ is a parallelogram,
   (iii) $MSTQ$ is a parallelogram.

Parallelograms with special properties

1. Rectangle

If one of the angles of a parallelogram is a right angle, then the other angles too are right angles. Such a parallelogram is a rectangle.

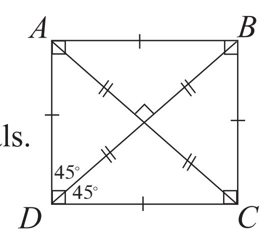
Apart from the properties of a parallelogram, a rectangle also has the following properties.
(i) All the vertex angles are right angles.
(ii) The diagonals are equal in length.



2. Square

A rectangle with two equal adjacent sides is a square. Apart from the properties of a rectangle, a square also has the following properties.

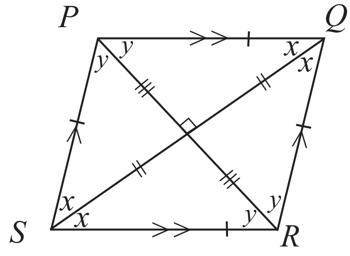
(i) All the sides are equal in length.
(ii) The diagonals bisect each other at right angles.
(iii) The angles at the vertices are bisected by the diagonals.



3. Rhombus

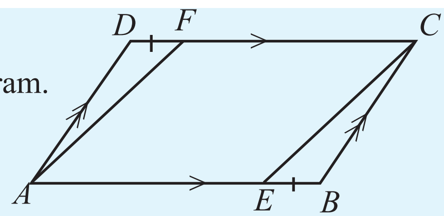
When two adjacent sides of a parallelogram are equal in length, then all four sides are equal in length. Such a parallelogram is called a rhombus. Apart from the properties of a parallelogram, a rhombus also has the following properties.

(i) All the sides are equal to each other.
(ii) The diagonals bisect each other at right angles.
(iii) The angles at the vertices are bisected by the diagonals.



Miscellaneous Exercise

1. $ABCD$ in the figure is a parallelogram. If $DF = EB$, prove that $AECF$ is a parallelogram.



2. In the triangle ABC, the bisector of $A\hat{B}C$ meets the side AC at the point P. The straight line through A drawn parallel to BC, meets BP produced at D such that $BP = PD$.

(i) Prove that $\Delta BCP \equiv \Delta ADP$
(ii) Show that $ABCD$ is a rhombus.
(iii) If AC = 18 cm and BD = 24 cm, find the length of AB.

3. In the triangle ABC, the midpoints of the sides AB and AC are respectively X and Y. The straight line drawn through C, parallel to AB, meets XY produced at Z. Prove that,

(i) $\Delta AXY \equiv \Delta CYZ$
(ii) $BCZX$ is a parallelogram.

4. In the parallelogram $ABCD$, the midpoints of the sides AB, BC, CD and AD are P, Q, R and S respectively. Prove that ,

(i) $\Delta ASP \equiv \Delta CQR$,
(ii) $PQRS$ is a parallelogram.

18 Sets

By studying this lesson, you will be able to

- identify methods of describing sets
- identify the regions in a Venn diagram when at most two sets are represented in the Venn diagram and solve problems using the formula relating the number of elements in these sets.

Set Notation

You have learnt three methods of describing sets before. They are

- the descriptive method
- the method of listing elements
- the Venn diagram method.

Let A be the set of all multiples of 3 between 1 and 10. Then, A can be denoted using the above three methods as follows:

- As a description,

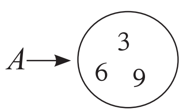
 A = {multiples of 3 between 1 and 10} or

 A = set of all multiples of 3 between 1 and 10.

- Listing the elements

$A = \{3, 6, 9\}$

- In a Venn diagram,



18.1 Set Builder Method

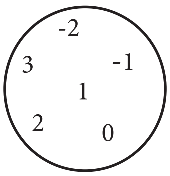
Set builder method is another method to describe sets. The set of all multiples of 3 between 1 and 10 can be denoted using the set builder method as follows:

$$A = \{x : x \text{ is a multiple of 3 and } 1 < x < 10\}.$$

The symbol x here is like a variable. You may use any symbol in place of x. The statement after the colon describes how the elements should be. There are several ways a set can be represented using the set builder method. For example, the following are three different ways of denoting the set $A = \{1, 2\}$ using the set builder method.

$A = \{ x : (x-1)(x-2) = 0 \}$
$A = \{ y : y \in \mathbb{Z} \text{ and } 1 \leq y \leq 2 \}$
$A = \{ n \in \mathbb{Z} \ : \ 0 < n \leq 2 \}$

Look at the following table for more examples on the set builder method.

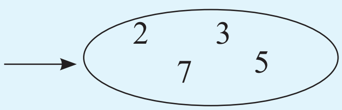
Set	Set builder form
A = {Positive integers less than 10}	$A = \{x : x \in \mathbb{Z}^+ \text{ and } 0 < x < 10\}$ or $A = \{x \in \mathbb{Z}^+ : 0 < x < 10\}$
$B = \{16, 25, 36, 49\}$	$B = \{ x : x$ is a perfect square. $16 \leq x \leq 49\}$
C 	$C = \{ x : x \in \mathbb{Z}, -2 \leq x \leq 3 \}$ or $C = \{x \in \mathbb{Z} : -2 \leq x \leq 3 \}$

Exercise 18.1

1. Describe the set of all positive integers from10 to 15 using
(i) the descriptive method
(ii) the listing method
(iii) a Venn diagram
(iv) the set builder method.

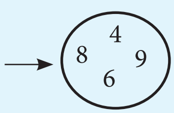
2. Describe each of the following sets using the descriptive method.
(i) A = {3, 6, 9, 12}

(ii) B



(iii) $C = \{ x : x$ is a perfect square and $10 < x < 100 \}$

3. Describe each of the following sets using the descriptive method.
i. X = {All letters in the word "ANURADHAPURA"}
ii. $A = \{x : x$ is a prime number and $10 < x < 20\}$
iii. B



4. Describe each of the following sets using a Venn diagram.

(i) $A = \{7, 14, 21, 28\}$

(ii) $B = \{\text{Vowels in the English alphabet}\}$

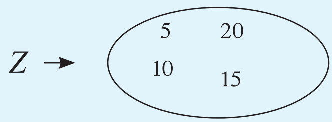
(iii) $Y = \{x \in \mathbb{Z} \ : x^2 = 4\}$

5. Describe each of the following sets using the set builder method.

(i) $X = \{\text{Odd numbers between 1 and 10}\}$

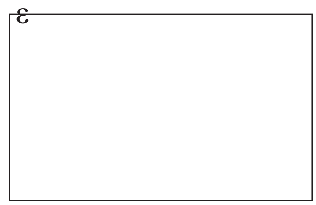
(ii) $Y = \{0, 1, 2, 3\}$

(iii)



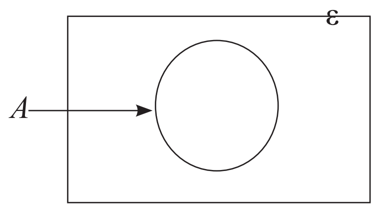
18. 2 Regions in a Venn Diagram

In Venn diagrams, the universal set is represented by a rectangular region and is denoted by ε.

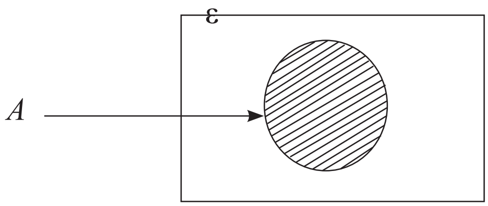


The subsets of the universal set are represented by circular (or elliptical) regions. These subsets give rise to several regions in the Venn diagram that represents the universal set. Now let us investigate these regions.

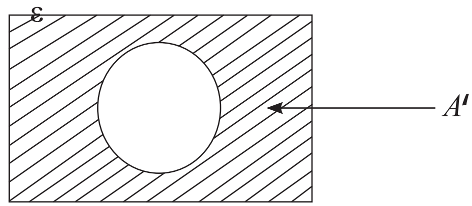
1. When there is one subset represented in the universal set.



The subset A divides the universal set into two regions which are shaded in the next figure.The subsets corresponding to these regions can be denoted by A and A' using set notation.



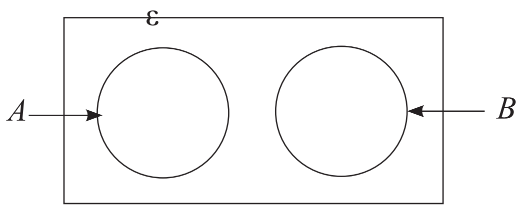
A



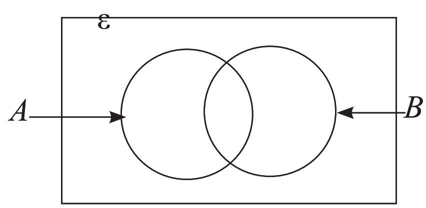
Complement of $A = A'$

2. When two subsets are represented in the Venn diagram

Let A and B be the two subsets. When there are no common elements in A and B, that is, when $A \cap B = \emptyset$, and when there are common elements in A and B, that is when $A \cap B \neq \emptyset$, the relevant Venn diagrams are shown below.



$A \cap B = \emptyset$



$A \cap B \neq \emptyset$

Before investigating the regions let us recall the following definitions.

A' = Set of elements not in A

$A \cap B$ = Set of elements belonging to both A and B

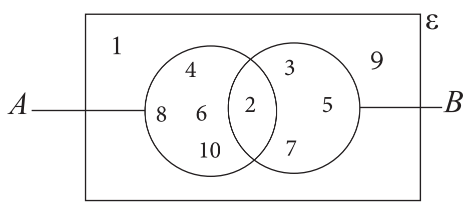
$A \cup B$ = Set of elements belonging to A or B (or both)

As an example, let us take $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 4, 6, 8, 10\}$ and

$B = \{2, 3, 5, 7\}$

Then, we can represent the above information in a Venn diagram as given below.



According to the given information, it is clear that

$A' = \{1, 3, 5, 7, 9\}$

$A \cap B = \{2\}$ and

$A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10\}$

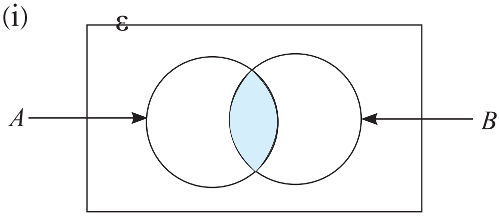
Also, when we observe the Venn diagram we can see that

$(A \cup B)' = \{1, 9\}$ and

$(A \cap B)' = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$

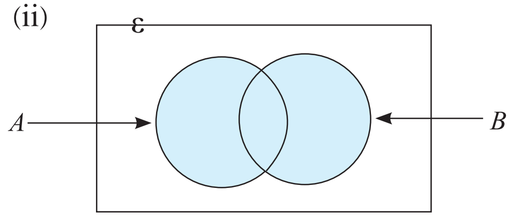
Two subsets of a universal set represented in a Venn diagram give rise to several regions. Some such regions and the way they can be written down using set complement, set intersection and set union are shown below.

(i)



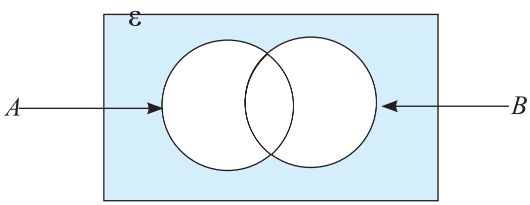
$A \cap B$

(ii)



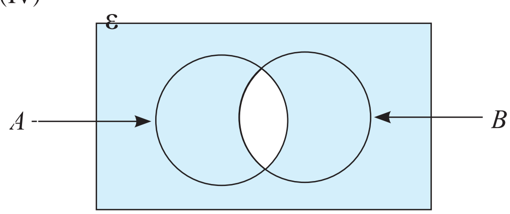
$A \cup B$

(iii)



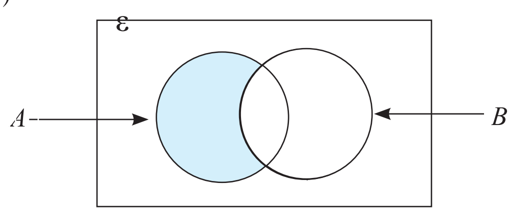
$(A \cup B)'$

(iv)$'$



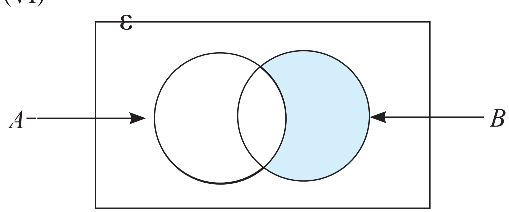
$(A \cap B)'$

(v)



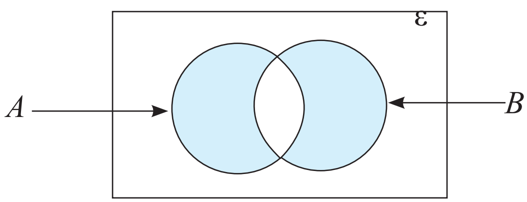
$A \cap B'$

(vi)



$A' \cap B$

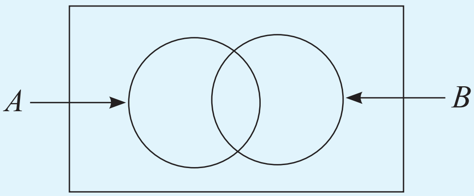
For the example we discussed above, we have that
$A \cap B' = \{4, 6, 8, 10\}$ and $A' \cap B = \{3, 5, 7\}$
Also, the Venn diagram given below is obtained by the Venn diagrams (v) and (vi)



$\left(A \cap B'\right) \cup \left(A' \cap B\right)$

Exercise 18.2

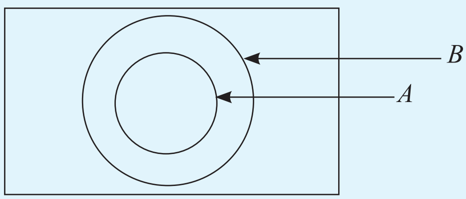
1. *a.* Shade the region denoted by each of the following sets in separate Venn diagrams.



(i) $A'\cap B'$ (ii) $A' \cup B'$
(iii) $(A\cap B)'$ (iv) $(A\cup B)'$
(v) $(A\cap \mathrm{B}) \cup (A\cup B)'$ (vi) $(A\cap B')'$
(vii) $(A'\cap B)'$ (viii) $(A \cup B')'$
(ix) $(A' \cup B)'$

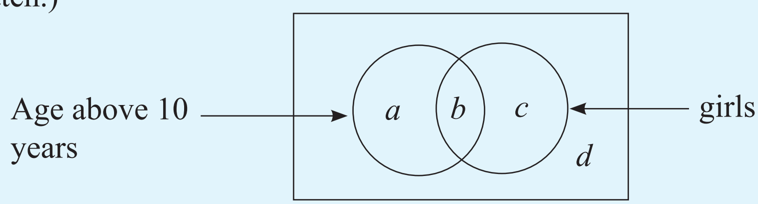
b. By investigating the regions you shades in part (a) above, find all pairs of equal sets.

2. Shown below is the Venn diagram of two sets A and B where $A\subset B$. In 6 copies of this Venn diagram, shade each of the given 6 regions.



(i) $A\cap B$ (ii) $A\cup B$
(iii) $A'\cap B$ (iv) $A'\cup B$
(v) $(A\cup B)'$ (vi) $(A'\cap B)'$

3. The information on the children in a society is shown in the following Venn diagram. (The letters *a, b, c* and *d* indicate the regions in which the letters are written.)



Describe in words the regions indicated by each of the letters *a, b, c* and *d*. For example, the boys whose ages are above 10 years are indicated by *a*.

4. Let $\varepsilon = \{1, 2, 3, 4, 5, 6, 7\}$

$A'\cap B = \{4, 5\}$

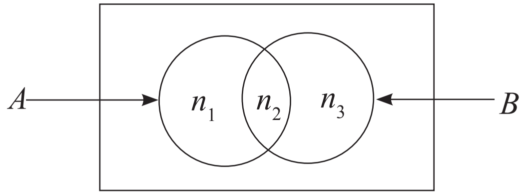
$A\cap B = \{3\}$

$(A\cup B)' = \{1\}$

Include the above information in a suitable Venn diagram and hence find A, $A\cup B$ and $B'\cap A$.

18.3 Relationship between the numbers of elements in two sets

Shown below is a Venn diagram with two subsets A and B of the universal set such that $A \cap B \neq \emptyset$. Here n_1, n_2 and n_3 denote the number of elements in the respective regions. (Though it is expected to write the elements inside the regions, we have written down the number of elements for ease.)



Let us the denote the number of elements belonging to the subset A by n(A) etc. Then, from the figure, we see that

$$n(A) = n_1 + n_2$$
$$n(B) = n_2 + n_3$$
$$n(A \cap B) = n_2$$
$$n(A \cup B) = n_1 + n_2 + n_3$$

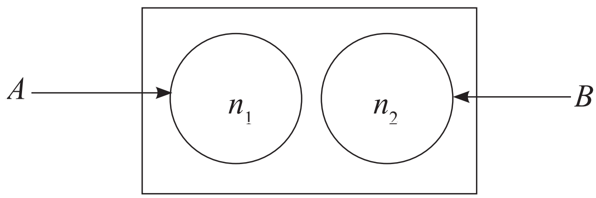
Now, we write, $n(A \cup B) = \underbrace{n_1 + n_2} + \underbrace{n_2 + n_3} - n_2$

$$= n(A) + n(B) - n(A \cap B)$$

Thus, we obtain the formula

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

The Venn diagram when the two subsets A and B are disjoint (i.e.,when $A \cap B = \emptyset$) is shown below.



In this case, we see that

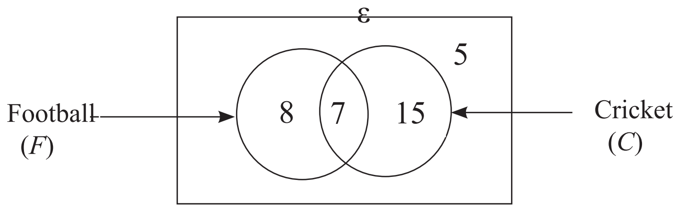
$$n(A) = n_1$$
$$n(B) = n_2$$
$$n(A \cup B) = n_1 + n_2$$

Thus, when $A \cap B = \emptyset$ we obtain the formula

$$\boxed{n(A \cup B) = n(A) + n(B)}$$

Example 1

Information on the numbers of students who participate in football and cricket are shown in the following diagram (In this case too, the numbers in the regions indicate the numbers of students in the regions).

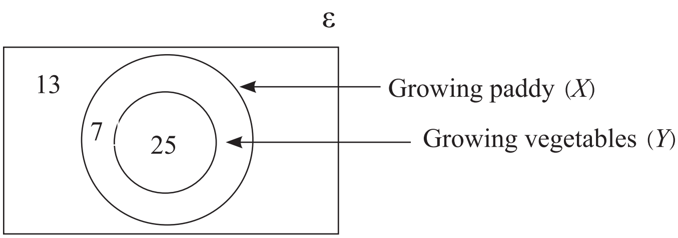


How many students are

1. participating in football? $n(F) = 8 + 7 = 15$
2. participating in cricket? $n(C) = 7 + 15 = 22$
3. participating in both sports (both football and cricket)? $n(F \cap C) = 7$
4. participating in cricket only? $n(C \cap F') = 15$
5. participating in football only? $n(F \cap C') = 8$
6. participating in football or cricket (or both)? $n(F \cup C) = 8 + 7 + 15 = 30$
7. not participating in football? $n(F') = 15 + 5 = 20$
8. not participating in cricket? $n(C') = 8 + 5 = 13$
9. participating in exactly in one of the two sports? $n\{(F \cap C') \cup (F' \cap C)\}$ $= 8 + 15 = 23$
10. Participating in neither of the two sports? $n(F \cup C)' = 5$

Example 2

The information obtained through a survey from a set of farmers in a certain village about the kind of crop they grow in their farms is shown in the following Venn diagram.



(a) How many farmers are growing

1. vegetables? $n\,(Y) = 25$
2. paddy? $n\,(X) = 7 + 25 = 32$
3. only paddy? $n\,(Y' \cap X) = 7$
4. only vegetables? $n(X' \cap Y) = 0$
5. both vegetables and paddy? $n\,(X \cap Y) = 25$
6. vegetables or paddy? $n\,(X \cup Y) = 7 + 25 = 32$
7. neither of the two crops? $n\,(X \cup Y)' = 13$

(b) How many farmers were surveyed?

$$n\,(\varepsilon) = 13 + 7 + 25 = 45$$

Exercise 18.3

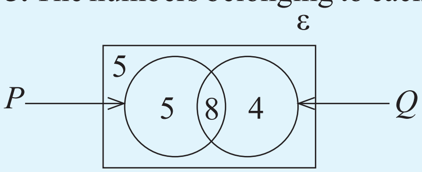
1. Find $n(A \cup B)$ if $n\,(A) = 35$, $n\,(B) = 24$, $n(A \cap B) = 11$.

2. Find $n(Y)$ if $n\,(X) = 16$, $n\,(X \cap Y) = 5$, $n\,(X \cup Y) = 29$.

3. Find $n(P \cap Q)$ if $n\,(P) = 70$, $n\,(Q) = 55$, $n(P \cup Q) = 110$.

4. Find $n(A \cap B)$ if $n(A) = 19$, $n(B) = 16$, $n\,(A \cup B) = 35$. What is special about the sets A and B?

5. The numbers belonging to each region is indicated in the following Venn diagram.



Find $n(P)$, $n(Q)$, $n(P \cap Q)$, $n(P \cup Q)$ and verify the formula

$$n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$$

6. In a sports club of 60 members, 30 play cricket, 25 play elle and 15 play both.
(i) Include this information in a Venn diagram.
(ii) How many members do not play either of the above two sports?
(iii) How many members play elle, but not cricket?

7. Out of 30 who attended a party, 12 ate Kavum, 20 ate kokis and 5 did not eat either of these. Represent this information in a Venn diagram.
(i) Find the number that ate both these food items.
(ii) How many of them ate only one of these two food items?

8. Out of 40 students in a class, 21 do not like to listen to the radio, 10 do not like to watch TV, and 8 do not like either of the two activities.
(i) Represent this information in a Venn diagram.
(ii) How many of the students like both activities?
(iii) How many of the students like only to watch TV?

9. From a group of 35 children who participated in a game, 19 were boys and 17 were above 15 years. 6 of the girls who participated were aged below 15 years.
   (i) Represent this information in a Venn diagram.
   (ii) How many boys were above 15 years?

10. From a group of 80 who went on an trip, 50% were wearing hats but were not wearing wrist watches. 40% of the group were wearing wrist watches, out of which 30 were also wearing hats.
   (i) Represent this information in a Venn diagram.
   (ii) How many of the group were not wearing either of the above mentioned two items?

11. In a certain village, 36 farmers grow potatoes and 18 farmers grow only chillies. Furthermore, the number of those who do not grow potatoes is 24 and the number of those who do not grow chillies is 26.
   (i) Represent this information in a Venn diagram.
   (ii) How many of the farmers grow neither of the two crops?
   (iii) How many of the farmers grow both crops?

12. A survey conducted in a certain village on 80 randomly selected households revealed the following information.
   * 5 households had neither water supply nor electricity.
   * 30 households had no electricity.

The number of households having only water supply was 7 more than the number that had both water supply and electricity.
   (i) Represent this information in a Venn diagram.
   (ii) How many households had both water supply and electricity?
   (iii) How many households had electricity but not water supply?
   (iv) How many households had no water supply?
   (v) How many households has exactly one of these facilities?