MATHEMATICS

Grade 10

Part - II

Educational Publications Department

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Contents

19 Logarithms I

By studying this lesson you will be able to

simplify numerical expressions using logarithms.

Indices

The result of multiplying 2 by itself four times is expressed as 2^4.
That is, $2 \times 2 \times 2 \times 2 = 2^4$.

Therefore, the value of 2^4 is 16.

Similarly, $3 \times 3 \times 3 = 3^3 = 27$.

Expressions such as 2^4 and 3^3 are called powers. The base of 2^4 is 2 while the index is 4. Do the following exercise to review the facts that you have learnt so far regarding indices.

Review Exercise

1. For each term in box A select the term in box B which is equal to it and join them together.

$a \times a$	5^{-1}
a^{-2}	$a \times a \times b \times b$
$5 \times 5 \times 5 \times 5 \times 5$	x
a	5^5
$a^2 b^2$	$\frac{a}{b}$
5^1	a^2
$\frac{1}{5}$	$\frac{1}{a^2}$
$5 \times 5^{\circ}$	1
x°	a^1
$5^3 \times 5^2$	5
ab^{-1}	

2. Fill in the blanks.

(i) $\frac{1}{8}=\frac{1}{2^{....}}=2^{....}$ (ii) $\frac{1}{100}=\frac{1}{10^{....}}=10^{....}$ (iii) $\frac{1}{125}=\frac{1}{5^{....}}=5^{....}$

(iv) $\frac{1}{81}=\frac{1}{3^{....}}=3^{....}$ (v) $0.01=\frac{1}{....}=\frac{1}{10^{....}}=....$ (vi) $0.001=\frac{1}{....}=\frac{1}{....}=....$

3. Simplify each of the following expressions.

(i) $a^2\times a^3$ (ii) $x^5\times x$ (iii) $\frac{x^5\times x^7}{x^{11}}$

(iv) $\frac{a^3\times a^5}{a^2\times a^6}$ (v) $\frac{p^3\times p^{-1}}{p}$ (vi) $\frac{x^{\circ}\times x^5}{x}$

4. Simplify each of the following expressions and find its value.

(i) $2^2\times 2^3$ (ii) $\frac{3^7}{3^4}$ (iii) $\frac{3^2\times 3^8}{3^5}$

(iv) $\frac{5^3\times 5^0}{5}$ (v) $\frac{10^2\times 10^3}{10\times 10^4}$ (vi) $\frac{2^5\times 2^3}{2^6\times 2^2}$

19.1 Logarithms

Let us now consider how simplifications are facilitated by using the properties of indices. To do this, let us use the table of powers of 2 given below.

Power of 2	2°	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Value	1	2	4	8	16	32	64	128	256	512	1024

Let us consider how the value of $\frac{64\times 512}{128}$ is found using this table.

First, let us write these numbers as powers of the same base.

$$\frac{64\times 512}{128}=\frac{2^6\times 2^9}{2^7}\quad\text{(According to the table)}$$

$$=2^{6+9-7}\quad\text{(Using the laws of indices)}$$

$$=2^8$$

$$=\underline{\underline{256}}\quad\text{(According to the table)}$$

It can be seen that the above simplification was done easily and concisely by using the laws of indices. In the above example it was possible to write the numbers as powers of 2. Any expression containing the multiplication and division of numbers can easily be simplified using the logarithm tables. John Napier (1550 A.D. -1617 A.D.) a Scottish mathematician is bestowed with the honour of introducing the logarithm tables first. Briggs another mathematician who was a contemporary of Napier developed logarithms further. Although the use of the logarithm tables has reduced in recent years due to the widespread use of calculators, it is very important to learn the mathematical concepts related to logarithms.

Index form and logarithm form

We know that $2^3 = 8$. Here 8 is expressed as a power with base 2 and index 3. Such expressions are defined as " index form". This can also be expressed as the logarithm of 8 to base 2 is 3. This is written as $\log_2 8 = 3$, which is defined as the "logarithm form". It must be clear to you that the same statement is written in index form and in logarithm form.

Accordingly, since $2^3 = 8$, we also have $\log_2 8 = 3$.
Similarly, $\log_2 8 = 3$, means $2^3 = 8$.

Let us consider several other examples.
- Since $3^2 = 9$, the logarithm of 9 to base 3 is 2. That is, $\log_3 9 = 2$.
- Since $5^1 = 5$, the logarithm of 5 to base 5 is 1. That is, $\log_5 5 = 1$.
- Since $10^3 = 1000$, the logarithm of 1000 to base 10 is 3. That is, $\log_{10} 1000 = 3$.

In general, for a positive number a,

$\boxed{\text{If } a^x = N, \text{ then } \log_a N = x}$ $\boxed{\text{If } \log_a N = x, \text{ then } a^x = N}$

$a^x = N$ is considered as the index form and $\log_a N = x$ is considered as the logarithm form. Here a and N take only positive values. (Since any power of a positive number is positive, in the above relationship, when a is positive, N is also positive.) Accordingly, when considering logarithms, the base always takes a positive value.

Let us now identify several properties of logarithms.

(i) For any base, the logarithm of the base value itself is 1.

That is, $\log_a a = 1$.
This is because $a^1 = a$.
For example, $\log_2 2 = 1$ and $\log_{10} 10 = 1$.

(ii) The logarithm of 1 to any base (other than 1) is 0.

That is, $\log_a 1 = 0$.
This is because $a^0 = 1$.
For example, $\log_2 1 = 0$ and $\log_{10} 1 = 0$.

We observe that so far we have obtained positive values for the logarithms. However, logarithms can also be negative. The logarithm of a number between 0 and 1 is always negative.
For example,

$$\text{since } 2^{-3} = \frac{1}{2^3} = \frac{1}{8}, \text{ we obtain, } \log_2\left(\frac{1}{8}\right) = -3,$$

$$\text{since } 0.01 = \frac{1}{100} = 10^{-2}, \text{ we obtain, } \log_{10}\left(\frac{1}{100}\right) = -2,$$

$$\text{and since } 0.5 = \frac{5}{10} = 2^{-1}, \text{ we obtain, } \log_2(0.5) = -1,$$

Now let us consider how equations involving logarithms are solved.

Example 1

Find the value represented by x in each of the following.

(i) $\log_2 64 = x$ (ii) $\log_x 81 = 4$ (iii) $\log_5 x = 2$

i) $\log_2 64 = x$

$2^x = 64$ (In index form)

$2^x = 2^6$

$\therefore \underline{\underline{x = 6}}$

(ii) $\log_x 81 = 4$

$x^4 = 81$

$x^4 = 3^4$

$x = \pm 3$

$x = +3$ or -3

Since the base of a logarithm cannot be negative

$\underline{\underline{x = +3}}$

(iii) $\log_5 x = 2$

$x = 5^2$

$\underline{\underline{x = 25}}$

Exercise 19.1

1. Write each of the following expressions in logarithm form.
(i) The logarithm of 32 to base 2 is 5.
(ii) The logartihm of 1000 to base 10 is 3.
(iii) The logarithm of x to base 2 is y.
(iv) The logarithm of q to base p is r.
(v) The logarithm of r to base q is p.

2. Express each of the following in index form.

(i) $\log_5 125 = 3$ (ii) $\log_{10} 100\,000 = 5$ (iii) $\log_a x = y$

(iv) $\log_p a = q$ (v) $\log_a 1 = 0$ (vi) $\log_m m = 1$

3. Express each of the following in logarithm form.

(i) $2^8 = 256$ (ii) $10^4 = 10000$ (iii) $7^3 = 343$

(iv) $20^2 = 400$ (v) $a^x = y$ (vi) $p^a = q$

4. Solve each of the following equations.

(i) $\log_3 243 = x$ (ii) $\log_{10} 100 = x$ (iii) $\log_6 216 = x$

(iv) $\log_x 25 = 2$ (v) $\log_x 64 = 6$ (vi) $\log_x 10 = 1$

(vii) $\log_3 x = 2$ (viii) $\log_{10} x = 4$ (ix) $\log_8 x = 2$

5. (i) Write 64 as a power of four different bases.

(ii) Find four distinct pairs of values for x and y such that $\log_x 64 = y$.

19.2 Laws of logarithms

Let us recall again how the value of 16 × 32 can be obtained by writing it in index form.

$$\begin{aligned} 16 \times 32 &= 2^4 \times 2^5 \\ &= 2^{4+5} \\ &= 2^9 \end{aligned}$$

Let us now consider $16 \times 32 = 2^{4+5}$.

Let us convert this to logarithm form.

$$\begin{aligned} 16 \times 32 &= 2^{4+5} \quad \text{(Index form)} \\ \therefore \log_2(16 \times 32) &= 4 + 5 \quad \text{(Logarithm form)} \\ &= \log_2 16 + \log_2 32 \quad (\text{Since } 4 = \log_2 16 \text{ and } 5 = \log_2 32) \end{aligned}$$

Similarly, since $27 \times 81 = 3^3 \times 3^4 = 3^{3+4}$,

$$\begin{aligned} \log_3(27 \times 81) &= 3 + 4. \\ &= \log_3 27 + \log_3 81 \quad (\text{Since } 3 = \log_3 27 \text{ and } 4 = \log_3 81) \end{aligned}$$

In the same manner we can write,

$$\log_{10}(10 \times 100) = \log_{10} 10 + \log_{10} 100 \text{ and}$$
$$\log_5(125 \times 25) = \log_5 125 + \log_5 25.$$

As seen above, when multiplying powers, an important fact about the behaviour of logarithms is highlighted. This is true in general for any product of powers and is expressed as follows.

$$\boxed{\log_a(mn) = \log_a m + \log_a n}$$

This statement can also be expressed as "the logarithm of a product is equal to the sum of the logarithms".

Such a formula exists for the logarithm of a quotient too. Let us investigate this now.

Let us consider the following example.

Let us recall how the value of 128 ÷ 16 is obtained by converting it into index form.

$$\frac{128}{16} = \frac{2^7}{2^4} \quad \text{(Representing as powers of 2)}$$

$$= 2^{7-4} \quad \text{(Applying the laws of indices)}$$

$$\therefore \log_2\left(\frac{128}{16}\right) = 7 - 4 \quad \text{(Writing in logarithm form)}$$

Now, since $128 = 2^7$, we obtain $7 = \log_2 128$,

and since $16 = 2^4$, we obtain $4 = \log_2 16$.

Accordingly, $\log_2\left(\frac{128}{16}\right) = 7 - 4 = \log_2 128 - \log_2 16$.

Similarly, $\log_5(125 \div 5) = \log_5 125 - \log_5 5$

and $\log_{10}\left(\frac{1000}{100}\right) = \log_{10} 1000 - \log_{10} 100$.

As seen above, when dividing powers, an important fact about the behaviour of logarithms is highlighted. This is true in general for any quotient of powers and is expressed as follows.

$$\boxed{\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n}$$

These properties are called "Laws of Logarithms."

Now let us learn how to solve problems using these laws of logarithms by considering the following examples.

Example 1

1. Find the value of each of the expressions given below.

   (i) $\log_4 32 + \log_4 2$ (ii) $\log_5 15 - \log_5 3$

(i) $\log_4 32 + \log_4 2 = \log_4(32 \times 2)$

$= \log_4 64$

$= \underline{\underline{3}}$ $\quad (64 = 4^3)$

(ii) $\log_5 15 - \log_5 3 = \log_5\left(\frac{15}{3}\right)$

$= \log_5 5$

$= \underline{\underline{1}}$

Example 2

Evaluate

$$\log_{10}25 + \log_{10}8 - \log_{10}2$$

$$\log_{10} 25 + \log_{10} 8 - \log_{10} 2 = \log_{10}\left(\frac{25 \times \cancel{8}^{4}}{\cancel{2}^{1}}\right)$$

$$= \log_{10} 100$$

$$= \underline{\underline{2}}$$

$\log_{10} 100 = x$

$10^x = 100$

$10^x = 10^2$

$\therefore x = 2$

Example 3

Express in terms of $\log_a 2$ and $\log_a 3$.

(i) $\log_a 6$ (ii) $\log_a 18$

(i) $6 = 2 \times 3$

$\log_a 6 = \log_a(2 \times 3)$

$= \underline{\underline{\log_a 2 + \log_a 3}}$

(ii) $18 = 2 \times 3 \times 3$

$\log_a 18 = \log_a(2 \times 3 \times 3)$

$= \log_a 2 + \log_a 3 + \log_a 3$

$= \underline{\underline{\log_a 2 + 2\log_a 3}}$

Example 4

Solve:

$$\log_a 5 + \log_a x = \log_a 3 + \log_a 10 - \log_a 2$$

$$\log_a 5 + \log_a x = \log_a 3 + \log_a 10 - \log_a 2$$

$$\Rightarrow \log_a(5 \times x) = \log_a\left(\frac{3 \times 10}{2}\right)$$

$$\therefore \quad 5x = \frac{3 \times \cancel{10}^{5}}{\cancel{2}_{1}}$$

$$\Rightarrow 5x = 15$$

$$\underline{\underline{x = 3}}$$

Now, do the following exercise by applying the laws of logarithms.

Exercise 19.2

1. Simplify and express the answer as a single logarithm.

(i) $\log_2 10 + \log_2 5$ (ii) $\log_3 8 + \log_3 5$ (iii) $\log_2 7 + \log_2 3 + \log_2 5$

(iv) $\log_6 20 - \log_6 4$ (v) $\log_a 10 - \log_a 2 - \log_a 5$ (vi) $\log_{10} 6 + \log_{10} 2 - \log_{10} 3$

2. Find the value of each of the following expressions.

(i) $\log_2 4 + \log_2 8$ (ii) $\log_3 27 - \log_3 3$

(iii) $\log_{10} 20 + \log_{10} 2 - \log_{10} 4$ (iv) $\log_2 80 - \log_2 15 + \log_2 12$

(v) $\log_{10} 20 + \log_{10} 10 - \log_{10} 2$ (vi) $\log_5 20 + \log_5 4 - \log_5 16$

3. Write the following expressions in terms of $\log_a 5$ and $\log_a 3$.

(i) $\log_a 15$ (ii) $\log\left(\frac{5}{3}\right)$ (iii) $\log_a\left(\frac{5}{3}\right)$

(iv) $\log_a 45$ (v) $\log_a 75$ (vi) $\log_a(225)$

4. Solve the following equations.

(i) $\log_2 5 + \log_2 3 = \log_2 x$ (ii) $\log_a 10 + \log_a x = \log_a 30$

(iii) $\log_3 20 + \log_3 x = \log_3 4 + \log_3 10$ (iv) $\log_a 15 - \log_a 3 = \log_a x$

(v) $\log_{10} 8 + \log_{10} x - \log_{10} 2 = \log_{10} 12$ (vi) $\log_5 24 - \log_5 4 = \log_5 2 + \log_5 x$

Summary

$$\log_a(mn) = \log_a m + \log_a n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a a = 1 \text{ and } \log_a 1 = 0 \ (a \neq 1)$$

Miscellaneous Exercise

1. Evaluate the following.

(i) $\log_3 27 + \log_2 8$ (ii) $\log_3 243 - \log_3 27$ (iii) $\log_2 16 \times \log_3 9$

(iv) $\dfrac{\log_{10} 10}{\log_2 32}$ (v) $\log_a 5 + \log_a 3 - \log_a 15$

2. If $\log_2 24 = x$, express $\log_2 48$ in terms of x.

3. Verify each of the following equations.

(i) $\log_a\left(\frac{9}{10}\right)+\log_a\left(\frac{25}{81}\right)=\log_a 5-\log_a 18$

(ii) $\log_5 1+\log_5 20-\log_5 8+\log_5 2=1$

(iii) $\log_{10} 2+\log_{10} 3-1=\log_{10} 0.6$

4. Evaluate.

(i) $\log_{10} 200 + \log_{10}300 - \log_{10}60$

(ii) $\log_{10}\left(\frac{12}{5}\right)+\log_{10}\left(\frac{25}{21}\right)-\log_{10}\left(\frac{2}{7}\right)$

5. Solve.

(i) $\log_{10}x - \log_{10}2 = \log_{10}3 - \log_{10}4 + 1$

(ii) $\log_2 12 - \log_2 3 = \log_2 x + 1$

20 Logarithms II

By studying this lesson you will be able to

- simplify expressions involving the product and quotient of numbers greater than 1, using the logarithms table
- identify the keys $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, $\boxed{=}$, $\boxed{(}$ and $\boxed{)}$ on a calculator.

Logarithms Table

Let us recall several facts we have learnt earlier regarding logarithms.
Since $10^0 = 1$, we obtain $\log_{10} 1 = 0$. That is, the logarithm of 1 to base 10 is 0.
Since $10^1 = 10$, we obtain $\log_{10} 10 = 1$. That is, the logarithm of 10 to base 10 is 1.
Since $10^2 = 100$, we obtain $\log_{10} 100 = 2$. That is, the logarithm of 100 to base 10 is 2.
Since $10^3 = 1000$, we obtain $\log_{10} 1000 = 3$. That is, the logarithm of 1000 to base 10 is 3.
The following table has been prepared accordingly.

Number	1	10	100	1 000	10 000
Logarithm to base 10	0	1	2	3	4

This table provides the logarithm of the numbers 1, 10, 100, 1000 and 10 000 to base ten. Logarithm values exist for numbers between 0 and 1, between 1 and 10, between 10 and 100 etc also. These values are not whole numbers. The Scottish mathematician Henry Briggs who lived about four centuries ago prepared the logarithms table by calculating these values. He included only the logarithms of numbers from 1 up to 10. Below is given part of this logarithms table.

N	0	1	2	3	4	5	6	7	8	9	Mean difference 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0334	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	0334	1106	3	7	10	14	17	21	24	33	31
13	1139	1173	1206	1239	1271	1303	1335	1367	0334	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	0334	0732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25

The numbers represented in the leftmost column under N by 10, 11, 12, …, 99 are 1.0, 1.1, 1.2, … , 9.9 which are from 1 up to 10. The decimal point (.) which should be placed in these numbers has not been indicated in the logarithms table. (This has been done to simplify the table). However, when using the table, the decimal point should be placed in the correct position. In the uppermost row, the numbers 0, 1, 2, 3, …, 9 have been written from left to right, and in the same row, on the right, the numbers 1, 2, 3, …, 9 have been written under the mean difference.

As an example the 29^{th} row of the table is given below. In this row, the value in the 6^{th} column is 4713. As mentioned earlier, the decimal point of the numbers is not written. But, when using the table, the decimal point should be placed in the correct position. That is, the value is 0.4713.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	11	12

This means that, the logarithm of 2.96 to base 10 is 0.4713. This also means that $10^{0.4713} = 2.96$. That is, when 2.96 is written as a power of ten it is $10^{0.4713}$. Using the logarithms table we can find the logarithm of numbers consisting of up to 4 digits. When writing logarithms to base 10, instead of writing $\log_{10}$ we write lg in short. Example: $\log_{10} 100 = 2$ is written as lg 100 = 2.

Note that to find the logarithm of 2.9, it must be written as 2.90. The value corresponding to the 29^{th} row and the column indicated by 0 must be taken as the logarithm of 2.90; that is 0.4624.
This is written as, $\log_{10} 2.9 = 0.4624$ or lg 2.9 = 0.4624.
Writing this in index form we obtain $2.9 = 10^{0.4624}$.

Note: Here the logarithm of a number is an approximate value.

20.1 The logarithm of a number with up to two decimal places lying between 1 and 10

Let us identify how lg 4.58 is obtained using the logarithms table. The value that belongs to the row containing 45 which are the first two digits of 4.58 and the column containing 8, which is the remaining digit, is 0.6609. This value is the required logarithm.
That is,
the logarithm of 4.58 = lg 4.58 = 0.6609
Writing this in index form we obtain $4.58 = 10^{0.6609}$.

	0	1	2	3	4	5	6	7	8	9	1	2	3
45									6609				

Example 1

Find the logarithm of each of the following numbers using the logarithms table. Indicate the relevant index form too.

(i) 6.85 (ii) 3.4 (iii) 8

(i) lg 6.85 = 0.8357, in index form $6.85 = 10^{0.8357}$

(ii) lg 3.4 = 0.5315, in index form $3.4 = 10^{0.5315}$ (Writing 3.4 as 3.40)

(iii) lg 8 = 0.9031, in index form $8 = 10^{0.9031}$

Exercise 20.1

1. Using the logarithms table, find the logarithm of each of the following numbers. Indicate the relevant index form too.

(i) 7.32 (ii) 1.05 (iii) 9.99 (iv) 5.8 (v) 9.2
(vi) 3.1 (vii) 4 (viii) 7 (ix) 1 (x) 1.01

20.2 The logarithm of a number with up to three decimal places lying between 1 and 10

We now know how to obtain the logarithm of a number with up to two decimal places which lies between 1 and 10. Next let us consider how the logarithm of a number with three decimal places which lies between 1 and 10 is found.

Let us identify how the logarithm of 5.075 which is a number with three decimal places is found using the table. The number 7050 is obtained from the table as the value which belongs to the row containing 50, which are the first two digits of 5.075, and the column containing 7 which is the third digit of this number. The mean difference in the same row, under the column containing 5, which is the fourth digit of 5.075 is 4.

						Mean difference								
			7	8	9	1	2	3	4	5	6	7	8	9
50			7050							4				

Now add 4 to 7050. Then since 7050 + 4 = 7054, we obtain lg 5.075 = 0.7054. The index form of this is $5.075 = 10^{0.7054}$.

Example 2

Using the logarithms table, find the logarithm of each of the following numbers and then write each number in index form.

(i) 1.099 (ii) 5.875 (iii) 9.071

(i) $\lg 1.099 = 0.0411$, in index form $1.099 = 10^{0.0411}$.

(ii) $\lg 5.875 = 0.7690$, in index form $5.875 = 10^{0.7690}$.

(iii) $\lg 9.071 = 0.9576$, in index form $9.071 = 10^{0.9576}$.

Exercise 20.2

Using the logarithms table, find the logarithm of each of the following numbers and then write each number in index form.

(i) 1.254 (ii) 3.752 (iii) 2.837 (iv) 8.032 (v) 9.998 (vi) 7.543

20.3 The logarithm of numbers which are greater than 10

Although only the logarithms of numbers from 1 up to 10 are included in the logarithms table, the logarithm of any other number (up to four digits or rounded off up to four digits) can also be found using the table. Now let us consider the strategy that is used in such a case.

Example 1

Find the logarithm of 54.37

Method (i): $\lg 54.37 = \lg (5.437 \times 10^1)$ (Representing in scientific notation)

$= \lg 5.437 + \lg 10^1$ (Applying the laws of logarithms)

$= 0.7354 + 1$ (From the logarithms table and since $\lg 10 = 1$)

$= \underline{\underline{1.7354}}$

Method (ii): Using indices

$54.37 = 5.437 \times 10^1$

$= 10^{0.7354} \times 10^1$ (Writing the value that is obtained from the table 0.7354 as an index of 10)

$= 10^{1.7354}$

$\therefore \lg 54.37 = \underline{\underline{1.7354}}$

Example 2

Find the logarithm of each of the following numbers.

(i) 8.583 (ii) 85.83 (iii) 858.3 (iv) 8583

(i) $\lg 8.583 = \lg (8.583 \times 10^0) = \lg 8.583 + \lg 10^0 = 0.9337 + 0 = 0.9337$

(ii) $\lg 85.83 = \lg (8.583 \times 10^1) = \lg 8.583 + \lg 10^1 = 0.9337 + 1 = 1.9337$

(iii) $\lg 858.3 = \lg (8.583 \times 10^2) = \lg 8.583 + \lg 10^2 = 0.9337 + 2 = 2.9337$

(iv) $\lg 8583 = \lg (8.583 \times 10^3) = \lg 8.583 + \lg 10^3 = 0.9337 + 3 = 3.9337$

(The decimal part of these logarithms remains the same since what is obtained from the table in each instance is the value in the 85^{th} row, the 8^{th} column and the 3^{rd} mean difference column).

In the logarithm 1.9337 of 85.83 in the above example, the decimal part of the logarithm is called the **mantissa** and the whole number 1 which is on the left of the decimal point is called the **characteristic** of the logarithm.

Observe the following table.

Number	**Number of digits in the integral part**	**Scientific Notation**	**Logarithm**	**Characteristic of the logarithm**
8.583	1	8.583×10^0	0.9337	0
85.83	2	8.583×10^1	1.9337	1
858.3	3	8.583×10^2	2.9337	2

According to the table, the characteristic of the logarithm of a number is the index of the power of ten, in the expression obtained when the number is expressed in scientific notation.
For numbers which are larger than 1, the characteristic of the logarithm is one less than the number of digits in the integral part of the number. Accordingly, for numbers such as 5.673 which have just one digit in the integral part, the characteristic is 0.

Example 3

Find the logarithm of each number using the logarithms table. Write each number in index form too.

(i) 69.34 (ii) 957.1 (iii) 1248

(i) $\lg\ 69.34 = 1.8409$, in index form $69.34 = 10^{1.8409}$

(ii) $\lg\ 957.1 = 2.9809$, in index form $957.1 = 10^{2.9809}$

(iii) $\lg\ 1248 = 3.0962$, in index form $1248 = 10^{3.0962}$

Exercise 20.3

1. Find the logarithm of each of the following numbers using the logarithms table, and write the numbers in index form too.

(i) 59.1 (ii) 100.2 (iii) 95.41 (iv) 1412 (v) 592.1 (vi) 890

2. If $10^{0.8939} = 7.832$, find each of the following values.

(i) lg 7.832 (ii) lg 78.32 (iii) lg 7832

20.4 Antilogarithm

According to the logarithms table, lg 59.3 = 1.7731. That is, the logarithm of 59.3 is 1.7731 . Another way of saying this is that 1.7731 is the logarithm of 59.3. Then we say that the antilogarithm of 1.7731 is 59.3. We write this as antilog 1.7731 = 59.3.

Now let us consider how the antilogarithm is obtained when the mean difference section is also involved.

Example 1

Find antilog 0.8436 using the logarithms table.

											Mean difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
69	- - -	- - -	- - -	- - -	- - -	- - -	- - -	8432								4			

antilog 0.8436 = 6.976

We can describe how the antilogarithm of 0.8436 was found using the above table as follows. Since this value is not in the table, we consider the number in the table which is less than this and closest to it. This value is 8432 which lies in the 69th row and 7th column. The difference 4 (= 8436 – 8432) is in the 6th column of the mean difference section. Accordingly, the required antilogarithm is 6.976 .

(Since the characteristic of 0.8436 is 0 there is only one digit in the integral part of the antilog)

When the characteristic of the logarithm is 0, the antilogarithm can be written directly from the table as in the above example, as a number between 1 and 10. However when the characteristic is greater than 0, the antilog can be found as shown in the following example.

Example 2

Find the value of antilog 1.8436

antilog 1.8436 $= 6.976 \times 10^1$ (6.971 from the mantissa and 1 in 10^1 from the characteristic)

$= 69.76$

Example 3

Find using the logarithms table.

(i) antilog 1.5432 (ii) antilog 2.5432 (iii) antilog 3.5432

(i) antilog $1.5432 = 3.493 \times 10^1$
$= \underline{\underline{34.93}}$

(ii) antilog $2.5432 = 3.493 \times 10^2$
$= \underline{\underline{349.3}}$

(iii) antilog $3.5432 = 3.493 \times 10^3$
$= \underline{\underline{3493}}$

Exercise 20.4

1. Find the following values using the logarithms table.

(i) antilog 0.7350 (ii) antilog 2.4337 (iii) antilog 3.5419

(iv) antilog 1.0072 (v) antilog 2.9114 (vi) antilog 3.8413

2. If $\lg x = 0.7845$,

(i) find the value of x.

(ii) find the value of x by expressing antilog 1.7854 in scientific notation.

(iii) find the value of antilog 2.7854.

(iv) If $\lg 10y = 0.7845$, find the value of y.

20.5 Multiplying and dividing numbers greater than 1 using the logarithms table

We know the two laws of logarithms $\lg (MN) = \lg M + \lg N$ and $\lg \left(\frac{M}{N}\right) = \lg M - \lg N$.

Now let us see how numbers can be multiplied and divided using the knowledge gained on logarithms up to this point, and the above laws.

Example 1

Evaluate using the logarithms table.

(i) 4.975×10.31 (ii) $53.21 \div 4.97$

Let us take $P = 4.975 \times 10.31$

$$\begin{aligned}
\text{Then, } \lg P &= \lg (4.975 \times 10.31) \\
&= \lg 4.975 + \lg 10.31 && \text{(Logarithm laws)} \\
&= 0.6968 + 1.0132 && \text{(From the logarithms table)} \\
&= 1.7100 \\
\therefore P &= \text{antilog } 1.7100 \\
&= 51.28 \\
4.975 \times 10.31 &= \underline{\underline{51.28}}
\end{aligned}$$

This product can also be obtained using the index laws

$$\begin{aligned}
4.975 \times 10.31 &= 10^{0.6968} \times 10^{1.0132} && \text{(From the logarithms table)} \\
&= 10^{1.7100} && \text{(Sum of the indices)} \\
&= 10^{0.7100} \times 10^{1} \\
&= 5.128 \times 10^{1} && \text{(Antilog of 0.7100 using the tables)} \\
&= \underline{\underline{51.28}}
\end{aligned}$$

(ii) $53.21 \div 4.97$

Let us take $P = 53.21 \div 4.97$

Then $\lg P = \lg (53.21 \div 4.97)$
$= \lg 53.21 - \lg 4.97$
$= 1.7260 - 0.6964$
$= 1.0296$

$\therefore P = \text{antilog } 1.0296$
$= \underline{\underline{10.71}}$

Simplifying using indices

$53.21 \div 4.97 = 10^{1.7260} \div 10^{0.6964}$
$= 10^{1.7260 - 0.6964}$
$= 10^{1.0296}$
$= 1.071 \times 10^{1}$
$= 10.71$

The simplification of an expression which involves both a product and a quotient is given in the following example.

Example 2

Using the logarithms table, find the value of $\dfrac{594.2 \times 9.275}{84.21}$

Let us take $P = \dfrac{594.2 \times 9.275}{84.21}$

$$\therefore \lg P = \lg\left(\frac{594.2 \times 9.275}{84.21}\right)$$
$$= \lg(594.2 \times 9.275) - \lg 84.21$$
$$= \lg 594.2 + \lg 9.275 - \lg 84.21$$
$$= 2.7739 + 0.9673 - 1.9254$$
$$= 1.8158$$
$$\therefore P = \text{antilog } 1.8158$$
$$= 6.543 \times 10^{1}$$
$$= \underline{\underline{65.43}}$$

Exercise 20.5

1. Find the value using the logarithms table.

(i) 54.3×1.75 (ii) 323.8×2.832 (iii) $54.1 \times 27.15 \times 43$

(iv) $523.2 \div 93.75$ (v) $43.17 \div 8.931$ (vi) $\dfrac{73.1 \times 25.41}{18.32}$

(vii) $\dfrac{85.72 \times 58.1}{29.73}$ (viii) $\dfrac{112.8 \times 73.45}{82.11}$ (ix) $\dfrac{953.1 \times 457}{23.25 \times 99.8}$

2. The circumference of a circle is given by the formula $c = 2\pi r$. If it is given that $\pi = 3.142$ and that $r = 10.5$ cm, find the value of c using the logarithms table.

3. The area of the curved surface of a cylinder is given by $A = 2\pi rh$. If it is given that $\pi = 3.142$, $r = 5.31$ cm and $h = 20$ cm, find the value of A using the logarithms table.

20.6 The calculator

The calculator is an outstanding invention of the 19th century which is used to perform calculations quickly and accurately.

There are two types of calculators, namely the ordinary calculator and the scientific calculator. In an ordinary calculator, the calculations are done according to the order in which the operations are given. However, in a scientific calculator, the operations are performed according to mathematical principles. A calculator consists of a keyboard to perform operations and a screen to display the answers.
The following table provides information on the task performed by each of the given keys in a calculator.

Key	Result of operating the keys
$\boxed{\text{ON}}$	Switch on the calculator
$\boxed{\text{OFF}}$	Switch off the calculator
$\boxed{\text{CE}}$	Erase the last number or operation entered
$\boxed{\text{AC}}$	Clear the screen
$\boxed{+}\boxed{-}\boxed{\times}\boxed{\div}$	Perform the relevant mathematical operation
$\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{8}\boxed{7}$ $\boxed{9}\boxed{2}\boxed{1}\boxed{0}$	Use the relevant digit
$\boxed{=}$	Show the result of the operations on the screen
$\boxed{\cdot}$	Placing the period as required in a decimal number
$\boxed{(}$	Start including the required part within brackets
$\boxed{)}$	End the part within brackets

Example 1

Write down in order, the keys that should be operated to perform each of the following calculations using a scientific calculator. Write down the result that appears on the screen too.

(i) $46 + 127$ (ii) $59 - 27$ (iii) $5.4 + 4.1 - 0.7$
(iv) 7.5×23 (v) $(2.7 + 42.3) \div 15$

(i) $\boxed{\text{ON}}\boxed{4}\boxed{6}\boxed{+}\boxed{1}\boxed{2}\boxed{7}$ $\boxed{=}$ $\boxed{173}$

(ii) $\boxed{\text{ON}}\boxed{5}\boxed{9}\boxed{-}\boxed{2}\boxed{7}$ $\boxed{=}$ $\boxed{32}$

(iii) $\boxed{\text{ON}}\boxed{5}\boxed{\cdot}\boxed{4}\boxed{+}\boxed{4}\boxed{\cdot}\boxed{1}\boxed{-}\boxed{0}\boxed{\cdot}\boxed{7}$ $\boxed{=}$ $\boxed{8.8}$

(iv) $\boxed{\text{ON}}\boxed{7}\boxed{\cdot}\boxed{5}\boxed{\times}\boxed{2}\boxed{3}$ $\boxed{=}$ $\boxed{172.5}$

(v) $\boxed{\text{ON}}\boxed{(}\boxed{2}\boxed{\cdot}\boxed{7}\boxed{+}\boxed{4}\boxed{2}\boxed{\cdot}\boxed{3}\boxed{)}\boxed{\div}\boxed{1}\boxed{5}$ $\boxed{=}$ $\boxed{3}$

Exercise 20.6

1. Write down in order, the keys of the scientific calculator that need to be used to perform the following calculations. Write down the result that is obtained on the screen too.

(i) $543 + 275 \times 17$ (ii) $2003 - 125 \times 3$ (iii) $25.1 + 3.04 \div 1.1$

(iv) $57.3 \times 1.75 + 45.3$ (v) $49.5 \div 15 + 12$ (vi) $(32.1 \times 4.3) + 1.5$

2. Simplify each of the following expressions using the logarithms table. Find the value of each expression using a scientific calculator too. Examine up to which decimal place the values obtained by the two methods are equal.

(i) 42.7×39.25 (ii) $514.1 \div 31.7$ (iii) $\dfrac{372.1 \times 4.3}{59.25}$

(iv) $\dfrac{753 \times 1.4}{101.5}$ (v) $(12.5 \times 62.4) \div 253.2$

Miscellaneous Exercise

1. Find the value of $\log_4 64 + \log_3 81 - \log_5 5 + 1$.

2. Find the value of each of the following expressions if $\lg 6.143 = 0.7884$

(i) $10^{0.7884}$ (ii) $10^{1.7884}$ (iii) $10^{2.7884}$

3. Find the value of each of the following expressions if $10^{0.6582} = 4.552$

(i) $\lg 4.552$ (ii) $\lg 45.52$ (iii) $\lg 455.2$

4. Find the value of each of the following expressions if antilog $1.6443 = 44.08$

(i) Antilog 0.6443 (ii) Antilog 2.6443 (iii) Antilog 3.6443

5. (i) If $\lg a = x$ and $\lg b = 2x$, express $\lg (ab)$ in terms of x.

(ii) If $\lg x = 0.9451$ and $\lg y = 0.8710$, find the value of $\lg\left(\dfrac{x}{y}\right)$

6. Simplify the following using the logarithms table. Examine the accuracy of your solution using a scientific calculator.

(i) $\dfrac{38.72 \times 1.003}{5.1}$ (ii) $\dfrac{5.432 \times 989.1}{379.1}$ (iii) $\dfrac{785.8}{27.2 \times 3.8}$

(iv) $\dfrac{75.23 \times 131.2}{5.74 \times 95.2}$ (v) $\dfrac{5.743 \times 83.21 \times 5.91}{12.75 \times 4.875}$ (vi) $\dfrac{573 \times 2.123 \times 6.1}{9.875 \times 54.21}$

21 Graphs

By studying this lesson you will be able to

- find the gradient of the graph of a straight line,
- draw the graph of a function of the form $y = ax^2 + b$.

Graph of a function of the form $y = mx + c$

The graph of a function of the form $y = mx + c$ is a straight line. The coefficient of x, which is m, represents the gradient of the line and the constant term c represents the intercept of the graph.

Review Exercise

1. Write down the gradient and intercept of the straight line represented by each of the following equations.

(i) $y = 3x + 2$ (ii) $y = -3x + 2$ (iii) $y = 5x - 3$

(iv) $y = 4x$ (v) $y = -5x$ (vi) $y = \frac{1}{2}x - 3$

(vii) $y = \frac{1}{2}x + 3$ (viii) $y = \frac{-2}{3}x - 1$ (ix) $2y = 4x + 5$

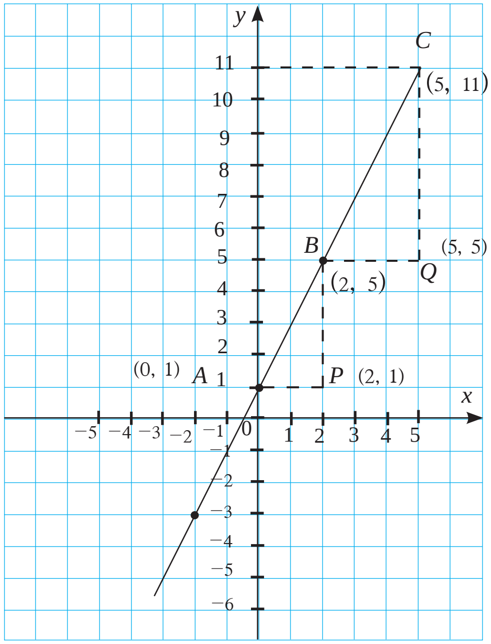
(x) $2y - x = 5$ (xi) $2y + 3 = 2x$ (xii) $\frac{1}{3}y - 5 = x$

21.1 Geometrical description of the gradient of a straight line

We defined the coefficient m of x in the equation $y = mx + c$ as the gradient of the straight line. Now by considering an example, let us see how the value of m is represented geometrically. To do this, let us consider the straight line given by $y = 2x + 1$. Let us use the following table of values to draw its graph.

x	-2	0	2
$y\ (= 2x + 1)$	-3	1	5

Let us mark any three points on the straight line. For example, let us take the three points as A (0, 1), B (2, 5) and C (5, 11).



First let us consider the points A and B.

Let us draw a line from A, parallel to the x – axis, and a line from B, parallel to the y – axis, and name the point of intersection of these lines as P. It is clear that the coordinates of the point P are (2, 1).

Also, length of $AP = 2 - 0$
$= 2$

length of $BP = 5 - 1$
$= 4$

Now for the points A and B, $\dfrac{\text{Vertical distance}}{\text{Horizontal distance}} = \dfrac{BP}{AP} = \dfrac{4}{2} = 2.$

We already know the gradiant of the straight line $y = 2x + 1$ is 2.

The quotient $\dfrac{\text{Vertical Distance}}{\text{Horizontal distance}}$ for the point A and B is also 2.

Now let us consider another case.

As the second case let us consider the points B and C.

Let us draw a line from B, parallel to the x - axis, and a line from C, parallel to the y - axis and name the point of intersection of these two lines as Q.

Then the coordinates of Q are (5, 5).

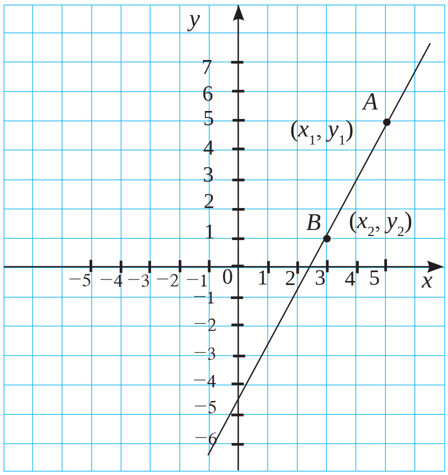
Length of $BQ = 5 - 2$
$= 3$

$$\text{Length of } CQ = 11 - 5$$
$$= 6$$

Now, for the points B and C, $\dfrac{\text{Vertical distance}}{\text{Horizontal distance}} = \dfrac{CQ}{BQ} = \dfrac{6}{3} = 2$.

In both instances, the ratio of the vertical distance to the horizontal distance between the two points under consideration is the gradient 2 of the straight line.

Accordingly, let us develop a formula to find the gradient of a straight line using its graph. Let us consider any straight line with equation $y = mx + c$.



Let us consider any two points A (x_1, y_1) and B (x_2, y_2) on the straight line.
Since these two points lie on the straight line,

$$y_1 = mx_1 + c \quad \text{——} \quad (1)$$
$$y_2 = mx_2 + c \quad \text{——} \quad (2)$$

From (1) and (2) $\quad y_1 - y_2 = mx_1 - mx_2$

$$\therefore y_1 - y_2 = m(x_1 - x_2)$$

$$\therefore \frac{y_1 - y_2}{x_1 - x_2} = m$$

$$\therefore m = \frac{y_1 - y_2}{x_1 - x_2}$$

$\therefore$ The gradient of the straight line $= \dfrac{y_1 - y_2}{x_1 - x_2}$

Example 1

The coordinates of two points on a straight line are (3, 10) and (2, 6). Find the gradient of the straight line.

$$\text{Gradient of the straight line} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{10-6}{3-2}$$

$$= \frac{4}{1}$$

$$= \underline{\underline{4}}$$

Example 2

The coordinates of two points on a straight line are (6, 3) and (2, 5). Find the gradient of the straight line.

$$\text{Gradient of the straight line} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{3-5}{6-2}$$

$$= \frac{-2}{4}$$

$$= -\underline{\underline{\frac{1}{2}}}$$

Example 3

Find the gradient of the straight line that passes through the points (–2, 4) and (1, –2).

$$\text{Gradient of the straight line} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{4-(-2)}{-2-1}$$

$$= \frac{4+2}{-3}$$

$$= \frac{6}{-3}$$

$$= \underline{\underline{-2}}$$

Exercise 21.1

1. Calculate the gradient of the straight line which passes through each pair of points.

(i) (4, 6) (2, 2) (ii) (6, 2) (4, 3) (iii) (1, –2) (0, 7) (iv) (–2, –3) (2, 5)
(v) (4, 5) (–8, –4) (vi) (6, –4) (2, 2) (vii) (1, –4) (–2, –7) (viii) (4, 6) (–2,–9)

21.2 Finding the equation of a straight line when the intercept of its graph and the coordinates of a point on the graph are given

Example 1

The intercept of the graph of a straight line is 3. The coordinates of a point on the graph is (2, 7). Write the equation of the straight line.

The equation of a straight line graph with gradient m and intercept c is $y = mx + c$.

By substituing the value of the intercept and the coordinates of the point on the graph into the equation of the function we obtain,

$$y = mx + c$$
$$7 = 2m + 3$$
$$7 - 3 = 2m$$
$$4 = 2m$$
$$m = \frac{4}{2}$$
$$m = 2$$

By substituting $c = 3$ and $m = 2$ into the equation we obtain $y = 2x + 3$.

$$\underline{\underline{y = 2x + 3}}$$

Exercise 21.2

1. For each graph with the given intercept and passing through the given point, write down the equation of the corresponding function.

(i) Intercept = 1and (3, 10) (ii) Intercept = 2 and (3, 3)
(iii) Intercept = 5 and (2, 1) (iv) Intercept = 0 and (3, 12)
(v) Intercept = – 4 and (3, 8) (vi) Intercept = –5 and (–2, –9)

21.3 Finding the equation of a straight line which passes through two given points

Let us find the equation of the straight line which passes through the points (1, 7) and (3, 15). To obtain the equation, let us find the gradient and the intercept of the graph.

Let us first find the gradient of the straight line, using the coordinates (1, 7) and (3, 15) of the two points on the line.

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m = \frac{7-15}{1-3}$$

$$m = \frac{-8}{-2}$$

$$m = 4$$

Let us substitute the value of m and the coordinates of one of the points into the equation $y = mx + c$. Thereby we can find the value of c.

$$x = 1 \quad y = 7 \quad m = 4$$

$$y = mx + c$$

$$7 = 4 \times 1 + c$$

$$7 - 4 = c$$

$$3 = c$$

$$c = 3$$

$m = 4$ and $c = 3$

The gradient of the graph is 4 and the intercept is 3.

Therefore, the required equation is $y = 4x + 3$.

Example 1

Find the equation of the straight line passing through the points (4, 3) and (2, –1).

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m = \frac{3-(-1)}{4-2}$$

$$m = \frac{4}{2}$$

$$m = 2$$

Now let us substitute the coordinates of the point (2, –1) and the gradient into the equation $y = mx + c$.

$$x = 2 \quad y = -1 \quad m = 2$$

$$\begin{aligned} y &= mx + c \\ -1 &= 2 \times 2 + c \\ -1 &= 4 + c \\ -1 - 4 &= c \\ -5 &= c \\ c &= -5 \end{aligned}$$

$\therefore$ The equation of the straight line is $y = 2x - 5$.

Exercise 21.3

1. Find the equation of each of the straight line graphs that passes through the given points.

(i) (1, 7) (2, 10) (ii) (3, –1) (–2, 9) (iii) (4, 3) (8, 4) (iv) (2, –5) (–2, 7)

(v) (–1, –8) (3, 12) (vi) (–5, 1) (10, –5) (vii) $\left(\frac{2}{3}, \frac{2}{3}\right)\left(1, 1\frac{1}{3}\right)$ (viii) (2, 2) (0, –4)

21.4 Graphs of functions of the form $y = ax^2$

Now let us identify several basic properties of graphs of functions of the form $y = ax^2$. Here a is a non-zero number. Here y is defined as the function. y can be considered as a function which is determined by ax^2.

First let us draw the graph of $y = x^2$.
To do this, let us proceed according to the following steps.

Step 1
Preparing a table of values to find the y values corresponding to given x values of the function.

$y = x^2$

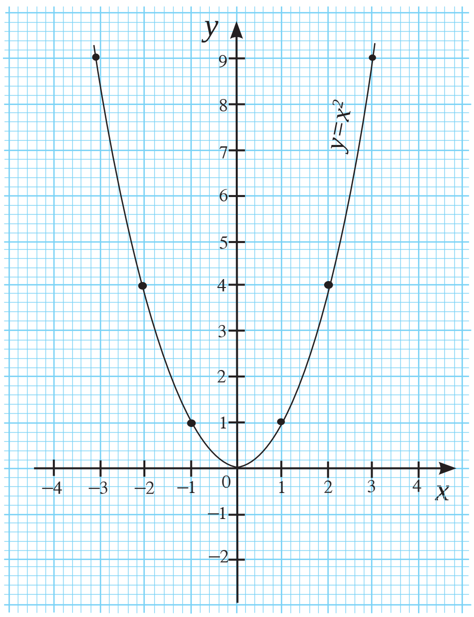
x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
y	9	4	1	0	1	4	9

Using the table of values, let us obtain the coordinates of the points required to draw the graph of the function.
(–3, 9), (–2, 4), (–1, 1), (0, 0), (1, 1), (2, 4), (3, 9)

Step 2
Preparing a Cartesian coordinate plane to mark the coordinates that were obtained.

The maximum value that the x coordinate takes in the pairs that were obtained is +3 and the minimum value is –3. The maximum value that the y coordinate takes is 9 and the minimum value is 0.
Let us draw the x and y axes according to a suitable scale, on the piece of paper that is to be used to draw the graph, so that the x - axis can be calibrated from –3 to 3 and the y - axis can be calibrated from 0 to 9.

Step 3
Drawing the graph of the function.
Let us mark the seven points on the coordinate plane that has been prepared.
Next let us join the points that have been marked so that a smooth curve is obtained.
This smooth curve is the graph of the function $y = x^2$.



The curve that is obtained as the graph of a function of the form $y = ax^2$ is defined as a parabola.
Let us identify several properties of the graph of the function $y = x^2$ by considering the graph that was drawn.

For the function $y = x^2$,

- the graph is symmetric about the y - axis. Therefore, the y - axis is the axis of symmetry of the graph and the equation of the axis of symmetry is $x = 0$.
- when the value of x increases negatively (i.e., –3 to 0) the function decreases positively and when the value of x increases positively (i.e., 0 to +3) the function increases positively.

To identify the common properties of the graphs of functions of the form $y = ax^2$ where $a > 0$, let us draw the graphs of the functions $y = x^2$, $y = 3x^2$ and $y = \frac{1}{2}x^2$ on the same coordinate plane.

$y = 3x^2$

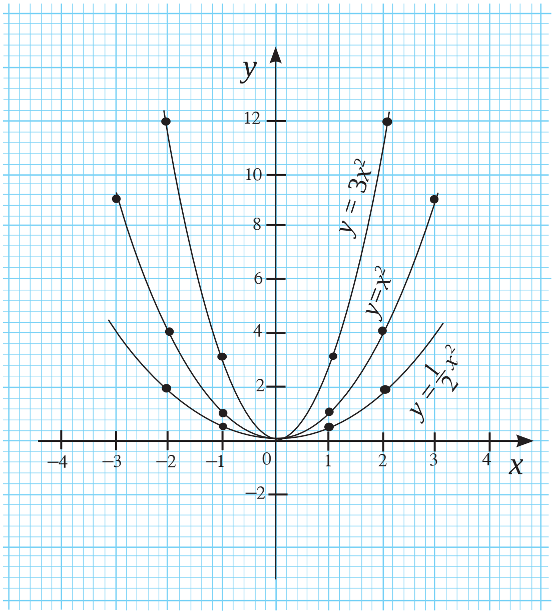
x	-2	-1	0	1	2
x^2	4	1	0	1	4
$3x^2$	12	3	0	3	12
y	12	3	0	3	12

$(-2, 12), (-1, 3), (0, 0), (1, 3), (2, 12)$

$y = \frac{1}{2}x^2$

x	-2	-1	0	1	2
x^2	4	1	0	1	4
$\frac{1}{2}x^2$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2
y	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2

$(-2, 2), (-1, \frac{1}{2}), (0, 0), (1, \frac{1}{2}), (2, 2)$



Let us identify several common properties of the graphs of functions of the form $y = ax^2$, where $a > 0$, by considering the above graphs.

- The graph is a parabola with a minimum point.
- The coordinates of the minimum point are (0, 0).
- The graph is symmetric about the y - axis.
- The equation of the axis of symmetry is $x = 0$.

- The minimum value of the function (i,e., value of y) is 0.
- The function decreases when the value of x increases negatively (increasing along negative values) and reaches the minimum value at $x = 0$.
- The function increases from 0 when the value of x increases positively (increasing along positive values).

To identify the common properties of graphs of functions of the form $y = ax^2$ where $a < 0$, let us draw the graphs of the functions $y = -x^2$, $y = -2x^2$ and $y = -\frac{1}{2}x^2$ on the same coordinate plane.

$y = -x^2$

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-x^2$	-9	-4	-1	0	-1	-4	-9
y	-9	-4	-1	0	-1	-4	-9

$(-3, -9)$ $(-2, -4)$ $(-1, -1)$ $(0, 0)$ $(1, -1)$ $(2,-4)$ $(3,-9)$

$y = -2x^2$

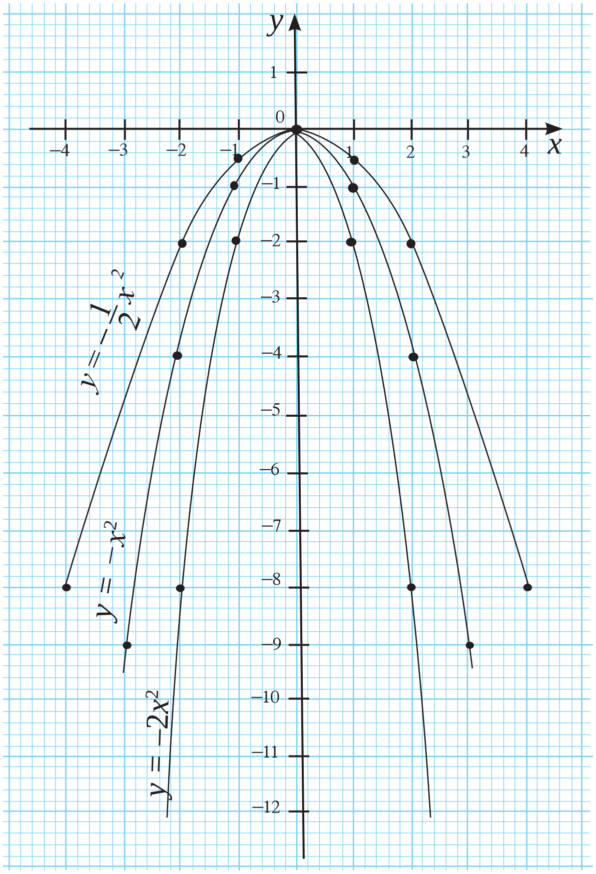
x	-2	-1	0	1	2
x^2	4	1	0	1	4
$-x^2$	-8	-2	0	-2	-8
y	-8	-2	0	-2	-8

$(-2, -8)$ $(-1, -2)$ $(0, 0)$ $(1, -2)$ $(2, -8)$

$y = -\frac{1}{2}x^2$

x	-4	-2	0	2	4
x^2	16	4	0	4	16
$-\frac{1}{2}x^2$	-8	-2	0	-2	-8
y	-8	-2	0	-2	-8

$(-4, -8)$, $(-2, -2)$, $(0, 0)$, $(2, -2)$, $(4, -8)$



Let us identify the common properties of the graphs of functions of the form $y = ax^2$, when a is negative ($a < 0$), by considering the above graphs.

- The graph is a parabola with a maximum point.
- The coordinates of the maximum point are (0, 0).
- The maximum value of the function is 0.
- The graph is symmetric about the y - axis.
- The equation of the axis of symmetry is $x = 0$.
- The function is increasing when x is negatively increasing and reaches the maximum point when $x = 0$.
- The function is decreasing when x is positively increasing.

Let us identify the basic properties of the graphs of functions of the form $y = ax^2$, by considering the graphs that have been drawn. Here a is any non-zero value.

For functions of the form $y = ax^2$,

- the graph is a parabola.
- the graph is symmetric about the y - axis.
  Therefore, the equation of the axis of symmetry of the graph is $x = 0$.
- the coordinates of the turning point (i.e., the maximum or minimum point) of the graph are (0, 0).
- when the coefficient of x takes a “positive” value, the graph is a parabola with a minimum point.
- when the coefficient of x takes a “negative” value, the graph is a parabola with a maximum point.

Example 1

By examining the function, write down for the graph of the function $y = \frac{2}{3}x^2$,

(i) the equation of the axis of symmetry,
(ii) the coordinates of the turning point,
(iii) whether the turning point is a maximum or a minimum point.

(i) The equation of the axis of symmetry is $x = 0$.
(ii) The coordinates of the turning point are (0, 0).
(iii) Since the coefficient of x^2 in the function is a positive value, the graph has a minimum point.

Example 2

By examining the function, write down for the graph of the function $y = -4x^2$,

(i) the equation of the axis of symmetry,
(ii) the coordinates of the turning point,
(iii) whether the turning point is a maximum or a minimum point.

Since the function is of the form $y = ax^2$,

(i) the equation of the axis of symmetry is $x = 0$.
(ii) the coordinates of the turning point are (0, 0).
(iii) since the coefficient of x^2 in the function is a negative value, the graph has a maximum point.

Exercise 21.4

1. Complete the following table by examining the function

Function	Coordinates of the turning point	Minimum value of y	Maximum value of y	Equation of the axis of symmetry
$y = 5x^2$				
$y = -\frac{1}{3}x^2$				
$y = -\frac{2}{3}x^2$				
$y = \frac{3}{4}x^2$				
$y = -7x^2$				

2. Incomplete tables of values prepared to draw the graphs of the functions $y = \frac{1}{3}x^2$ and $y = -\frac{1}{4}x^2$ are given below.

$y = \frac{1}{3}x^2$

x	-6	-3	0	3	6
y	12	___	0	3	___

$y = -\frac{1}{4}x^2$

x	-4	-2	0	2	4
y	-4	-1	0	___	___

(i) Complete the tables and draw the graphs separately.

(ii) For the functions, write down

(*a*) the equation of the axis of symmetry of the graph,

(*b*) the coordinates of the turning point of the graph,

(*c*) the maximum or minimum value.

3. (i) Using values of x such that $-3 \leq x \leq 3$, prepare a suitable table of values to draw the graphs of the functions $y = 2x^2$, $y = 4x^2$, $y = -\frac{1}{3}x^2$ and $y = -3x^2$.

(ii) Draw the graphs on a suitable coordinate plane.

(iii) For each of the graphs write down

(*a*) the equation of the axis of symmetry

(*b*) the coordinates of the turning point

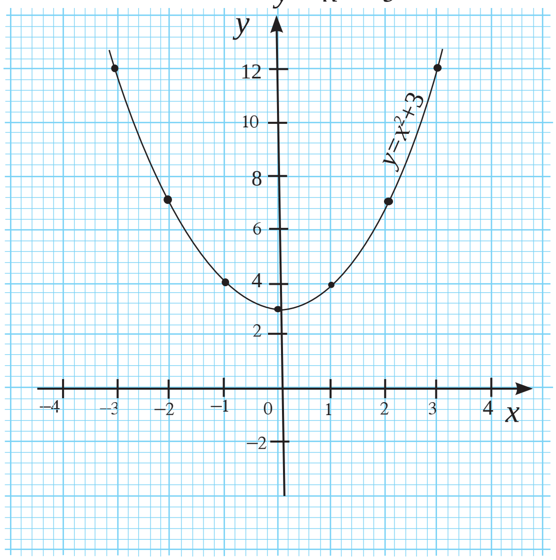
(*c*) the maximum or minimum value of the function.

21.5 Graph of a function of the form $y = ax^2 + b$

Let us draw the graph of the function $y = x^2 + 3$ to identify several basic properties of the graph of a function of the form $y = ax^2 + b$ (Here $a \neq 0$).

x	– 3	– 2	– 1	0	1	2	3
x^2	9	4	1	0	1	4	9
+3	+3	+3	+3	+3	+3	+3	+3
y	12	7	4	3	4	7	12

$y = x^2 + 3$



The graph of the function $y = x^2 + 3$ is a parabola with a minimum point.

For the function $y = x^2 + 3$,

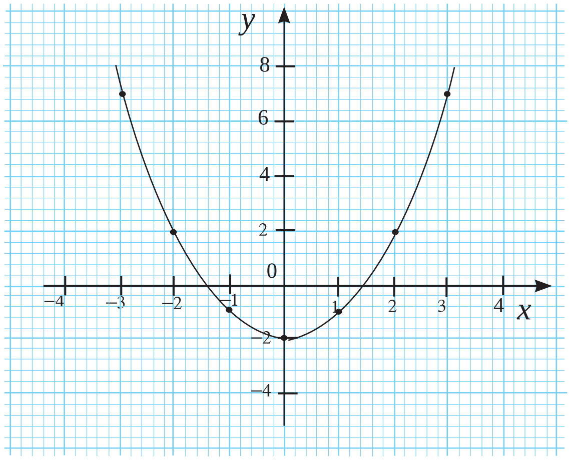
- the equation of the axis of symmetry of the graph is $x = 0$.
- the graph has a minimum point, with coordinates (0, 3).
- the minimum value of the y coordinates of the points on the graph is 3. Therefore, the minimum value of the function is 3.

Let us draw the graph of the function $y = x^2 - 2$ to identify the properties of the graph of a function of the form $y = ax^2 + b$ when the value of b is negative.

$y = x^2 - 2$

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
–2	–2	–2	–2	–2	–2	–2	–2
y	7	2	–1	–2	–1	2	7

(–3, 7) (–2, 2) (1, –1) (0, –2) (1, –1) (2, 2) (3, 7)



The graph of the function $y = x^2 - 2$ is a parabola with a minimum value.
For the function $y = x^2 - 2$,

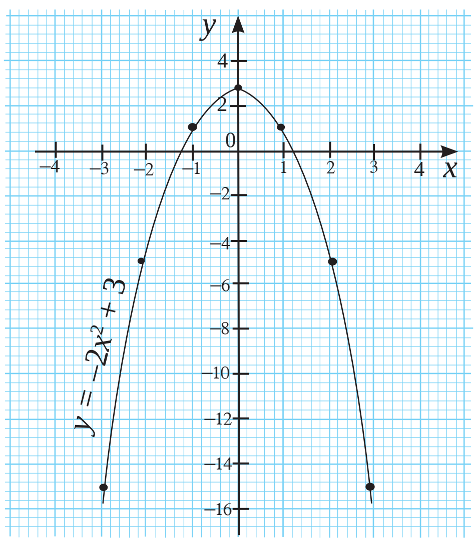
- the equation of the axis of symmetry is $x = 0$.
- the coordinates of the turning point are $(0, -2)$.
- the minimum value of the y coordinates of the points on the graph of the function is -2. Therefore, the minimum value of the function is -2.

Let us draw the graph of the function $y = -2x^2 + 3$ to identify the properties of the graph of a function of the form $y = ax^2 + b$. When the value of a is negative.

$y = -2x^2 + 3$

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-2x^2$	-18	-8	-2	0	-2	-8	-18
$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
y	-15	-5	$+1$	$+3$	$+1$	-5	-15

$(-3, -15)$ $(-2, -5)$ $(-1, 1)$ $(0, 3)$ $(1, 1)$ $(2, -5)$ $(3, -15)$



The graph of the function $y = -2x^2 + 3$ is a parabola with a maximum point.
For the function $y = -2x^2 + 3$,

- the equation of the axis of symmetry of the graph is $x = 0$.
- the coordinates of the turning point of the graph are (0, 3).
- the maximum value of the points on the graph of the function is 3.

Therefore, the maximum value of the function is 3.

Let us identify several common properties of the graphs of functions of the form $y = ax^2 + b$, by examining the graphs that were drawn of functions of this form.

The graph of a function of the form $y = ax^2 + b$,

- is a parabola with a minimum point when a is a positive value.
- is a parabola with a maximum point when a is a negative value.
- the equation of the axis of symmetry of the graph is $x = 0$.
- the coordinates of the maximum or minimum point (turning point) is $(0, b)$.
- the maximum or minimum value of the function is b.

Example 1

Write down for the graph of the function $y = 3x^2 - 5$,

(i) the equation of the axis of symmetry,
(ii) the coordinates of the turning point,
(iii) the maximum or minimum value of the function.

(i) Since the graphs of functions of the form $y = ax^2 + b$ are parabolas which are symmetric about the y - axis, the equation of the axis of symmetry of the graph of the function $y = 3x^2 - 5$ is $x = 0$.

(ii) Since the coordinates of the turning point of the graphs of functions of the form $y = ax^2 + b$ are $(0, b)$, the coordinates of the turning point of the graph of the function $y = 3x^2 - 5$ are $(0, -5)$.

(iii) Since the coefficient of x^2 in $y = 3x^2 - 5$ is positive, the graph has a minimum value. The minimum value of the function is -5.

Example 2

Write down for the function $y = 4 - 2x^2$,

(i) the equation of the axis of symmetry of the graph,
(ii) the coordinates of the turning point of the graph,
(iii) the maximum or minimum value of the function.

(i) The equation of the axis of symmetry of the graph of $y = 4 - 2x^2$, is $x = 0$.

(ii) The coordinates of the turning point are (0, 4).

(iii) Since the coefficient of x^2 in $y = 4 - 2x^2$ is negative, the graph has a maximum value. The maximum value of the function is 4.

Exercise 21.5

1. Without drawing the graphs of the functions of the form $y = ax^2 + b$ given below, complete the following table.

Function	**Equation of the axis of symmetry**	**Coordinates of the turning point of the graph**	**Whether the graph has a maximum or minimum value**	**The maximum or minimum value of the function**
$y = 3x^2 + 4$				
$y = 3 - 4x^2$				
$y = \frac{3}{2}x^2 + 4$				
$y = \frac{3}{2}x^2 - 5$				
$y = 2x^2 - \frac{1}{3}$				

2. Incomplete tables of values prepared to draw the graphs of the functions $y = 2x^2 - 4$ and $y = -x^2 + 5$ are given below.

$y = 2x^2 - 4$

x	–2	–1	0	1	2
y	4	___	___	–2	4

$y = -x^2 + 5$

x	–3	–2	–1	0	1	2	3
y	–4	___	+ 4	+ 5	___	+ 1	–4

(i) Complete each table and draw the corresponding graph. For each graph write down

(*a*) the equation of the axis of symmetry of the graph,

(*b*) the coordinates of the turning point,

(*c*) the maximum or minimum value of the function.

3. For each of the functions given in (a) to (d) below, prepare a table of values to draw the graph of the function, considering the integral values of x in the range $-3 \leq x \leq 3$.

For each function,

(i) draw the corresponding graph

(ii) write down the equation of the axis of symmetry of the graph

(iii) indicate the turning point on the graph and write down whether it is a maximum or a minimum

(iv) write down the maximum or minimum value of the function.

(*a*) $y = x^2 + 4$

(*b*) $y = 4 - x^2$

(*c*) $y = -(2x^2 + 3)$

(*d*) $y = 4x^2 - 5$

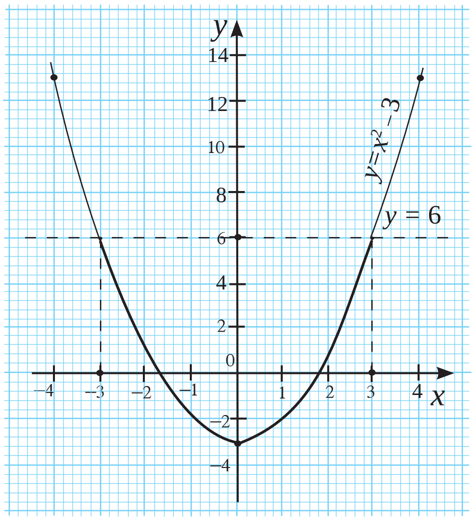
21.6 Finding the interval of values of x corresponding to an interval of values of y, for a function of the form $y = ax^2 + b$

Let us identify how the interval of values of x corresponding to an interval of values of y is found for a function with a minimum value, by considering the graph of the function $y = x^2 - 3$. Let us find the interval of values of x for which the value of the function is less than 6; that is $y < 6$. Let us first draw the graph of $y = x^2 - 3$.

$y = x^2 - 3$

x	-4	-3	-2	-1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
y	13	6	1	-2	-3	-2	1	6	13

$(-4, 13)$ $(-3, 6)$ $(-2, 1)$ $(-1, -2)$ $(0, -3)$ $(1, -2)$ $(2, 1)$ $(3, 6)$ $(4, 13)$



To identify the region of the graph, belonging to the interval of values $y < 6$, let us draw the straight line $y = 6$. In the region of the graph below the line $y = 6$, the y coordinate takes values less than 6. This region of the graph has been indicated by a dark line.

Let us draw two lines parallel to the y - axis, from the points of intersection of the graph and the line $y = 6$, up to the x - axis. Let us mark the two points at which these two lines meet the x - axis (-3 and $+3$).

The interval of values of x between these two points is the interval of values of x for which $y < 6$. That is, when the value of x is greater than -3 and less than $+3$, the value of $y < 6$. Therefore, the interval of values of x for which the function $y = x^2 - 3$ satisfies the condition $y < 6$ is $-3 < x < 3$.

Example 1

For the function $y = x^2 - 4$,

(i) find the values of x for which $y \geq 0$.

(ii) what is the interval of values of x for which the function is increasing positively?

(iii) what is the interval of values of x for which the function is decreasing positively?

(iv) what is the interval of values of x for which the function is increasing negatively?

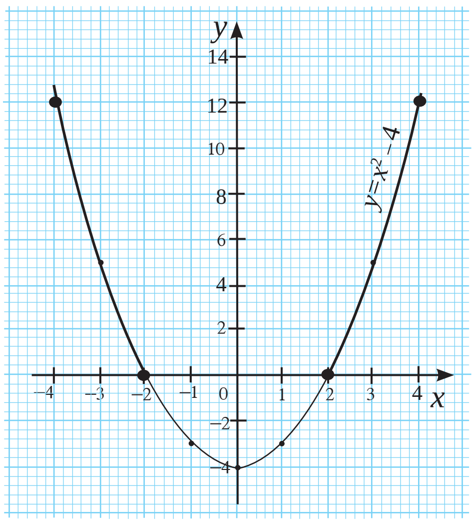
(v) what is the interval of values of x for which the function is decreasing negatively?

First let us draw the graph.

$$y = x^2 - 4$$

x	−4	−3	−2	−1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
−4	−4	−4	−4	−4	−4	−4	−4	−4	−4
y	12	5	0	−3	−4	−3	0	5	12

(–4, 12) (–3, 5) (–2, 0) (–1, –3) (0, –4) (1, –3) (2, 0) (3, 5) (4, 12)



(i) The portion of the graph for which $y \geq 0$, is the portion on and above the straight line $y = 0$.

The corresponding values of x are those which are less than or equal to –2 or greater than or equal to +2. That is, $x \leq -2$ or $x \geq 2$

(ii) $x > 2$.

(iii) $x < -2$.

(iv) $0 < x < 2$.

(v) $-2 < x < 0$.

Exercise 21.6

1. Draw the graph of $y = 3 - 2x^2$ and find the interval of values of x for which $y \geq 1$.

2. Draw the graph of $y = 2x^2 - 4$ and find,
(i) the interval of values of x for which $y < -3$.
(ii) the interval of values of x for which the function increases negatively.
(iii) the interval of values of x for which the function increases positively.
(iii) the interval of values of x for which the function decreases positively.
(iv) the interval of values of x for which the function decreases negatively.

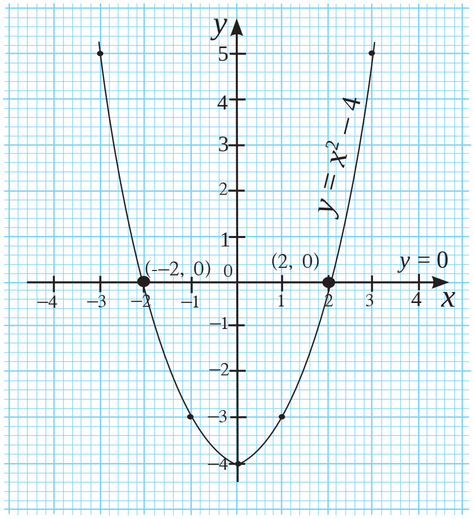
21.7 Finding the roots of an equation of the form $ax^2 + b = 0$ by considering the graph of a function of the form $y = ax^2 + b$

Let us consider for example how the roots of the equation $x^2 - 4 = 0$ are found. To do this, we need to first draw the graph of the function $y = x^2 - 4$.

$y = x^2 - 4$

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
-4	-4	-4	-4	-4	-4	-4	-4
y	5	0	-3	-4	-3	0	5

$(-3, 5)$ $(-2, 0)$ $(-1, -3)$ $(0, -4)$ $(1, -3)$ $(2, 0)$ $(3, 5)$



The two points at which the graph of the function $y = x^2 - 4$ cuts the x - axis are $x = -2$ and $x = +2$. That is, when $x = -2$ and $x = +2$, the y coordinate of the graph is 0. Therefore, when $x = -2$ and $x = +2$, we have $x^2 - 4 = 0$. That is, $x = -2$ and $x = +2$ satisfy the equation $x^2 - 4 = 0$. To put this in another way, the roots of the equation $x^2 - 4 = 0$ are 2 and -2.

Exercise 21.7

1. Complete the following table of values to draw the graph of the function $y = 9 - 4x^2$.

x	-2	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	1	2
y	-7	5	8	9	____	5	-7

(i) Using the table, draw the graph of the function $y = 9 - 4x^2$

(ii) Using the graph, find the roots of the equation $9 - 4x^2 = 0$

2. Prepare a table of values with $-3 \leq x \leq 3$ to draw the graph of the function $y = x^2 - 1$.

(i) Draw the graph of $y = x^2 - 1$.

(ii) Using the graph, find the roots of $x^2 - 1 = 0$.

3. Prepare a table of values with $-3 \leq x \leq 3$ to draw the graph of the function $y = 4 - x^2$.

(i) Draw the graph of $y = 4 - x^2$.

(ii) Using the graph, find the roots of $4 - x^2 = 0$.

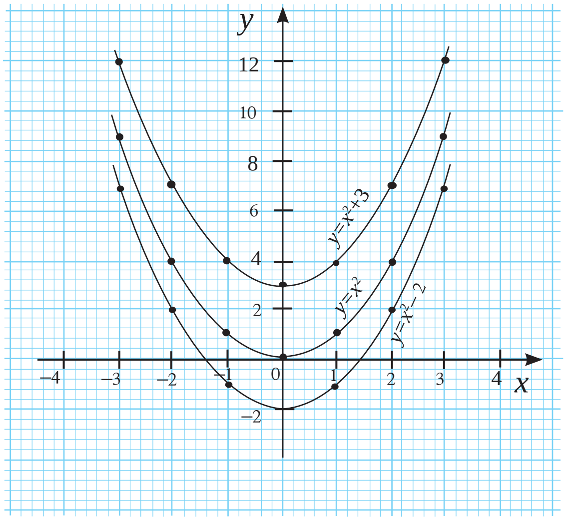
4. Prepare a suitable table of values and draw the graph of $y = x^2 - 9$.

(i) Draw the graph of $y = x^2 - 9$.

(ii) Using the graph, find the roots of $9 - x^2 = 0$.

21.8 Verticle displacement of graphs of functions of the form $y = ax^2 + b$

Consider the graphs given below which you have studied earlier.



• Observe that,

by translating the graph of $y = x^2$ by 3 units vertically upwards, the graph corresponding to the function with the equation $y = x^2 + 3$ is obtained and also by translating the graph of $y = x^2$ by 2 units vertically downwards, the graph corresponding to the function with the equation $y = x^2 - 2$ is obtained.

Observe the following table.

Equation of the graph	Minimum point	Axis of symmetry
$y = x^2$	(0, 0)	$x = 0$
$y = x^2 + 3$	(0, 3)	$x = 0$
$y = x^2 - 2$	(0, –2)	$x = 0$

Accordingly,

- if we translate the graph of $y = x^2$ by 6 units vertically upwards, the equation of the graph of the corresponding function will be $y = x^2 + 6$.
- if we translate the graph of $y = x^2$ by 4 units vertically downwards, the equation of the corresponding function of the graph will be $y = x^2 - 4$.
- in general, the equation of the graph obtained by translating the graph of a function of the form $y = ax^2 + b$ vertically upwards or downwards by c units is respectively $y = ax^2 + b + c$ or $y = ax^2 + b - c$.

Exercise 21.8

1. If the graph of the function $y = x^2 + 2$,

(i) moves upwards along the y axis by 2 units

(ii) moves downwards along the y axis by 2 units

write the equation of the graph.

2. If the graph of the function $y = -x^2$,

(i) moves upwards along the y axis by 3 units

(ii) moves downwards along the y axis by 3 units

write the equation of the graph.

3. If the graph of the function $y = 2x^2 + 5$,

(i) moves upwards along the y axis by 6 units

(ii) moves downwards along the y axis by 6 units

write the equation of the graph.

Miscellaneous Exercise

1. The coordinates of two points on a straight line graph are (0,3) and (3, 1).

(i) Calculate the gradient of the graph.

(ii) Find the intercept of the graph.

(iii) Write down the function of the graph.

2. Without drawing the graph, providing reasons show that the points (–1, –3), (2, 4) and (4, 6) line on the same straight line graph.

3. Without drawing the graph, providing reasons show that the points (–2, –8), (0, –2), (3, 7) and (2, 4) lie on the same straight line graph.

4. Draw the graph of the function $y = \frac{x^2}{2} - 3$.

(i) Using the graph, find the interval of values of x for which $y \geq 1\frac{1}{2}$.

(ii) Using the graph, find the interval of values of x for which the value of the function is less than –1.

5. Construct the table of values to draw the graph of the function $y = 3 - 2x^2$ in the interval $-2 \leq x \leq 2$.

(i) Using the table of values, draw the graph of $y = 3 - 2x^2$.

(ii) Using the graph, find the roots of the equation $3 - 2x^2 = 0$.

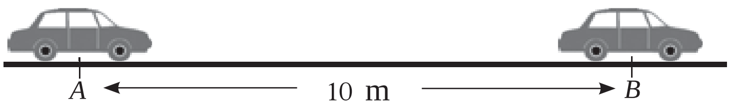
(iii) Write down the equation of the graphs which is obtained when the above graph is shifted upwards by 2 units.

22 Rate

By studying this lesson you will be able to

- solve problems related to distance, time and speed
- represent information related to distance and time graphically
- solve problems related to liquid volumes, time and rate.

22.1 Speed



Let us assume that a battery operated toy car takes 5 seconds to travel from point *A* to point *B* which is 10 m away.

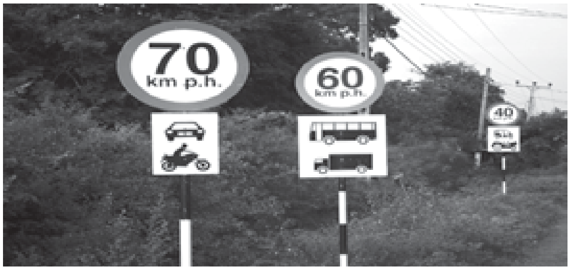
Then the distance that the car has travelled during 5 seconds is 10 m. If the distance that the car moves forward during each second is the same from the moment it starts, then the distance it travels during each second is $\frac{10}{5}$ metres, that is, 2 metres. Accordingly, as the car moves forward from *A*, the rate at which the distance changes with respect to time is 2 metres per second. We can define this value as the speed with which the car travels from *A* to *B*.

If the distance travelled by an object in motion is a constant per unit of time, then the object is said to be travelling with uniform speed. Further, the speed of the object is then the distance travelled per unit of time. From this point on, only objects which travel with uniform speed will be considered in this lesson.

However, in reality, vehicles that travel on the main road are usually unable to maintain a uniform speed throughout the whole journey due to the traffic on the road and various other reasons. The instrument called the speedometer gives the speed of a vehicle at any given instance.

The speed denoted by the speedometer in the figure can be written as 80 kmph. It can also be written as 80 km /h or as 80 kmh^{-1}.

As you travel along a main road, you may observe road signs with 40 kmph and 60 kmph written on them to indicate speed limits. Try to recall that heavy vehicles such as lorries carry a board at the back with 40 kmph written on it.



For an object that is moving with uniform speed, the relationship between the three quantities, namely the distance travelled, the time taken and the speed can be written as follows.

$$\boxed{\text{Speed} = \frac{\text{Distance travelled}}{\text{Time taken}}}$$

This relationship can also be written in the following simple form (without fractions).

$$\boxed{\text{Distance} = \text{Speed} \times \text{Time}}$$

Example 1

A feather floating on air with uniform speed, drifts 100 m in 20 seconds. Calculate the speed with which the feather drifts.

$$\text{Speed with which it drifts} = \frac{\text{Distance it drifts}}{\text{time}}$$

$$= \frac{100\ \text{m}}{20\ \text{s}}$$

$$= \underline{\underline{5\ \text{ms}^{-1}}}$$

Example 2

Calculate the distance travelled in one minute by a bird that flies at a uniform speed of 5 ms^{-1}.

$$\text{Distance it flies} = \text{speed} \times \text{time}$$

$$= 5\ \text{ms}^{-1} \times 60\ \text{s}$$

$$= \underline{\underline{300\ \text{m}}}$$

Example 3

Calculate the time it takes for a car to travel 150 km on a highway, at a uniform speed of 60 kmh^{-1}.

$$\text{Time taken} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{150\ \text{km}}{60\ \text{kmh}^{-1}}$$

$$= \underline{\underline{2\frac{1}{2}\ \text{h}}}$$

Example 4

How far does a motorcycle travel along a main road in 5 seconds, if its speedometer displays a constant speed of 36 kmh^{-1} during this period?

Here, the speed has been given in kilometres per hour. Let us convert it to metres per second.

Since the speed is 36 kmh^{-1},

distance travelled during an hour = 36 km

= 36×1000 m

However, 1 hour = 60×60 seconds

$\therefore$ Distance travelled in 60×60 seconds = 36×1000 m

Distance travelled in 1 second = $\dfrac{36 \times 1000}{60 \times 60}$ m

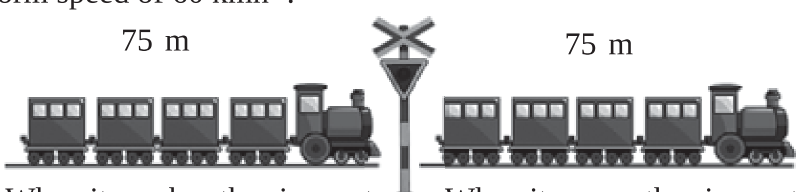
$\therefore$ Distance travelled by the motorcycle in one second = 10 m

$\therefore$ Distance travelled in 5 seconds = 10×5 m

= $\underline{\underline{50\ \text{m}}}$

Example 5

How long does it take a train which is 75 m long to pass a signpost, if it is travelling at a uniform speed of 60 kmh^{-1}?



The distance travelled by the train as it passes the signpost = 75 m

First, let us find the speed in terms of metres per second.

The speed of the train is 60 kmh^{-1}.

$\therefore$ Distance travelled in one hour = 60 km

Distance travelled in one hour = 60×1000 m

Distance travelled in one second = $\dfrac{60 \times 1000}{60 \times 60}$ m

$= \dfrac{50}{3}$ m

$\therefore$ Speed of the train $= \dfrac{50}{3} ms^{-1}$

Since time $= \dfrac{\text{distance}}{\text{speed}}$,

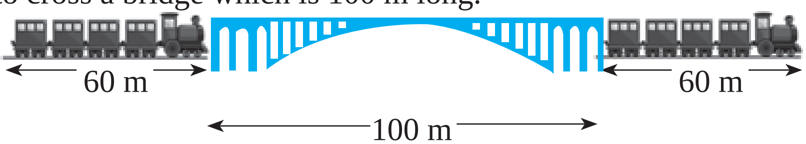
time taken by the train to pass the signpost $= 75 \div \dfrac{50}{3}$ seconds

$= 75 \times \dfrac{3}{50}$ seconds

$= \underline{\underline{4.5 \text{ seconds}}}$

Example 6

Find the time it takes for a train of length 60 m travelling at a uniform speed of 72 kmh^{-1} to cross a bridge which is 100 m long.



Here, the time taken for the train to travel a distance of 160 m needs to be found. For this, let us first find the speed in metres per second.

$72 \text{ kmh}^{-1} = \dfrac{72 \times 1000}{60 \times 60} ms^{-1}$

$= 20 ms^{-1}$

The total distance travelled in crossing the bridge = 100 m + 60 m

= 160 m

Distance travelled by the train in 1 second = 20 m

That is, time taken to travel 20 m = 1 second

$\therefore$ Time taken to travel 160 m $= \dfrac{1}{20} \times 160$ seconds

$= \underline{\underline{8 \text{ seconds}}}$

Average Speed

A vehicle travelling along a main road is usually unable to maintain the same speed throughout the journey. The concept of average speed is important in such situations. The value obtained when the total distance travelled by an object is divided by the total time taken is called the average speed.

Example 1

An intercity bus took $\frac{1}{2}$ an hour to travel the first 25 km of a journey. If it took the bus 1 hour to cover the remaining 80 km of the journey, find the average speed of the bus.

Total distance travelled by the bus $= 25 + 80$ km

$= 105$ km

Total time taken for the journey $= \frac{1}{2} + 1$ h

$= 1\frac{1}{2}$ h

The average speed of the bus $= 105 \text{ km} \div 1\frac{1}{2}$ h

$= 105 \times \frac{2}{3} \text{ kmh}^{-1}$

$= \underline{\underline{70 \text{ kmh}^{-1}}}$

Exercise 22.1

1. Calculate the speed of an aircraft which flies 1200 km in 4 hours with uniform speed.
2. If a child runs 200 m in 40 seconds at a uniform speed, find his speed in kilometres per hour.
3. On a certain day, an electric train moving at a uniform speed, took 6 hours to travel a distance of 300 km. On another day, the train took 8 hours to travel the same distance. Find the difference between the speeds at which the train travelled during the two days.
4. How long will it take an aircraft which travels at a uniform speed of 300 kmh^{-1} to fly 4500 km?
5. Find the distance in metres that a car which travels at a uniform speed of 48 kmh^{-1}, covers during 30 seconds.

6. A bus travels for 15 minutes at a speed of 40 kmh^{-1} and then it travels a further 30 minutes at a speed of 70 kmh^{-1}. Calculate the average speed of the bus.

7. If the time taken by a train to pass a signpost is 10 seconds when it is travelling at a uniform speed of 54 kmh^{-1}, find the length of the train.

8. Find the time it takes for a train of length 60 m travelling at a speed of 72 kmh^{-1} to pass a 100 m long platform.

9. A train leaves city *A* at 0800 h and travels at a uniform speed of 60 kmh^{-1} towards city *B*. Another train leaves city *B* at the same instance and travels at a uniform speed of 40 kmh^{-1} towards city *A*. If the distance between the two cities *A* and *B* is 100 km, calculate the time at which the two trains pass each other.

10. Two motorcyclists, who start their journeys at the same instance from two different cities, travel with uniform speeds of 40 kmh^{-1} and 50 km^{-1} respectively towards each other. If they meet each other $\frac{1}{2}$ an hour after commencing their journeys, find the distance between the two cities.

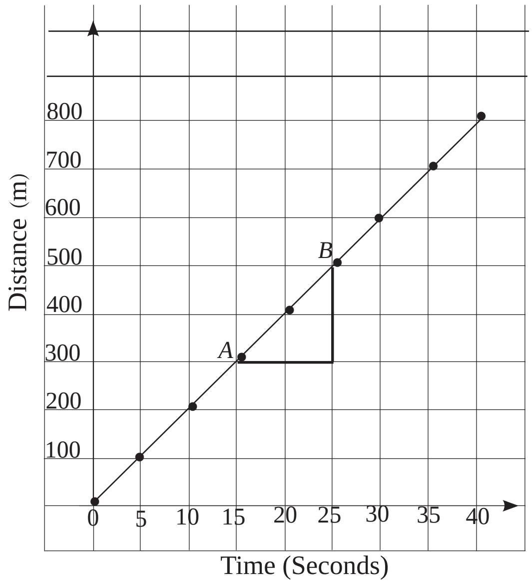
22.2 Distance - Time Graphs

A graph can be used to illustrate the change in the distance travelled by an object in motion, with respect to time. In such a graph, the *x* axis represents the time and the y axis represents the distance travelled. A graph of this form is called a **distance-time graph**.

A table prepared with the information collected by observing the motion of a satellite travelling with uniform speed is given below.

Time that has passed from the commencement of the journey (seconds)	5	10	15	20	25	30	35	40
Distance from the starting point (metres)	100	200	300	400	500	600	700	800

The distance-time graph drawn with this information is given below.



The speed of the satellite can be calculated by dividing the total distance travelled by the total time taken.

$$\text{Speed of the satellite} = \frac{800\text{ m}}{40\text{ s}}$$

$$= 20\text{ ms}^{-1}$$

Observe that the gradient of the straight line $AB = \frac{500\text{-}300}{25\text{-}15} = \frac{200}{10} = 20$

Since the satellite is travelling with uniform speed, the speed can also be obtained by considering the distance travelled per unit of time.

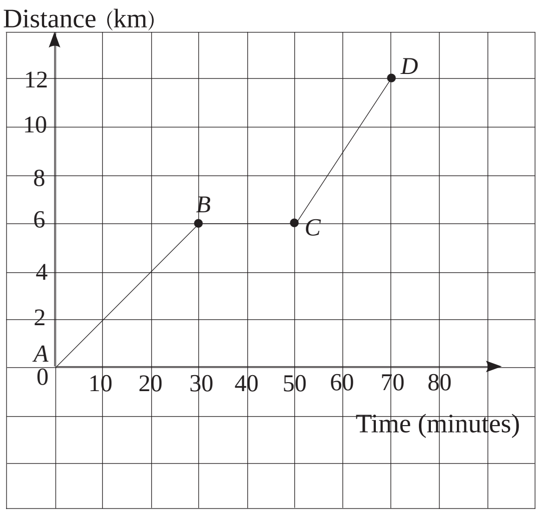
Accordingly, you can observe that the gradient of the graph and the speed of the satellite are equal. Therefore, for an object moving with uniform speed, a straight line is obtained as the distance-time graph, and the speed of the object can be obtained from the gradient of this line.

Gradient of the distance-time graph = Speed of the object in motion

Example 1

A distance-time graph illustrating the motion of Nimal who cycled to his friend's house and then returned back home after spending some time with his friend is given below.

(i) Calculate the speed at which Nimal cycled to his friend's house.
(ii) Calculate the speed at which Nimal returned home.



According to the above graph,

the distance from Nimal's house to his friend's house $= 6$ km

time taken by Nimal to cycle to his friend's house $= 30$ minutes

$= \frac{1}{2}$ h

$\therefore$ The speed at which Nimal cycled to his friend's house $= \dfrac{6\text{ km}}{\frac{1}{2}\text{ h}}$

$= \underline{\underline{12\text{ kmh}^{-1}}}$

The distance is the same during the period that Nimal spent time with his friend

Amount of time Nimal spent at his friend's house $=$ 20 minutes

Time taken for Nimal to return home $=$ 20 minutes

$= \frac{1}{3}$ h

Speed at which Nimal cycled back home $= \dfrac{6\text{ km}}{\frac{1}{3}\text{ h}}$

$= \underline{\underline{18\text{ kmh}^{-1}}}$

Exercise 22.2

1. The following table provides information on the distance travelled by a car moving at a uniform speed along a highway, and the time taken for the journey.

Time (hours)	0	1	2	3	4	5	6
Distance (km)	0	60	120	180	240	300	360

(i) Draw a distance-time graph with the above information.
(ii) Find the gradient of the graph.
(iii) Hence calculate the speed of the car.

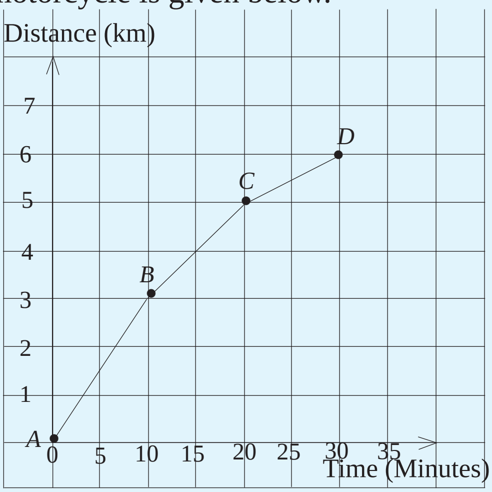
2. The change in distance with time of an object in motion is given in the following table.

Time (s)	0	2	4	6	8	10
Distance (m)	0	6	12	18	24	30

(i) Draw a distance-time graph with the above information.
(ii) Find the gradient of the graph.
(iii) Hence calculate the speed of the object.

3. A coach, moving with uniform speed from the commencement of its journey, travels a distance of 60 km in 2 hours. It then travels another 40 km in 2 hours, also with uniform speed, and reaches its destination. Represent the motion of the coach in a distance-time graph.

4. A distance-time graph of the motion of a man who travels from his home to the city on his motorcycle is given below.



(i) How far is it from his home to the city?
(ii) How long did it take him to reach the city?
(iii) Calculate his average speed.
(iv) Separately calculate the speeds at which he travelled from *A* to *B*, from *B* to *C* and from *C* to *D*.

22.3 Volume and Time

We defined speed as the distance travelled per unit of time. Another way of saying this is that speed is the rate of change of distance with respect to time. This idea of rate can also be used to describe various other processes that we come across in day to day life. Let us consider the example of water flowing out of a tap. If we collect the water that flows out from a tap during periods of one second each, and if by measuring we discover that the volume of water that flows out during each second is a constant, then we say that the water flows out at a uniform rate. Further, we call this constant value the rate at which water flows out from the tap.

When time is measured in seconds and the volume of water is measured in litres, the unit of the rate of flow is litres per second (ls^{-1}).

Suppose it takes 20 minutes for a tank of capacity 1000 l to be filled completely using a pipe through which water flows at a uniform rate.

Then, the volume of water that flowed out of the pipe during 20 minutes $= 1000\, l$

$\therefore$ The amount of water that flowed out during 1 minute $= \dfrac{1000\, l}{20}$

$= 50\, l$

Accordingly, the amount of water that flows out of the pipe per unit of time, that is, during one minute, is 50 litres. Therefore, we can express the rate at which water flows out of the pipe as 50 litres per minute.

$$\boxed{\text{Rate of change of volume} = \frac{\text{Change of volume}}{\text{Time}}}$$

This can also be represented as follows.

$$\boxed{\text{Change of volume} = \text{Rate of change of volume} \times \text{Time}}$$

Example 1

The time taken for 30 litres of petrol to be pumped into a car through a pump at a certain petrol shed was 60 seconds. Find the rate at which petrol flows out of the pump.

$$\text{Rate at which petrol flows out of the pump} = \frac{\text{Volume of petrol}}{\text{Time}}$$

$$= \frac{30\,l}{60\,s}$$

$$= \underline{\underline{\frac{1}{2}\,l\,\mathrm{s}^{-1}}}$$

Example 2

The length, breadth and height of a cuboid shaped indoor water tank are 2 m, $1\frac{1}{2}$ m and 1 m respectively. On an occasion when the tank was completely filled with water, it took 50 minutes for the tank to be emptied by a pipe. Find the rate at which water flowed out through the pipe. (Assume that the water flowed through the pipe uniformly)

$$\text{Volume of the tank} = 2\text{ m} \times 1\frac{1}{2}\text{ m} \times 1\text{ m}$$

$$= 2 \times \frac{3}{2} \times 1\text{ m}^3$$

$$= 3\text{ m}^3$$

Since $1\text{ m}^3 = 1000\ l$,

the volume of water that can be filled into the tank $= 3 \times 1000\ l$

$= 3000\ l$

$$\therefore \quad \text{Rate at which water flowed out through the pipe} = \frac{\text{capacity of the tank}}{\text{time}}$$

$$= \frac{3000\ l}{50\ \text{minutes}}$$

$$= \underline{\underline{\text{60 litres per minute}}}$$

Example 3

A saline solution was administered to a patient at a rate of 0.2 mls^{-1}. Calculate the time it takes for 450 ml of saline solution to be administered.

$$\text{Since rate} = \frac{\text{volume}}{\text{time}}$$

$$\text{Time} = \frac{\text{Volume of Saline}}{\text{Rate of administration}}$$

$$= \frac{450\text{ml}}{0.2\text{mls}^{-1}}$$

$$= 2250 \text{ seconds}$$

$$= \frac{2250}{60} \text{ minutes}$$

$$= 37\frac{1}{2} \text{ minutes}$$

Exercise 22.3

1. A cuboid shaped tank built to provide water to a housing scheme is of length 3 m, breadth 2m and height 1.5 m.

(i) Calculate the volume of the tank.

(ii) How many litres is the volume equal to?

(iii) How much time will it take to fill this tank completely using a pipe through which water flows at a uniform rate of 300 litres per minute?

2. If it took 40 minutes to completely fill a cube shaped tank of side length 2 m using a pipe, what is the rate at which water flows through the pipe in litres per minute? (Hint: $1 \text{ m}^3 = 1000 \text{ l}$)

3. How long will it take to fill a fish tank of length 80 cm, breadth 60 cm and height 40 cm using a pipe through which water flows at a uniform rate of 6 *l* per minute? (Hint: $1 \text{ cm}^3 = 1 \text{ ml}$)

4. The volume of a tank at a water distribution centre is 1800 m^3. If water is distributed from this tank at a rate of $500\ ls^{-1}$, how many minutes will it take to empty half the tank?

5. It took 40 minutes to fill an empty tank using a pump through which petrol flows at a uniform rate of 120 litres per minute. Find the capacity of the tank.

Summary

- Speed $= \dfrac{\text{Distance travelled by the object}}{\text{Time taken}}$

- Rate of change of volume $= \dfrac{\text{Change of volume}}{\text{Time}}$

Miscellaneous Exercise

1. A cylindrical water tank of cross-sectional area 0.5 m^2 is filled to a height of 70 cm in 1 minute and 10 seconds by a pipe through which water flows at a uniform rate. Calculate the rate at which water flows out of the pipe.

2. The distance between railway stations *X* and *Y* is 420 km. A train leaves station *X* at 7.00 p.m. and travels towards station *Y* with a uniform speed of 100 kmh^{-1}. An hour later, another train leaves station *Y* and travels towards station *X* with a uniform speed of 60 kmh^{-1}. At what time do the two trains pass each other?

3. The railway stations *A* and *B* are 300 km apart. A certain train takes 12 hours to travel from *A* to *B* and then back to *A*, after spending 2 hours at *B*. Another train leaves station *A* ten hours after the first train left *A*, and travels towards *B* at the same uniform speed. How far has the second train travelled when the two trains pass each other?

23 Formulae

By studying this lesson you will be able to

- change the subject of a formula which involves squares and roots.
- find the value of an unknown term in a formula when the values of the other unknown terms are given.

You may recall that a formula expresses the relationship that exists between two or more physical quantities.
If the area of a rectangle is denoted by A, then A can be expressed in terms of the length a and breadth b of the rectangle as $A = a \times b$.

A is called the subject of this formula. The subject of a formula can be changed if required.
If the above formula is written in the form $b = \frac{A}{a}$, then the subject is b.

Do the following exercise to recall the facts you have learnt earlier about changing the subject of a formula.

Review Exercise

1. Make u the subject of the formula $v = u + at$.

2. Make F the subject of the formula $C = \frac{5}{9}(F - 32)$.

3. Consider the formula $l = a + (n-1)d$.
(i) Make a the subject of the formula.
(ii) Make d the subject of the formula.
(iii) Make n the subject of the formula.
(iv) Find the value of d when $l = 24$, $a = 3$ and $n = 8$.

4. Consider the formula $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$.

(i) Make r_1 the subject of the above formula.
(ii) Find the value of r_1 when $R = 4$ and $r_2 = 6$.

23.1 Changing the subject of formulae containing squares and square roots

Given below is the formula for the area of a circle. Here A denotes the area and r denotes the radius of the circle.

$$A = \pi r^2$$

Let us consider how r is made the subject of this formula.
Let us first make r^2 the subject of the formula.
That is, $r^2 = \frac{A}{\pi}$.
Now, to make r the subject of the formula, let us take the square root of both sides.
This can be expressed as $r = \pm\sqrt{\frac{A}{\pi}}$.

Since $\sqrt{\ }$ denotes the positive square root value, remember that the signs + and – should be written in front of the symbol $\sqrt{\ }$ to denote the square root. In this example, since r represents the radius of the circle, it is positive. Therefore, we can ignore the negative value. However, when the meanings of the unknowns are not known (or have not been given), the correct form is to have both signs.

Now let us consider how the subject of a formula which contains a square root is changed. For this, let us consider the formula $t = 2\pi\sqrt{\frac{l}{g}}$.

Let us see how l is made the subject of this formula.
Let us first keep the term with the square root sign on one side of the equality symbol, and move all the other terms to the other side.

$$\frac{t}{2\pi} = \sqrt{\frac{l}{g}}$$

Let us now square both sides.

$$\left(\frac{t}{2\pi}\right)^2 = \sqrt{\left(\frac{l}{g}\right)}^{\,2}$$

Then $\frac{t^2}{4\pi^2} = \frac{l}{g}$

Now l can easily be made the subject of the formula.

$$\frac{gt^2}{4\pi^2} = l$$

That is,

$$l = \frac{gt^2}{4\pi^2}$$

Exercise 23.1

1. Make the unknown term within brackets the subject of the relevant formula.

(i) $v^2 - u^2 = 2as$ (u) (ii) $a^2 + b^2 = c^2$ (b)

(iii) $v = \frac{1}{3}\pi r^2 h$ (r) (iv) $v = \frac{a^2 h}{3}$ (a)

(v) $A = \pi(R^2 - r^2)$ (r) (vi) $E = \frac{1}{2}m(v^2 - u^2)$ (u)

2. Make the unknown term within brackets the subject of the relevant formula.

(i) $T = 2\pi\sqrt{\frac{l}{g}}$ (g) (ii) $\theta = \left(\frac{3rt}{m}\right)^{1/2}$ (m)

(iii) $4\sqrt{p} = q$ (p) (iv) $S = a + \sqrt{b}$ (b)

(v) $v = w\sqrt{a^2 - x^2}$ (a) (vi) $A = \pi r\sqrt{h^2 + r^2}$ (h)

23.2 Substitution

If the values of all the unknowns in a formula except for one are given, then by substituting these values in the formula, the value of the remaining unknown can be found.

Given below is the formula for the volume (v) of a cone, in terms of its radius (r) and its height (h).

$$v = \frac{1}{3}\pi r^2 h$$

Find the value of r when $v = 132$ and $h = 14$.

Let us first make r the subject of the formula.

$$\frac{3v}{\pi h} = r^2$$

$$\therefore \quad r = \sqrt{\frac{3v}{\pi h}}. \qquad \text{(since } r \text{ is positive)}$$

Now let us substitute the known values.

$$r = \sqrt{\frac{3 \times 132}{\frac{22}{\cancel{7}_1} \times \cancel{14}_2}}$$

$$r = \sqrt{9}$$

$$\underline{\underline{r = 3}}$$

To solve this problem, it is not necessary to first make r the subject of the formula.

The known values can be substituted first, and then r can be made the subject. This is done as follows.

$$v = \frac{1}{3}\pi r^2 h$$

$$132 = \frac{1}{3} \times \frac{22}{\cancel{7}_1} \times r^2 \times \cancel{14}_2$$

$$\frac{132 \times 3}{22 \times 2} = r^2$$

$$r^2 = 9$$

$$\underline{\underline{r = 3}}$$

The same answer is obtained by both methods. Therefore, either of the above two methods can be used to find the value of an unknown term.

However, there are many benefits of knowing how to change the subject of a formula. For example, when it is required to find the base radius of several cones of different volumes, if r is made the subject of the above formula for the volume of a cone, then it will be very easy to perform the required calculations to find the different radii.
Also, it is necessary to change the subject of the formula if these calculations are being done using a calculator or a computer.

Exercise 23.2

1. Consider the formula $v^2 = u^2 + 2as$.
(i) Find the value of a when $v = 10$, $u = 0$ and $s = 10$.
(ii) Find the value of s when $v = 10$, $u = 5$ and $a = 2$.
(iii) Find the value of u when $v = 10$, $a = 3$ and $s = 6$.

2. If $x = \sqrt{y + z}$,
(i) find the value of x when $y = 6$ and $z = 10$.
(ii) find the value of y when $x = 5$ and $z = 5$.

3. If $k^2 = lm$, find the value of k when $l = 9$ and $m = 4$.

4. Consider the formula $s = ut + \frac{1}{2}at^2$.
(i) Find the value of t when $u = 0$, $a = 5$ and $s = 250$.
(ii) Find the value of t when $u = 5$, $a = 10$ and $s = 30$.

5. Find the value of t in the formula $t = 2\pi\sqrt{\frac{l}{g}}$, when $l = 490$, $g = 10$ and $\pi = \frac{22}{7}$.

Miscellaneous Exercises

1. The relationship between the base radius r, the height h and the volume V of a cylinder is given by $V = \pi r^2 h$. If a cylindrical shaped water tank of base radius 50 cm is filled with water to a height of 70 cm, find the volume of water in the tank. Take $\pi = \frac{22}{7}$.

2. The surface area A of a sphere, expressed in terms of the radius r is given by the formula $A = 4\pi r^2$.

(i) Express the radius of the sphere in terms of the surface area.

(ii) If the surface area of a sphere is 616 cm^2, find its radius. Take $\pi = \frac{22}{7}$.

3. The kinetic energy E of an object of mass m travelling with velocity v is given by the formula $E = \frac{1}{2}mv^2$.

(i) Express the velocity of the object in terms of its kinetic energy and mass.

(ii) Find the kinetic energy of an object of mass 2.4 kg when it is travelling with a velocity of 3 ms^{-1}.

4. If the length of the hypotenuse of a right angled triangle is x, and the lengths of the other two sides are a and b respectively, then according to Pythagoras' theorem, $x = \sqrt{a^2 + b^2}$. Find b when x = 25 cm and a = 24 cm.

5. The energy that a moving object possesses is given by the formula $E = mgh + \frac{1}{2}mv^2$. In this formula, E denotes the energy of the object, m its mass, v its velocity and h the height of its position.

(i) Express the mass of the object in terms of the other quantities.

(ii) Express the velocity of the object in terms of the other quantities.

(iii) The energy possessed by an object of mass 3 kg when it is 5 m above the ground is 153 N. Find the velocity of this object at this moment.
Take g = 10 ms^{-1}.

24 Arithmetic Progressions

By studying this lesson, you will be able to

recognize arithmetic progressions and solve problems related to arithmetic progressions.

In earlier grades, you have learned various number patterns. A number pattern when indicated as a list is called a sequence of numbers or simply a sequence. Let us consider the following sequence.

3, 8, 13, 18, ...

In this sequence, the first term is 3, the second term is 8, the third term is 13 etc. A feature of this sequence is that, considering any two consecutive terms in the sequence, when the first term is subtracted from the second term, a constant value is obtained. In this case, the constant value is 5.

A similar sequence is shown below.

8, 5, 2, –1, –4, ...

In this sequence too, when the first term is subtracted from the second term, for any pair of consecutive terms, a constant value is obtained. In this case, the constant value is –3.

Sequences having this feature are called arithmetic progressions. The constant value obtained by subtracting any term from the term right after that term is called the **common difference**, and is usually denoted by *d*.

> An arithmetic progression is a sequence of numbers such that a constant value is obtained when any term is subtracted from the term right after that term.

The common difference *d*, of an arithmetic progression can be found as follows:

common difference (*d*) = (any term other than the first term) – (the preceding term)

Review Exercise

1. Determine whether each of the following sequences is an arithmetic progression.

(i) 9, 11, 13, 16, ...

(ii) – 8, – 5, – 1, 2, ...

(iii) 2.5, 2.55, 2.555, 2.5555, ...

(iv) $5\frac{1}{2}, 5\frac{3}{4}, 6, 6\frac{1}{2}, \ldots$

(v) 1, −1, 1, −1, ...

2. Find the common difference of each of the following arithmetic progressions.

(i) 12, 17, 22, ...

(ii) 10, 6, 2, ...

(iii) −5, −1, 3, ...

(iv) −2, −8, −14, ...

(v) 2.5, 4, 5.5, ...

24.1 n^{th} term of an Arithmetic Progression

The following notation is used to denote the terms of an arithmetic progression.

$T_1 = 1^{st}$ term

$T_2 = 2^{nd}$ term

$T_3 = 3^{rd}$ term etc.

For example, for the arithmetic progression 6, 8, 10, 12, 14,...

we may write $T_1 = 6,\ T_2 = 8,\ T_3 = 10,\ T_4 = 12,\ T_5 = 14$ etc .

What is the 25^{th} term of this progression? In other words, what is the value of T_{25}? It is clear that if you continue writing the terms according to the above pattern, the 25^{th} term appears when you write 25 terms. If you do this, you will get 54 as the 25^{th} term. That is, $T_{25} = 54$.

Now, if you require to find the 500^{th} term of this progression, how would you find it? For this you would have to write down 500 terms following the pattern, which is quite a tedious task. Let us see how we can derive a formula that can be used to find any term of an arithmetic progression rather easily.

Let us illustrate this derivation using the above arithmetic progression 6, 8, 10, 12, ... For this progression, the first term is 6 and the common difference is 2. Observe carefully how the terms of this progression have been written in terms of the first term and the common difference in the following table.

Term	**Value of the term**	**Value of the term in terms of the first term and the common difference**
T_1	6	$6 \qquad = 6 + (1 - 1) \times 2$
T_2	8	$6 + 2 \qquad = 6 + (2 - 1) \times 2$
T_3	10	$6 + 2 + 2 \qquad = 6 + (3 - 1) \times 2$
T_4	12	$6 + 2 + 2 + 2 = 6 + (4 - 1) \times 2$
...	...	... ...

Now, according to the pattern in the table,

$$
\begin{aligned}
T_{500} &= 6 + (500 - 1) \times 2 \\
&= 6 + 499 \times 2 \\
&= 6 + 998 \\
&= 1004
\end{aligned}
$$

Hence, the 500th term is 1004.

Can you generalize the above pattern further? In other words, can you find a formula for the n^{th} term T_n in terms of the first term a and the common difference d? For this, look at the expression $T_{500} = 6 + (500 - 1) \times 2$ again, where 6 is the 1st term and 2 is the common difference.

You can see that if you follow the above pattern to obtain the n^{th} term T_n of the arithmetic progression with first term a and common difference d, you obtain $T_n = a + (n - 1)\, d$. In this formula, according to our notation, T_n denotes the n^{th} term. Hence, the n^{th} term T_n of the arithmetic progression with first term a and common difference d is given by

$$\boxed{T_n = a + (n - 1)\, d}$$

The importance of this formula is that it gives the relationship between the four unknowns a, d, n and T_n. This formula can be used to find the value of any one of the four unknowns when the values of the other 3 unknowns in an arithmetic progression are known.

Now let us consider how problems on arithmetic progressions are solved using this formula.

Example 1 (Finding T_n when a, d and n are known)

Find the 15th term of the arithmetic progression 3, 7, 11, 15....

Here, $a = 3$, $d = 7 - 3 = 4$, $n = 15$. Substitute these values in $T_n = a + (n-1)d$ to get

$$T_{15} = 3 + (15 - 1) \times 4$$
$$= 3 + 56$$
$$= 59$$

Therefore the 15th term is 59.

Example 2 (Finding a when d, n and T_n are known)

Find the first term of the arithmetic progression with common difference 4 and 26th term equal to 105.

Here, $d = 4$ and $T_n = 105$ when $n = 26$.

Substituting these in $T_n = a + (n-1)\, d$ we get,

$$T_{26} = a + (26 - 1) \times 4$$
$$105 = a + (26 - 1) \times 4$$
$$\therefore 105 - 100 = a$$
$$\therefore \quad a = 5$$

Therefore the first term is 5.

Example 3 (Finding d when a, n and T_n are known)

Find the common difference of the arithmetic progression with 1st term – 32 and 12th term 1.

Here, $a = -32$ and $T_n = 1$ when $n = 12$

Substituting these in $T_n = a + (n-1)\, d$, we get

$$1 = -32 + (12 - 1) \times d$$
$$\therefore \quad 33 = 11 \times d$$
$$\therefore \quad \frac{33}{11} = d$$
$$\therefore \quad d = 3$$

Therefore the common difference is 3.

Example 4 (Finding n when a, d and T_n are known)

Find which term is – 65 in the arithmetic progression 30, 25, 10, ...

Here, $a = 30$, $d = -5$, $T_n = -65$

Substituting these in $T_n = a + (n-1)\, d$,

$$-65 = 30 \times (n-1) \times (-5)$$
$$-65 = 30 - 5n + 5$$
$$-65 - 35 = -5n$$

$$\frac{-100}{-5} = n$$

$\therefore\ n = 20.\quad \therefore -65$ is the 20^{th} term

For an arithmetic progression, when the values of 2 unknowns out of a, d, n and T_n are not given, the values of these two unknowns can be found by solving two simultaneous equations, when sufficient information has been provided.

Example 5

The 7^{th} and 12^{th} terms of an arithmetic progression are 38 and 63 respectively.

Find, (i) the first term and the common difference

(ii) the 20^{th} term.

(i) Since $T_n = 38$ when $n = 7$ and $T_n = 63$ when $n = 12$, we get, by substituting in $T_n = a + (n-1)\,d$,

$T_7 = a + (7-1) \times d$

$38 = a + 6d$ ——— (1)

$T_{12} = a + (12-1) \times d$

$63 = a + 11d$ ——— (2)

Now, let us solve (1) and (2)

(2) – (1) gives

$63 - 38 = a + 11d - (a + 6d)$

$25 = a + 11d - a - 6d$

$\therefore\ 25 = 5d$

$\therefore\ 5 = d$

Substitute $d = 5$, in (1) to get

$38 = a + 6 \times 5$

$\therefore\ 38 - 30 = a$

$\therefore\ a = 8$

$\therefore$ the first term is 8 and the common difference is 5.

(ii) Now that we know the first term and the common difference of the progression, let us substitute $a = 8$, $d = 5$ and $n = 20$ in $T_n = a + (n-1)\,d$.

$T_{20} = 8 + (20-1) \times 5$

$= 8 + 19 \times 5$

$= 8 + 95$

$= 103$

$\therefore$ the 20^{th} term is 103.

Example 6

In a certain sequence, the n^{th} term T_n is given by $T_n = 3n + 4$.

(i) Write down the first 4 terms of this sequence.

(ii) Write down an expression for the $(n-1)^{th}$ term of this sequence and hence show that this is an arithmetic progression.

(iii) Which term is 169 in this arithmetic progression?

(iv) Show that no term in this progression is equal to 95.

(i) We have $T_n = 3n + 4$.

When $n = 1$, $T_1 = 3 \times 1 + 4 = 7$
When $n = 2$, $T_2 = 3 \times 2 + 4 = 10$
When $n = 3$, $T_3 = 3 \times 3 + 4 = 13$
When $n = 4$, $T_4 = 3 \times 4 + 4 = 16$
$\therefore$ the first four terms are 7, 10, 13 and 16, respectively.

(ii) Replacing n by $n - 1$ in $T_n = 3n + 4$, we get

$$T_{n-1} = 3(n-1) + 4$$
$$= 3n - 3 + 4$$
$$= 3n + 1$$
$$T_n - T_{n-1} = (3n + 4) - (3n + 1)$$
$$= 3$$
$$= \text{constant}$$

$\therefore$ the sequence is an arithmetic progression.

(iii) It is given that $T_n = 169$.

Substituting in $T_n = 3n + 4$, we get

$$169 = 3n + 4$$
$$169 - 4 = 3n$$
$$\frac{165}{3} = n$$
$$55 = n$$

$\therefore$ 169 is the 55^{th} term.

(iv) If there is a term with value 95, then there should be a positive integer n such that $T_n = 95$.

Then, $95 = 3n + 4$
$$3n = 95 - 4$$
$$= 91$$
$$\therefore\ n = \frac{91}{3}$$

n is not an integer

$\therefore$ no term is equal to 95.

Exercise 24.1

1. Find the first 5 terms of the arithmetic progression for each of the following situations.

(a) $a = 5;\ d = 2$ **(b)** $a = -3;\ d = 4$

(c) $a = 4.5;\ d = 2.5$ **(d)** $a = 10\frac{1}{4};\ d = -\frac{1}{2}$

(e) $a = 2x;\ d = x + 3$

2. Find the indicated term of each of the following arithmetic progressions.

(a) 13, 15, 17, ... (10^{th} term) **(b)** 40, 38, 36, ... (21^{st} term)

(c) –2, –7, –12, ... (15^{th} term) **(d)** –3, 2, 7, ... (20^{th} term)

(e) 6.5, 8, 9.5, ... (12^{th} term) **(f)** $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, ... (11^{th} term)

(g) $12\frac{1}{2}$, 12, $11\frac{1}{2}$, ... (18^{th} term)

3. (*a*) Using the given information, find the first term of the relevant arithmetic progression for each of the following situations.

(i) $d = 5;\ T_{21} = 101$

(ii) $d = -3;\ T_{35} = -113$

(iii) $d = 2\frac{1}{2}\ ;\ T_{37} = 93$

(*b*) Using the given information find the common difference of the relevant arithmetic progression for each of the following situations.

(i) $a = 60;\ T_{15} = 102$

(ii) $a = -30;\ T_{35} = -25$

(iii) $a = 4\frac{1}{4};\ T_{37} = -7\frac{3}{4}$

(*c*) For each of the following situations, find the number of terms (n) of the relevant arithmetic progression.

(i) $a = 9;\ d = 4;\ T_n = 69$

(ii) $a = -20;\ d = \frac{1}{2}\ ;\ T_n = 35$

(iii) $a = 7\ ;\ d = \frac{1}{2}\ ;T_n = 27$

4. For each of the following arithmetic progressions find the n^{th} term in the simplest form.

(i) 7, 12, 17, 22, ...

(ii) –15, –12, –9, – 6, ...

(iii) $3\frac{1}{4}$, 4, $4\frac{3}{4}$, ...

(iv) 67, 64, 61, ...

5. Find the

(i) first three terms

(ii) common difference

(iii) 15^{th} term

for each of the arithmetic progressions with n^{th} term given by

(i) $2n + 1$ (ii) $5n – 1$ (iii) $8 – n$ (iv) $20 – 5n$

6. Between 1 and 150, how many multiples of

(i) 2

(ii) 3

(iii) 5

are there?

7. (i) In an arithmetic progression, the third term is 7 and the sixth term is 13. Find the first term of this progression.

(ii) In an arithmetic progression, the fifth term is 34 and the fifteenth term is 9. Which term in this progression is – 6?

(iii) In an arithmetic progression, the fifth term is 22 and the tenth term is 47. Show that the fifteenth term is six times the third term.

(iv) In an arithmetic progression, the sum of the third and sixth terms is 42 and the sum of the second and tenth terms is 54. Which term in this progression is 63? Show further that no term in this progression is 30.

(v) In an arithmetic progression, the second term is 10 and the value of the twelvth term is 12 more than that of the tenth term. Find the first term, the common difference and the 21^{st} term of this progression.

(vi) Which term is 52 more than the 7^{th} term of the arithmetic progression 3, 7, 11, ...

24.2 Sum of the first *n* terms of an arithmetic progression

Consider the arithmetic progression 3, 5, 7, 9, 11, 13, 15, 17, ... in which the first 8 terms are written down. The sum of these 8 terms is

$$3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 80.$$

We denote by S_n the sum of the first n terms of an arithmetic progression. In this notation, the sum of the first 8 terms of the above arithmetic progression can be written as

$$S_8 = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$$

$$S_8 = \underline{\underline{80}}$$

When we have to find the sum of a large number of terms, this lengthy way of adding all the terms one by one is a tedious task. In order to overcome this problem, let us derive a formula that can be used to find the sum of the initial terms of an arithmetic progression.

Considering the above example, we rewrite

$S_8 = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$ ——— ①

and again by reversing the terms we write

$S_8 = 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3$ ——— ②

Now from ① and ② we may write, rearranging the terms,

$$2\,S_8 = (3 + 17) + (5 + 15) + (7 + 13) + (9 + 11) + (11 + 9) + (13 + 7) + (15 + 5) + (17 + 3)$$

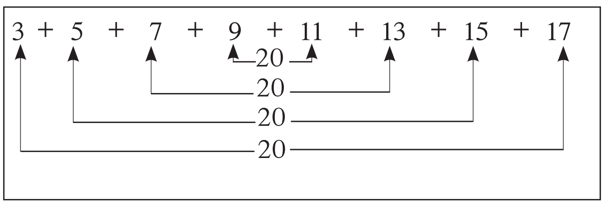
$$2\,S_8 = 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20$$

$$\therefore\ 2\,S_8 = 8 \times 20 \quad \text{(there are 8 terms of 20)}$$

$$\therefore\ S_8 = \frac{8}{2} \times 20$$

$$= 80$$

We may illustrate the above method as follows.



There are 8 terms in the progression.

The first term and the last term add to 20.

The second and the penultimate term add to 20.

Continuing this, we see that the sum of the first 8 terms of the progression is a sum of 4 pairs.

The number of such pairs is one half the total number of terms to be added.

Then the sum of all the terms is the product,

$$\frac{\text{(number of terms)}}{2}\,\text{(first term + last term)}$$

$$\therefore\ S_8 = \frac{8}{2}[3 + 17]$$

Now, let us derive a formula using the above process, for the sum of the first n terms, S_n, of an arithmetic progression with first term a, common difference d and last (n^{th}) term l.

We may write,

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + ... (l - 2d) + (l - d) + l$ ——— ①

and then reversing the terms,

$S_n = l + (l - d) + (l - 2d) + (l - 3d) + ... (a + 2d) + (a + d) + a$ ——— ②

Now ① and ② gives

$$2S_n = (a + l) + (a + l) + (a + l) + ... (a + l) + (a + l) + (a + l)$$

$$2S_n = n(a + l) \quad \text{[there are } n \text{ number of } a + l \text{ terms]}$$

$$\therefore \boxed{S_n = \frac{n}{2}(a+l)}\ .$$

As an example, let us find using the above formula, the sum of all integers from 1 to 100. The relevant progression is 1, 2, 3, 4, ...98, 99, 100, where $a = 1$, $l = 100$ and $n = 100$.

$$\therefore \quad S_{100} = \frac{100}{2}(1+100)$$

$$S_{100} = 50\,(101)$$

$$\therefore S_{100} = \underline{\underline{5050.}}$$

$\therefore$ the sum of the integers from 1 to 100 is 5050.

The above formula can be used to find the sum of an arithmetic progression given the first term (a), the number of terms (n) and the last term (l).

Now we derive a formula for the sum, given the first term (a), the number of terms (n) and the common difference (d).

For this purpose, in the formula

$$S_n = \frac{n}{2}(a+l)\ ,$$

we substitute for l, which is actually T_n, the n^{th} term given by $T_n = a + (n - 1)\,d$. we then get,

$$S_n = \frac{n}{2}\{a+a+(n-1)d\},$$

and this simplifies to

$$S_n = \frac{n}{2}\{2a+(n-1)d\}\cdot$$

Therefore, to find the sum S_n of the first n terms of an arithmetic progression with first term a and common difference d, use the formula

$$\boxed{S_n = \frac{n}{2}\{2a+(n-1)d\}}$$

As an example, let us find the sum of the first 30 terms of the arithmetic progression 2, 4, 6, 8, ... In this progression, $a = 2$ $d = 2$ and $n = 30$.

Substituting in $S_n = \frac{n}{2}\{2a + (n-1)d\}$

we get

$$S_{30} = \frac{30}{2}\{2\times 2 + (30-1)\times 2\}$$
$$= \frac{30}{2}\{4 + 29\times 2\}$$
$$= \frac{30}{2}\{62\}$$
$$= 15\times 62$$
$$\therefore\ S_{30} = 930$$

$\therefore$ the sum of the first 30 terms is 930.

In summary, we could use

- ★ $S_n = \frac{n}{2}(a+l)$ when the first term, the number of terms and the last term are known
- ★ $S_n = \frac{n}{2}\{2a + (n-1)d\}$ when the first term, the number of terms and the common difference are known

Let us look at some examples now.

Example 1 Find the sum of the first 12 terms of the arithmetic progression 5, 10, 15, 20, ...

Here $a = 5$, $d = 5$ and $n = 12$. Substituting these in $S_n = \frac{n}{2}\{2a + (n-1)d\}$,we get,

$$S_{12} = \frac{12}{2}\{2\times 5 + (12-1)\times 5\}$$
$$= \frac{12}{2}\{10 + 11\times 5\}$$
$$= 6\ \{10 + 55\}$$
$$= 6\times 65$$
$$= 390$$

$\therefore$ the sum of the first 12 terms is 390.

Example 2 An arithmetic progression has 16 terms. The first term and the last term are 75 and 0 respectively. Find its sum if the common difference is – 5.

Here $n = 16$, $a = 75$, $d = -5$, and $l = 0$.

Substituting in $S_n = \frac{n}{2}(a+l)$,

$$S_{16} = \frac{16}{2}(75+0)$$
$$= \frac{16}{2} \times 75$$
$$= 8 \times 75$$
$$= 600$$

$\therefore$ the sum of the arithmetic progression is 600 .

Example 3 Find the sum of the arithmetic progression 70, 66, 62, 58, ..., 2

Here $a = 70$, $l = 2$, and $d = -4$.
First we have to find the number of terms in the progression.
Using the formula $T_n = a + (n-1)d$, we get,

$$2 = 70 + (n-1)\times(-4)$$
$$2 = 70 - 4n + 4$$
$$2 - 74 = -4n$$
$$\frac{-72}{-4} = n$$
$$18 = n$$

The progression has 18 terms and its sum can be found by using the formula

$$S_n = \frac{n}{2}(a+l).$$
$$S_{18} = \frac{18}{2}(70+2)$$
$$= \frac{18}{2}\times 72$$
$$= 9 \times 72$$
$$= 648$$

$\therefore$ the sum of the terms of the series is 648 .

The formula $S_n = \frac{n}{2}(a+l)$ is in 4 unknowns, namely S_n, *n*, *a* and *l*. Whenever three of these unknowns are given, the remaining unknown can be found by using the formula. The case for the formula $S_n = \frac{n}{2}\{2a+(n-1)d\}$ is similar.

Let us look at some examples to illustrate such situations.

Example 4

The first term, the last term and the sum of all the terms of an arithmetic progression are 12, 99 and 1665 respectively. Find the number of terms, the common difference and the sum of the first 15 terms.

Here $a = 12,\ l = 99,\ S_n = 1665$.

We use $S_n = \frac{n}{2}(a+l)$ to find *n*. Substitution gives

$$1665 = \frac{n}{2}(12+99)$$

$$3330 = n \times 111$$

$$\frac{3330}{111} = n$$

$$30 = n$$

$\therefore$ there are 30 terms in the progression.

Substituting,

$T_n = l = 99$, $a = 12$ and $n = 30$ in the formula $T_n = a+(n-1)d$,

$$99 = 12+(30-1)d$$

$$99-12 = 29d$$

$$d = \frac{87}{29}$$

$$= 3$$

$\therefore$ the common difference is 3.

Finally, in order to find the sum of the first 15 terms, substitute $n = 15$, $a = 12$, $d = 3$ in $S_n = \frac{n}{2}\{2a+(n-1)d\}$ to get

$$S_{15} = \frac{15}{2}\{2\times12+(15-1)\times3\}$$
$$= \frac{15}{2}\{24+14\times3\}$$
$$= \frac{15}{2}\{24+42\}$$
$$= \frac{15}{2}\{66\}$$
$$= 15\times33$$
$$S_{15} = \underline{\underline{495}}$$

$\therefore$ the sum of the first 15 terms is 495.

Example 5

How many terms (starting from the 1st term) of the arithmetic progression 13, 11, 9, ..., add to 40?

Substitute $a = 13$, $d = -2$, $S_n = 40$.

$$S_n = \frac{n}{2}\{2a+(n-1)d\}$$
$$40 = \frac{n}{2}\{2\times13+(n-1)\times(-2)\}$$
$$80 = n\{26-2n+2\}$$
$$80 = 28n-2n^2$$
$$2n^2-28n+80 = 0$$
$$n^2-14n+40 = 0$$
$$(n-10)(n-4) = 0$$
$$n-10=0 \text{ or } n-4=0$$
$$n=10 \text{ or } n=4.$$

Note:

Here there are two acceptable solutions for n.

Sum of the first four terms when n is 4 $= 13 + 11 + 9 + 7 = 40$

Sum of the first ten terms when n is 10 $= 13+11+9+7+5+3+1+-1+-3+-5$

$= \underline{\underline{40}}$

Here, both values are acceptable. Therefore 4 or 10 terms can be taken for the sum to be 40.

Exercise 24.2

1. For each of the following cases, find the sum of the relevant arithmetic progression using the given data.

(i) $a = 2,\ l = 62$ and $n = 31$

(ii) $a = 95,\ l = 10$ and $n = 12$

(iii) $a = 7\frac{1}{2},\ d = \frac{1}{2}$ and $n = 15$

(iv) $a = 3.25,\ d = 1.7$ and $n = 21$

2. Find the sum of the indicated number of terms in each of the following progressions

(i) 3, 7, 9, ... first 11 terms

(ii) –10, –9. 7, –9.4, ... first 20 terms

(iii) $1, 1\frac{3}{4}, 2.5, \ldots$ first 17 terms

(iv) 67, 65, 63, ... first 12 terms

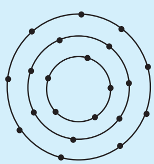
3. (i) Find the number of odd numbers between 2 and 180 and then find their sum.

(ii) Find the number of positive terms divisible by 5 below 200 and then find their sum.

(iii) Find the number of terms between 3 and 200, whose remainder upon division by 4 is 1, and then find their sum.

(iv) Find the sum of the terms between 5 and 170 which are not divisible by 3.

4. The sum of the first 4 terms of an arithmetic progression is 36 and the 11th term is 43. Find the first term and the common difference of this progression and also find the sum of the first 15 terms.

5. 

The figure shows the way some small light bulbs have been connected in a circular pattern in the first three rings of a decoration. The last ring in this decoration contains 35 bulbs.

(i) How many rings are there in the decoration?

(ii) How many bulbs have been used in total?

(iii) If one bulb costs 50 rupees, how much money was spent on the bulbs in total?

6. The installments per month and the number of months interest needs to be paid for a loan of Rs. 50 000 taken from two financial institutions P and Q are as given below:

P: 11 000, 10 000, 9 000, … for 11 months

Q: 14 000, 15 000, 16 000, … for 8 months

To which institution does one have to pay less interest? Give reasons.

7. A father opens a bank account for his daughter by depositing Rs. 500 on her tenth birthday. Every month, he deposits in that account an amount of money equal to the amount he deposited in the previous month plus a constant amount. Find the constant amount of money the father should deposit so that the total amount in the account, without interest, at her 18th birthday is Rs.504 000.

8. The n^{th} terms of an arithmetic progression is given by $T_n = 63 - 2n$.

(i) Write down the first four terms.
(ii) Find the sum of the first 21 terms.
(iii) Find the 21st term.
(iv) Find the number of terms that should be added to get a sum of 336.

9. Find the number of terms of an arithmetic progression needed to get the indicated sum in each of the following cases:

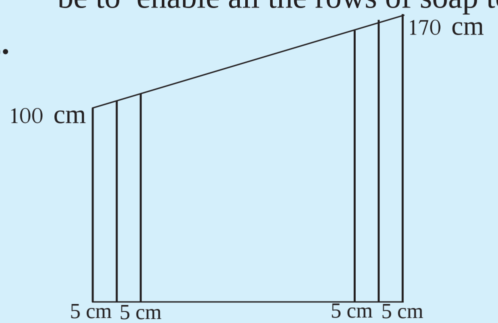
(i) $a = 7$, $l = 10$, $S_n = 34$
(ii) $a = 63$, $d = 3$, $S_n = 345$

Miscellaneous Exercises

1. In a shop, bars of soap are stacked on top of each other on a rack in such a way that the bottom row has 24 bars, the row above that has 21 bars, and the row above that has 18 bars and so on.

(i) Find the number of bars in the 8^{th} row from the bottom.
(ii) If the top row has 3 bars of soap, find the total number of rows and the total number of bars of soap.
(iii) If the bars of soap are 5 cm wide, find the minimum height the rack should be to enable all the rows of soap to be placed on it.

2.



The figure shows a sketch of one part of a gate having two equal parts, which is made by fixing planks of wood together. Each plank is 5 cm wide. The shortest plank is 100 cm long and each plank fixed after the shortest is 5 cm longer than the previous plank. The longest plank is 170 cm long.

(i) Find the number of planks used for one part of the gate.
(ii) Find the minimum width of the gate.

(iii) Find the total length of all the planks used for the gate.

(iv) If one 30 cm long plank costs Rs. 50, then find the total cost of planks required for the gate.

3. The sum of the first n terms of a series is given by $S_n = n^2 - 8n$.

(i) Write the first term.

(ii) Find the sum of the first two terms.

(iii) Find the common difference.

(iv) How many terms, starting from the first term, add up to 180?

4. The pages numbered 3, 5, 7,...of a magazine are printed in a special pink colour. Thushan reads 5 pages of the magazine on the first day and thereafter, on each day, he reads 3 more pages than he read on the previous day.

(i) Find the number of pages he completes reading at the end of the fifth day.

(ii) Find the number of pages he completes reading at the end of the seventh day.

(iii) If Thushan finishes reading the whole magazine in 10 days, find the number of pages there are in the magazine.

(iv) What is the maximum number of pink pages the magazine can have?

(v) Thushan claims that the last page he read is pink. Determine the truth of his statement.

25 Algebraic Inequalities

By studying this lesson you will be able to

- solve inequalities and representing the solutions on a number line,
- represent inequalities on a coordinate plane.

Let us recall what has been learnt earlier about inequalities by considering the following examples.

Example 1

Solve the inequality $x + 20 > 50$ and
 (i) write down the set of integral values that x can take.
 (ii) represent the integral values that x can take on a number line.

$$x + 20 > 50$$
$$x > 50 - 20$$
$$x > 30$$

(i) $\{31, 32, 33, 34, ...\}$

(ii) 

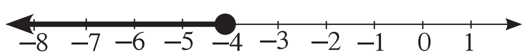
Example 2

Solve the inequality $-3x \geq 12$ and represent all the values that x can take on a number line.

$-3x \geq 12$ (When an inequality is divided by a negative number, the inequality sign changes)

$$\frac{-3x}{-3} \leq \frac{12}{-3}$$

$$x \leq -4$$



Review Exercise

1. Solve each of the following inequalities.

(i) $x+4>11$ (ii) $y+3\geq 0$ (iii) $p-5<2$ (iv) $p-3>-1$

(v) $a+5\leq 1$ (vi) $5y<12$ (vii) $-2x\geq 10$ (viii) $-3y<-9$

(ix) $\frac{-2x}{3}>6$

2. Solve each of the following inequalities and represent the solutions on a number line.

(i) $x+3\geq 1$ (ii) $y-4<-1$ (iii) $3x>-3$

(iv) $\frac{x}{2}\leq 0$ (v) $-5y>10$ (vi) $-4x\geq 12$

3. For each of the following inequalities, one of the values of x which satisfies the inequality is given within the brackets. Select that value and underline it.

(i) $x+3>7\ (4,7)$ (ii) $x-3<2\ (1,6)$ (iii) $3x>7\left(2.3,\frac{8}{3}\right)$

(iv) $-2x<8\ (-5,3)$ (v) $5-x>6\ (12,-2)$

4. (i) Solve the inequality $x+1>-2$ and write down the smallest integral value that x can take.

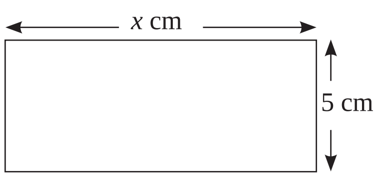
(ii) Solve the inequality $-3y>15$ and write the largest integral value that y can take.

5. Solve the inequalities $x+3>1$ and $2x\leq 12$ and represent all the solutions on a number line.

25.1 Inequalities of the form $ax+b \gtreqless c$

Example 1

Nimal who constructed a rectangular structure of breadth 5 cm as shown in the figure using a 30 cm long piece of wire, saved a small piece of the wire.



If the length of the rectangle is taken as x, an inequality in terms of x, involving the perimeter of the rectangular structure is given by $2x+10<30$. On a number line, represent all possible values that x can take if $x>5$.

$$2x + 10 < 30$$
$$2x + 10 - 10 < 30 - 10$$
$$2x < 20$$
$$\frac{2x}{2} < \frac{20}{2}$$
$$x < 10$$

Example 2

Solve the inequality $3 - 2x \le 9$ and on a number line, represent all the possible values that x can take.

$$3 - 2x \le 9$$
$$\cancel{3} - 2x - \cancel{3} \le 9 - 3$$
$$-2x \le 6$$
$$\frac{-2x}{-2} \ge \frac{6}{-2}$$
$$\underline{\underline{x \ge -3}}$$

Exercise 25.1

1. Solve each of the following inequalities.

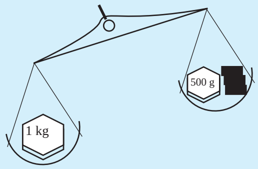
(i) $4x + 1 > 5$ (ii) $5x - 3 < 7$ (iii) $3 + 2p \ge 1$ (iv) $7x + 9 < -5$

(v) $-2y - 5 > 1$ (vi) $3 - 4x \ge 3$ (vii) $8 - 4y < 0$ (viii) $2(3 - x) > 10$

2. Solve each of the following inequalities and write down the set of integral solutions of the inequality.

(i) $5x + 1 > -4$ (ii) $3y - 1 \ge 2$ (iii) $-2p - 4 < 0$ (iv) $7 - 4p > 3$

3. Rs. 100 is sufficient to buy 3 mangoes and 2 mandarins. If the price of a mango is Rs. 20 and the price of a mandarin is taken to be Rs. y, then an inequality $60 + 2y \le 100$ in y can be written. Solve this inequality and find the maximum value that the price of a mandarin can take.

4. 

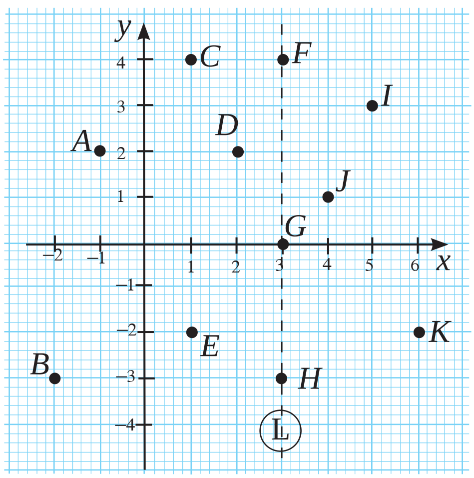
Nimal who placed a 1 kg standard weight in one pan of a balance scale, placed a 500 g standard weight and three cakes of soap of the same type on the other pan. He observed that the pan with the 1 kg standard weight dipped below the other pan.

If the mass of a cake of soap is taken as p grammes, an inequality $1000 > 500 + 3p$ in terms of p can be written. Find the maximum integral value that the mass of a cake of soap can be.

25.2 Regions represented by inequalities of the form $y \gtreqless a$ and $x \gtreqless b$

Regions which are separated by a line parallel to the *y* - axis

The points *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I*, *J* and *K* and the straight line (L) drawn parallel to the y axis are represented on the Cartesian plane given in the figure.



Consider the following tables and the related properties.

Points lying on the line (L)	***x* coordinate**	***y* coordinate**
F	3	4
G	3	0
H	3	–3

- The x coordinate of the points that lie on the line (L) is equal to 3.
- Therefore the straight line (L) is named $x = 3$.
- The x coordinate of any point that lies on the straight line $x = 3$ is equal to 3.

Points lying to the right of the line (L)	x coordinate	y coordinate
I	5	3
J	4	1
K	6	–2

- The x coordinate of each of the points that lie to the right of the straight line (L) is greater than 3.
- Therefor the region to the right of the straight line (L) is named $x > 3$.
- The x coordinate of any point that belongs to the region $x > 3$ is greater than 3.

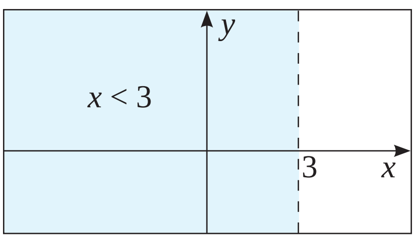
Points lying to the left of the line (L)	x coordinate	y coordinate
A	–1	2
B	–2	–3
C	1	4
D	2	2
E	1	–2

- The x coordinate of the points that lie to the left of the straight line (L) is less than 3.
- Therefore the region to the left of the straight line (L) is named $x < 3$.
- The x coordinate of any point that belongs to the region $x < 3$ is less than 3.

It is clear that the Cartesian plane illustrated in the above example is divided into three specific regions, namely $x < 3$, $x = 3$ and $x > 3$ by the straight line $x = 3$.

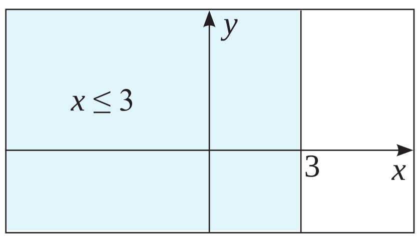
Now let us see how these regions are represented in a Cartesian plane.

The region $x < 3$



The straight line $x = 3$ is indicated by a dashed line. This means that the points such that $x = 3$ do not belong to the region $x < 3$.

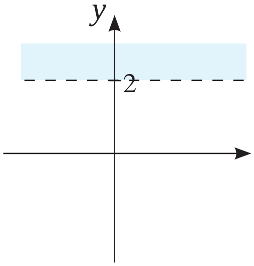
The region $x \leq 3$



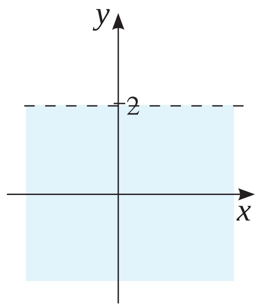
The line $x = 3$ has been indicated by a thick line. This means that both the regions $x < 3$ and $x = 3$ belong to the shaded region. Therefore this region is named $x \leq 3$.

A few more examples to illustrat the regions on a Cartesian plane separated by a line parallel to the x axis are given below.

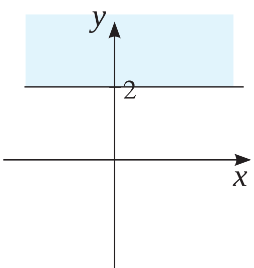
$y > 2$



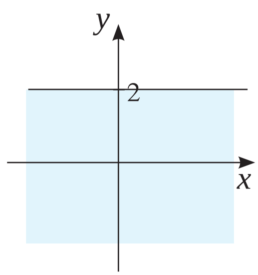
$y < 2$



$y \geq 2$



$y \leq 2$



Exercise 25.2

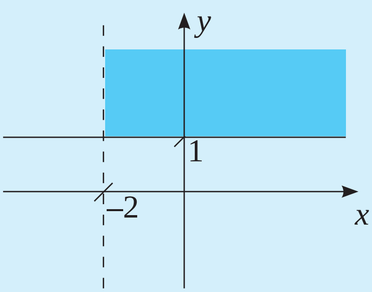
1. Write down the coordinates of three points belonging to the region $x < -2$.

2. Write down the coordinates of three points belonging to the region $x > -1$.

3. Write down the coordinates of three points belonging to both the regions $x > 1$ and $y < -2$.

4. Which of the following points belong to both the regions $x \leq -2$ and $y > 0$.
$A = (-3, 0)$ $B = (-2, 1)$ $C = (-1, 4)$

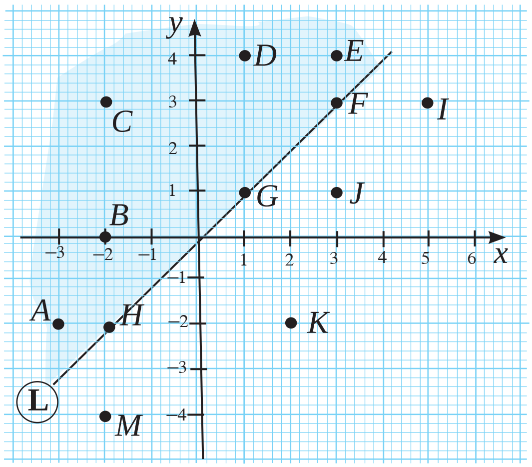
5. Write the two inequalities relevant to the shaded region.



6. In a Cartesian plane, shade the region satisfying the four inequalities $x > 1$, $x \geq 3$ $y \leq 2$, and $y > -1$.

25.3 Inequalities of the form $y \gtrless x$

The points *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I*, *J*, *K* and *M* and the straight line (L) are represented on the Cartesian plane given in the figure.



Points lying on the line (L)	*x* coordinate	*y* coordinate
F	3	3
G	1	1
H	– 2	– 2

- The *y* coordinate of each of the points that lie on the line (L) is equal to the corresponding *x* coordinate.
- Therefore the line (L) is named $y = x$.

Points belonging to the shaded region	*x* coordinate	*y* coordinate
A	–3	–2
B	–2	0
C	–2	3
D	1	4
E	3	4

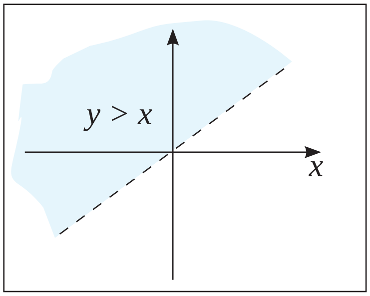
- The *y* coordinate of each of the points in the shaded region is greater than the corresponding *x* coordinate.
- Therefore the shaded region is named $y > x$.

Points belonging to the unshaded region	x coordinate	y coordinate
I	5	3
J	3	1
K	2	–2
M	–2	–4

- The y coordinate of each of the points in the unshaded region is less than the x coordinate.
- Therefore the unshaded region is named $y < x$.

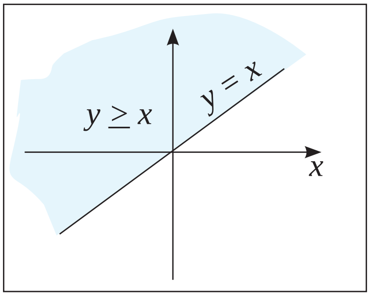
Now let us see how a few more inequalities are represented on a Cartesian plane.

(i) $y > x$



What is meant by indicating $y = x$ by a dashed line is that the points satisfying $y = x$ do not belong to the shaded region $y > x$.

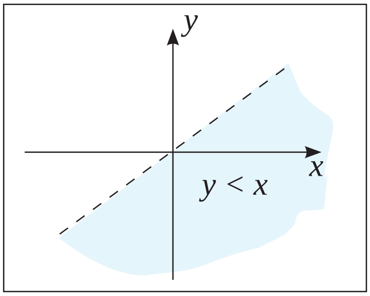
(ii) $y \geq x$



What is meant by indicating $y = x$ by a thick line is that the points satisfying $y = x$ belong to the shaded region $y \geq x$.

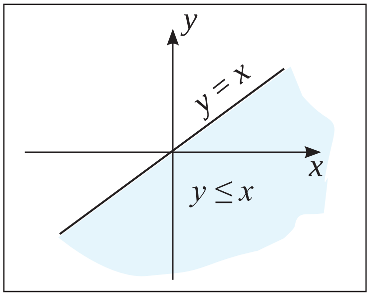
(iii)

$y < x$



(iv)

$y \leq x$



Exercise 25.3

1. Write down the coordinates of 3 points that belong to the region $y = x$.

2. Which of the following points belong to the region $y \geq x$?
$A = (5, 5)$ $B = (-3, -2)$ $C = (0, -1)$

3. Write down the coordinates of three points which satisfy both the inequalities $y < -2$ and $y > x$.

4. In a Cartesian plane, shade the area common to both the regions represented by the inequalities $x \geq 0$ and $y > x$.

5. Write down the coordinates of 3 points which satisfy the three inequalities $x < 3$, $y > 0$ and $y < x$.

26 Frequency Distributions

By studying this lesson you will be able to

- find the mean of grouped data.

Grouped Data

Data collected through a survey on the number of family members that reside in each of the houses in a certain housing scheme is given below.

4, 5, 2, 7, 4, 3, 6, 8, 9, 5, 5, 4, 4, 6, 3
8, 4, 5, 6, 4, 6, 5, 5, 4, 2, 4, 5, 3, 5, 7
5, 5, 7, 5, 3, 5, 7, 5, 4, 5, 6, 4, 4, 6, 4

The greatest value of this data set is 9 and the least value is 2. The value that is obtained when the least value is subtracted from the greatest value is defined as the **range**.

$\therefore$ The range of the given data set $= 9 - 2$
$= 7$

When the range of a data set is small, as in the above case, the information related to the data can be tabulated as follows. A table of the following form is called a frequency distribution.

Number of family members residing in the house	**Frequency (Number of families)**
2	2
3	4
4	12
5	14
6	6
7	4
8	2
9	1

Let us consider another example.

Information on the marks obtained for mathematics at the term test by the grade 10 students of a certain school is given below.

25, 12, 65, 40, 32, 84, 52, 65, 32, 09
70, 53, 67, 56, 65, 48, 20, 17, 08, 43
52, 68, 73, 25, 39, 42, 61, 22, 37, 45
36, 65, 24, 53, 46, 18, 39, 54, 26, 35
27, 94, 59, 87, 72

In this case, the greatest value is 94 while the least value is 8.

Accordingly, the range of the data set $= 94 - 8$
$= 86$

Since the range of this data set is large, if we tabulate this information under each value from 8 to 94, we will obtain a very long table.

When the range of the data set is large, as in the above case, it is convenient to divide the data into groups and tabulate the information.

Let us now see how a set of data is separated into groups (class intervals).

A frequency distribution prepared by separating the above data into class intervals is given below.

Class Interval	Frequency
8 - 16	3
17 - 25	7
26 - 34	4
35 - 43	8
44 - 52	5
53 - 61	6
62 - 70	7
71 - 79	2
80 - 88	2
89 - 97	1

What is meant by stating that the frequency of the class interval 8 - 16 is 3, is that there are 3 values (data) that lie in the interval from 8 to 16.

The greatest frequency of this distribution is 8. It corresponds to the class interval 35 - 43. This class interval is named the **modal class**.

A frequency distribution such as the above which is expressed using class intervals is called a **grouped frequency distribution**.

When preparing a grouped frequency distribution, the class intervals are formed so that there are about 10 class intervals.
Observe that all the class intervals in this example are of the same size.That is, the **size** of each class interval is 9.

The initial class interval of this distribution is 8 - 16, and the next class interval is 17 -25. The relevant data are test marks. Since there are no marks between 16 and 17, the class intervals have been organized such that when the first class interval ends with 16, the next class interval starts with 17.

Now let us consider how the mean of a grouped frequency distribution such as the above distribution is found. For this, the mid-value of each class interval has to be found first.

26.1 Mid-value of a class interval

Let us find the mid-value of the class interval 8 - 16 in the above example.

It can be found as follows; $\frac{8+16}{2} = 12$.

Accordingly, the mid-value of the class interval 8 – 16 is 12.
The mid-value of a class interval is found by adding together the upper value and the lower value of the class interval and then dividing this sum by 2. The mid-value of every class interval can be found in this manner.

The mid-value of a class interval is considered as the representative value of the data in that class interval when calculations are performed.

Class Interval	Mid-value	Frequency
8 - 16	12	3
17 - 25	21	7
26 - 34	30	4
35 - 43	39	8
44 - 52	48	5
53 - 61	57	6
62 - 70	66	7
71 - 79	75	2
80 - 88	84	2
89 - 97	93	1

A grouped frequency distribution prepared with the data collected on the ages of the staff members in a certain office is given below.

Age of the staff member (Years)	20 - 25	25 - 30	30 - 35	35 - 40	40 - 45	45 - 50	50 - 55	55 - 60
Number of staff members	5	3	3	5	4	2	2	1

Let us recall how the class intervals were written in the example discussed earlier on the mathematics marks of the grade 10 students. The first class interval was written as 8 – 16 and the second class interval was written as 17 – 25. Since there were no marks between 16 and 17, it was appropriate to separate the class intervals in this manner. However in this example, the first class interval is written as 20 – 25, and the second class interval is written as 25 – 30. That is, the second class interval starts with the same value that the first class interval ends. The reason for this is because the data collected here are the ages of people. Since there can be people whose ages lies between 25 and 26 years, the second class interval should commence with the value with which the previous class interval ended.

The ages 20 or greater but less than 25 belong to the class interval 20 – 25. Accordingly the age 25 years belongs to the class interval 25 – 30 years.

Data such as length, weight, etc,. Which can take any value within a range are called continuous data. Data such as no.of students, marks, etc,. Which take only integral values are called discrete data.

Given below is the grouped frequency distribution of the ages of the staff members, together with the mid-values of the class intervals.

Class Interval	**Mid-value**	**Frequency**
20 - 25	22.5	5
25 - 30	27.5	3
30 - 35	32.5	3
35 - 40	37.5	5
40 - 45	42.5	4
45 - 50	47.5	2
50 - 55	52.5	2
55 - 60	57.5	1

Exercise 26.1

1. The marks obtained by several grade 10 students of a certain school has been grouped and tabulated as follows.

Class Interval	Mid-value	Frequency
11 - 20	15.5	1
21 - 30		7
31 - 40		9
41 - 50		8
51 - 60		10
61 - 70		7
71 - 80		4
81 - 90		2
91 - 100		2

(i) Complete the mid-value column.
(ii) What is the size of each class interval?
(iii) What is the modal class?

2. The data (height to the nearest centimeter) obtained by measuring the heights of the children in a certain class is given below

Class Interval	Mid-value	Frequency
140 - 145		5
145 - 150		8
150 - 155		15
155 - 160		7
160 - 165		8
165 - 170		6

(i) Copy the table and complete the mid-value column.
(ii) By using the table, find the number of children in the class whose heights are less than 150 cm.
(iii) To which class interval do the heights of the most number of students belong?

3. A grouped frequency distribution prepared using the information on the number of students who were present in a certain school during the first term is given below.

Class Interval (Number who were present)	**Mid-value**	**Frequency (Number of days)**
531 - 550		4
551 - 570		10
571 - 590		21
591 - 610		12
611 - 630		10

(i) Copy the table and complete the mid-value column.
(ii) On how many days were there less than 591 students present?
(iii) On how many days were there more than 570 students present?
(iv) How many days was the school in session during the given term?

4. The information obtained from a test conducted to determine the lifetime of a certain type of light bulb is given below.

Time the bulb remained lit (hours)	**Mid-value**	**Number of bulbs**
100 - 200		5
200 - 300		12
300 - 400		25
400 - 500		30
500 - 600		16
600 - 700		12

(i) Copy the table and complete the mid-value column.
(ii) How many bulbs burned out in less than 400 hours?
(iii) How many bulbs were used to conduct this test?
(Assume that the number of hours each bulb that was used remained lit was between 100 hours and 700 hours)

26.2 Calculating the mean of grouped data

When calculating the mean of grouped data, the mid-value of a class interval is taken as the value which represents the whole class interval. Let us now consider how the mean of grouped data is calculated using the mid-values.

Example 1

The following grouped frequency distribution gives the marks that 40 students received out of 25 on a mathematics test.

Class Interval (Marks)	04 - 08	08 - 12	12 - 16	16 - 20	20 - 24
Frequency	3	7	15	11	4

By using the above information, let us construct a table which contains a mid-value column and a column consisting of the product of the mid-values and the corresponding frequencies. Here x denotes the mid-value and f denotes the frequency.

Class Interval	Mid-value x	Frequency f	fx
04 - 08	6	3	18
08 - 12	10	7	70
12 - 16	14	15	210
16 - 20	18	11	198
20 - 24	22	4	88
		$\Sigma f = 40$	$\Sigma fx = 584$

Here, Σf denotes the total number of children, fx denotes the product of the mid-value x and the relevant frequency f, and Σfx denotes the sum of the values in the fx column.Then the mean is defined by $\frac{\sum fx}{\sum f}$.

That is,

$$\boxed{\text{Mean} = \frac{\sum fx}{\sum f}}$$

$\therefore$ For the above grouped frequency distribution,

$$\begin{aligned}\text{Mean} &= \frac{\Sigma fx}{\Sigma f}\\ &= \frac{584}{40}\\ &= \underline{\underline{14.6}}\end{aligned}$$

Therefore, the mean mark obtained by the students is 14.6

Exercise 26.2

1. A frequency distribution prepared using the data on the quantity of beans that was brought to a certain vegetable collection centre on a certain day by 40 farmers is given below. This information was obtained through a survey conducted on the quantities of vegetables that are brought to the vegetable collection centre by farmers.

Mass (kg)	14 - 18	18 - 22	22 - 26	26 - 30	30 - 34
Number of Farmers	3	7	15	11	4

(i) Calculate the mean quantity of beans that was brought by the farmers.
(ii) Accordingly, what is the quantity of beans that can be expected to be brought to this centre during a 10 day period?

2. Information on the number of shirts produced each day of a month by a certain garment factory is given in the following frequency distribution.

Number of shirts	01 - 15	16 - 30	31 - 45	46 - 60	61 - 75
Number of days	4	8	6	8	4

(i) Calculate the mean number of shirts that is produced per day according to the above information.
(ii) Based on the mean, find the number of shirts that can be expected to be produced during three months.

3. A frequencey distribution of the marks received for an assignment by 30 students in a certain class is given below.

Class Interval	1 - 10	11 - 20	21 - 30	31 - 40	41 - 50
Frequency	2	9	13	4	2

(i) What is the size of a class interval?
(ii) What is the modal class?
(iii) Find the mean mark obtained by a student in the class.

4. The age groups that the teachers serving in a certain educational division belong to are given in the following table.

Age (Years)	21 - 26	26 - 31	31 - 35	36 - 41	41 - 46	46 - 51	51 - 56
Frequency	11	32	51	40	27	18	6

(i) How many teachers are serving in this educational division?
(ii) To which age group does the most number of teachers belong?
(iii) Calculate the mean age of a teacher serving in this educational division based on the given information.

5. The information obtained by measuring the circumference of the tree trunks stacked in a certain lorry is given below.

Circumference of a tree trunk (cm)	0 - 25	25 - 50	50 - 75	75 - 100	100 - 125
Frequency	8	10	12	20	18

(i) Find the modal class of this distribution.
(ii) Using the above information, calculate the mean circumference of a tree trunk that was stacked in the lorry.

26.3 Calculating the mean using the assumed mean

The class intervals of a grouped frequency distribution may sometimes contain large mid-values. In such situations, finding the mean using the above method may not be easy. Let us consider a more suitable method of finding the mean of a distribution of this type, through an example.

That is, by considering a simple example, let us explain how the mean is calculated using the assumed mean.

Example 1

The following table contains data on the number of units of water that was consumed during a month by 70 families who receive water from a certain water scheme.

Class interval	12 - 14	15 - 17	18 - 20	21 - 23	24 - 26	27 - 29
Number of families	5	9	11	26	11	8

Calculate the mean number of water units that was consumed by a family, to the nearest whole number.

Let us first find the mid-values which represent each of the class intervals.
Let us now assume that the mid-value 22 of the class interval 21 – 23 is the mean. That is, let us take 22 to be the assumed mean. Now let us find the deviation of each mid-value from the assumed mean by subtracting the assumed mean from each mid-value.

That is, $\boxed{\text{deviation} = \text{mid-value} - \text{assumed mean}}$

Let us now tabulate this information. We denote the deviation by *d*.

Class Interval	Mid-value x	Deviation d	Frequency f	fx
12 - 14	13	–9	5	–45
15 - 17	16	–6	9	–54
18 - 20	19	–3	11	–33
21 - 23	$\boxed{22}$	0	26	0
24 - 26	25	3	11	33
27 - 29	28	6	8	48
			$\Sigma f = 70$	$\Sigma fd = 81 - 132$ $= -51$

Here Σf denotes the total number of families, fd the product of the deviation and the corresponding frequency, and Σfd the sum of the values in the fd column.

The mean is obtained by,

Mean = Assumed Mean + Mean of the Deviations

Accordingly, for the above example,

$$\text{Mean} = 22 + \left(\frac{-51}{70}\right)$$
$$= 22 - 0.728$$
$$= 21.272$$
$$\approx \underline{\underline{21}}$$

It is easy to find the deviations by taking the assumed mean to be either the mid-value of the modal class or the mid-value of the median class.

If the assumed mean is denoted by A and the deviations are denoted by d, then the mean of the frequency distribution is given by $A + \frac{\Sigma fd}{\Sigma f}$

That is, $\boxed{\text{Actual mean} = A + \frac{\Sigma fd}{\Sigma f}}$

Exercise 26.3

1. Information on the ages of 100 viewers of a certain television programme is given in the following table.

Age (Years)	5 - 15	15 - 25	25 - 35	35 - 45	45 - 55	55 - 65	65 - 75
Number of viewers	7	16	25	31	14	5	2

(i) What is the modal class of the above frequency distribution?
(ii) Find the number of viewers whose ages are less than 25 years, as a percentage of the total number of viewers.
(iii) Find the mean age of a viewer of this programme to the nearest whole number, by taking the mid-value of the class interval 35 - 45 as the assumed mean.

2. The following table has been prepared using the number of days of leave that the staff of a private organization took during a year.

Number of days of leave	0 - 6	6 - 12	12 - 18	18 - 24	24 - 30	30 - 36	36 - 42
Number of staff	5	15	20	11	8	6	5

(i) What is the modal class of this distribution?
(ii) If gifts are to be given to those who took less than 6 days leave, what percentage of the total staff will receive gifts?
(iii) Find the mean number of days of leave taken by a staff member of this organization by using the mid-value of the class interval 18 – 24 as the assumed mean.
(iv) According to the answer to (iii) above, how many 'man days' of work can the organization expect to lose during a year?

3. A distribution of the marks obtained by 240 students at an examination is given below.

Interval of marks	0 - 8	9 - 17	18 - 26	27 - 35	36 - 44	45 - 53	54 - 62	63 - 71	72 - 80
Frequency	15	18	39	39	48	33	23	14	11

(i) To which class interval does the greatest number of students belong?

(ii) By taking the mid-value of the modal class as the assumed mean, find the mean mark of a student.

(iii) If the 30% who have obtained the lowest marks are to be given remedial lessons, determine the mark below which a student would be selected to follow these lessons.

(iv) If the top 20% are to be awarded distinctions, above which mark should this selection be made?

4. A table with information on the amount of rice that was sold during a period of 90 days at a cooperative store is given below.

Amount of rice sold in a day (kg)	151 - 175	176 - 200	201 - 225	226 - 250	251 - 275	276 - 300	301 - 325	326 - 350	351 - 375
Number of days	5	7	7	10	21	16	10	8	6

(i) Write down the modal class of this distribution.

(ii) By taking the mid-value of the modal class as the assumed mean, calculate to the nearest kilogramme, the mean amount of rice sold in a day during this period.

(iii) If this pattern of sales is expected to continue during the next two months too, estimate the amount of rice that should be stored to be sufficient for the next 60 days.

(iv) What is the probability of the amount of rice sold during a day in this period exceeding 300 kg?

5. Frequency distributions of the marks obtained by two groups of 100 students each for a mathematics test is given in the following table.

Class Interval	1 - 10	11 - 20	21 - 30	31 - 40	41 - 50	51 - 60	61 - 70	71 - 80	81 - 90
Number of students in group A	4	8	18	24	16	14	10	4	2
Number of students in group B	7	9	17	26	14	15	8	3	1

(i) What is the maximum mark that a student may have obtained in this test?

(ii) Find the mean mark of a student in each group by taking the mid-value of the class interval 41 – 50 as the assumed mean.

(iii) Thereby determine which group of students performed better at the test.

6. A frequency distribution containing information on the number of electricity units used during a certain month by hundred households is given below.

Number of electricity units	31 - 40	41 - 50	51 - 60	61 - 70	71 - 80	81 - 90	91 - 100
Number of households	5	12	26	34	18	3	2

(i) What is the modal class of this distribution?

(ii) Find the mean number of electricity units used by a household during this month, by taking the assumed mean to be the mid-value of the class interval 61 – 70.

(iii) When the number of electricity units used lies in the interval 61 – 90, the Electricity Board charges Rs.14 per unit. Accordingly, what is the income that the Electricity Board can expect to earn from these 100 households during that month?

7. Information on the monthly telephone bills of the customers of a certain private telephone company, obtained by conducting a survey in a particular region is given below.

Monthly telephone bill (Rs.)	100-250	250-400	400-550	550-700	700-850	850-1000	1000-1150	1150-1300
Number of customers	2	5	7	15	20	10	8	3

(i) What is the modal class of this distribution?

(ii) Find the mean monthly telephone bill of a customer in this region by taking the mid-value of the class interval 550 – 700 as the assumed mean.

(iii) According to the above mean, what is the income that the company can expect to earn in a month from 1000 customers who use this type of telephone service?

(iv) If those whose monthly bill exceed Rs. 850 are given an opportunity to win a prize through a special draw, show that more than 30% of the customers in this group are entitled to this.

8. The following table contains information on the speed of the vehicles that passed a certain location during a period of two hours. (The interval 30 – 40 denotes the interval of speeds which are greater than 30 kmh^{-1} but less than or equal to 40 kmh^{-1}.)

Speed (kmh^{-1})	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80	80 - 90
Number of Vehicles	5	7	12	16	15	3	2

(i) What is the modal class of this distribution?

(ii) If those who exceed the speed limit of 70 kmh^{-1} are taken to courts, find the percentage of drivers who are taken to courts for exceeding this speed limit.

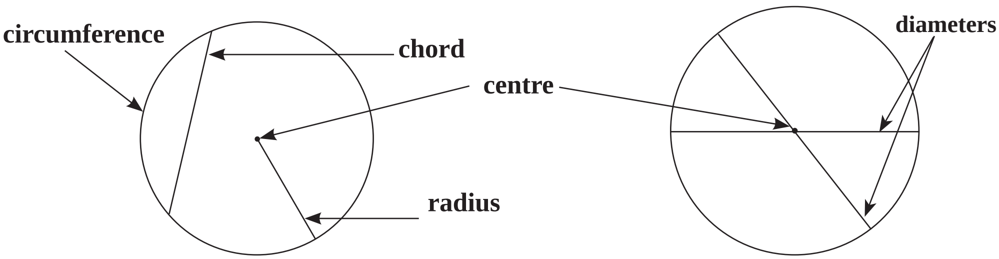
(iii) Find the mean speed of a vehicle which passes this location by taking the mid-value of the class interval 50 – 60 as the assumed mean.

(iv) What is that distance that can be covered during a period of two hours by travelling at the mean speed?

27 Chords of a Circle

By studying this lesson you will be able to

- solve problems by using the theorem that the straight line joining the midpoint of a chord of a circle to the centre is perpendicular to the chord, and
- solve problems by using the theorem that the perpendicular drawn from the centre of a circle to a chord bisects the chord.



A straight line segment drawn from the centre of a circle to a point on the circle is called a **radius** (Observe the above figure). The length of this line segment is the same irrespective of the point on the circumference that is selected. Therefore, such a line segment is called the radius of the circle. The length of the radius is also called the **radius**.

A straight line segment joining two points on a circle is defined as a **chord**.

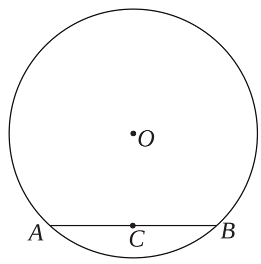
A chord which passes through the centre of the circle is called a **diameter**. All the diameters of a circle are equal in length. This length is also called the diameter. A diameter is the longest chord of a circle. The length of a diameter is twice the length of the radius.

27.1 The straight line segment drawn from the centre of a circle to the midpoint of a chord

Activity 1

- On a piece of paper, construct a circle of radius 3 cm using a pair of compasses and name its centre as O. Draw a chord AB which is not a diameter of the circle.
- By measuring the length using a ruler, mark the midpoint of the chord as C and join OC.

- Measure the magnitude of the angle $O\hat{C}A$ (or $O\hat{C}B$) with the aid of a protractor. Observe that this angle is 90°, that is, that OC and AB are perpendicular to each other.



- Draw several more chords of different lengths within this circle itself and observe that the straight line joining the midpoint of each chord to the centre is perpendicular to the respective chord.
- Draw several circles of different radii and do the above activity with these circles too.

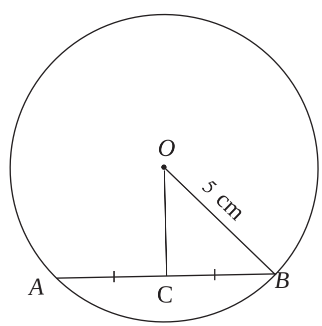
Discuss what you learnt through this activity with the other students in the class. The relationship that was identified is a theorem related to the chords of a circle.

Theorem
The straight line joining the centre of the circle to the midpoint of a chord is perpendicular to the chord.

Let us now consider how calculations are done using the above theorem.

Example 1

AB is a chord of a circle with centre O and radius 5 cm. The midpoint of AB is C. If AB = 8 cm, find the length of OC.
Let us represent this information in a figure.



$O\hat{C}B = 90°$ (Since the straight line joining the centre of the circle to the midpoint of a chord is perpendicular to the chord)

$\therefore$ OCB is a right angled triangle.

Let us find the length of OC by applying Pythagoras' theorem.

Now, $BC = \frac{8}{2} = 4$ cm (Since C is the midpoint of AB)

Also $OB = 5$ cm (OB is the radius of the circle)

$OB^2 = OC^2 + CB^2$ (Pythagoras' theorem)

$$\therefore \quad 5^2 = OC^2 + 4^2$$

$$25 = OC^2 + 16$$

$$25 - 16 = OC^2$$

$$OC^2 = 9$$

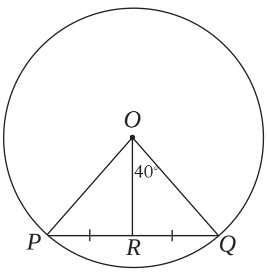
$$\therefore \quad OC = \sqrt{9}$$

$$= 3$$

$\therefore$ The length of OC is 3 cm.

Example 2

PQ is a chord of a circle with centre O. The midpoint of PQ *is* R. If $Q\hat{O}R = 40°$, find $O\hat{P}R$.



$O\hat{R}Q = 90°$ (Since the straight line joining the centre of the circle to the midpoint of a chord is perpendicular to the chord)

Since the sum of the interior angles of a triangle is 180°,

$$O\hat{Q}R = 180° - (40° + 90°)$$

$$\therefore O\hat{Q}R = 50°$$

Now let us consider the triangle OPQ

$OQ = OP$ (Radii of the same circle)

$\therefore$ OPQ is an isosceles triangle.

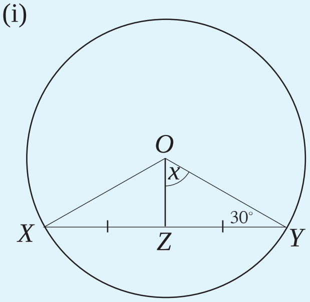
$$\therefore O\hat{P}R = O\hat{Q}R$$

$$\therefore O\hat{P}R = \underline{\underline{50°}}$$

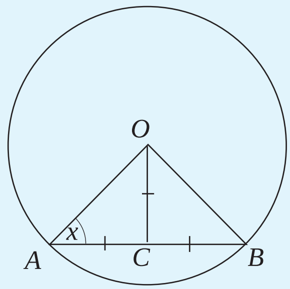
Exercise 27.1

1. Find the value of x in each of the following figures using the given data. In each figure, O denotes the centre of the circle.

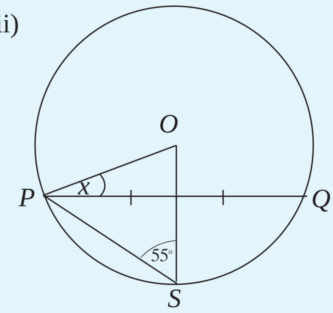
(i)



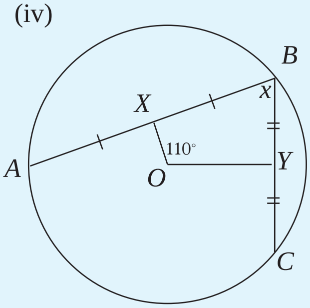
(ii)



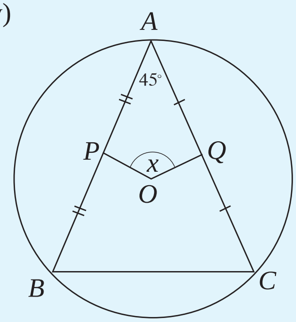
(iii)



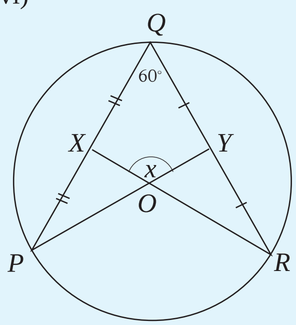
(iv)



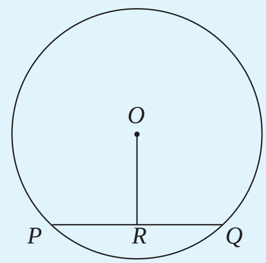
(v)



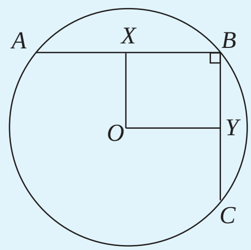
(vi)

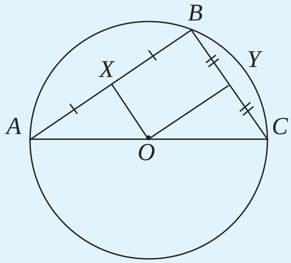


2. PQ is a chord of a circle with centre O. The midpoint of the chord is R. If PQ=12 cm and OR = 8 cm, find the radius of the circle.



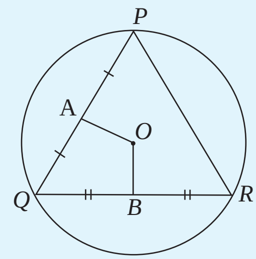
3. AB and BC are two chords of a circle with centre O which are perpendicular to each other. AB = 12 cm and BC = 8 cm. The mid-points of AB and BC are X and Y respectively. Find the perimeter of the quadrilateral $OXBY$.



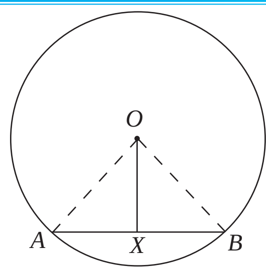
4. 

AB and *BC* are two chords of a circle with centre *O* which are perpendicular to each other. The mid-points of *AB* and *BC* are *X* and *Y* respectively. If *AB* = 8 cm and *BC* = 6 cm, find the perimeter of the quadrilateral *BXOY*.

5. The vertices *P*, *Q* and *R* of the triangle *PQR* lie on a circle with centre *O*. The midpoints of *PQ* and *QR* are *A* and *B* respectively. If *PQ* = 16 cm, *OA* = 6 cm and $OB = \sqrt{19}$ cm, calculate the length of *QR*.



27.2 The formal proof of the theorem that "the straight line joining the midpoint of a chord of a circle to the centre is perpendicular to the chord"



Data : *AB* is a chord of a circle with centre *O*. The midpoint of *AB* is *X*.

To be proved: *OX* is perpendicular to *AB*.

Construction: Join *OA*, *OB*.

Proof : In the triangles *OXA* and *OXB*,

$AO = BO$ (Radii of the same circle)

$AX = XB$ (Since *X* is the midpoint of *AB*)

OX is a common side.

$\therefore\ OXA\Delta \equiv OXB\Delta$ (SSS)

$\therefore\ O\hat{X}A = O\hat{X}B$

But, $O\hat{X}A + O\hat{X}B = 180^{o}$

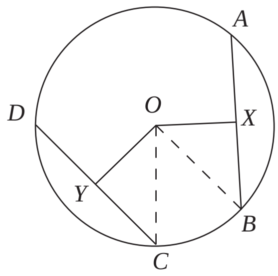
$\therefore\ 2O\hat{X}A = 180^{o}$

$O\hat{X}A = 90^{o}$

$\therefore$ *OX* is perpendicular to *AB*

Let us now consider how riders are proved using the above theorem.

Example 1



AB and *CD* are two equal chords of a circle of centre *O*. The midpoints of *AB* and *CD* are *X* and *Y* respectively. Show that *OX* = *OY*.

To show that *OX* = *OY*, let us first show that the two triangles *OXB* and *OYC* are congruent to each other under the conditions of hypotenuse-side.

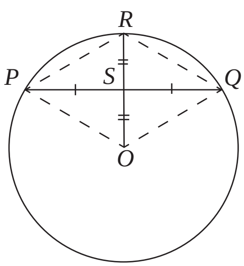
In the triangles *OXB* and *OYC*,
since *X* is the midpoint of *AB* and *Y* is the midpoint of *CD*,
$O\hat{X}B = 90^\circ$ and $O\hat{Y}C = 90^\circ$.
Since they are radii of the same circle, *OB* = *OC*.
Also, since it is given that *AB* = *CD*, we obtain $\frac{1}{2}AB = \frac{1}{2}CD$.

Therefore, since *X* and *Y* are the midpoints of the chords *AB* and *CD*, we obtain *XB* = *YC*.

$\therefore \Delta OXB \equiv \Delta OYC$ (Hypotenuse-side)

$\therefore \quad OX = OY$ (Corresponding elements of congruent triangles are equal)

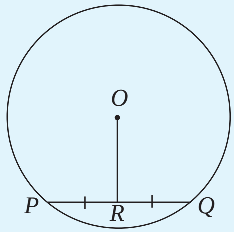
Example 2



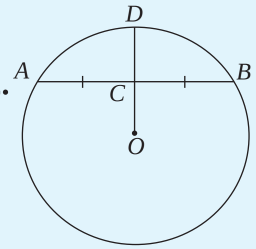
S is the midpoint of the chord *PQ* of the circle with centre *O*. *OS* produced meets the circle at *R*. If *RS* = *SO*, show that *OPRQ* is a rhombus.

PS = *SQ* (Since *S* is the midpoint of the chord *PQ*)
RS = *SO* (Given)
$\therefore$ *OPRQ* is a parallelogram.(Since the diagonals bisect each other)
$P\hat{S}O = 90^\circ$ (Since the straight line joining the midpoint of a chord of a circle to the centre is perpendicular to the chord)
That is *PQ* and *RO* bisect each other perpendicularly.
$\therefore$ *PRQO* is a rhombus.

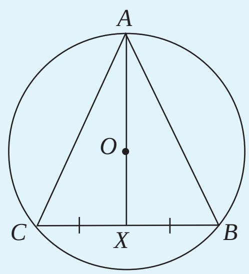
Exercise 27.2

1. 

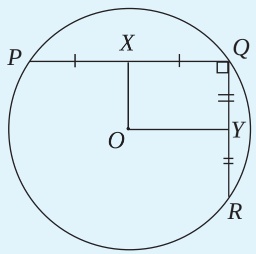
R is the midpoint of the chord PQ of the circle with centre O. If $R\hat{O}Q = 45^\circ$, show that $RQ = OR$.

2. 

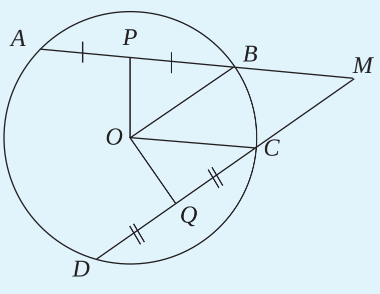
AB is a chord of the circle with centre O. Its midpoint is C. When OC is produced it meets the circle at D. Show that $AD = DB$.

3. 

The vertices A,B and C of the triangle ABC lie on a circle with centre O. The midpoint of BC is X. If O lies on AX, show that $AB = AC$.

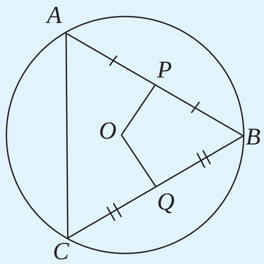
4. 

PQ and QR are two chords of a circle with centre O which are perpendicular to each other. The mid-points of PQ and QR are X and Y respectively. Show that $OXQY$ is a rectangle.

5. 

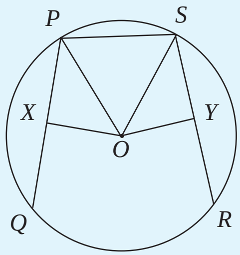
AB and CD are two chords of a circle with centre O. The midpoints of AB and CD are P and Q respectively. The chords AB and DC produced meet at M. Show that $P\hat{O}Q$ and $P\hat{M}Q$ are supplementary angles.

6.



The midpoints of the chords *AB* and *BC* of the circle with centre *O* are *P* and *Q* respectively.
Show that $P\hat{O}Q = B\hat{A}C + A\hat{C}B$.

7.



PQ and *RS* are two equal chords of a circle with centre *O*. The midpoints of *PQ* and *RS* are *X* and *Y* respectively. Show that $X\hat{P}S = Y\hat{S}P$.

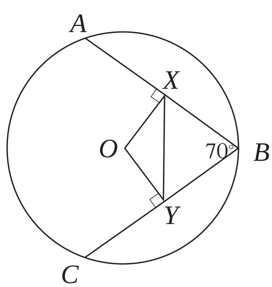
27.3 Converse of the theorem and its applications

The theorem studied above states that the straight line joining the midpoint of a chord of a circle to the centre is perpendicular to the chord. The converse of this theorem is also true. It is expressed as a theorem as follows.

Theorem: The perpendicular drawn from the centre of a circle to a chord bisects the chord.

Now let us consider a couple of examples of calculations done using the theorem that "the perpendicular drawn from the centre of a circle to a chord bisects the chord".

Example 1



AB and *BC* are two equal chords of a circle with centre *O*. The perpendiculars drawn from *O* to the two chords are *OX* and *OY* respectively. If $X\hat{B}Y = 70^\circ$, then find the magnitude of $B\hat{X}Y$.

Since $OX \perp AB$ and $OY \perp BC$,
X is the midpoint of *AB* and *Y* is the midpoint of *BC*.

Also, since it has been given that *AB* = *BC*, we obtain that *XB* = *YB*.
Therefore, *BXY* is an isosceles triangle.

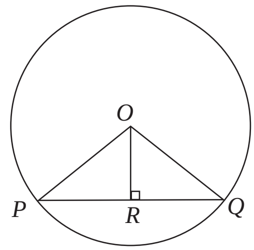
$$\therefore B\hat{X}Y = B\hat{Y}X.$$

$$\therefore B\hat{X}Y = \frac{180^\circ - 70^\circ}{2}$$

$$= \underline{\underline{55^\circ}}$$

Example 2

OR is the perpendicular drawn from a circle with centre *O* to the chord *PQ*. If *OR*= 3 cm and *PQ* = 8 cm, find the radius of the circle.



Since $OR \perp PQ$, *R* is the midpoint of *PQ*.

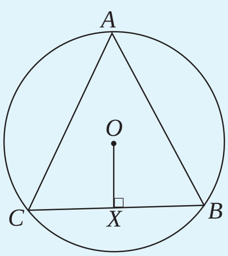
$$\therefore \; PR = \frac{8}{2} = 4 \text{ cm}.$$

Now, by applying Pythagoras' theorem to the triangle *ORP* ,

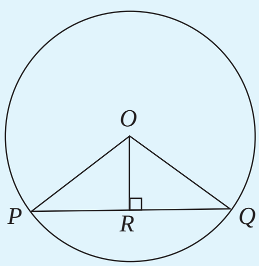
$$OP^2 = OR^2 + PR^2$$
$$= 3^2 + 4^2$$
$$= 25$$
$$\therefore OP = \sqrt{25}$$
$$= 5$$

$\therefore$ <u>Therefore, the radius of the circle is 5 cm</u>.

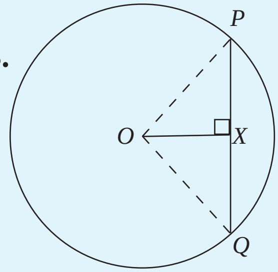
Exercise 27.3



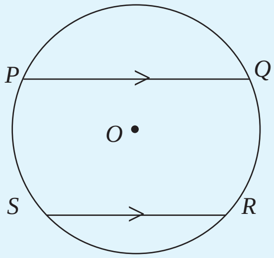
1. *ABC* is an equilateral triangle. The vertices *A*,*B* and *C* lie on the circle with centre *O*. The perpendicular drawn from *O* to *BC* is *OX*. If *XB* = 6 cm, find the perimeter of the triangle *ABC*.



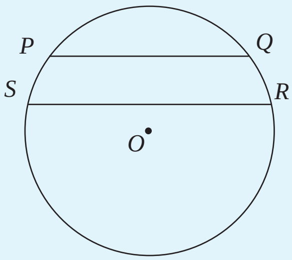
2. *PQ* is a chord of a circle with centre *O*. The perpendicular drawn from *O* to *PQ* is *OR*. If *PQ* = 12 cm and *OR* = 8 cm, find the perimeter of the triangle *OPQ*.

3. 

PQ is a chord of a circle with centre *O*. The perpendicular drawn from *O* to *PQ* is *OX*. If *PQ* = 6 cm and the radius of the circle is 5 cm, find the length of *OX*.

4. 

PQ and *RS* are two parallel chords of a circle which lie on opposite sides of the centre *O*. The radius of the circle is 10 cm. If *PQ* = 16 cm and *SR* = 12 cm, find the distance between the two chords.

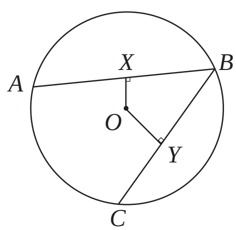
5. 

PQ and *RS* are two parallel chords of the circle with centre *O* as shown in the figure. The radius of the circle is 10 cm. If *PQ* = 12 cm and *SR* = 16 cm, find the distance between the two chords.

27.4 Proving riders using the theorem that "the perpendicular drawn from the centre of a circle to a chord bisects the chord"

Example 1

AB and *BC* are two equal chords of a circle with centre *O*. The perpendiculars drawn from *O* to the two chords are *OX* and *OY* respectively. Prove that *OX* = *OY*.



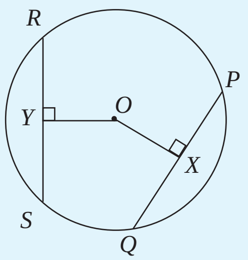
We will prove that *OX* = *OY* by showing that the right triangles *OXB* and *OYB* are congruent under the conditions of hypotenuse-side.
OB is the common side of these two triangles.

Since $AB = BC$, by the above theorem we have that $BX = YB$.

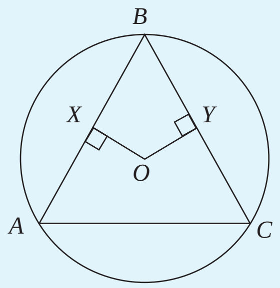
$\therefore \Delta OXB \equiv \Delta AYB$ (Hypotenuse-side)

$\therefore OX = OY$. (The remaining elements of congruent triangels are equal to each other.)

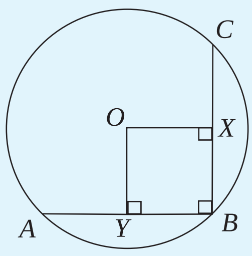
Exercise 27.4

1. 

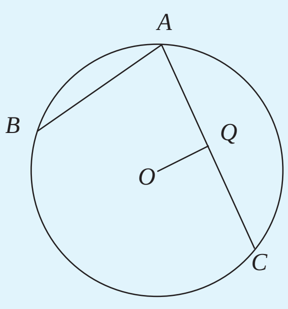
PQ and RS are two chords of a circle with centre O. The perpendiculars drawn from O to PQ and RS are OX and OY respectively. If $OX = OY$ prove that $PQ = RS$.

2. 

The vertices A, B and C of the triangle ABC lie on the circle with centre O. The perpendiculars drawn from O to AB and BC are OX and OY respectively. If $AX = CY$, then prove that $B\hat{A}C = B\hat{C}A$.

3. 

AB and BC are two equal chords of a circle with centre O which are perpendicular to each other. Prove that $OXBY$ is a square.

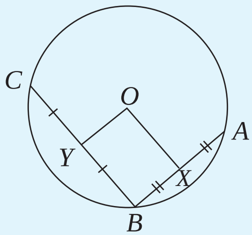
4. 

AB and AC are two chords of a circle with centre O. The perpendicular drawn from O to AC is OQ. If $2\,AB = AC$, prove that $AB = AQ$.

Miscellaneous Exercises

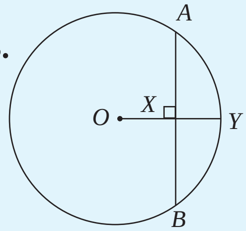
1. A chord of a circle lies 8 cm from the centre. If the length of the chord is 12 cm, find the radius of the circle.

2.



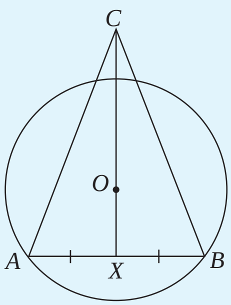
The radius of a circle with centre O is 5 cm. The lengths of the chords AB and BC are 6 cm and 8 cm respectively. The midpoints of the chords are X and Y respectively. Find the perimeter of the quadrilateral $OXBY$.

3.



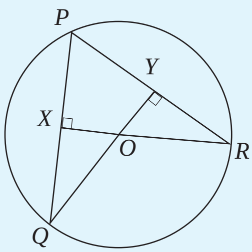
The length of the chord AB of the circle with centre O is 8 cm. The perpendicular drawn from O to the chord, intersects the chord at X, and meets the circle at Y. If $XY = 3$ cm, find the radius of the circle.

4.



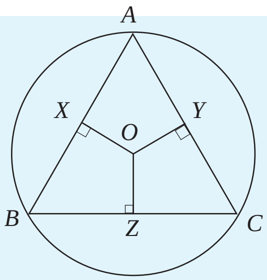
AB is a chord of a circle with centre O. The midpoint of the chord is X. The point C lies on the line drawn from X through the point O. Prove that $AC = BC$.

5.

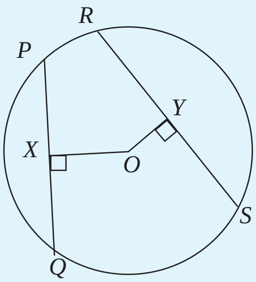


PQ and PR are two chords of a circle with centre O. The perpendiculars drawn from O to PQ and PR are OX and OY respectively. If XR and QY are straight lines, prove that $PQ = PR$.

6. A chord of length 24 cm lies 5 cm away from the centre of the circle. Another chord lies 12 cm from the centre. Find its length.

7. 

The vertices A,B and C of the equilateral triangle ABC lie on the circle with centre O. The perpendiculars drawn from O to AB, AC and BC are OX, OY and OZ respectively. Prove that $OX = OY = OZ$.

8. 

PQ and RS are two chords of a circle with centre O. The perpendiculars drawn from O to PQ and RS are OX and OY respectively. Show that $PQ^2 - RS^2 = 4\,OY^2 - 4OX^2$.

28 Constructions

By studying this lesson you will be able to

- construct the four basic loci
- construct parallel lines
- construct triangles with the given information.

28.1 Construction of the basic loci

The path of a point in motion is defined as its locus. Several examples of loci that can be observed in our environment are given below.

1. The path of a fruit which falls from a tree.
2. The path of the pointed end of a clock hand.
3. The path of a planet orbiting around the sun.
4. The path of a pendulum in a pendulum clock.
5. The path of a ball that is hit by a bat.

In this lesson we will only be considering loci in a plane.

Note:

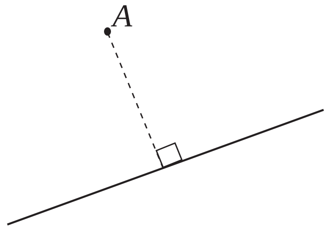
Before considering the construction of loci, direct your attention to the following facts.

1. Distance between two points:

Let us consider two points A and B that lie on a plane. What is meant by the distance between the two points is the length of the straight line segment joining the two points.

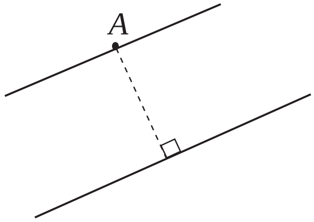
2. Distance from a point to a straight line:

Let us consider a given point A and a given straight line. What is meant by the distance from A to the straight line is the shortest distance from A to the straight line. This shortest distance is the perpendicular distance from A to the straight line.



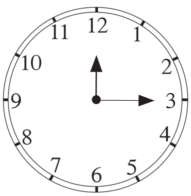
3. Distance between two parallel lines

Consider the following two parallel lines. Let us consider any point A on one of the lines. The perpendicular distance from A to the other straight line is said to be the distance between the two lines. Since the two lines are parallel, irrespective of where the point A is located on the line, this distance remains the same.



Now let us consider the 4 basic loci.

1. Constructing the locus of a point moving at a constant distance from a fixed point



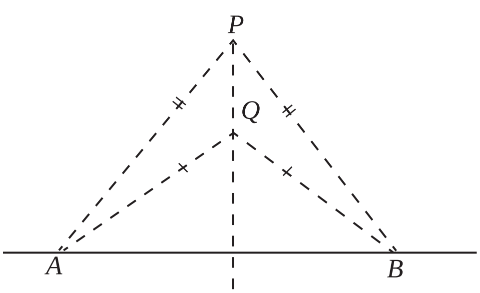
The pointed end of each hand on the clock face in the figure is always located at a constant distance from the centre of the clock, which is the location at which the hand is fixed to the clock. You will be able to observe when the clock is working, that the path of the pointed end of each hand is a circle. The point where the hands are fixed to the clock is the centre of these circles, and the radius of each circle is the length of the relevant hand. Observe here that the pointed end of each hand is travelling at a constant distance from a fixed point. That particular constant distance is the length of the hand.

The locus of a point moving at a constant distance from a fixed point is a circle.

Let us see how a circle is constructed.

Mark a point. Take the radius of the circle that you want to construct to the pair of compasses using the ruler and keep the point of the pair of compasses on the point you marked. Now draw the circle.

2. Constructing the locus of a point moving at an equal distance from two fixed points



As shown in the figure, the point P is at an equal distance from the two points A and B. Further, Q is another point which is at an equal distance from A and B. There are a large number of points such as these, which are at an equal distance from A and B. Observe what is obtained when all these points are joined together.

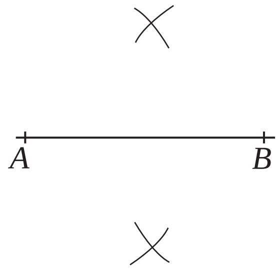
It is clear that the straight line that is obtained when all these points are joined together, passes through the midpoint of the line joining *A* and *B*, and is perpendicular to *AB*.

The locus of a point moving at an equal distance from two fixed points is the perpendicular bisector of the straight line joining the two points.

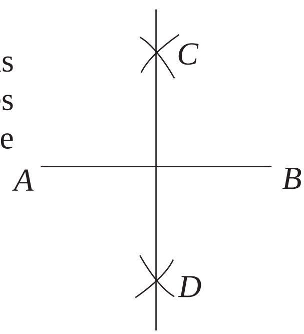
Now let us consider how this locus, that is, the perpendicular bisector of the line segment *AB* is constructed.

Mark two points and name them as *A* and *B*.

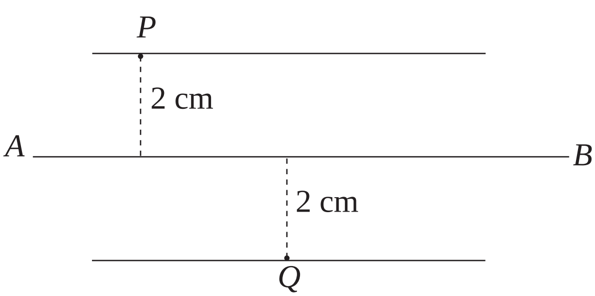
Step 1: Draw the line segment *AB*. On the pair of compasses, take a length which is a little more than half the length of *AB*, and taking *A* and *B* as the centres and the length on the pair of compasses as the radius, draw two arcs which intersect each other (as shown in the figure).



Step 2: Name the intersection points of the two arcs as *C* and *D* and draw the straight line which passes through these two points. This straight line is the required locus.



3. Constructing the locus of a point moving at a constant distance from a straight line



The figure illustrates a pair of straight lines drawn parallel to the straight line *AB* on opposite sides of *AB*. Each of these lines is at a constant distance of 2 cm from *AB*. Conversely, if a point lies at a distance of 2 cm from *AB*, then it is clear that this point must lie on one of the above two lines.

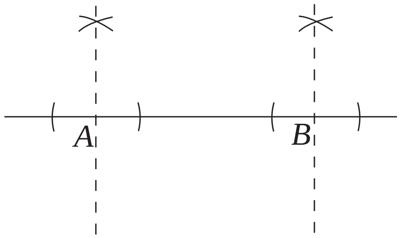
Accordingly, the locus of a point which lies 2 cm from the straight line AB is one of two straight lines which are parallel to AB and lie on opposite sides of AB 2 cm from it.

The locus of a point moving at a constant distance from a given straight line is a line parallel to the given straight line, at the given constant distance from the straight line, which may lie on either side of the straight line.

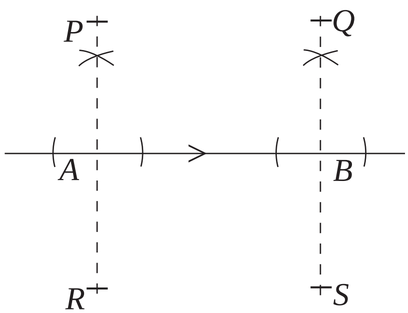
Now let us consider how a pair of lines parallel to a given straight line, which is the locus under consideration, is constructed.

Draw a straight line segment using a straight edge. Select two points A and B on this straight line.

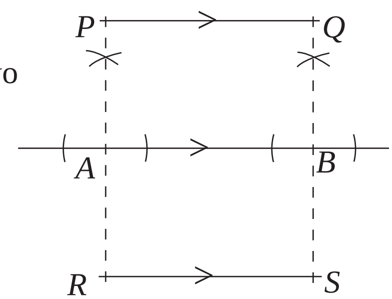
Step 1: At the points A and B, construct two lines perpendicular to the given line.



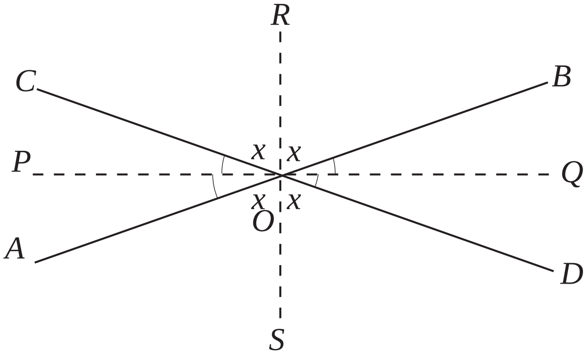
Step 2: On each of these two perpendicular lines, mark two points at the required distance (say 5 cm), on either side of the given straight line, and name them P, Q, R and S as shown in the figure.



Step 3: Draw the straight lines PQ and RS. These two straight lines are the required locus.



4. Constructing the locus of a point moving at an equal distance from two intersecting straight lines



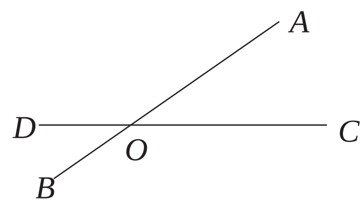
The straight lines AB and CD in the figure intersect at O. The straight line PQ has been drawn such that the angle $A\hat{O}C$ (and $B\hat{O}D$) is divided into two equal angles. The line PQ is called the bisector of the angle $A\hat{O}C$ (or $B\hat{O}D$). Similarly, the straight line RS has been drawn such that the angle $C\hat{O}B$ (and $A\hat{O}D$) is divided into two equal angles. The line RS is called the bisector of the angle $C\hat{O}B$ (or $A\hat{O}D$).

Can you see that the distance from any point on the line PQ to the lines AB and CD is equal? Understand that similarly, the distance from any point on the line RS to the lines AB and CD is also equal. Do you see that conversely, if a point is at an equal distance from the lines AB and CD, then it must lie on either PQ or RS?

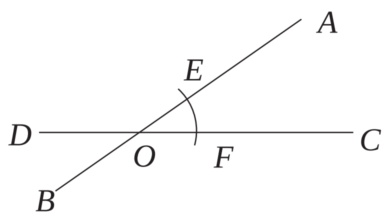
The locus of a point moving at an equal distance from two intersecting straight lines is a bisector of the angle formed at the intersection point of the lines.

Now let us consider how this locus is constructed.

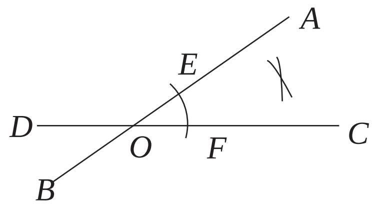
Let the two straight lines AB and CD intersect at the point O.



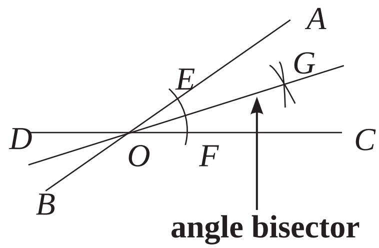
Step 1: Using the pair of compasses, draw an arc with centre O such that it intersects both BA and DC. Name the two points at which the arc intersects BA and DC as E and F respectively.



Step 2: Using the pair of compasses and taking E and F as centres, draw two intersecting arcs.



Step 3: Name the point of intersection of the two arcs as G, and draw the straight line which passes through the points O and G. Construct the other angle bisector in a similar manner.



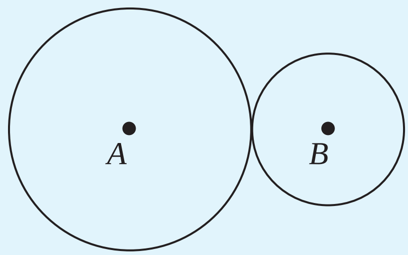
The required locus is one of these angle bisectors.

Exercise 28.1

1. If the length of the seconds hand of a clock is 3.5 cm, construct the path of the pointed end of this hand.

2. If the maximum distance between a cow and a tree to which the cow has been tied with a rope is 5 m, construct the path along which the cow can travel such that the distance between the tree and the cow will be at its maximum.

3. A is the centre of a fixed cogwheel of radius 3 cm, and B is the centre of a revolving cogwheel of radius 2 cm. Construct the locus of B as the smaller cogwheel revolves around the larger cogwheel of centre A.



4. (i) Construct a straight line segment PQ such that PQ = 5 cm.
Construct two circles of radius 3 cm each with P and Q as centres.
(ii) Name the points of intersection of the two circles as X and Y and join XY.
(iii) Name the point of intersection of the straight lines PQ and XY as S and measure and write down the lengths of PS and QS.
(iv) Measure and write down the magnitudes of $P\hat{S}X$ and $Q\hat{S}X$.
(v) Describe the locus represented by XY.

5. Construct the straight line segment AB such that AB = 7 cm and divide it into four equal parts.

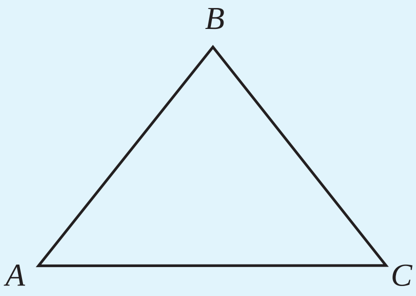
6. Draw the angle $B\hat{A}C$ such that $AB = 5$cm and $B\hat{A}C = 40^{\circ}$. Construct the locus of the points which are equi-distant from A and B and name the point of intersection of this locus and the straight line AC as D.

7. (i) Draw an acute triangle and name it ABC.

(ii) Construct the locus of a point which is equi-distant from A and C.

(iii) Construct the locus of a point which is equi-distant from A and B.

(iv) Name the point of intersection of these two loci as O. What can you say about the distance from O to the points A, B and C?



8. Draw a straight line segment KL. Construct the locus of a point which is 2.5 cm from this line.

9. Contruct a rectangle of length 5 cm and breadth 3 cm. Construct the locus of a point which lies outside the rectangle at a distance of 2 cm from the sides of the rectangle.

10. Using the protractor draw the following angles and construct their bisectors.

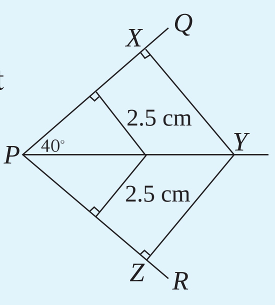
(i) 60° (ii) 90° (iii) 120°

11. Based on the information in the figure,

(i) name the locus of the points which are equi-distant from PQ and PR.

(ii) write down a relationship between XY and YZ.

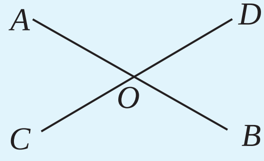
(iii) What is the magnitude of $R\hat{P}Y$?



12. The straight lines AB and CD in the figure intersect at O.

(i) Construct the locus of the points equi-distant from AB and CD.

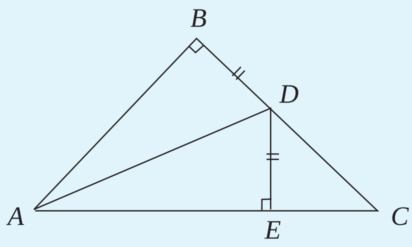
(ii) What is the magnitude of the angle between the two lines which form this locus?



13. In the given figure, $A\hat{B}C = A\hat{E}D = 90^\circ$ and $BD = DE$.

(i) Name the locus of the points which are equi-distant from AB and AC.

(ii) If $A\hat{C}B = 40^\circ$, what are the magnitudes of $B\hat{A}D$ and $C\hat{A}D$?



28.2 Construction of triangles

A triangle has three sides and three angles. The sides and the angles are called the elements of the triangle. Let us study three instances when a triangle can be constructed with the information given on the magnitude of three elements of a triangle.

1. When the lengths of the three sides of a triangle are given

Example 1

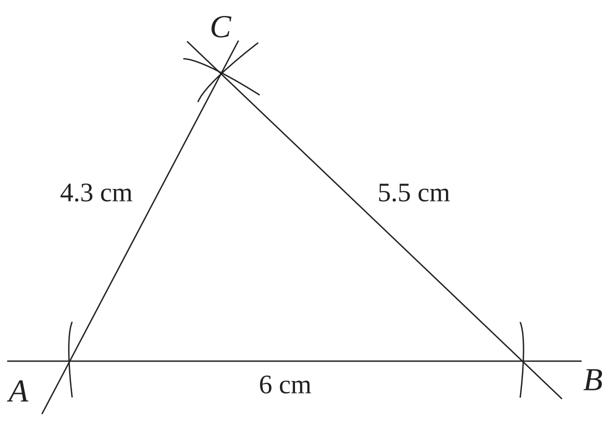
Construct the triangle ABC such that $AB = 6$ cm, $BC = 5.5$ cm and $AC = 4.3$ cm.

Step 1: Draw a straight line segment of length 6 cm and name it AB.

Step 2: Take B as the centre and draw a circular arc of radius 5.5 cm (of sufficient length).

Step 3: Draw another circular arc of radius 4.3 cm with centre A, such that it intersects the arc drawn in step 2 above.

Step 4: Name the point of intersection of the two arcs as C, and by joining AC and BC, complete the triangle ABC.



2. When the lengths of two sides and the magnitude of the included angle are given

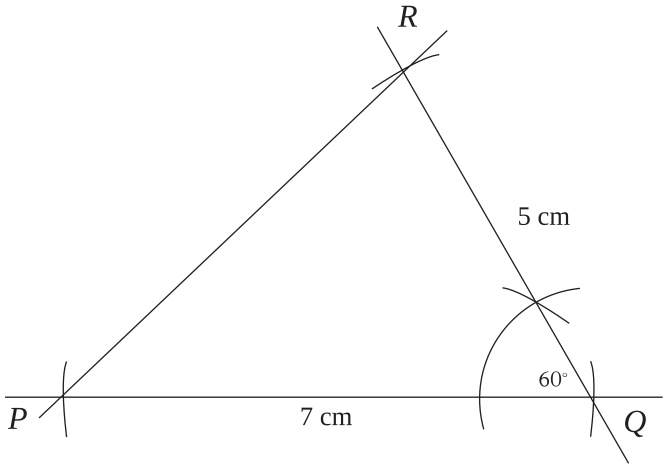
Example 2

Construct the triangle PQR such that $PQ = 7$ cm, $QR = 5$ cm and $P\hat{Q}R = 60^\circ$.

Step 1: Construct an angle of 60 degrees and name its vertex Q. The sides of the angle should be longer than the given lengths of the triangle.

Step 2: Mark a straight line segment QP of length 7 cm on one side of the angle, and a straight line segment QR of length 5 cm on the other side of the angle. (See the figure)

Step 3: Complete the triangle PQR by joining PR.



3. When the magnitudes of two angles and the length of a side are given

Example 3

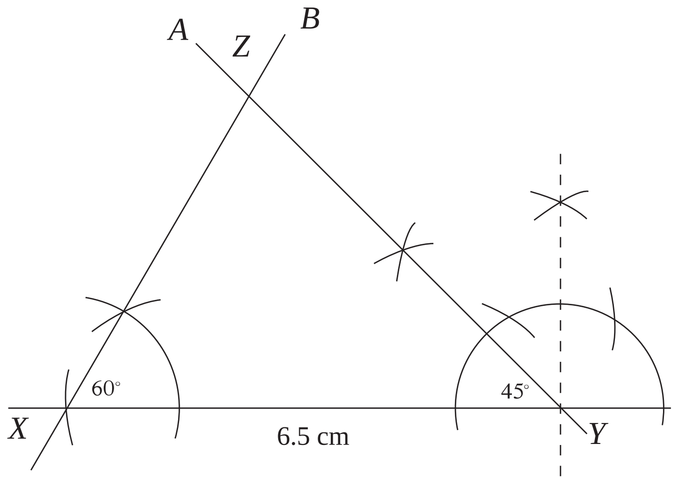
Construct the triangle XYZ such that $XY = 6.5$ cm, $X\hat{Y}Z = 45^\circ$ and $Y\hat{X}Z = 60^\circ$.

Step 1: Construct a straight line segment of length 6.5 cm and name it XY.

Step 2: Construct the angle $X\hat{Y}A$ at the point Y, such that $X\hat{Y}A = 45^\circ$

Step 3: Construct the angle $Y\hat{X}B$ at the point X, such that $Y\hat{X}B = 60^\circ$.

Step 4: Name the intersection point of YA and XB as Z. Then XYZ is the required triangle.



Exercise 28.2

1. Construct the equilateral triangle ABC of side length 6 cm.

2. Construct the isosceles triangle PQR, such that $PQ = 8$ cm and $PR = QR = 6$ cm.

3. (i) Construct the triangle KLM where $KL = 7.2$ cm, $LM = 6.5$ cm and $KM = 5$ cm.
(ii) Measure the magnitude of each angle in the triangle and write it down.

4. (i) Construct the triangle ABC where $AB = 6$ cm, $A\hat{B}C = 90^\circ$ and $BC = 4$ cm.
(ii) Measure and write down the length of the side AC.
(iii) Write down a relationship between the sides AB, BC and AC.
(iv) Thereby find an approximate value for $\sqrt{52}$.

5. (i) Construct the triangle XYZ such that $XY = 5$ cm, $X\hat{Y}Z = 75^\circ$ and $YZ = 6$ cm.
(ii) Measure and write down the length of the side XZ.
(iii) Measure and write down the magnitude of $Y\hat{X}Z$.

6. (i) Construct the triangle SRT such that $RS = 6.5$ cm, $S\hat{R}T = 120^\circ$ and $RT = 5$ cm.
(ii) Construct a straight line through T parallel to SR.

7. Construct the triangle DEF such that $DE = 6.8$ cm, $D\hat{E}F = 60^\circ$ and $E\hat{D}F = 90^\circ$.

8. (i) Construct the triangle ABC such that $AB = 6$ cm, $A\hat{B}C = 105^\circ$ and $BC = 4.5$ cm.
(ii) Thereby construct the parallelogram $ABCD$.
(iii) Measure the length of the diagonal AC and write it down.

9. (i) Construct the triangle PQR such that $QR = 7$ cm, $Q\hat{R}P = 60^\circ$ and $Q\hat{P}R = 75^\circ$
(ii) Construct the perpendicular from P to QR and name the foot of the perpendicular as S.
(iii) Measure and write down the length of PS.

10. (i) Construct the triangle KLM such that $KL = 6.5$ cm, $K\hat{L}M = 75^\circ$ and $LM = 5$ cm.
(ii) Construct the quadrilateral $KLMN$ by finding the point N which is equidistant from K and M and is such that $MN = 4$ cm.
(iii) Measure and write down the magnitude of $L\hat{K}N$.

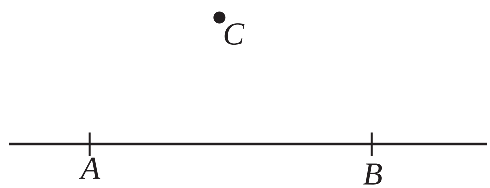
28.3 Constructions related to parallel lines

You have learnt in a previous grade how to construct parallel lines using a set square and a straight edge.

Now let us learn how to construct parallel lines using a straight edge and a pair of compasses.

1. Constructing a line parallel to a given straight line through an external point

Method 1
Let us assume that the straight line is AB and the external point is C.



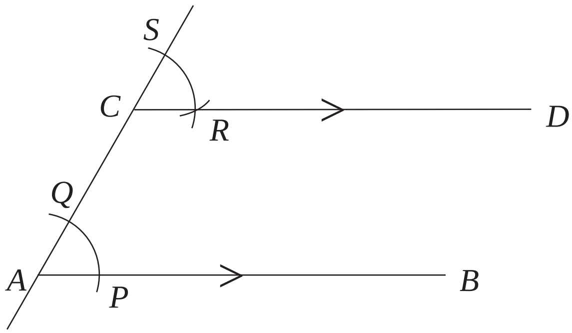
Step 1: Draw the straight line passing through the points A and C.

Step 2: Draw an arc on $B\hat{A}C$ taking A as the centre. Name this arc PQ.

Step 3: Taking the same radius, (that is, without changing the position of the pair of compasses), draw another arc with C as the centre, such that it intersects AC produced at S as shown in the figure.

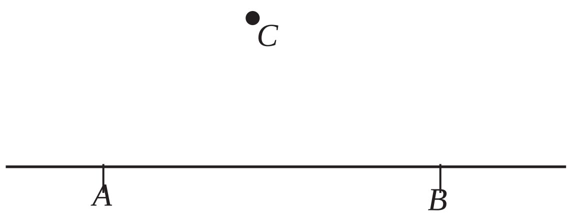
Step 4: Mark RS on the second arc as shown in the figure, such that it is equal in length to PQ.

Step 5: Draw the straight line CD such that is passes through the point R. Since the angle $R\hat{C}S$ which is then formed and $B\hat{A}C$ are corresponding angles which are equal to each other, the straight lines AB and CD are parallel to each other.



Method 2

Let us assume that the straight line is *AB* and the external point is *C*.



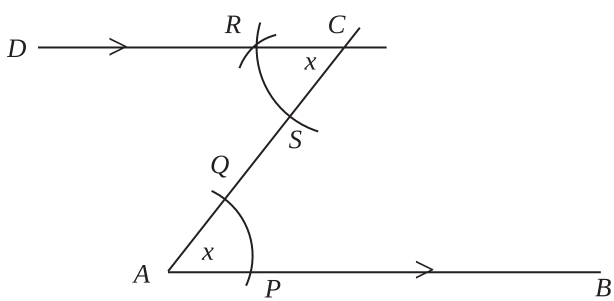
Step 1: Join *AC*.

Step 2: Draw an arc on $B\hat{A}C$, taking *A* as the centre. Name this arc *PQ*.

Step 3: Taking the same radius, draw another arc with *C* as the centre such that it intersects *AC* at the point *S* as shown in the figure.

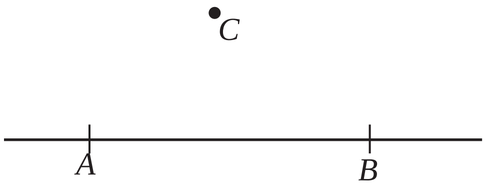
Step 4: Mark the point *R* on this arc such that *RS* is equal in length to *PQ*.

Step 5: Draw the straight line *CD* such that is passes through the point *R*. Since the angle $R\hat{C}S$ which is then formed and $B\hat{A}C$ are alternate angles which are equal to each other, the straight lines *AB* and *DC* are parallel to each other.



Method 3

Let us assume that the straight line is AB and the external point is C.



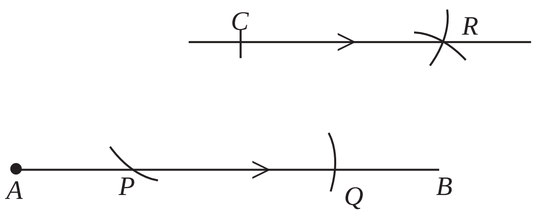
Step 1: Using a pair of compasses draw an arc with centre C such that it intersects AB. Name the point of intersection as P.

Step 2: Draw another arc with centre P and the same radius as that of the previous arc (i.e., keeping the radius CP unchanged), such that it intersects AB. Name the intersection point as Q.

Step 3: Draw another arc with centre Q and the same radius as before, in the direction of C.

Step 4: Now draw another arc with centre C and the same radius as before, such that it intersects the arc in step 3. Name the intersection point of the arcs as R.

Step 5: Join CR. Then CR is parallel to AB.



Activity

Do the following activity to further understand about constructions related to parallel lines.

1. Construct an angle of 60° and name the vertex as A. On one arm (side) of the angle mark point B such that AB = 8 cm. Mark point C on the other arm (side) such that AC = 5 cm. Now using the pair of compasses complete the parallelogram $ABDC$.

2. Draw two parallel lines such that the distance between the lines is 4 cm. Mark the points A and B on one line such that AB = 7 cm. Mark point D on the other line so that AD is 5 cm. Now complete the parallelogram $ABCD$.

3. Draw two parallel lines such that the distance between the lines is 4 cm. Mark the points A and B on one line such that AB is 7 cm. Mark point C on the other line such that BC = 5 cm. Now mark point D on the same line which C is on, such that CD = 4 cm. Then complete the quadrilateral $ABCD$ and observe that it is a trapezium.

Exercise 28.3

1. Draw an acute angle and name it $A\hat{B}C$. Construct a straight line segment which is parallel to AB and which passes through the point C.

2. Draw an obtuse angle and name it $P\hat{Q}R$. Construct a straight line segment which is parallel to PQ and which passes through the point R.

3. Construct a square of side length 6 cm.

4. Construct a rectangle of length 6.5 cm and breadth 4 cm. Name it as $ABCD$. Draw its diagonal AC and construct two straight line segments through the points B and D such that each is parallel to AC.

5. Construct the parallelogram $ABCD$ such that AB = 6 cm, $A\hat{B}C = 120^\circ$ and BC = 5 cm.

6. Construct the rhombus $KLMN$ such that KL = 7 cm and $K\hat{L}M = 60^\circ$.

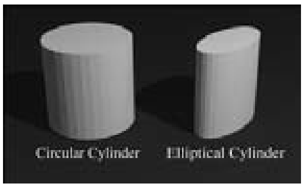
7. (i) Construct a circle of radius 3 cm and name its centre O.
(ii) Construct a chord of the above circle of length 4 cm and name it PQ.
(iii) Join PO and produce it to meet the circle again at R.
(iv) Construct a line through R parallel to PQ.

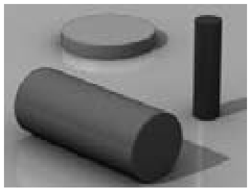
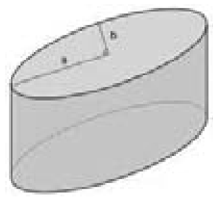
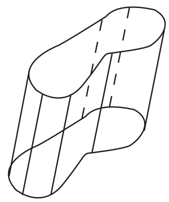
29 Surface Area and Volume

By studying this lesson you will be able to

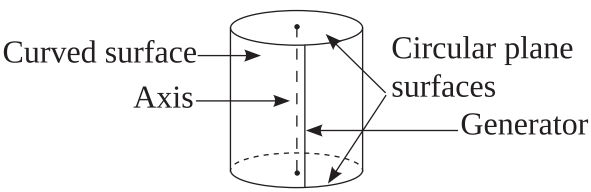
- calculate the surface area and volume of a right circular cylinder
- calculate the surface area and volume of a right triangular prism

The Cylinder



The cross sections of the solids given above are uniform, and the plane surfaces at the two ends are parallel to each other. Solids having such shapes are defined in general as cylinders.



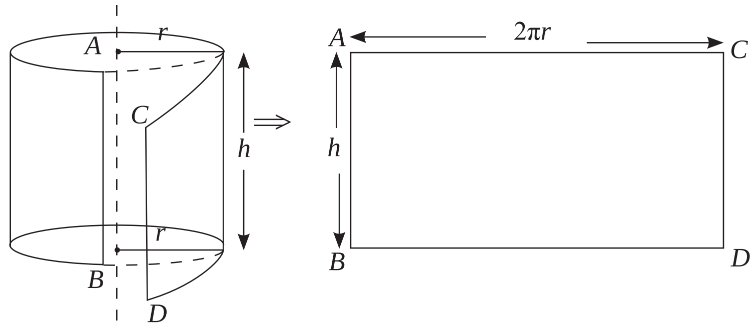
The cylinder in the figure has two circular plane surfaces at the top and the bottom. Apart from these, it also has a curved surface. The radii of the two circular plane surfaces are equal to each other. Therefore, the areas of these two plane surfaces are equal. The straight line joining the centres of these two circles is called the axis of the cylinder. Any straight line on the curved surface which is parallel to the axis of the cylinder is called a generator of the cylinder.

The axis of the cylinder is perpendicular to the two circular plane surfaces. Therefore, such a cylinder is called a **right circular cylinder**. (There are cylinders which are not right circular. However such cylinders will not be discussed here.) What is meant by the term “right” is that the axis of the cylinder is perpendicular to the two plane surfaces. What is meant by the term “circular” is that any cross section which is perpendicular to the axis is circular in shape.

The radius of a plane face of the cylinder is usually denoted by r and the length of the axis is usually denoted by h. The radius r of the circular faces is called the radius of the cylinder and the length of the axis h is called the height of the cylinder.

29.1 Surface area of a right circular cylinder

When the radius and the height of a cylinder are given, the areas of the three surface parts of the cylinder need to be added together to obtain the total surface area. The areas of the two circular faces at the two ends can be calculated using the formula for the area of a circle. A mechanism such as the following can be used to calculate the area of the curved surface



When the curved surface of the cylinder is cut along a generator as shown in the figure and opened out, a rectangle is obtained. The length of one side of this rectangle is equal to the height (h) of the cylinder while the length of the other side is equal to the perimeter of one of its circular plane faces.

The area of this rectangle is equal to the area of the curved surface of the cylinder. Accordingly, an expression for the curved surface of the cylinder can be developed in the following manner.

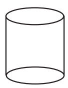
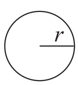
Area of the curved surface of the cylinder = Length of one side of the rectangle × Length of the other side of the rectangle

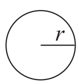
$= 2\pi r \times h$

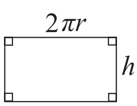
$\therefore$ The area of curved surface of the cylinder is $\underline{\underline{2\pi rh}}$.

Now we can find the total surface area of the cylinder in the following manner.

Total surface area of the cylinder = Area of top face + Area of bottom face + Area of curved surface

 = 

 + 

 + 

$\text{A} = \pi r^2 + \pi r^2 + 2\pi rh$

$$\boxed{\text{A} = 2\pi r^2 + 2\pi rh}$$

Note:

(i) Exterior surface area of a cylinder without a lid $= \pi r^2 + 2\pi rh$

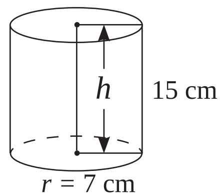
(ii) Exterior surface area of a cylinder without a base or a lid $= 2\pi rh$

Let us now consider several solved problems related to the surface area of a cylinder. In this lesson the value of π is taken as $\frac{22}{7}$ which is an approximate value for π.

Example 1

Determine the following for a cylindrical shaped solid log of base radius 7 cm and height 15 cm.

(i) Area of one plane face
(ii) Area of the curved surface
(iii) Total surface area



(i) Area of one plane face $= \pi r^2$

$= \frac{22}{7} \times 7 \times 7 \ \text{cm}^2$

$= \underline{\underline{154 \text{ cm}^2}}$

(ii) Area of the curved surface $= 2\pi rh$

$= 2 \times \frac{22}{7} \times 7 \times 15 \ \text{cm}^2$

$= \underline{\underline{660 \text{ cm}^2}}$

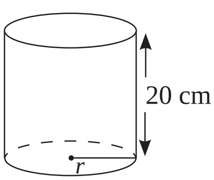
(iii) Total surface area $= 2\pi r^2 + 2\pi rh$

$= 2 \times (154) + 660 \text{ cm}^2$

$= 308 + 660 \text{cm}^2$

$= \underline{\underline{968 \text{ cm}^2}}$

Example 2



The circumference of the base of a cylindrical vessel without a lid, of height 20 cm is 88 cm.

(i) Find the radius of the base.
(ii) Find the total exterior surface area.

Let us denote the base radius by *r* and the height by *h*.

(i) Circumference of the base $= 2\pi r$

$$\therefore 2\pi r = 88$$

$$\therefore \quad r = \frac{88}{2\pi} = \frac{88 \times 7}{2 \times 22}$$

$$\therefore \text{ The radius } \quad r = \underline{\underline{14\text{ cm}}}$$

(ii) Total surface area $= \pi r^2 + 2\pi rh$

$$= \frac{22}{7} \times 14 \times 14 + 2 \times \frac{22}{7} \times 14 \times 20$$

$$= 616 + 1760$$

$$\therefore \text{ The total surface area } = \underline{\underline{2376\text{ cm}^2}}$$

Example 3

The surface area of a solid metal cylinder is 2442 cm^2 while the sum of its radius and height is 37 cm.

(i) Find the radius of the cylinder.

(ii) Find the area of the curved surface of the cylinder.

Let us denote the radius of the cross section by *r* and the height by *h*.

(i) Sum of the radius and the height $= 37\text{ cm}$

That is, $\quad r + h = 37\text{cm}$

The total surface area, $2\pi r^2 + 2\pi rh = 2442\text{cm}^2$

$$\therefore 2\pi r(r + h) = 2442$$

$$\therefore 2\pi r(37) = 2442 \quad \text{(By substituting for } r + h\text{)}$$

$$\therefore r = \frac{2442 \times 7}{2 \times 22 \times 37}$$

$$= 10.5$$

$$\therefore \text{The radius } \; r = \underline{\underline{10.5\text{ cm}}}$$

(ii) $$r + h = 37 \text{ cm}$$

Since $r = 10.5\text{cm},\ h = 37 - 10.5$ cm

$= 26.5$ cm

$\therefore$ The area of the curved surface $= 2\pi rh$

$$= 2 \times \frac{22}{7} \times 10.5 \times 26.5 \text{ cm}^2$$

$$= \underline{\underline{1749 \text{ cm}^2}}$$

Exercise 29.1

1. The radius of a cylinder is 7 cm and its height is 12 cm.
 (i) Find the area of the two circular faces.
 (ii) Find the area of the curved surface.
 (iii) Find the total surface area.

2. Find the area of the metal sheet that is required to make 200 cylindrical tins of radius 3.5 cm and height 10 cm, without lids.

3. The total surface area of a cylindrical vessel with a lid is 5412 cm^2. If the area of the curved surface is 2640 cm^2,
 (i) find the total surface area of the two circular faces.
 (ii) find the radius of the cylinder.
 (iii) find the height of the cylinder.

4. The base of a cylindrical vessel with a lid, which has been produced using a thin metal sheet, has a circumference of 88 cm. If the area of the curved surface is 1078 cm^2, find the height of the vessel.

5. The area of the curved surface of a cylindrical tin with a lid is 990 cm^2.
 (i) If the height of the tin is 15 cm, find the base radius.
 (ii) Find the total area of the two circular faces.
 (iii) Find the total surface area.

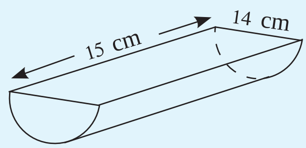
6. It is possible to paint an area of 13.5m^2 with one litre of a certain type of paint. The roof over the verandah of a certain house rests on 10 cylindrical pillars, each of height 3 m and diameter 28 cm. It has been decided to paint all these pillars.
 (i) Find the area of the curved surface of these 10 pillars to the nearest square metre.
 (ii) Find the required amount of paint in litres.
 (iii) If one litre of paint is Rs. 450, find the amount that has to be spent on the paint.

7. It is required to cover the total surface area of the curved surface of a right cylindrical shaped food container of radius 7 cm and height 10 cm with a label.

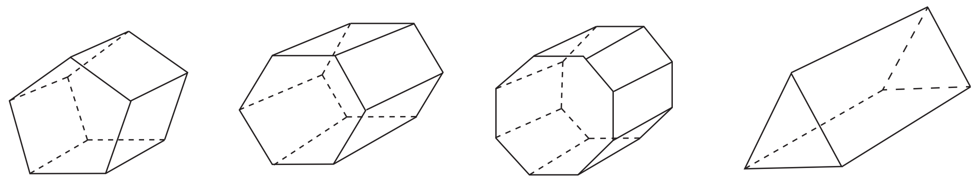
(i) How many labels can be cut out from a thin piece of paper of length 180 cm and breadth 90 cm such that the waste is minimized? Find the area of the piece of paper that goes waste.

(ii) Calculate how many such pieces of paper are required to cut out labels for 1200 containers of the above type.

8. The figure illustrates one half of a solid cylinder. Calculate the total surface area using the given information.

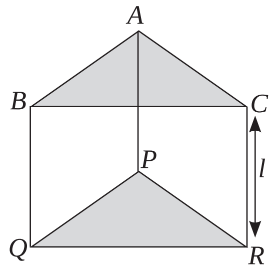


Prisms



The properties given below are common to the solids illustrated in the above figure.
- The cross section is uniform.
- The cross section takes the shape of a polygon.
- The side faces are rectangles.
- The faces at the two ends are perpendicular to the side faces.

Solids with these properties are called **right prisms**. From these right prisms, we will pay further attention to the one with a triangular cross section.



The figure illustrates a right prism with a triangular cross section. Here,

(1) ABC and PQR represent the two triangular faces at the two ends of the prism.

(2) the three rectangular side faces are represented by $BQRC$, $CRPA$ and $APQB$ (These faces are also called lateral faces).

(3) The distance between the two triangular faces is named the length or the height of a the prism and is denoted by l.

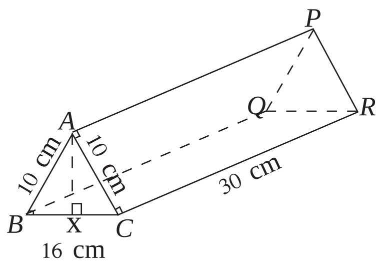
(4) The total surface area of the prism is the sum of the areas of the pair of triangular faces and the three rectangular faces.

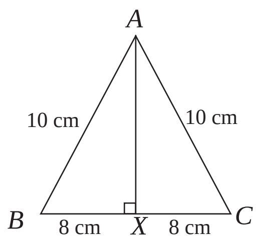
29.2 The surface area of a right prism with a triangular cross section

Example 1

Let us consider how the total surface area of the right prism with a cross section the shape of an isosceles triangle which is given below is found, using the given data.

Let us first find the area of the triangular face *ABC*. For this, let us find the perpendicular distance from A to *BC*.
According to the properties of isosceles triangles, if the midpoint of *BC* is *X*, then $AX \perp BC$. Now applying Pythogaras' theorem to the triangle *AXC*,





$$AC^2 = AX^2 + XC^2$$
$$10^2 = AX^2 + 8^2$$
$$100 - 64 = AX^2$$
$$\therefore\ 36 = AX^2$$
$$\therefore\ AX = \sqrt{36} \quad \text{(Since lengths cannot take negative values)}$$
$$\therefore\ AX = 6\text{ cm}$$

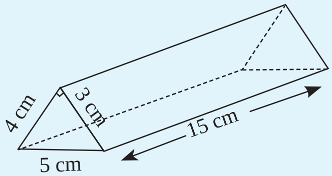
Accordingly, the area of the triangular face *ABC* $= \frac{1}{2} \times 16\text{ cm} \times 6\text{ cm} = 48\text{ cm}^2$

$\therefore$ The area of the two triangular faces *ABC* and *PQR* $= 2 \times 48\text{ cm}^2 = 96\text{ cm}^2$
The area of the rectangular face *ACRP* $= 10\text{ cm} \times 30\text{ cm} = 300\text{ cm}^2$
The area of the rectangular face *APQB* $= 10\text{ cm} \times 30\text{ cm} = 300\text{ cm}^2$
The area of the rectangular face *BCRQ* $= 16\text{ cm} \times 30\text{ cm} = 480\text{ cm}^2$
$\therefore$ The total surface area $= 96 + 300 + 300 + 480\text{ cm}^2$
$= \underline{\underline{1176\text{ cm}^2}}$

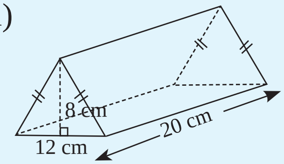
Exercise 29.2

1. Find the total surface area of each of the following prisms.

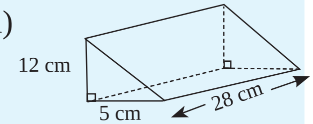
(i)



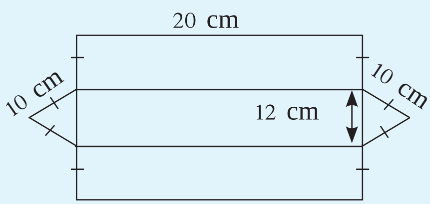
(ii)



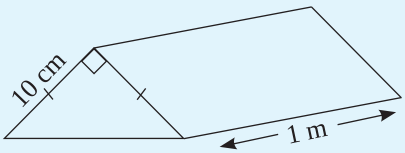
(iii)



2. Find the total surface area of the right prism with a triangular cross section that can be made with the following net with the given measurements.

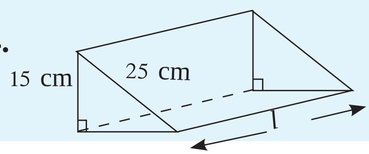


3.



Find the surface area of the prism in the figure.

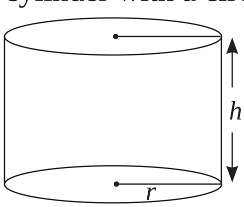
4.



If the total surface area of the solid wooden prism in the figure is $2100 cm^2$, find the length of the prism (l).

29.3 Volume of a cylinder

Recall what you have learnt in previous grades about calculating the volume of solids with a uniform cross section. You calculated the volume by multiplying the area of the cross section by the height. We can calculate the volume of a right cylinder with a circular cross section in the same manner.



Let us consider a right circular cylinder of base radius r and perpendicular height h. Let us denote its volume by V.

The volume of the cylinder = Area of the cross section $\times$ Height

$$= \pi r^2 \times h = \pi r^2 h$$

$$\boxed{\text{Volume of the cylinder } (V) = \pi r^2 h}$$

Let us now consider the following solved problems related to the volume of a right circular cylinder.

Example 1

Find the volume of a right circular cylinder of radius 14 cm and height 20 cm.

Here r = 14 cm

h = 20 cm

$\therefore$ The volume of the cylinder $= \pi r^2 h$

$$= \frac{22}{7} \times 14 \times 14 \times 20 \text{ cm}^3$$

$$= \underline{\underline{12320 \text{ cm}^3}}$$

Example 2

The volume of a cylindrical vessel of base area 346.5 cm^2 is 6930 cm^3.

(i) Find the radius of the cylinder.

(ii) Find the height of the cylinder.

(i)

Area of the base of a cylinder of radius r $= \pi r^2$

$$\therefore \pi r^2 = 346.5$$

$$\therefore r^2 = \frac{346.5}{22} \times 7$$

$$\therefore r^2 = 110.25$$

$$\therefore r = \pm 10.5 \quad \text{(Length cannot be negative)}$$

$$\therefore \text{The radius } (r) = \underline{\underline{10.5 \text{ cm}}}$$

(ii) Since the volume is 6930 cm^3

$$\pi r^2 h = 6\,930$$

$$346.5 \times h = 6\,930$$

$$\therefore h = \frac{6\,930}{346.5}$$

$$\therefore \underline{\underline{h = 20 \text{ cm}}}$$

Method 2

$$\pi r^2 h = 6930$$

$$\therefore \frac{22}{7} \times 10.5 \times 10.5 \times h = 6930$$

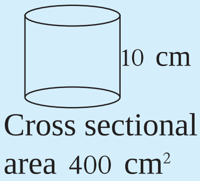
$$\therefore h = \frac{6930 \times 7}{22 \times 10.5 \times 10.5}$$

$$\therefore \text{Height} = \underline{\underline{20 \text{ cm}}}$$

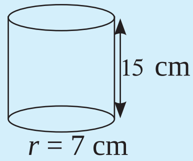
Exercise 29.3

1. Find the volume of each of the following cylinders, based on the data that is given.

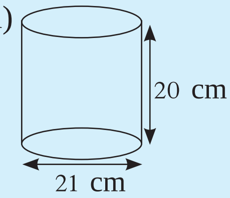
(i)



(ii)



(iii)



2. (i) Complete the following table by finding the cross sectional area and volume of three cylinders, each of radius 7 cm and height 8 cm, 16 cm and 24 cm respectively.

Base radius	Cross sectional area	Height	Volume
(*a*) 7 cm (*b*) 7 cm (*c*) 7 cm		8 cm 16 cm 24 cm	

(ii) By considering the data in the above completed table, explain how the volume of a cylinder changes when the radius is a constant and the height is doubled and tripled.

3. (i) Complete the following table by finding the area of the cross section and the volume of three cylinders, each of height 20 cm and of radius 7 cm, 14 cm and 21 cm respectively.

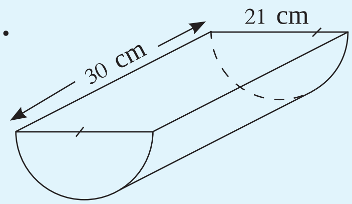
Base radius	Cross sectional area	Height	Volume
(*a*) 7 cm (*b*) 14 cm (*c*) 21 cm		20 cm 20 cm 20 cm	

(ii) By considering the data in the above completed table, explain how the volume of a cylinder changes when the height is a constant and the radius is doubled and tripled.

4. The diameter of a cylindrical shaped vessel is 28 cm. If the vessel contains a volume of 6160 cm^3 of water, find the height of the water level.

5. A rectangular metal sheet is of length 22 cm and breadth 11 cm. Draw the two cylinders that can be constructed with this sheet such that each side forms the curved edge, mark the measurements on the figure, and find the volume of each cylinder.

6. Find the volume of a right circular cylinder of diameter 20 cm and curved surface area 1000 cm^2.

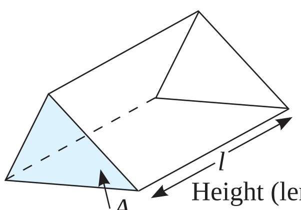
7. 

Calculate how many solid metal cylinders of height 21 cm and radius 3.5 cm can be made without wastage by heating the solid semi-cylindrical metal object with the measurements given in the figure.

8. A cylindrical vessel of radius 14 cm has been filled with water to a height of 30 cm. What is the minimum number of times that a cylindrical vessel of radius 7 cm and height 10 cm should be used to remove all the water in the given vessel?

29.4 Volume of a prism

Let us consider how the volume of a prism with a triangular cross section which you identified in 28.2 above is found.



We know that the volume of a right solid with a uniform cross section is equal to the product of the area of the cross section and the height (length).

We can use the above principle to find the volume of the right prism with a uniform triangular cross section given in the figure.

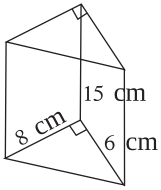
Then,

Volume of the prism $=$ Area of the cross section $\times$ Perpendicular height (length) $V = A\,l$

Note:
When the area A of the triangular cross section is not given directly, it is necessary to calculate it using the data related to the triangular cross section given in the problem.

Now consider the solved problems related to the volume of a prism given below.

Example 1



Based on the information in the figure,
(i) find the area of the cross section of the prism.
(ii) find the volume of the prism.

(i) Area of the cross section $= \frac{1}{2} \times 6 \times 8 = \underline{\underline{24 \text{ cm}^2}}$

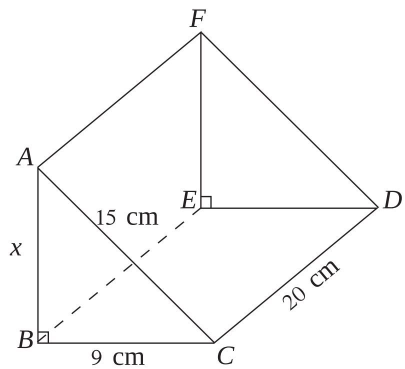
(ii) Volume of the prism $=$ Area of the cross section $\times$ Height

$= 24 \text{ cm}^2 \times 15 \text{ cm}$

$= \underline{\underline{360 \text{ cm}^3}}$

Example 2

A prism with a cross section the shape of a right angled triangle is shown in the figure.
(i) Find the length denoted by x in the cross section.
(ii) Find the area of the cross section.
(iii) Find the volume of the prism.



(i) By applying Pythagoras' theorem to the triangle ABC.

$$AC^2 = AB^2 + BC^2$$
$$15^2 = x^2 + 9^2$$
$$225 = x^2 + 81$$
$$225 - 81 = x^2$$
$$\sqrt{144} = x$$
$$x = \underline{\underline{12 \text{ cm}}}$$

(ii) Area of the cross section

$$= \frac{1}{2} \times 9 \times 12$$
$$= \underline{\underline{54 \text{ cm}^2}}$$

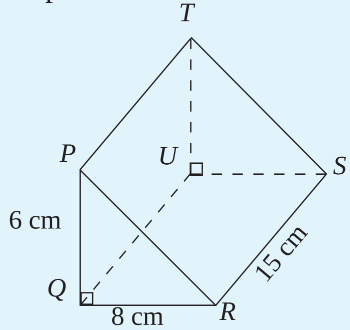
(iii) Volume of the prism

$$= 54 \times 20$$
$$= \underline{\underline{1080 \text{ cm}^3}}$$

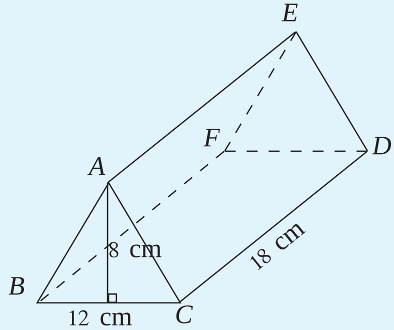
Exercise 29.4

1. Using the data that is marked on the prisms illustrated below find the volume of each prism.

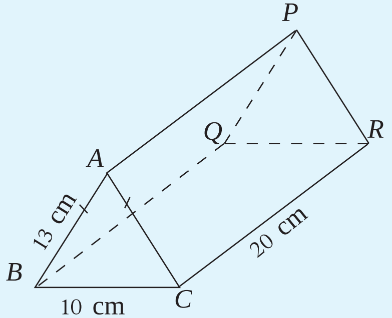
(i)



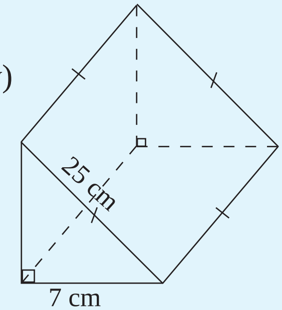
(ii)



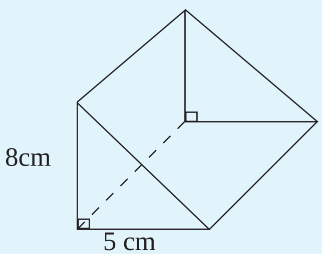
(iii)



(iv)

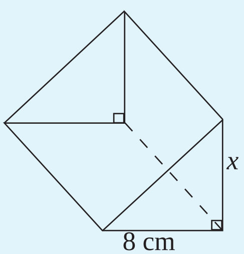


2. (i)

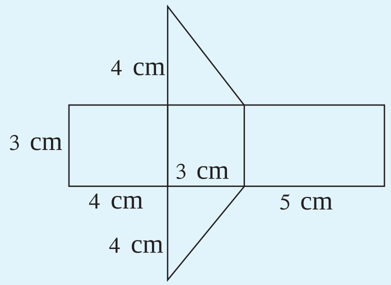


If the volume of the prism is 400 cm^3, find its length.

(ii)



Find the value of x if the height of the prism of volume 288 cm^3 given in the figure is 12 cm.

3. 

Find the volume of the prism that can be constructed using this net.

4. Water has been filled to a height of 8 cm of a cuboid shaped vessel of length 30 cm and breadth 20 cm. If the level of the water in the vessel rises by 2 cm when a right triangular prism of cross sectional area 60 cm^2 is dropped carefully into the water, find the perpendicular height of the prism.

5. A water tank, the shape of a prism with a triangular cross section of area 800 cm^2, is filled with water to a height of 30 cm. If this water is poured into a cuboid shaped vessel of length 60 cm and height 20 cm without wastage, to what height will the water level rise?

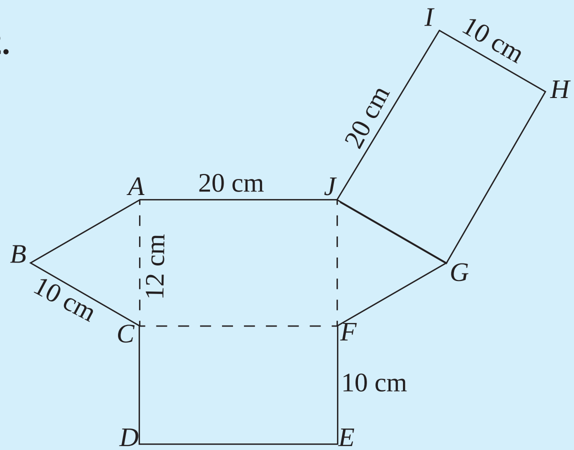
Summary

For a right circular cylinder of base radius r and height h,

- the total surface area $= 2\pi r^2 + 2\pi rh$
- the volume $= \pi r^2 h$

Miscellaneous Exercise

1. A cylindrical shaped log is of radius 14 cm and height 25 cm.
   (i) Find the total surface area.
   (ii) Find the volume.

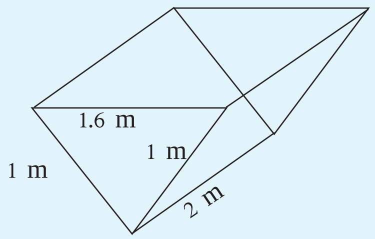
2. 

A sketch has been provided of a net with measurements, that has been drawn on a thick piece of paper such that it can be used to make a right triangular prism by folding it along the dotted lines.

(i) With which edge does the edge GH coincide?
(ii) With which vertex does the vertex H coincide?

(iii) Find the area of a triangular face of the prism that is made.
(iv) Find the total surface area and the volume of the prism.

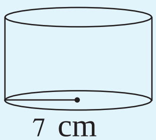
3.



A fish tank in the shape of a prism with a triangular cross section having the dimensions given in the figure has been made with cement in Dayan's garden.

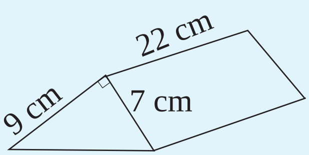
(i) Find the interior surface area of this tank.
(ii) Find in litres, the amount of water that is required to fill the tank completely.
(iii) If a pipe through which water flows at a rate of 20*l* per minute is used to fill the tank completely, find how much time is required to fill the tank.
(iv) Dayan now decides to build a new tank which takes the shape of a semi - circular cylinder of length 1 m, having the same volume as the above tank. Suggest suitable measurements for this tank.

4.



The volume of a cylinder of radius 7 cm and height h is 3080 cm^3.
i) Find the height of the cylinder.
ii) Find the surface area of the cylinder.

5.



A vessel, the shape of the prism in the figure is completely filled with water. All the water in this vessel is poured into a vessel the shape of a right cylinder of radius 7 cm. To what height does the water level rise in the cylinder?

30 Probability

By studying this lesson you will be able to

- identify simple events and composite events
- find the probability of events which are not mutually exclusive
- find the probability of an event using a grid or a tree diagram

We know that we will get either heads or tails when we toss a coin. This observation is an example of a random experiment. The possible results are either 'heads' or 'tails'. However we cannot say with certainty what the outcome will be before observing it. Experiments in which we know the possible outcomes, but cannot say with certainty which outcome will occur are called **random experiments**. The set to which all the possible outcomes of a random experiment belong is called 'the sample space'. It is denoted by *S*.

The following table shows some examples for random experiments and the relevant sample spaces.

Random Experiment	Sample Space
1. Tossing a coin and recording the result	$S = \{\text{Heads, Tails}\}$
2. Rolling a die numbered from 1 to 6 and recording the number shown	$S = \{1, 2, 3, 4, 5, 6\}$
3. Throwing a ball at a target and recording the result.	$S = \{\text{Hits the target, Misses the target}\}$

Events

An event is a subset of the sample space. Consider the following examples.

Consider the random experiment of rolling a fair tetrahedronal die numbered from 1 to 4 and recording the result.

The sample space is $S = \{1, 2, 3, 4\}$.

Some subsets of this sample space are $\{3\}$, $\{2, 4\}$, $\{1, 2, 3\}$.

These subsets can be explained as follows:

$\{3\}$ denotes "The event of getting 3 as the result".

$\{2, 4\}$ denotes "The event of getting 2 or 4 as the result".

Also if "getting a number less than 4" is denoted by A, then it can be written as $A = \{1, 2, 3\}$.

An event is a subset of the sample space.

Simple events and composite events

Consider rolling an unbiased die numbered from 1 to 6. In this random experiment,

the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
Let us write some events relevant to the sample space.
$\{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 4\}, \{1, 3, 5\}, \{2, 3, 5\}, \{3, 4, 5, 6\}$

In the above events, $\{1\}$, $\{2\}$ and $\{3\}$ consist of only one outcome each. Such events are called **simple events**.

An event consisting of only one outcome is called a simple event.

Thus, $\{5\}$, $\{6\}$ are simple events.
Events which are not simple are called **composite events**. The events $\{1, 3\}$, $\{2, 4\}$, $\{1, 3, 5\}$ are composite events. These composite events can be further decomposed into subsets.

30.1 Equally likely outcomes

The sample space of tossing an unbiased coin is shown below.

S = {getting heads, getting tails}

Because the coin is unbiased, it is clear that the likelihood of getting either of these two outcomes is equal.

Let us consider another example.
There are 3 identical balls in a bag coloured red, white and black. Consider taking out one ball at random. The sample space is shown below.

S = {getting the red ball, getting the white ball, getting the black ball}

Because the balls are identical, it is clear that the likelihood of taking any one of the balls is equal.

If each of the outcomes in a random experiment has an equal likelihood of occuring, that experiment is called an experiment with **equally likely outcomes**.

Consider the experiment of "tossing an unbiased coin." As you have learnt in earlies grades, the probability of each equally likely outcome "getting heads" and "getting tails" in the sample space is $\frac{1}{2}$;

That is, the probability of heads occuring $= \frac{1}{2}$.

The probability of tails occuring $= \frac{1}{2}$.

Now let us consider a random experiment which is not an experiment with equally likely outcomes.

Amara planted a mango seed and observed whether a plant would grow within a week. The sample space is

$$S = \{\text{grow, not grow}\}.$$

Here there are no reasons to assume that the outcomes are equally likely. Here, taking the probability of a plant growing as $\frac{1}{2}$ is not correct.

In an instance where all the outcomes of a sample space are equally likely to occur, the probability of an event occuring is defined as below.

$$\text{Probability of the event occuring} = \frac{\text{Number of elements in the event}}{\text{Number of elements in the sample space}}$$

Let us denote the number of elements in the sample space S by $n\,(S)$ and the number of elements in the event A by $n\,(A)$. The probability of A occuring is denoted by $P(A)$. Then

$$\boxed{P\,(A) = \frac{n(A)}{n(S)}}$$

Example 1

With reference to an experiment in which a card is drawn randomly from a bag containing 5 identical cards numbered from 1 to 5,

(i) write the sample space and find $n(S)$.

(ii) write the elements of A and find $n(A)$ if A is the event of getting an even number

(iii) find the probability $P(A)$ that an even number is drawn.

It is clear that the outcomes are equally likely since the cards are identical.

(i) $S = \{1, 2, 3, 4, 5\}$. $\therefore$ $n\,(S) = 5$

(ii) $A = \{2, 4\}$. $\therefore$ $n\,(A) = 2$

(iii) $P(A) = \dfrac{n(A)}{n(S)}$

$= \dfrac{2}{5}$

Example 2

In an experiment of rolling an unbiased die with its faces marked 1, 2, 3, 4, 5, 6,
(i) find the probability that the number shown is 4.
(ii) find the probability that the number shown is odd.
(iii) find the probability that the number is greater than 2.

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Thus $n(S) = 6$.

(i) Probability of getting 4 $= \dfrac{1}{6}$

(ii) Since there are three (1, 3 and 5) odd numbers, the probability of getting an odd number $= \dfrac{3}{6} = \dfrac{1}{2}$

(iii) Since there are four (3, 4, 5 and 6) numbers greater than 2, the probability of getting a number greater than 2 $= \dfrac{4}{6} = \dfrac{2}{3}$

Exercise 30.1

1. Write down the sample space for each of the following random experiments.
(i) Recording the card drawn from a pack of 10 identical cards numbered 1 to 10.
(ii) A circular disk is divided into three identical sectors, one of which is coloured red, another blue and the other yellow. A pointer is fixed to the center of the disk and the disk is spun. The colour of the sector at which it stops is recorded.
(iii) Recording the number of runs scored by a batsman in a single delivery in a cricket match.

2. Determine whether each event given below is a simple or a composite event.
(i) When rolling a tetrahedronal die numbered 1 to 4,
(*a*) getting the number 3
(*b*) getting a side with an odd number
(ii) In a pack of 5 identical cards labelled *A*, *B*, *C*, *D* and *E*,
(*a*) drawing the card labelled *A*
(*b*) drawing a card labelled with a vowel

3. When randomly taking a card from a bag containing 8 identical cards numbered 1 to 8,
(*a*) if the event of getting a card with a number greater than 4 is *A*, then write down the elements in *A*.
(*b*) Write 5 simple events in the event *A*.

4. There is a bag containing 10 identical pieces of paper numbered from 1 to 10. A piece of paper is drawn at random.
(i) Write the sample space.
(ii) If the event of drawing a square number is X, then write the elements of X and the value of $n(X)$.
(iii) Find the probability $P(X)$ of getting a square number.

5. 3 of 5 identical beads are blue and the remaining two are red. A bead is drawn randomly.
(i) Write the sample space.
(ii) Find the probability of drawing a red bead.
(iii) Find the probability of drawing a blue bead.

6. There are toffees of the same size and shape, but of different brands in a box. The table below gives information on them.

	Mango Flavoured	Orange Flavoured
Brand A	2	1
Brand B	3	2

A toffee is drawn randomly from the box. Find the probability of getting:
(i) an orange flavoured toffee
(ii) a toffee of brand A
(iii) a toffee of brand B
(iv) a mango flavoured toffee of brand A
(v) an orange flavoured toffee of brand B

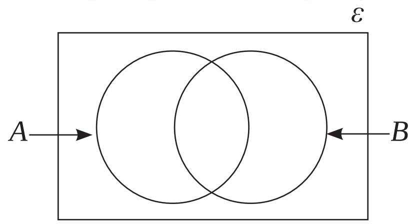
30.2 The intersection and union of two events

If A and B are two events, then their intersection $A \cap B$ and their union $A \cup B$ are also events.
For example, suppose there are 5 identical balls numbered 1, 2, 3, 4, 5 and one is picked at random. Then,

the sample set $S = \{1, 2, 3, 4, 5\}$.

If we denote the event of picking a ball with a number greater than 2 as A,
$A = \{3, 4, 5\}$.
If we denote the event of picking an even numbered ball as B,
$B = \{2, 4\}$.
Then, $A \cap B = \{4\}$. Here, $A \cap B$ represents the event of picking a ball in both the sets A and B; that is, a ball with an even number greater than 2.
Furthermore, $A \cup B = \{2, 3, 4, 5\}$. Here, $A \cup B$ denotes the event of picking a ball in either set A or in set B; that is, a ball with either an even number or a number greater than 2.

Now let us consider a relationship between the events A, B, $A \cup B$ and $A \cap B$ where A and B are any two events in a sample space with equally likely outcomes.



From our knowlege of sets, we have the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Divide each term by $n(S)$ to get

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}.$$

Since the outcomes are equally likely, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Thus, for any two events A and B in a sample space with equally likely outcomes, we have

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Therefore for the example discussed above, we have

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{5},$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{5},$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{5}$$

and $$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{4}{5}.$$

Also, $P(A) + P(B) - P(A \cap B) = \frac{3}{5} + \frac{2}{5} - \frac{1}{5}$

$$= \frac{4}{5}$$

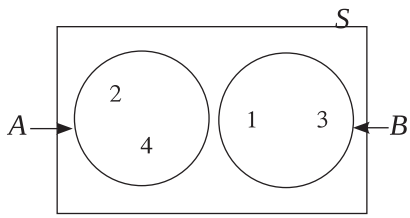
So the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ is true for this example.

Mutually exclusive events

A fair tetrahedronal die with its sides numbered from 1 to 4 is rolled. Let us denote the event that it rests on an even numbered face by A and the event that it rests on an odd numbered face by B.
That is, $A = \{2, 4\}$ and $B = \{1, 3\}$.
Then $A \cap B = \emptyset$. This means that A and B have no common elements.
That is, these two events do not occur together. Such events are said to be **mutually exclusive events**.

$$\boxed{\text{If } A \cap B = \emptyset. \text{ then } A \text{ and } B \text{ are mutually exclusive.}}$$

Now, let us find $A \cup B$ in the example we were talking about. Let us show the given information in a Venn diagram.



Then,

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{0}{4} = 0$$

Because $A \cap B = \emptyset$ when A and B are mutually exclusive, $P(A \cap B) = 0$

Therefore

If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$

Complement of an event

There is a pack of 5 identical cards numbered from 1 to 5. Consider the random experiment of drawing a card randomly.
The sample space here is $S = \{1, 2, 3, 4, 5\}$.
If A is the event of drawing an even numbered card, then $A = \{2, 4\}$.

If the event of A not happening, that is, the event of not drawing an even numbered card is B, then, $B = \{1, 3, 5\}$.
In the above experiment, if the event of drawing an even numbered card is A, then the event of drawing a card which is not even numbered is the complement of A. The complement of A is written as A' .
Therefore $A' = \{1, 3, 5\}$.

Here $A \cup A' = S$

Also $A \cap A' = \emptyset$.

Therefore, A and A' are mutually exclusive events.

These results are true for any event.

Accordingly $P(A \cup A') = P(A) + P(A')$

$$\therefore P(S) = P(A) + P(A')$$

$$\therefore 1 = P(A) + P(A') \qquad [\text{since } P(S) = 1]$$

$$\therefore P(A') = 1 - P(A)$$

$$\boxed{\text{For any event } A,\ P(A') = 1 - P(A)}$$

Example 1

For the events A and B of a random experiment,

$P(A) = \frac{2}{7}$ $P(B) = \frac{3}{7}$ and $P(A \cap B) = \frac{1}{14}$.

Find (i) $P(A \cup B)$ (ii) $P(A')$ (iii) $P(B')$ (iv) $P[(A \cap B)']$ (v) $P[(A \cup B)']$

(i) Using the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B) = \frac{2}{7} + \frac{3}{7} - \frac{1}{14}$$

$$= \frac{4}{14} + \frac{6}{14} - \frac{1}{14} = \underline{\underline{\frac{9}{14}}}$$

(ii) Using the formula $P(A') = 1 - P(A)$

$$P(A') = 1 - \frac{2}{7}$$

$$= \frac{7}{7} - \frac{2}{7}$$

$$= \underline{\underline{\frac{5}{7}}}$$

(iii) Using the formula $P(B') = 1 - P(B)$

$$P(B') = 1 - \frac{3}{7}$$

$$= \frac{7}{7} - \frac{3}{7}$$

$$= \underline{\underline{\frac{4}{7}}}$$

(iv) $P\left[(A\cap B)'\right]=1-P(A\cap B)$

$$= 1-\frac{1}{14}$$
$$= \frac{14}{14}-\frac{1}{14}$$
$$= \underline{\underline{\frac{13}{14}}}$$

(v) $P\left[(A\cup B)'\right]=1-P(A\cup B)$

$$= 1-\frac{9}{14}$$
$$= \frac{14}{14}-\frac{9}{14}$$
$$= \underline{\underline{\frac{5}{14}}}$$

Example 2

X and Y are two mutually exclusive events of a random experiment.

$P(X)=\frac{1}{6}$ and $P(Y)=\frac{7}{12}$.

Find (i) $P(X\cap Y)$ (ii) $P(X\cup Y)$

(i) Since X and Y are mutually exclusive events, $P(X\cap Y)=0$.

$$P(X\cup Y)=P(X)+P(Y)$$
$$=\frac{1}{6}+\frac{7}{12}$$
$$=\frac{2}{12}+\frac{7}{12}=\underline{\underline{\frac{9}{12}}}=\underline{\underline{\frac{3}{4}}}$$

Exercise 30.2

1. In a random experiment of rolling a fair die numbered from 1 to 6, let
A be the event that a prime number is obtained,
B be the event that a perfect square is obtained,
C be the event that a number greater than 4 is obtained and
D be the event that a multiple of 6 is obtained.

Select all pairs of mutually exclusive events.

2. Let X and Y be two events of a random experiment which are not mutually exclusive.

If $P(X)=\frac{1}{4}$, $P(Y)=\frac{5}{6}$ and $P(X\cap Y)=\frac{1}{6}$, find each of the following:

(i) $P(X\cap Y)$ (ii) $P(X')$ (iii) $P(Y')$ (iv) $P[(X\cap Y)']$ (v) $P[(X\cup Y)']$

3. A and B are two events of a random experiment.

$P(A)=\frac{2}{7}$ and $P(B')=\frac{1}{4}$. Find $P(A')$ and $P(B)$.

4. X and Y are two events of a random experiment. It is given that

$P(X) = \frac{1}{2}$, $P(Y) = \frac{1}{3}$ and $P(X \cup Y) = \frac{5}{6}$.

(i) Find $P(X \cap Y)$

(ii) There by show that X and Y are mutually exclusive.

5. X, Y and Z are three events of a random experiment. If

$P(X) = \frac{1}{6}$, $P(Y) = \frac{1}{9}$, $P(Z') = \frac{2}{3}$, $P(X \cap Y) = \frac{1}{18}$ and $P(X \cap Z) = \frac{1}{12}$.

Find the following:

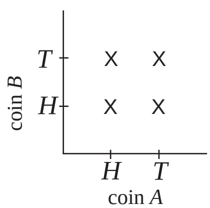
(i) $P(X)$ (ii) $P(Y')$ (iii) $P(Z)$ (iv) $P(X \cup Y)$ (v) $P[(X \cup Z)']$

30.3 Representation of the sample space in a grid

Consider a random experiment of tossing two unbiased identical coins A and B simultaneously. Let us denote the heads by H and the tails by T. The set of all possible outcomes of this experiment can be listed as follows:

Getting heads in both coins, (H, H)
Getting heads in coin A and tails in coin B, (H, T)
Getting tails in coin A and heads in coin B, (T, H)
Getting tails in both coins, (T, T)

Accordingly, the sample space can be written as $\{(H, H), (H, T), (T, H), (T, T)\}$. This sample space can be represented in a grid as follows:



Here, the outcomes are denoted by 'x'

The sample space of this experiment consists of four outcomes. Since the coins are unbiased and identical, all the possible outcomes are equally likely and thus the following probabilities are obtained:

(i) The probability that both coins show heads $= \frac{1}{4}$

(ii) The probability that coin A shows heads and coin B shows tails $= \frac{1}{4}$

(iii) The probability that one coin shows tails and the other coin shows heads $= \frac{2}{4}$

(iv) The probability that both coins show tails $= \frac{1}{4}$

Note: In the above random experiment all the outcomes were equally likely. Even though it is not compulsory to represent the outcomes in a grid, finding the probability by this method is not possible when the outcomes are not equally likely.

Example 1

Let us consider the experiment of tossing a coin and rolling a tetrahedronal die numbered from 1 to 4, and recording the faces touching the table.

(i) Show the sample space as a set of ordered pairs on a grid and then represent it on a grid.

(ii) Find the probability of getting

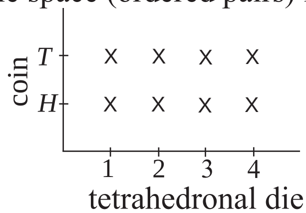
(*a*) 1 on the die.

(*b*) an even number on the die and tails on the coin.

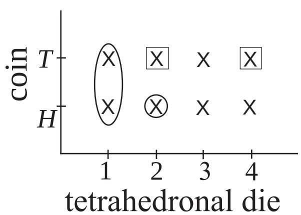
(*c*) 2 on the die and heads on the coin.

(i) $S = \{(1,H),(2,H),(3,H),(4,H),(1,T),(2,T),(3,T),(4,T)\}$

Let us show the sample space (ordered pairs) in a grid.



(ii) It is clear that all the results here are equally likely.



(*a*) In the above grid, the area marked by are the elements of the event of getting 1 on the die. There are 2 elements there. The total number of elements in the sample space is 8.

$\therefore$ The probability of getting 1 on the die $= \frac{2}{8} = \frac{1}{4}$

(*b*) In the above grid, the area marked by are the elements of the event of getting an even number on the die and tails on the coin. There are 2 such elements.

$\therefore$ The probability of getting an even number on the die and tails on the coin $\Big\} = \frac{2}{8} = \frac{1}{4}$

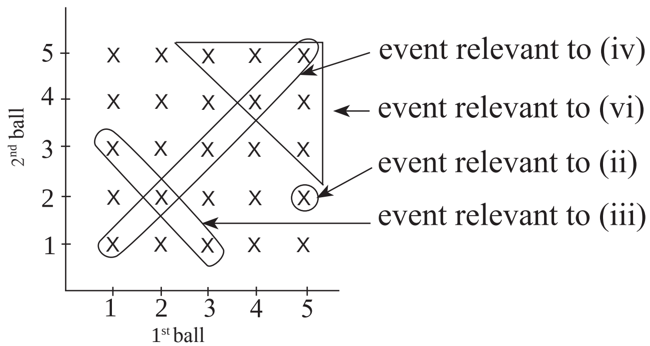
(*c*) In the above grid, the area marked by are the elements of the event of getting two on the die and heads on the coin. There is one such element.

$\therefore$ The probability of getting two on the dice and heads on the coin $= \frac{1}{8}$

Example 2

There are 5 identical balls numbered from 1 to 5 in a bag. A ball is taken from the bag randomly and the number is recorded. Then the ball is put back in the bag (with replacement) and again a ball is randomly taken for a second time and this number is also recorded.

(i) Show the relevant sample space in a grid.



(ii) Find the probability that the first ball is numbered 5 and the second ball is numbered 2.

$$\frac{1}{25}$$

(iii) Find the probability that the sum of the numbers on the two balls is 4.

$$\frac{3}{25}$$

(iv) Find the probability that the same ball is taken on both occassions.

$$\frac{5}{25} = \frac{1}{5}$$

(v) Are the events in (ii) and (iv) mutually exclusive?
Yes. This is because the two events have no common elements.

(vi) Find the probability that the sum of the numbers on the two balls taken is greater than 7.

$$\frac{6}{25}$$

(vii) Are the events in (iii) and (iv) mutually exclusive?

No.The reason is that there is an element common to both events; i.e., (2,2).

Exercise 30.3

1. A fair die numbered from 1 to 6 is rolled and an unbiased coin is tossed at the same time. The side on top is recorded in each case. Consider this experiment.

(*a*) Show the sample space in a grid.

(*b*) Use it to find the probabilities of the following events.

(i) Getting 1 on the die and heads on the coin.

(ii) Getting an even number on the die and heads on the coin.

(iii) Getting tails on the coin.

2. Two fair dice numbered from 1 to 6 are rolled simultaneously and the side shown is recorded. Consider this experiment.

(*a*) Show the sample space in a grid.

(*b*) Use it to find the probabilities of the following events.

(i) The sum of the two numbers being 5.

(ii) The sum of the two numbers being greater than 10.

(iii) The two numbers being the same.

(iv) Getting 3 on the first die.

3. A bag contains identical beads. There are 3 red beads, one blue bead, and 2 yellow beads. These are named R_1, R_2, R_3, B, Y_1, Y_2. A bead is taken randomly, its colour is recorded and then put back in the bag. A bead is randomly taken from the bag again and its colour is recorded.

(*a*) Show the sample space in a grid.

(*b*) Use it to find the probabilities of the following events.

(i) The first bead being red and the second bead being yellow.

(ii) Both beads being red.

(iii) Both beads being the same colour.

(iv) Getting at least one blue bead.

(v) Write all the pairs of mutually exclusive events in the above questions (i) – (iv).

4. There are 5 roads labelled *A, B, C, D,* and *E* that meet at a junction. Here, it is possible to enter or exit from any road. Draw a grid showing all the possible ways in which a person can enter and exit and find the probabilities of the following events. (Assume that all the possible outcomes are equally likely.)

(i) Entering from *A* and exiting from *B*.

(ii) Entering from *A* or *B* and exiting from *D*.

(iii) Entering from *E*

(iv) Entering and exiting from different roads.

5. A plant has 4 red flowers and 3 yellow flowers of identical shape and size. 2 butterflies, A and B, come to the flowers to drink nectar. It is possible for both to drink nectar from the same flower at the same time. Draw a grid of the sample space, showing all the possible ways the butterflies can pick flowers to get nectar and find the probabilities of the following events. (Assume that the butterflies pick flowers randomly.)

(i) Butterfly A picking a red flower and butterfly B picking a yellow flower.
(ii) Both butterflies picking flowers of the same colour.
(iii) Both butterflies picking flowers of different colours.
(iv) Both butterflies picking the same flower.

30.4 Independent events

Consider the following random experiments.

(i) Two unbiased coins are tossed simultaneously and the sides shown are recorded. It is clear that whatever the result of one coin toss is, it will not affect the result of the other.
(ii) There are two bags containing some identical balls. One ball is taken randomly from each bag. It is clear that the ball taken from one bag will not affect what ball is taken from the other bag.
(iii) During the germination of a few planted seeds, germination of one seed does not have an impact on the germination of the other seeds.

In this way, if one event does not affect another, the two events are called **independent**.
The independence of two events is defined as follows.

$$\boxed{\text{If } P(A\cap B)=P(A).P(B) \text{ then } A \text{ and } B \text{ are independent.}}$$

We have learnt that two mutually exclusive events are two events that do not occur together. But what is meant by 2 events being independent is that the occurrance of one event does not affect the occurrance of the other event.

Example 1

X and Y are two independent events. $P(X)=\frac{1}{3}$ and $P(Y)=\frac{1}{4}$, Find $P(X\cap Y)$, and $P(X\cup Y)$.

Because X and Y are independent events

$$P(X\cap Y) = P(X).P(Y).$$

$$\therefore \quad P(X\cap Y) = \frac{1}{3}\times\frac{1}{4}=\frac{1}{12}.$$

Using the formula $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$$P(X \cup Y) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12}$$

$$= \frac{4+3-1}{12}$$

$$P(X \cup Y) = \frac{6}{12} = \frac{1}{2}$$

Example 2

The probability that candidate A will pass an examination is $\frac{1}{5}$ and the probability that candidate B will pass is $\frac{3}{10}$. Assume that these events are independent and find the probabilities of the events below.

(i) Both of them passing

(ii) One of them passing.

Let us denote the event of candidate A passing as A and candidate B passing as B.

(i) The probability of both candidates A and B passing is $P(A \cap B)$.

Because they are independent events,

$$P(A \cap B) = P(A).\ P(B)$$

$$= \frac{1}{5} \times \frac{3}{10} = \frac{3}{50}$$

(ii) The probability that one of the candidates pass is $P(A \cup B)$ Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore\ P(A \cup B) = \frac{1}{5} + \frac{3}{10} - \frac{3}{50}$$

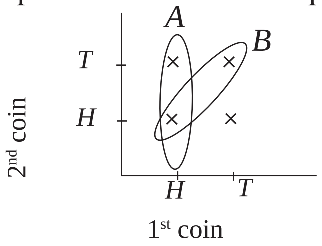
$$= \frac{10+15-3}{50}$$

$$= \frac{22}{50}$$

$$= \frac{11}{25}$$

Example 3

Two unbiased identical coins are tossed simultaneously. Let us represent the sample space of this random experiment in a grid.



Take the event of the first coin being heads as A. Take the event of both the coins showing the same side as B.

Here, the event of A or B does not affect the other event; therefore A and B are independent events.
Let us find the probabilities relevant to A and B.

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{4}$$

Further, $P(A).P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

That is, $P(A \cap B) = P(A).P(B)$

$\therefore$ A and B are independent events.

Exercise 30.4

1. X and Y are independent events. $P(X) = \frac{1}{2}$ and $P(X \cap Y) = \frac{1}{3}$
 (i) Find $P(Y)$.
 (ii) Find $P(X \cup Y)$.

2. An unbiased coin and a fair die numbered from 1 to 6 are tossed simultaneously.
 (*a*) Represent the sample space relevant to this experiment in a grid.
 (*b*) Take the event of the coin showing heads as A and the event of the die showing the number 4 as B. Show these events on the grid and find the probabilities of the events below.

 (i) $P(A)$ (ii) $P(B)$ (iii) $P(A \cap B)$ (iv) $P(A \cup B)$

3. A bag contains 3 red beads and 2 blue beads, all of which are otherwise identical. A bead is taken out randomly and its colour is recorded and then it is put back in the bag. A bead is randomly taken out again and its colour is recorded. Find the probabilities of the following events.

(i) Both of the beads being red.
(ii) The first bead being blue and the second bead being red.
(iii) The first bead being red and the second bead being blue.
(iv) Both beads being blue.

30.5 Tree Diagrams

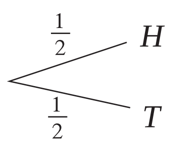
Tree diagrams can be used to find the probabilities of events of a random experiment. Let us consider the following example in order to understand this method.

Example 1

An unbiased coin is tossed twice and in each case the outcomes is recorded. Draw the relevant tree diagram and find the probabilities of the following events.

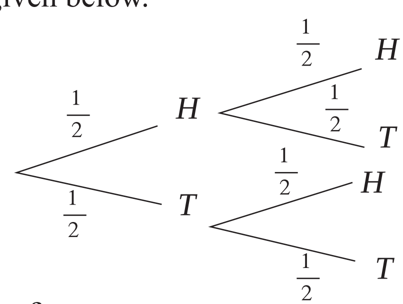
(i) Getting heads both times.
(ii) Getting the same side both times.
(iii) Getting tails at least once.
(vi) Getting heads the second time.

This experiment can be separated into two. That is, the first toss and the second toss. The two outcomes of the first toss can be represented in a tree diagram with two branches as shown below.



1st attempt

Here the corresponding probabilities are given on the branches. We know the probabilities are $\frac{1}{2}$ (since the coin is unbiased). For the second toss we can extend the tree diagram as given below.



first attempt second attempt

Here too the probabilities are given on the branches. As the 1^{st} and 2^{nd} attempts are independent, both the probabilities are $\frac{1}{2}$ each. There are 4 paths from start to end. That is,

(i) Heads in the 1^{st} attempt and heads in the 2^{nd} attempt

(ii) Heads in the 1^{st} attempt and tails in the 2^{nd} attempt

(iii) Tails in the 1^{st} attempt and heads in the 2^{nd} attempt

(iv) Tails in the 1^{st} attempt and tails in the 2^{nd} attempt

All the possible outcomes are represented above.
As the events given by the 1^{st} and the 2^{nd} attempts are independent, to find the probability of each outcome, we can take the product of the relevant probabilities.

i.e., P (Heads in the 1^{st} attempt, and heads in the 2^{nd} attempt)

$$= \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}$$

Likewise,

P (Heads in the 1^{st} attempt and tails in the 2^{nd} attempt) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

P (Tails in the 1^{st} attempt and tails in the 2^{nd} attempt) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

P (Tails in the 1^{st} attempt and heads in the 2^{nd} attempt) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Sample space of the experiment

$$\text{S} = \{(H,H), (H,T), (T,H), (T,T)\}$$

The probabilities in short are

$$P\,(H,H) = \frac{1}{4}$$

$$P\,(H,T) = \frac{1}{4}$$

$$P\,(T,H) = \frac{1}{4}$$

$$P\,(T,T) = \frac{1}{4}$$

Now let us answer the questions.

(i) P (getting heads both times) $= P\,(H, H)$

$$= \frac{1}{4}$$

(ii) P (getting the same side both times) $= \mathrm{P}((H, H) \text{ or } T, T))$

$= \mathrm{P}\,(H, H) + \mathrm{P}\,(T, T)$

(Because the two events are mutually exclusive)

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(iii) Getting tails at least once $= 1 - P$ (Getting heads both times)

$$= 1 - P(H, H)$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

(iv) Getting heads in the 2nd attempt $= P\,((H, H) \text{ or } (T, H))$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(Because the two events are mutually exclusive)

Exercise 30.5

1. A box contains 4 red pencils and two black pencils. A pencil is taken from the box at random and returned back to the box after its color is recorded, and then another pencil is taken out of the box. Depict the sample space as a tree diagram and hence find the probability that

(i) both pencils are red.

(ii) the two pencils are of opposite colours.

(iii) the two pencils are of the same colour.

2. Sarath and Sunith work at the same workplace. They use the bus for their daily travel. The probability that Sarath is late to work is $\frac{1}{3}$ and that for Sunith is $\frac{1}{4}$. Depict the relevant sample space in a tree diagram and hence find the probability that

(i) both are not late

(ii) only one person is late.

3. The probability that the shooter in a netball team shoots the ball correctly is $\frac{3}{5}$. Draw the sample space of two shots in a tree diagram and find the probability of

(i) shooting correctly both times

(ii) shooting correctly once

Miscellaneous Exercises

1. In a survey of 25 students conducted to see who likes to drink tea and coffee, it was found that 17 students like to drink tea, 15 like to drink coffee and 10 like to drink both tea and coffee.

(*a*) Draw a Venn diagram illustrating this information.

(*b*) Use it to find the probability that a student

(i) likes to drink only tea

(ii) likes to drink only one of these two types

(iii) likes to drink either tea or coffee

(iv) does not like to drink either tea or coffee.

2. In a mixed school, each student who studies in the biological stream and in the mathematics stream has to take either Exam P_1 or P_2. Below is the actual classification of the 100 students in the two streams.

Type Of Exam	Sex	Biological Stream	Mathematics Stream
P_1	Girl	10	5
	Boy	20	5
P_2	Girl	30	10
	Boy	15	5

If a student is picked randomly, find the probability that this student is

(i) a girl

(ii) following the mathematics stream

(iii) doing Exam P_1

(iv) a boy following the mathematics stream and doing exam P_2

(v) given that the student is a girl, what is the probability of her following the biological stream?

3. Following is a quote from an advertisement:

"Every one in 7 of the tickets in this lottery is a winning ticket."

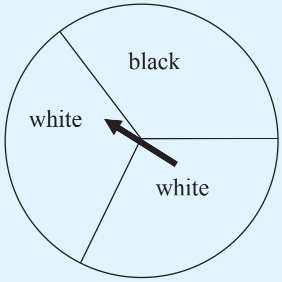
Hearing this, a person purchased two of the tickets of this lottery.

(*a*) Draw the relevant tree diagram.

(*b*) Find the probability of

(i) both tickets being winning tickets.

(ii) at least one of the tickets being a winning ticket.

4. 

A circular disk, as shown in the figure, is divided into three equal sectors. Two sectors are painted white and the third is painted black. A pointer at the centre is free to rotate. The pointer is given a spin and the colour of the sector at which the pointer stops is recorded.

Draw a tree diagram for two such spins and hence find the probability that the pointer stops in

(i) a white area in both instances

(ii) a black area at least once.

5. 10% of the candidates who applied for a job qualified through a competitive examination. Of those who qualified, 60% were selected in the first round. Find the probability that a randomly picked applicant is selected in the first round.

6. There are four choices for each question in a multiple choice question paper, out of which only one answer is correct. A student who is unsure of the answers for two of the questions chooses random answers. Draw a tree diagram and hence find the probability that

(i) the same choice number is selected for both questions.

(ii) both answers are correct

(iii) at least one answer is correct.

7. A and B are two government servants who work in the same office. They are entitled to a days leave on any one of the five working days in a week. Assuming that each of A and B gets his leave at random on a grid depict the sample space of all the possible ways in wihch they both can get leave during the five days of a week. Hence find the probability of each of the following events.

(i) A taking leave on Monday and B on Wednesday.

(ii) B taking leave on a day previous to A.

(iii) B taking leave on a day after A.

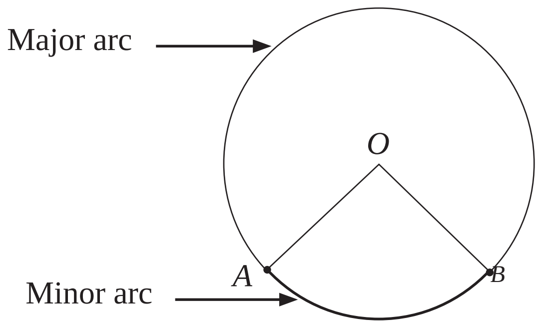
(iv) Both taking leave on the same day.

31 Angles in a Circle

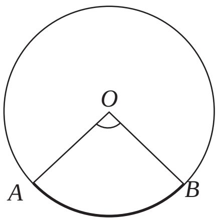
By studying this lesson you will be able to

identify and apply the theorems related to angles in a circle.

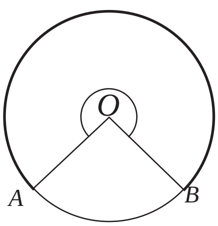
31.1 Angles subtended by an arc at the centre and on the circle



The circle in the figure is divided into two parts by the two points A and B on the circle. These two parts are called arcs. When the two points A and B are such that the straight line joining the two points passes through the centre of the circle, that is, when it is a diameter, then the two arcs are equal in length. When this is not the case, the two arcs are unequal in length. Then the shorter arc is called the **minor arc** and the longer arc is called the **major arc**.

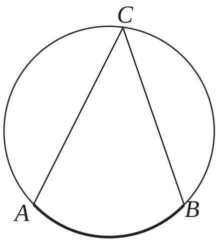


The angle $A\hat{O}B$ which is formed by joining the end points of the minor arc AB which is indicated by the thick line in the figure, to the centre, is defined as the angle subtended by the minor arc AB at the centre.



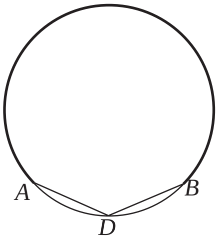
The reflex angle $A\hat{O}B$ which is formed by joining the end points of the major arc AB which is indicated by the thick line in the figure, to the centre, is defined as the angle subtended by the major arc AB at the centre.

Note: The angle subtended at the centre by a major arc is always a reflex angle.



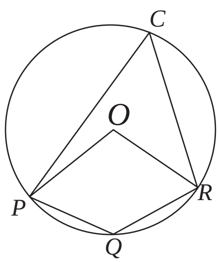
Let us assume that C is any point on the major arc AB.

The angle $A\hat{C}B$ is formed when the end points of the minor arc AB are joined to the point C on the major arc. That is, $A\hat{C}B$ is the angle that is subtended by the minor arc AB on the remaining part of the circle.



Similarly, the angle $A\hat{D}B$ in the given figure is defined as the angle that is subtended on the remaining part of the circle by the major arc AB.

Example 1



The centre of the circle in the given figure is O.

(*a*) Write down,

(i) the angle that is subtended by the minor arc PR on the remaining part of the circle.

(ii) the angle that is subtended at the centre by the minor arc PR.

(*b*) Write down,

(i) the angle that is subtended by the major arc PR on the remaining part of the circle.

(ii) the angle that is subtended at the centre by the major arc PR.

(*a*) (i) $P\hat{C}R$

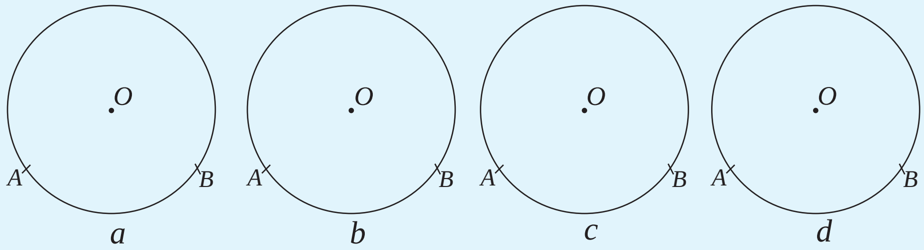
(ii) $P\hat{O}R$

(*b*) (i) $P\hat{Q}R$

(ii) reflex angle $P\hat{O}R$

Exercise 31.1

1. Copy the four circles in the figure given below onto your exercise book. The centre of each circle is denoted by O.



Mark each of the angles indicated below.

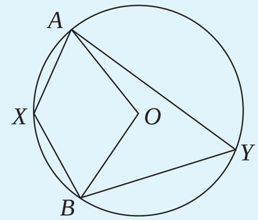
(i) An angle that is subtended on the remaining part of the circle by the minor arc in figure *a*.

(ii) The angle subtended at the centre by the minor arc in figure *b*.

(iii) An angle that is subtended on the remaining part of the circle by the major arc in figure *c*.

(iv) The angle subtended at the centre by the major arc in figure *d*.

2. According to the figure,



(i) write down for the minor arc AB,

(*a*) the angle subtended on the remaining part of the circle

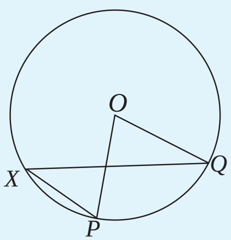
(*b*) the angle subtended at the centre.

(ii) write down for the major arc AB,

(*a*) the angle subtended on the remaining part of the circle

(*b*) the angle subtended at the centre.

3. The centre of the circle in the given figure is O. The point X is on the major arc PQ.



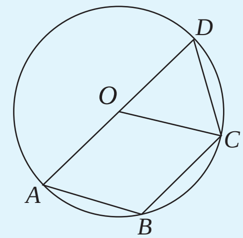
(i) Write down the angle that is subtended on the remaining part of the circle by the minor arc PQ.

(ii) Write down the angle that is subtended at the centre by the minor arc PQ.

(iii) Mark any point on the minor arc PQ and name it Y. Define the angle $P\hat{Y}Q$.

(iv) Write down the angle that is subtended at the centre of the circle by the major arc PQ.

4. The centre of the circle in the figure is O.



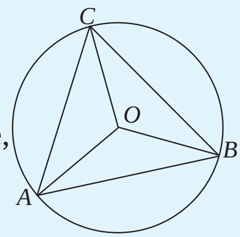
(i) Name an angle that is subtended on the remaining part of the circle by the minor arc AC.

(ii) Write down the angle that is subtended at the centre by the minor arc AC.

(iii) Write down an angle that is subtended on the remaining part of the circle by the major arc AC.

(iv) Write down the angle that is subtended at the centre by the major arc AC.

5. The centre of the circle in the figure is O.



(i) Write down for the minor arc AB,

(*a*) the angle subtended on the remaining part of the circle,

(*b*) the angle subtended at the centre.

(ii) Write down for the minor arc BC,

(*a*) the angle subtended on the remaining part of the circle

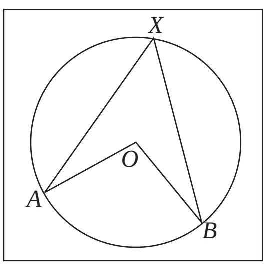
(*b*) the angle subtended at the centre.

31.2 The relationship between the angles subtended by an arc at the centre and on the circumference of a circle

Let us engage in the following activity to gain an understanding of the relationship between the angle subtended at the centre by an arc and the angle subtended on the remaining part of the circle by the same arc.

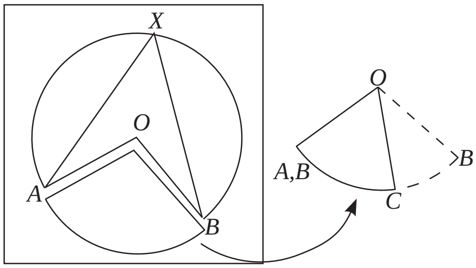
Activity

Draw a circle on a tissue paper and name its centre O. Mark two points on the circle such that a minor arc and a major arc are obtained. Name these two points as A and B.



Mark a point on the major arc and name it X.

Identify the angle subtended at the centre by the arc AB. This angle is $A\hat{O}B$. Cut out the sector of the circle AOB as shown in the figure.



- Fold the sector of the circle AOB into two such that OA and OB coincide, to obtain an angle which is exactly half the size of $A\hat{O}B$.
- Place this folded sector on $A\hat{X}B$ such that O overlaps with X and examine it.

You would have established the fact that the angle $A\hat{O}B$ subtended at the centre by the minor arc AB is twice the angle $A\hat{X}B$ subtended by this arc on the remaining part of the circle. In the same manner, mark arcs of different lengths on circles of varying radii and repeat this activity.

Through these activities you would have observed that the angle subtended at the centre by an arc is twice the angle subtended on the remaining part of the circle by the same arc.

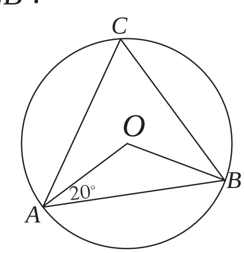
This result is given as a theorem below.

Theorem

The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc on the remaining part of the circle.

Now, by considering the following examples, let us see how calculations are performed using the above theorem.

The points A, B and C are on a circle of centre O. If $O\hat{A}B = 20^\circ$, let us find the magnitude of $A\hat{C}B$.



$OA = OB$ (The radii of a circle are equal)

$\therefore$ OAB is an isosceles triangle.

In an isosceles triangle, since angles opposite the equal sides are equal,

$$O\hat{A}B = O\hat{B}A$$

$$\therefore \; O\hat{B}A = 20^\circ. \qquad (\text{Since } O\hat{A}B = 20^\circ)$$

Since the sum of the interior angles of a triangle is equal to 180°,

$$A\hat{O}B + O\hat{A}B + O\hat{B}A = 180^\circ.$$

$$A\hat{O}B + 20^\circ + 20^\circ = 180^\circ$$

$$A\hat{O}B + 40^\circ = 180^\circ$$

$$A\hat{O}B = 180^\circ - 40^\circ$$

$$A\hat{O}B = 140^\circ$$

The angle subtended at the centre by the minor arc AB is $A\hat{O}B$. Since $A\hat{C}B$ is an angle subtended by this arc on the remaining part of the circle, according to the theorem,

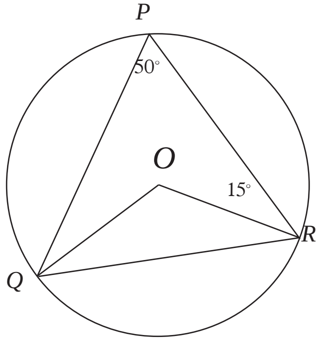
$$2\,A\hat{C}B = A\hat{O}B$$

$$\therefore A\hat{C}B = \frac{140^\circ}{2}$$

$$\therefore \underline{\underline{A\hat{C}B = 70^\circ}}$$

Example 1

The centre of the circle in the given figure is O. Using the information in the figure find $P\hat{Q}R$.



$Q\hat{O}R = 2\,Q\hat{P}R$ (The angle subtended at the centre by an arc of a circle is twice the angle subtended by the arc on the remaining part of the circle)

$\therefore\ Q\hat{O}R = 2 \times 50^\circ$

$= 100^\circ$

$O\hat{Q}R + O\hat{R}Q + Q\hat{O}R = 180^\circ$ (The sum of the interior angles of a triangle is 180°)

$\therefore\ O\hat{Q}R + O\hat{R}Q + 100 = 180^\circ$

$O\hat{Q}R + O\hat{R}Q = 80^\circ$ —— ①

$OQ = OR$ (The radii of a circle are equal)

$\therefore\ O\hat{Q}R = O\hat{R}Q$ (In an isosceles triangle, the angles opposite equal sides are equal)

According to ① $2\ O\hat{R}Q = 80^\circ$

$$O\hat{R}Q = \frac{80^\circ}{2}$$

$O\hat{R}Q = 40$

Now, $P\hat{R}Q = P\hat{R}O + O\hat{R}Q$

$P\hat{R}Q = 15^\circ + 40^\circ$

$P\hat{R}Q = 55^\circ$

$P\hat{Q}R + Q\hat{P}R + P\hat{R}Q = 180^\circ$ (The sum of the interior angles of a triangle is 180°)

$P\hat{Q}R + 50^\circ + 55^\circ = 180^\circ$

$P\hat{Q}R + 105^\circ = 180^\circ$

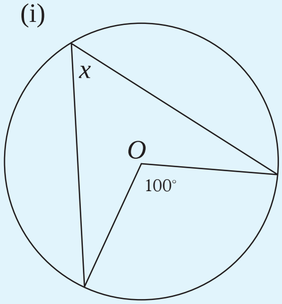
$P\hat{Q}R = 180^\circ - 105^\circ$

$P\hat{Q}R = \underline{\underline{75^\circ}}$

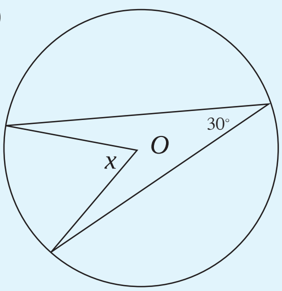
Exercise 31.2

1. The centre of each of the circles given below is O. Find the value of x based on the given data.

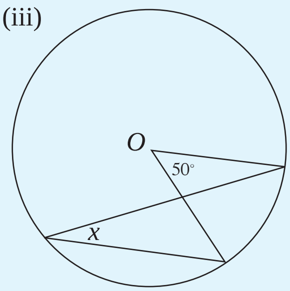
(i)



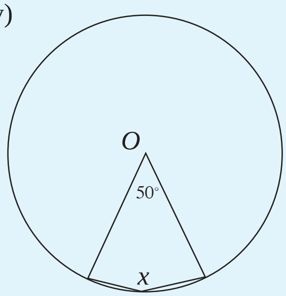
(ii)



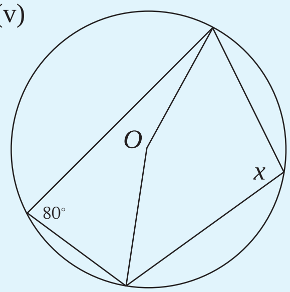
(iii)



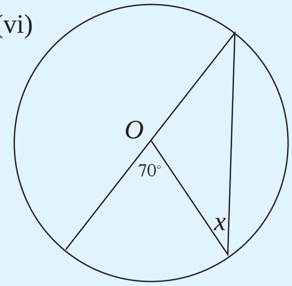
(iv)



(v)

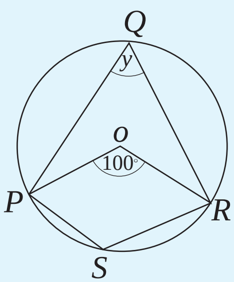


(vi)

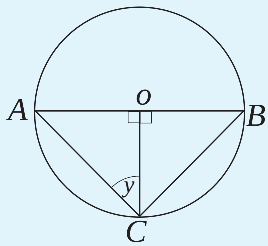


2. The centre of each of the following circles is denoted by *O*. Providing reasons, find the value of *y* based on the given data.

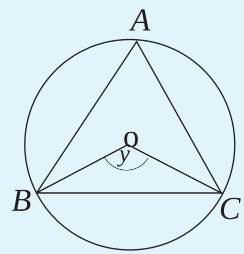
(i)



(ii)

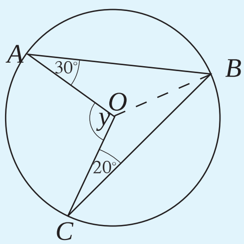


(iii)

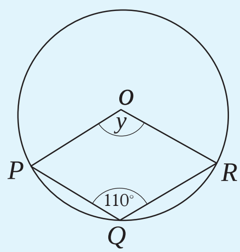


$A\hat{B}C = 70°$, $A\hat{C}B = 60°$

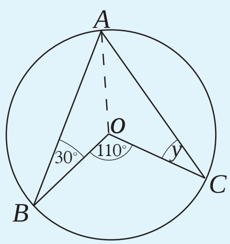
(iv)



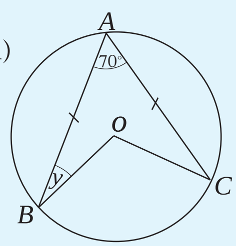
(v)



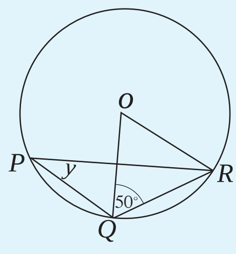
(vi)



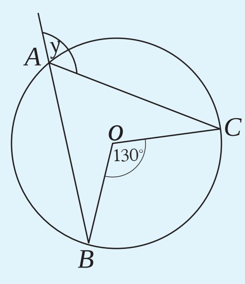
(vii)



(viii)

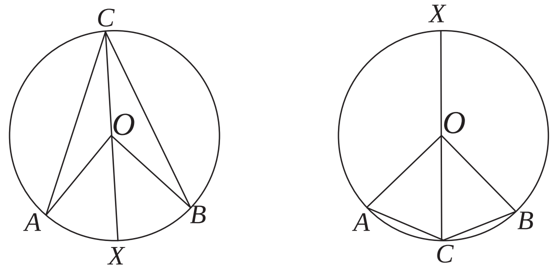


(ix)



31.3 Formal proof of the theorem

"The angle subtended at the centre by an arc of a circle is twice the angle subtended by the same arc on the remaining part of the circle".



Data : The points A, B and C lie on the circle of centre O.

To be proved : $A\hat{O}B = 2\,A\hat{C}B$.

Construction : The straight line CO is produced up to X

Proof : $OA = OC$ (Radii of the same circle)

$\therefore\ O\hat{A}C = O\hat{C}A$ —— ① (Since the angles opposite equal sides of an isosceles triangle are equal)

$O\hat{A}C + O\hat{C}A = X\hat{O}A$ —— ② (Exterior angle formed by producing a side of a triangle is equal to the sum of the interior opposite angles)

From ① and ②, $X\hat{O}A = 2\,O\hat{C}A$ —— ③

Similarly, $X\hat{O}B = 2\,O\hat{C}B$ —— ④

From ③ and ④, $\underbrace{X\hat{O}A + X\hat{O}B} = 2\ O\hat{C}A + O\hat{C}B$

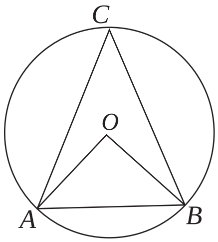
$$A\hat{O}B = 2\,\underbrace{(O\hat{C}A + O\hat{C}B)}$$

$$\underline{\underline{A\hat{O}B = 2\ A\hat{C}B}}$$

Now let us consider how the theorem which has been proved above can be used to prove other results (riders).

Example 1

The points A,B and C lie on a circle of centre O. If $A\hat{C}B + A\hat{B}C = A\hat{O}B$, show that $AC = AB$.



Proof: $A\hat{C}B + A\hat{B}C = A\hat{O}B$ —— 1 (Given)

$2\,A\hat{C}B = A\hat{O}B$ —— 2 (The angle subtended at the centre by an arc of a circle is twice the angle subtended by the same arc on the remaining part of the circle)

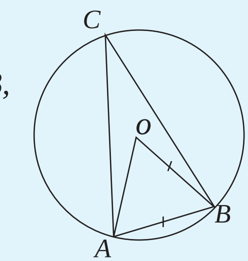
By 1 and 2

$$2\,A\hat{C}B = A\hat{C}B + A\hat{B}C$$
$$2\,A\hat{C}B - A\hat{C}B = A\hat{B}C$$
$$A\hat{C}B = A\hat{B}C$$
$$\underline{\underline{AC = AB}}$$

(In an isosceles triangle sides opposite equal angles are equal)

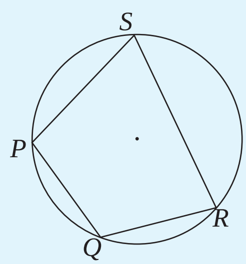
Exercise 31.3

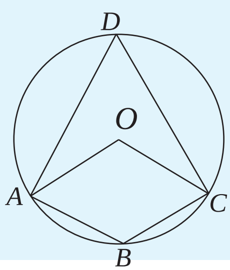
1. The points A, B and C lie on a circle of centre O. If $OB = AB$, show that $A\hat{C}B = 30^\circ$.



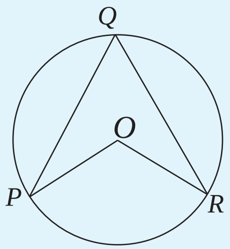
2. P, Q, R and S are points on a circle.

Prove that $P\hat{Q}R + P\hat{S}R = 180^\circ$.



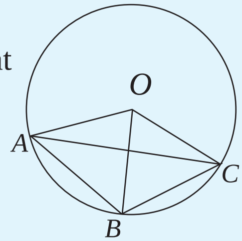
3. 

A, B, C and D are points on a circle of centre O. If $A\hat{O}C = A\hat{B}C$, show that $A\hat{D}C = 60^\circ$.

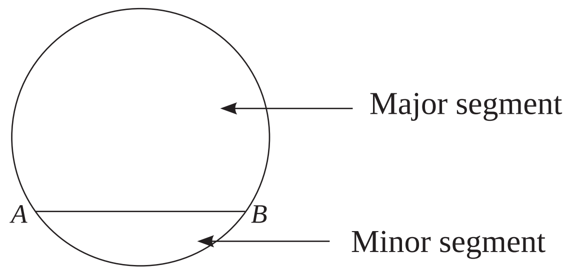
4. 

P, Q and R are points on the circumference of the circle of centre O. If $O\hat{P}Q = O\hat{R}Q$, show that $P\hat{O}R = 4\,O\hat{R}Q$. (Join O and Q)

5. The points A, B and C lie on a circle of centre O. Show that $A\hat{O}C = 2\,(B\hat{A}C + B\hat{C}A)$.

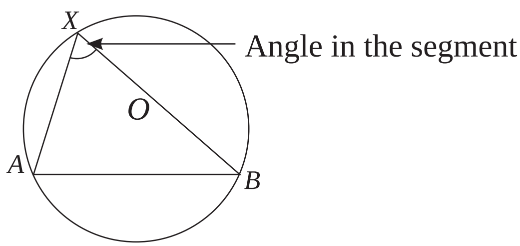


31.4 Relationship between the angles in the same segment

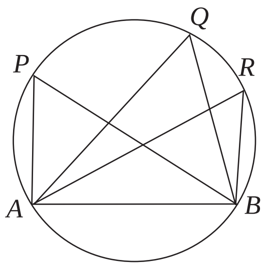


A circle and a chord AB of the circle are shown in the figure. The circle is divided into two regions by the chord.

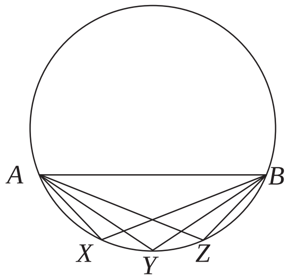
One is the region bounded by the chord and the major arc which is called the **major segment.** The other is the region bounded by the chord and the minor arc called the **minor segment.**



The angle formed by joining the end points of the chord AB to a point on the arc of a segment is defined as an angle in the segment. $A\hat{X}B$ is an angle in the segment AXB .



$A\hat{P}B$, $A\hat{Q}B$ and $A\hat{R}B$ in the figure are angles in the major segment. Therefore, $A\hat{P}B$, $A\hat{Q}B$ and $A\hat{R}B$ are called angles in the same segment.

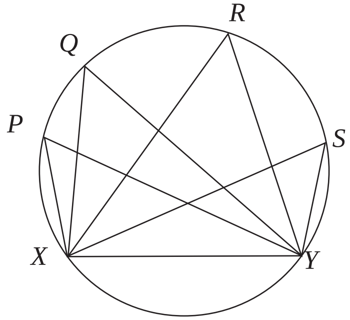


The angles $A\hat{X}B$, $A\hat{Y}B$ and $A\hat{Z}B$ in the figure are angles in the minor segment and hence belong to the same segment.

Let us identify the relationship between angles in the same segment through the following activity.

Activity

- Draw a circle on a piece of paper. Mark the points X and Y on the circle and draw the chord XY.
- Mark the points P, Q, R and S on the arc XY of the major segment.
- Join these points to the two end points of the chord XY. Then the angles $X\hat{P}Y$, $X\hat{Q}Y$, $X\hat{R}Y$ and $X\hat{S}Y$ which are angles in the same segment are obtained.

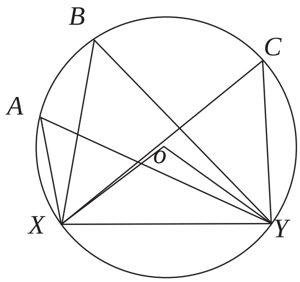


- Using a protractor, measure the angles that you have drawn in the same segment. Examine the magnitude of each angle.
- In the same manner, draw several angles in the minor segment, measure them and examine the values you obtain.

Through these activities you would have identified that angles in the same segment are equal in magnitude. This is given below as a theorem.

Theorem: The angles in the same segment of a circle are equal.

Let us establish this theorem through a geometric proof.



Data : The points A, B and C lie on the circle of centre O, on the same side of the chord XY.

To be proved : $X\hat{A}Y = X\hat{B}Y = X\hat{C}Y$

Construction : Join XO and YO.

Proof : The angle subtended at the centre by an arc of a circle is twice the angle subtended by the same arc on the remaining part of the circle.

$$\therefore X\hat{O}Y = 2\,X\hat{A}Y \text{ ——— } ①$$

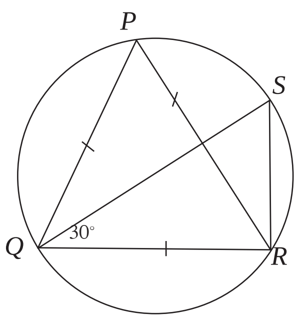
$$X\hat{O}Y = 2\,X\hat{B}Y \text{ ——— } ②$$

$$X\hat{O}Y = 2\,X\hat{C}Y \text{ ——— } ③$$

From ①, ② and ③, $2\,X\hat{A}Y = 2\,X\hat{B}Y = 2\,X\hat{C}Y$

$$\therefore \underline{\underline{X\hat{A}Y = X\hat{B}Y = X\hat{C}Y}}$$

Let us consider how calculations are done using the above theorem. Find $Q\hat{R}S$ using the information in the figure.



In the above figure, $PQ = QR = PR$ and $R\hat{Q}S = 30^\circ$. Let us find $Q\hat{R}S$.

Since $PQ = QR = PR$, (the triangle PQR is an equilateral triangle)

$$Q\hat{P}R = 60^\circ$$

$$Q\hat{P}R = Q\hat{S}R \text{ (Angles in the same segment are equal)}$$

$$\therefore Q\hat{S}R = 60^\circ$$

Since the sum of the interior angles of a triangle is 180°,

$$Q\hat{R}S + R\hat{Q}S + Q\hat{S}R = 180^\circ$$

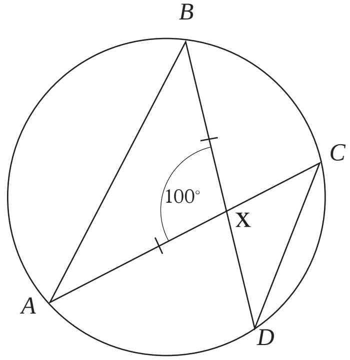
$$Q\hat{R}S = 180^\circ - (30^\circ + 60^\circ)$$

$$Q\hat{R}S = 180^\circ - 90^\circ$$

$$\underline{\underline{Q\hat{R}S = 90^\circ}}$$

Example 1

Find the magnitude of $B\hat{D}C$ using the information in the figure.



XAB is an isosceles triangle since *XB* = *XA*.

$\therefore \quad X\hat{B}A = X\hat{A}B$ —— (1) (In an isosceles triangle, the angles opposite equal sides are equal)

In the triangle *ABX*,

$X\hat{B}A + X\hat{A}B + A\hat{X}B = 180^\circ$ (The sum of the interior angles of a triangle is 180°)

$$X\hat{B}A + X\hat{A}B + 100^\circ = 180^\circ$$

$$X\hat{B}A + X\hat{A}B = 180^\circ - 100^\circ$$

$$X\hat{B}A + X\hat{A}B = 80^\circ$$

From (1), $2\,X\hat{A}B = 80^\circ$ (Since $X\hat{B}A = X\hat{A}B$)

$$\therefore X\hat{A}B = 40^\circ$$

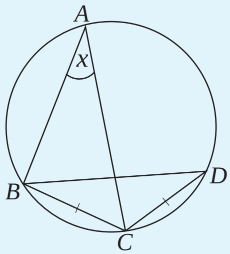
Since the angles in the same segment are equal,

$$B\hat{D}C = X\hat{A}B$$

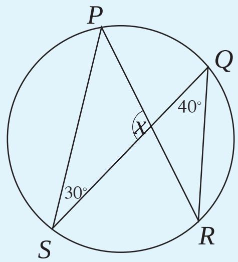
$$\therefore \underline{\underline{B\hat{D}C = 40^\circ}}$$

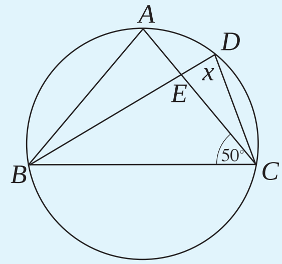
Exercise 31.4

Find the value of x in the following exercises.

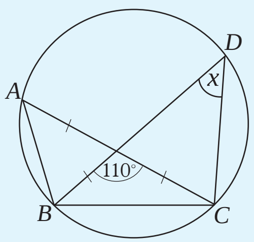
1. 

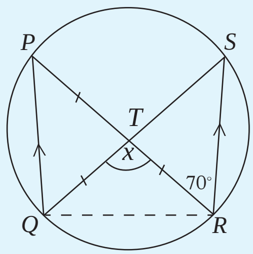
$B\hat{C}D = 110^\circ$

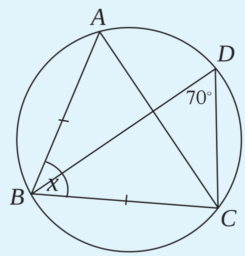
2. 

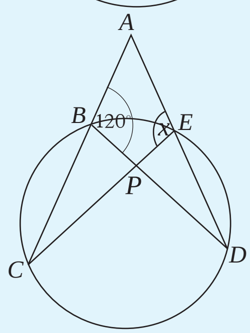
3. 

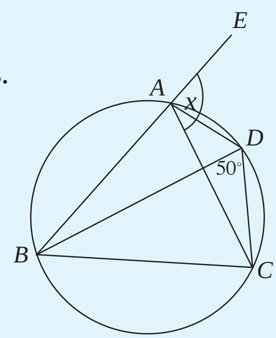
$AB = AC$

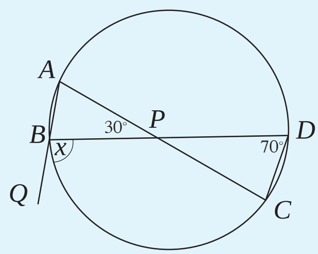
4. 

5. 

6. 

7. 

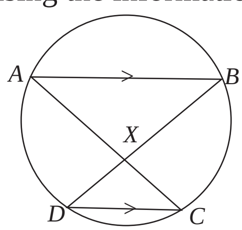
8. 

9. 

31.5 Proving riders using the theorem "Angles in the same segment of a circle are equal.

Example 1

Prove that $AC = BD$ using the information in the figure.



Proof: $A\hat{B}D = B\hat{D}C$ ($AB//DC$, alternate angles)

$A\hat{B}D = A\hat{C}D$ (Angles in the same segment)

$\therefore B\hat{D}C = A\hat{C}D$

Since the sides opposite equal angles in a triangle are equal, in the triangle XCD,

$XD = XC$

$B\hat{A}C = A\hat{C}D$ ($AB//CD$, alternate angles)

$A\hat{B}D = A\hat{C}D$ (Angles in the same segment)

$\therefore B\hat{A}C = A\hat{B}D$

Since the sides opposite equal angles in a triangle are equal,

$XA = XB$

$XC = XD$ (Proved)

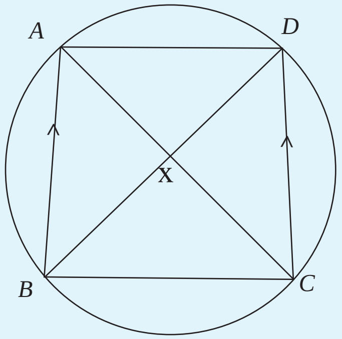
Using the axioms,

$$\underbrace{XA + XC} = \underbrace{XB + XD}$$

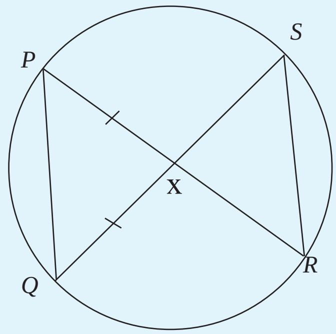
$$\therefore \underline{\underline{AC = BD}}$$

Exercise 31.5

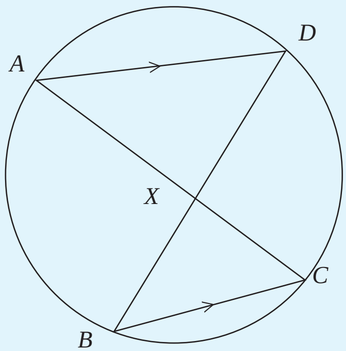
1. If $AB//DC$, show that .



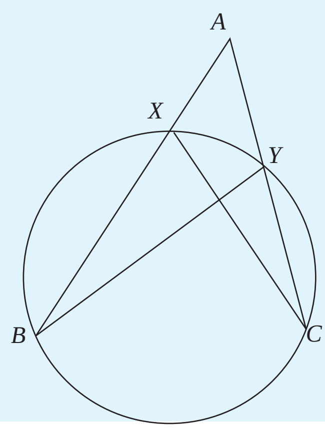
2. If $PX = QX$, show that $PQ//SR$.



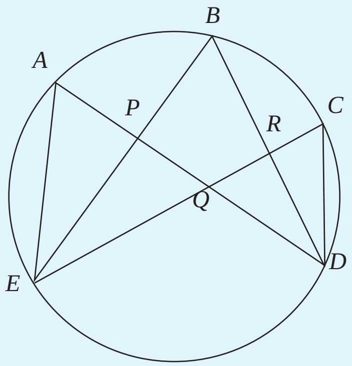
3. If $AD//BC$, show that $AX = DX$.



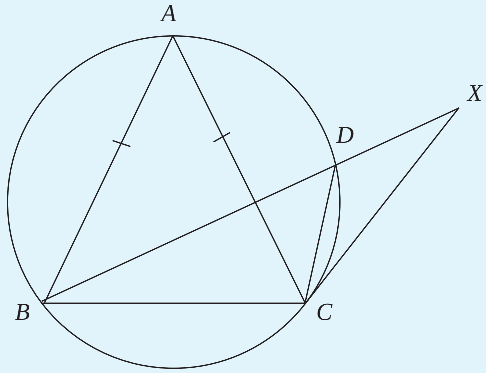
4. Show that $A\hat{X}C = A\hat{Y}B$.



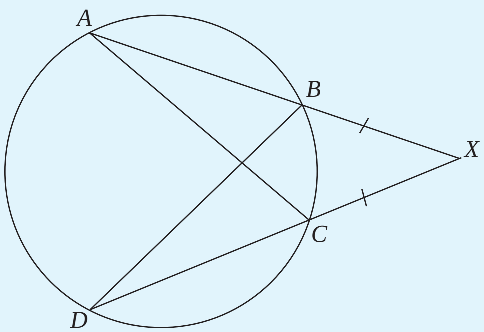
5. If $B\hat{P}Q = B\hat{R}Q$, show that BE is the bisector of $A\hat{E}C$.



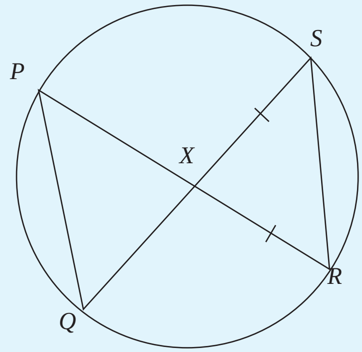
6. If $AB = AC$, show that $C\hat{D}X = 2\ A\hat{B}C$.



7. If $XB = XC$, show that $AC = BD$.

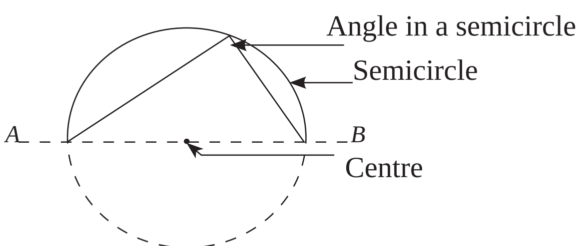


8. If $XS = XR$, show that $XP = XQ$.



31.6 Angles in a semicircle

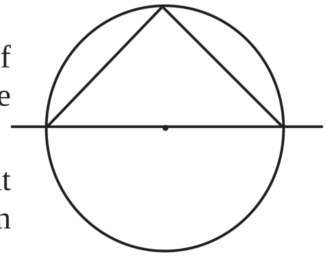
An arc of a circle which is exactly half a circle is defined as a semicircle.



By drawing a line through the centre of a circle, the circle is divided into two semicircles. The angle formed by joining a point on a semicircle to its end points is called an angle in the semicircle.

Let us engage in the following activity to identify the properties of angles in a semicircle.

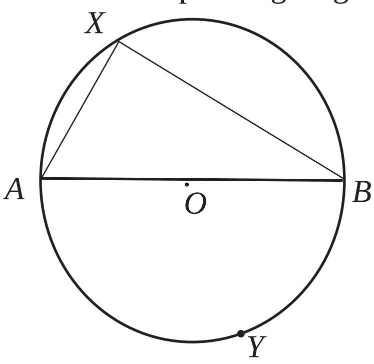
Activity 31.3



- Draw a circle on a piece of paper using a pair of compasses. Then draw a diameter of the circle. Now, the circle is divided into two semicircles.
- Mark a point on one of the semicircles. Join this point to the two end points of the semicircle. Then an angle in a semicircle is obtained.
- Using the protractor, measure the angle.

You would have observed that the angle in the semi-circle is 90°. In the above manner, draw several more circles and draw and measure angles in a semi-circle for these circles too. You will be able to identify through this activity that the angle in a semi-circle is always a right angle.

Let us establish the above relationship through a geometric proof.



Data: As shown in the figure, X and Y are points on the circle with centre O and AB is a diameter of the circle.

To be proved: $A\hat{X}B$ is a right angle.

Proof: $A\hat{O}B$, is the angle subtended at the centre by the arc AYB.

Since it is a semicircle, AOB is a diameter.

$$A\hat{O}B = 2 \text{ right angles} \quad \text{\_\_\_\_\_\_\_\_} \quad ①$$

$A\hat{X}B$ is an angle subtended on the remaining part of the circle by the chord AYB.

Since the angle subtended at the centre by an arc of a circle is twice the angle subtended by the same arc on the remaining part of the circle,

$$A\hat{O}B = 2\,A\hat{X}B \quad \text{\_\_\_\_\_\_\_\_} \quad ②$$

By ① and ②,

$2\,A\hat{X}B = 2 \times 2$ right angles

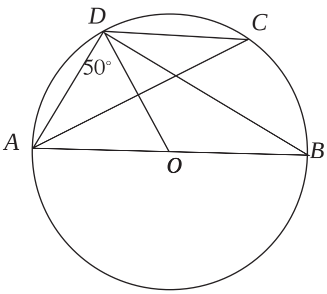
$\therefore A\hat{X}B = 1$ right angle

The relationship which has been established through the above proof is given below as a theorem.

Theorem: An angle in a semicircle is a right angle.

Let us identify how calculations are performed using the above theorem by considering the following examples.

Let us find the magnitude of $A\hat{C}D$ using the data in the figure of a circle with centre O.



$A\hat{D}B = 90^\circ$ (Angle in a semicircle)

$A\hat{D}B = A\hat{D}O + O\hat{D}B$

$\therefore A\hat{D}O + O\hat{D}B = 90^\circ$

$50^\circ + O\hat{D}B = 90^\circ$

$O\hat{D}B = 90^\circ - 50^\circ$

$O\hat{D}B = 40^\circ$

Since they are radii of the same circle,

$DO = OB$

Since the angles opposite equal sides of a triangle are equal,

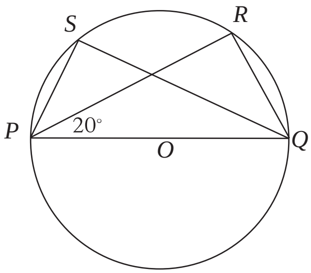
$D\hat{B}O = O\hat{D}B$

$\therefore D\hat{B}O = 40^\circ$

$D\hat{B}O = A\hat{C}D$ (Angles in the same segment)

$\therefore \underline{\underline{A\hat{C}D = 40^\circ}}$

Example 1



PQ is a diameter of the circle $PQRS$.

If $Q\hat{P}R = 20^\circ$ and $PS = QR$, find the magnitude of $R\hat{P}S$.

$P\hat{R}Q = 90^\circ$ (Angle in a semicircle)

$P\hat{Q}R + Q\hat{P}R + P\hat{R}Q = 180^\circ$ (The sum of the interior angles of a triangle is 180°)

$P\hat{Q}R + 20^\circ + 90^\circ = 180^\circ$

$P\hat{Q}R = 180^\circ - 110^\circ$

$P\hat{Q}R = 70^\circ$

Since PQ is a diameter,

$P\hat{S}Q = 90^\circ$ (Angle in a semicircle)

$P\hat{R}Q = 90^\circ$ (Angle in a semicircle)

$\therefore$ The triangles PSQ and PRQ are right angled triangles.

$\therefore$ In the triangles PSQ and PRQ,

$SP = RP$ (Given)

PQ is a common side.

$\therefore \Delta PSQ \equiv \Delta PRQ$ (RHS)

$\therefore S\hat{P}Q = P\hat{Q}R$ (Corresponding angles of congruent triangles)

$\therefore S\hat{P}Q = 70^\circ$

$R\hat{P}S + Q\hat{P}R = 70^\circ$

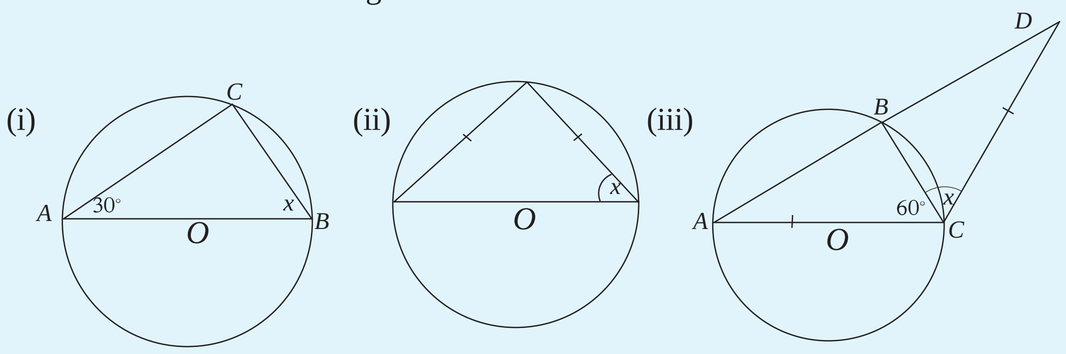
$R\hat{P}S + 20^\circ = 70^\circ$

$R\hat{P}S = 70^\circ - 20^\circ$

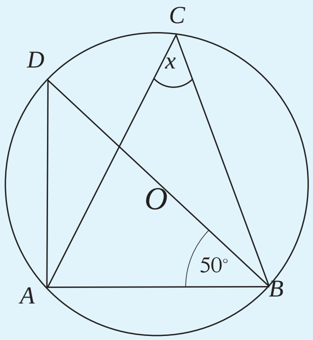
$\underline{\underline{R\hat{P}S = 50^\circ}}$

Exercise 31.6

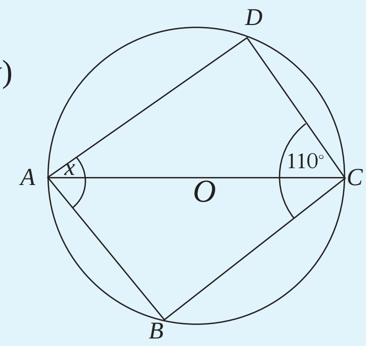
1. The centre of each of the following circles is denoted by O. Find the value of x based on the data in the figure.



(iv)



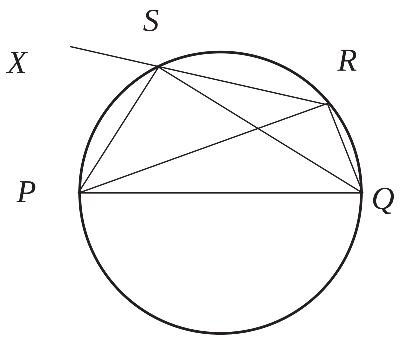
(v)



31.7 Proving riders using the theorem " The angle in a semicircle is a right angle"

Example 1

PQ is a diameter of the circle *PQRS*. The chord *RS* has been produced to *X*. Prove that $R\hat{P}Q + P\hat{S}X = 90^\circ$.



Proof:

$Q\hat{S}R + P\hat{S}Q + P\hat{S}X = 180^\circ$ (Sum of the angles on a straight line is 180°)

$P\hat{S}Q = 90^\circ$ (Angle in a semicircle is 90°)

$\therefore\ Q\hat{S}R + 90^\circ + P\hat{S}X = 180^\circ$

$Q\hat{S}R + P\hat{S}X = 180^\circ - 90^\circ$

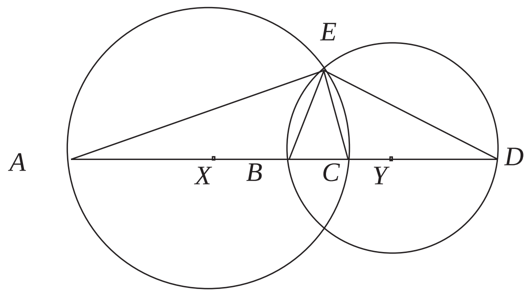
$Q\hat{S}R + P\hat{S}X = 90^\circ$

$Q\hat{S}R$ and $R\hat{P}Q$ are angles in the segment *PSRQ*.

$\therefore\ Q\hat{S}R = R\hat{P}Q$

$\therefore\ \underline{\underline{R\hat{P}Q + P\hat{S}X = 90^\circ}}$

Example 2



The centres of the given two circles are X and Y. Prove that $A\hat{E}B = C\hat{E}D$.

Proof:

Since AC passes through X, AC is a diameter of the circle with centre X.

$\therefore$ The arc AEC is a semicircle.

$\therefore A\hat{E}C = 90^\circ$ (Since the angle in a semicircle is a right angle)

$\therefore A\hat{E}B + B\hat{E}C = 90^\circ$ ——— ①

Since BD passes through the centre Y, BD is a diameter of the circle with centre Y.

$\therefore$ The arc BED is a semicircle.

$\therefore B\hat{E}D = 90^\circ$ (Since the angle in a semicircle is a right angle)

$C\hat{E}D + B\hat{E}C = 90^\circ$ ——— ②

$A\hat{E}B + B\hat{E}C = C\hat{E}D + B\hat{E}C$

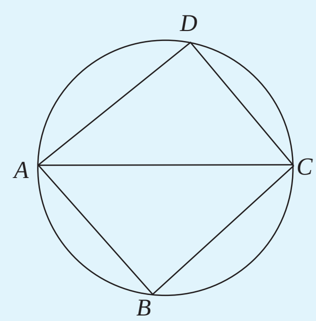
Subtracting $B\hat{E}C$ from both sides.

$\underline{\underline{A\hat{E}B = C\hat{E}D}}$

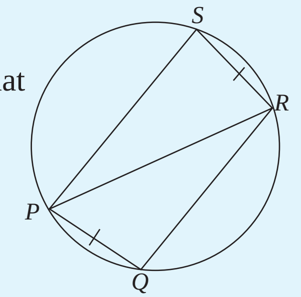
Exercise 31.7

1. AC is a diameter of the circle $ABCD$.

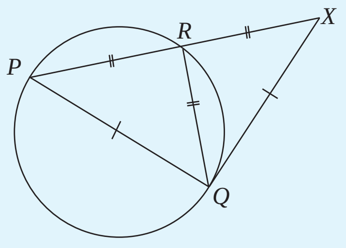
Show that $B\hat{A}D + B\hat{C}D = 180^\circ$.



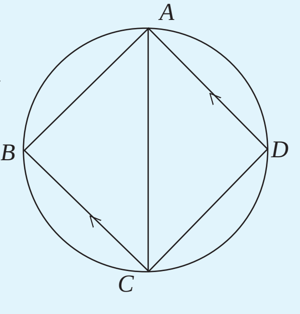
2. PR is a diameter of the circle $PQRS$. If $PQ = RS$, show that $PQRS$ is a rectangle.



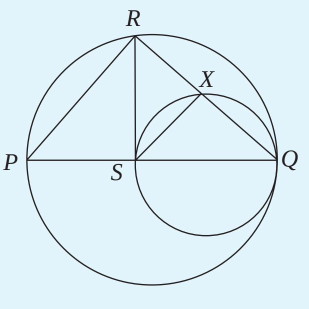
3. PQ is a diameter of the circle PQR. If $PQ = QX$ and $PR = QR = RX$, then show that $P\hat{Q}X = 90°$.



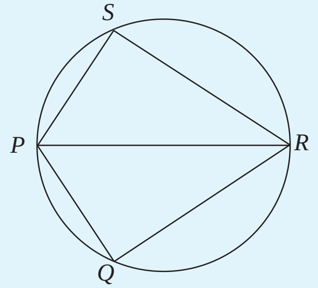
4. AC is a diameter of the circle $ABCD$. If $BC // AD$, show that $ABCD$ is a rectangle.

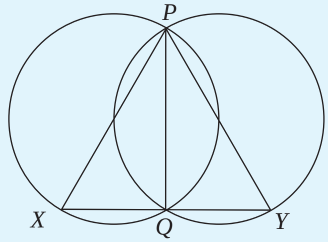


5. PSQ is a diameter of the larger circle and SQ is a diameter of the smaller circle. If RQ intersects the smaller circle at X, show that $P\hat{R}S = R\hat{S}X$.



6. PR is a diameter of the circle $PQRS$. If $S\hat{R}P = Q\hat{R}P$, show that $S\hat{P}R = Q\hat{P}R$.



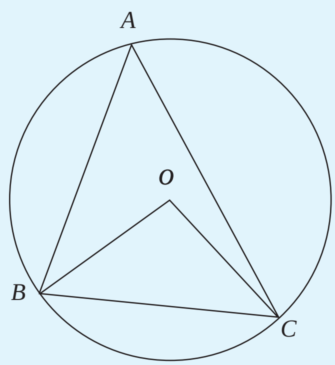


7. The two circles in the figure intersect at P and Q. PX and PY are diameters of the two circles. Show that XQY is a straight line.

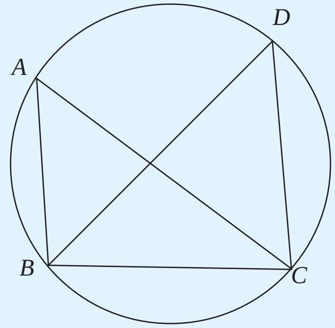
Miscellaneous Exercise

Mark the given data on the given figures and solve the problems.

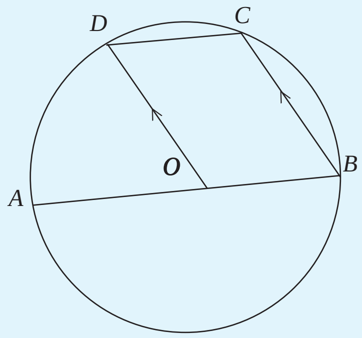
1. The centre of the circle ABC is O.
If $A\hat{B}O = O\hat{B}C$ and $A\hat{B}O = 40^{\circ}$, find the magnitude of $A\hat{C}O$.



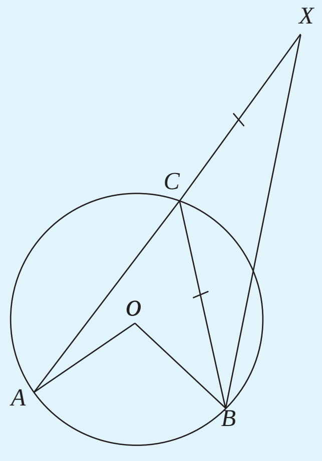
2. BD is a diameter of the circle $ABCD$. If $BC = CD$ and $A\hat{C}B = 35^{\circ}$, find the magnitude of $A\hat{B}C$



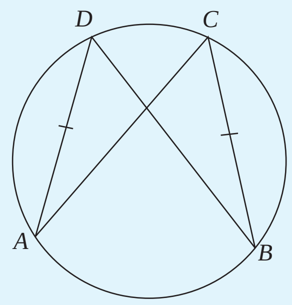
3. The centre of the circle $ABCD$ is O. If $BC//OD$ and $A\hat{B}C = 60°$, find the magnitude of the angle $B\hat{C}D$.



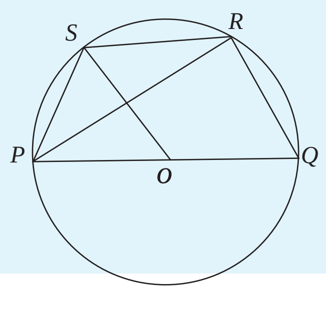
4. The centre of the circle ABC is O. AC has been produced to X such that $BC = CX$. Show that $A\hat{O}B = 4C\hat{B}X$.



5. The points A, B, C and D lie on the circle such that $AD = BC$. Show that $DB = CA$.



6. PQ is a diameter of the circle with centre O. Also, $QR//OS$. Show that $SR = SP$.



32 Scale Diagrams

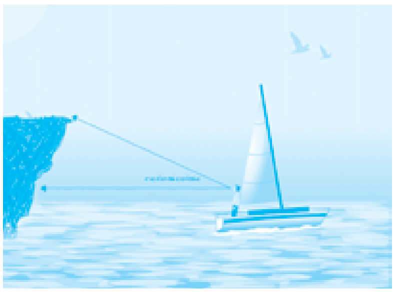
By studying this lesson, you will be able to

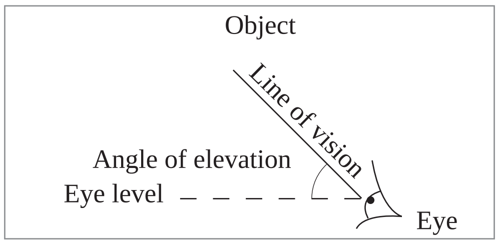
solve problems related to scale diagrams in a vertical plane.

32.1 Scale Diagrams

In the earlier grades, you used bearings and distances to indicate the position of a point on a horizontal plane.
In this lesson, you will learn how to find the position of a point on a vertical plane by drawing a scale diagram using angles of elevation and angles of depression.

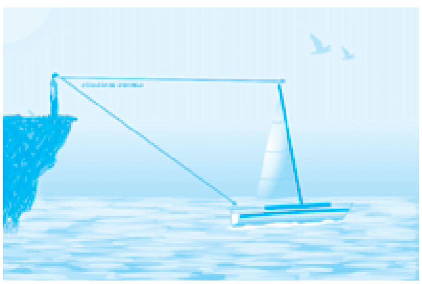
Angle of Elevation

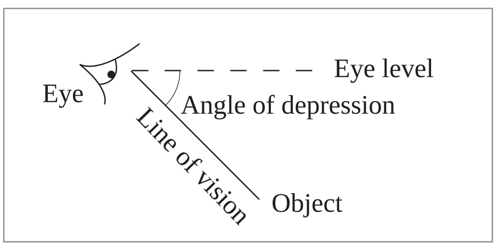




An angle of elevation is defined as the angle formed between the line of vision and the eye level (horizontal level) of an observer when the observer is looking at an object above the eye level.

Angle of Depression



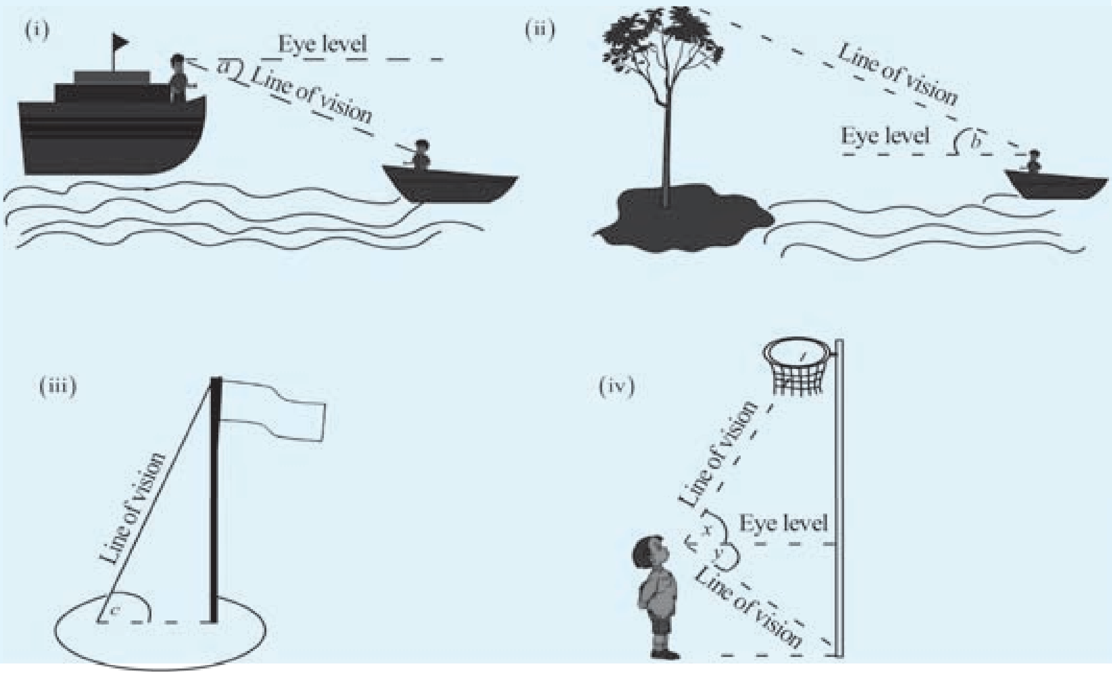


An angle of depression is defined as the angle formed between the line of vision and the eye level (horizontal level) of an observer when the observer is looking at an object below the eye level.

Note: Angles of elevation and angles of depression are always made with the horizontal level.

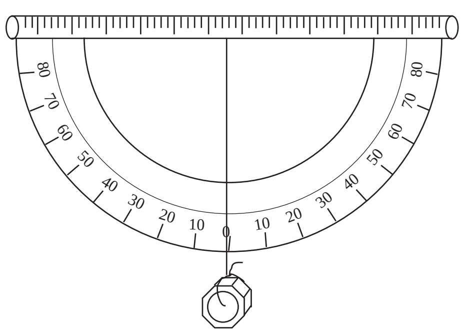
Exercise 32.1

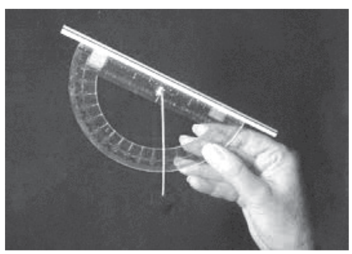
1. Write down whether the angle marked with a letter in each picture is an angle of elevation or an angle of depression.



32.2 Clinometer

When stating the position of an object in a vertical plane, we need to state the magnitude of the angle of elevation or depression. We may use a clinometer for the purpose of measuring these angles.





A simple clinometer can be made in the classroom as follows:

- Cut out a semicircle with a radius of about 10 cm from a piece of cardboard.
- Mark both ends of the curved edge as 90° and the middle of the curved edge as 0°.
- Calibrate the curved edge in both directions from 0°, 10° by 10° as shown in the figure.
- Fix a straw along the straight edge of the semicircle.
- Attach a small weight to one end of a piece of string longer than 10 cm and fix the other end to the center of the semicircle.

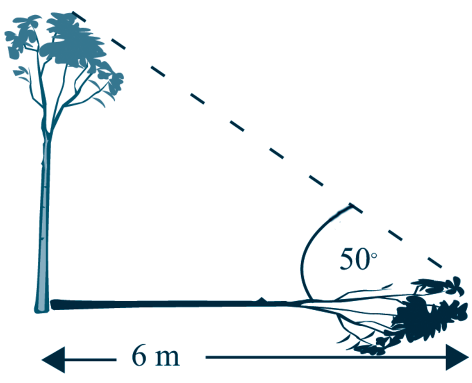
When the straw is horizontal, the string goes through 0°. When the straw is inclined at an angle of 45°, for example, the string goes through 45°, in which case the straw shows a 45° vertical inclination.
Angles of elevation and depression can be measured in this manner using a clinometer.

Exercise 32.2

1. From a suitable position find the angle of elevation of each of the points given below using the clinometer you made.
(i) The top of the flagpole at your school.
(ii) The top of a building.
(iii) The top of a tree in your school.

32.3 Scale diagrams in a vertical plane

Now let us consider several instances when scale diagrams are used to represent information in a vertical plane.
The figure below shows a tree and its shadow on the ground. Let us draw a scale diagram using the given information and hence find the height of the tree.

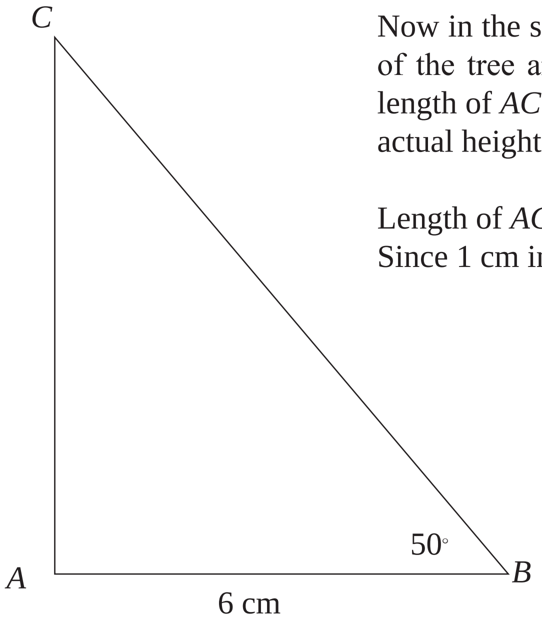


First we choose a suitable scale. Let 1 cm in the scale diagram represent an actual distance of 1 m.

In other words, let 1 cm represent 100 cm.

Therefore, the scale is 1:100.

Accordingly, we need to draw a line segment of 6 cm to represent 6 m. Let us represent this by a horizontal line segment AB, (see the figure below). Now let us draw an angle of 50° at B and complete the triangle ABC such that $B\hat{A}C = 90^\circ$ as shown in the figure.



Now in the scale diagram, AC represents the height of the tree and you can measure and find that the length of AC is 7.2 cm. Hence, we can calculate the actual height of the tree as follows:

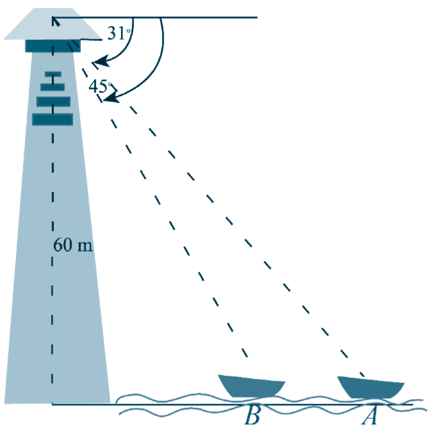
Length of AC = 7.2 cm.

Since 1 cm in the scale diagram represents 100 cm,

$$\begin{aligned}\text{height of the tree} &= 7.2\text{ cm} \times 100\\ &= 720\text{ cm}\\ &= \underline{\underline{7.2\text{ m}}}\end{aligned}$$

Example 1

From a sixty meter tall lighthouse a boat A is observed at sea with an angle of depression of 31° and another boat B with an angle of depression of 45° (see the figure). The two boats and the lighthouse are in the same vertical plane. Draw a scale diagram depicting the above information and find the distance between the boats A and B.



First let us draw a sketch diagram using the given information. Next let us choose a suitable scale. Let us represent 10 m by 1 cm.

Since 1 m = 100 cm, In the chosen scale, 1 cm represents 1000 cm.

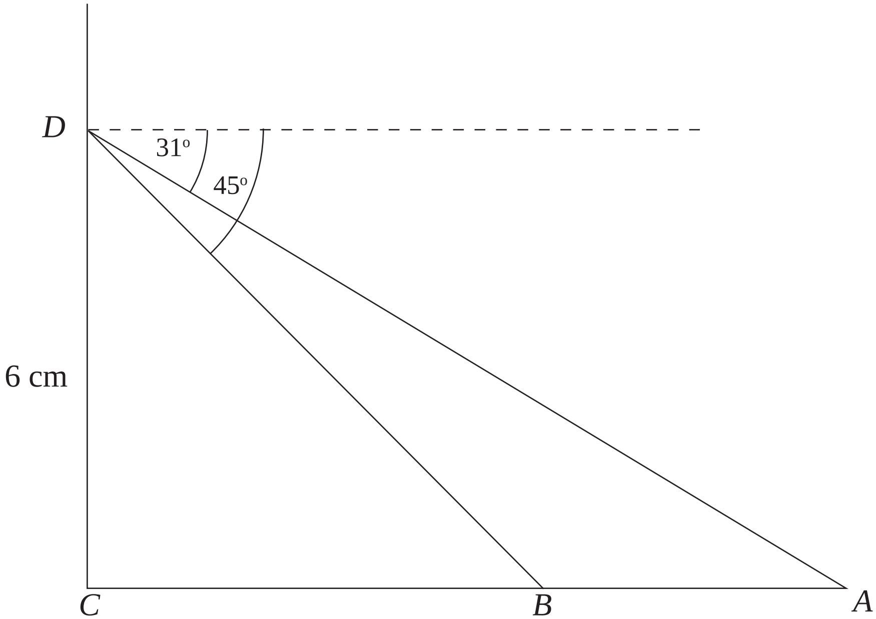
$\therefore$ the scale is 1: 1000.

Note: When drawing scale diagrams involving objects at great distances, a man's height being comparatively small can be ignored.

According to the scale, a 6 cm line must be drawn to represent the lighthouse's height. Denote this line by *CD*.

Now, let us draw the scale diagram.

- First draw a vertical line segment of length 6 cm and label it *CD*.
- Draw two lines perpendicular to *CD* at *C* and *D*.
- Draw an angle of depression of 31° at *D*. Extend this line to the point where it meets the line perpendicular to *CD* at *C* and label this point as *A*.
- Draw another angle of depression of 45° at *D*. Extend this line to the point where it meets the line perpendicular to *CD* at *C* and label this point as *B*.
- Now measure the length of *AB*. It should be 4 cm.



The distance between the two boats $= 4 \times 1000$ cm

$= 4000$ cm

$= \underline{\underline{40 \text{ m}}}.$

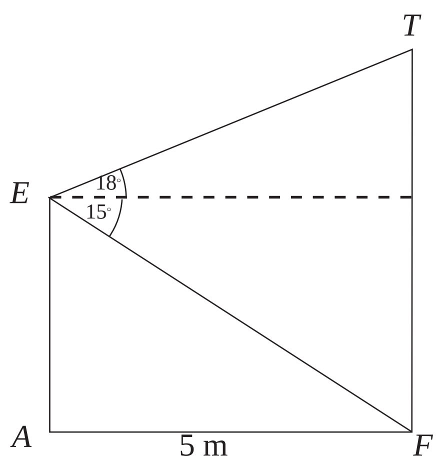
Example 2

In a horizontal playground, Dilini is standing at the location A, 5 m away from a netball goal post.

She can see the top of the goal post T, with an angle of elevation of 18° from her eye level E. She can see the base of the goal post F, from the same position with an angle of depression of 15°. Draw a scale diagram and find Dilini's height and the height of the goal post.

When a diagram is not given, it is best to draw a sketch diagram prior to drawing the scale diagram.

Sketch diagram:



Now, we have to choose a suitable scale for the scale diagram.
Let us take 2 cm to represents 1 m.
Then, 2 cm represent 100 cm.
$\therefore$ 1 cm represents 50 cm.
$\therefore$ the scale is 1: 50.
If 1 m is represented by 2 cm, 5 m is represented by 10 cm.

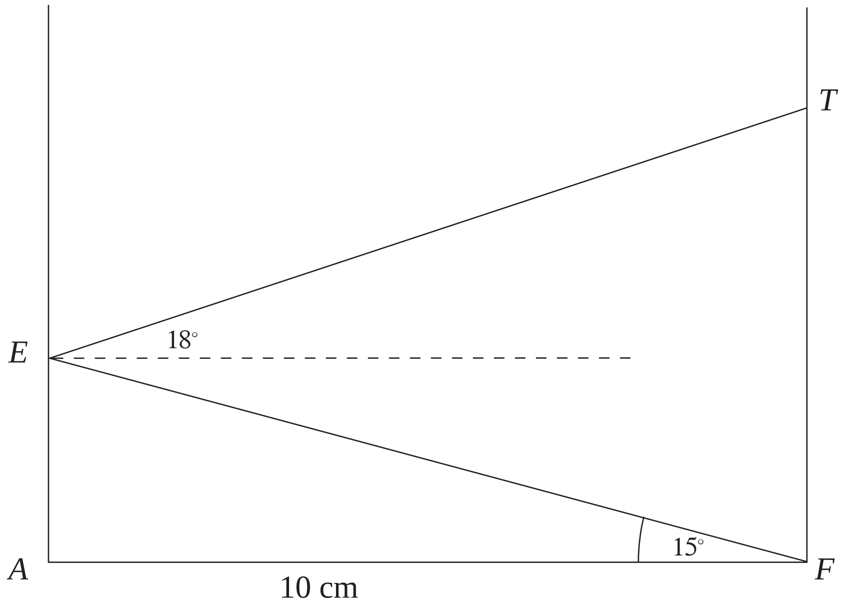
Note: Because the distance between the human and the pole is not large, we get a more accurate answer by taking the man's height into consideration when drawing the scale diagram.

Now let us draw the scale diagram.

- Because the distance between A and F is 5 m, draw a line segment of length 10 cm and label the two ends as A and F.
- Then, draw two lines at A and F, each perpendicular to AF.

- Because the point E is not yet determined, we cannot draw the angle of elevation or depression at E at once. Instead, we first find E as follows. Observe that $A\hat{F}E = 15^\circ$ because $A\hat{F}E$ and the angle of depression at E are alternate angles. Now draw an angle $A\hat{F}E$ of magnitude 15° at F such that E is on the line drawn perpendicular to AF at A.
- Now that we know point E, draw a line at E perpendicular AE.
- Draw the angle of elevation of 18° with the line drawn in the previous step. Label the point T which is the intersection point of the line of sight of this angle of elevation and the line drawn perpendicular to AF at F, as shown in the figure.
- Dilini's height is represented by AE and the height of the goal post is represented by TF.

Scale Drawing:



According to the scale diagram,

$AE = 2.6$ cm

$\therefore$ Dilini's height $= 2.6 \times 50$ cm

$= 130.0$ cm

$= \underline{\underline{1.3 \text{ m}}}$.

Also, from the scale diagram,

$TF = 6$ cm

$\therefore$ the height of the goal post $= 6 \times 50$ cm

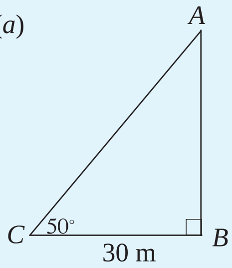
$= 300$ cm

$= \underline{\underline{3 \text{ m}}}$.

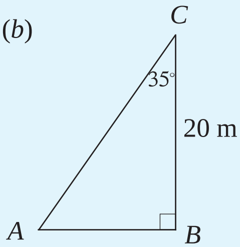
Exercise 32.3

1. Draw scale diagrams according to the given information and find the length AB.

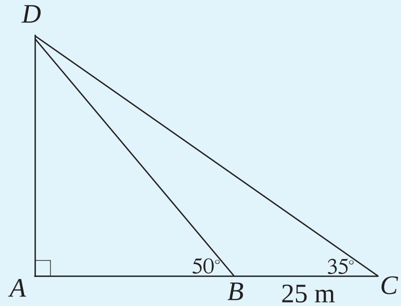
(*a*)



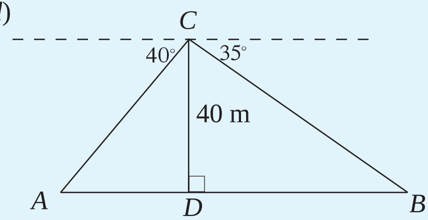
(*b*)



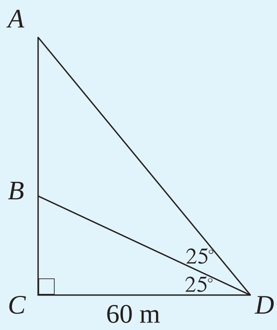
(*c*)



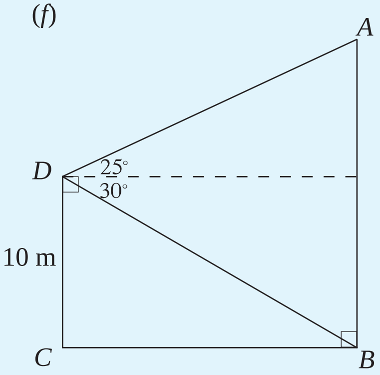
(*d*)



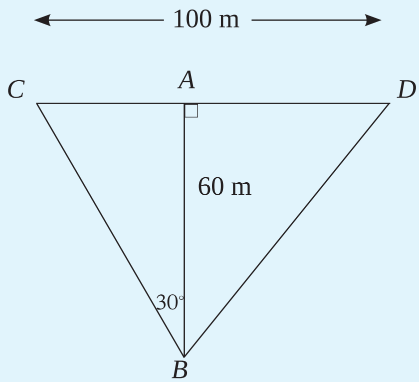
(*e*)



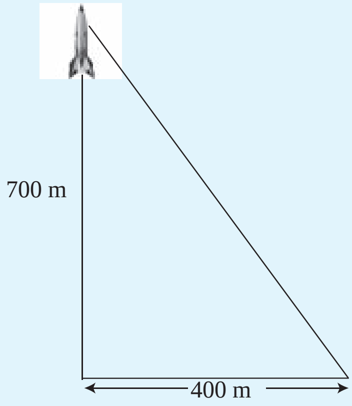
(*f*)



(*g*)



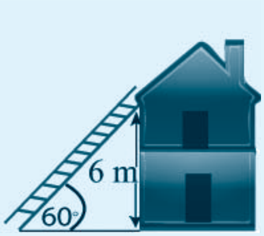
2. A person observes a rocket from a point 400 m horizontally away from the launching pad when the rocket has travelled 700 m vertically up from the launching pad. Using a scale diagram, find the angle of elevation of the rocket.



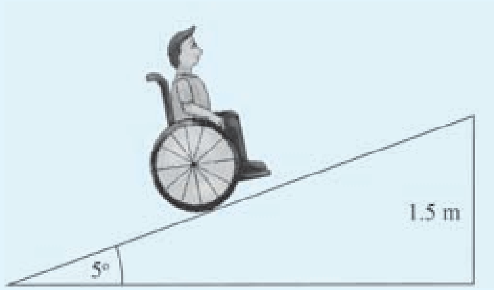
3. A ladder leaning against a wall is shown in the figure. Draw a scale diagram using the given information and find

(i) the length of the ladder and

(ii) the distance from the foot of the ladder to the wall.



4. A ramp for the use of wheelchairs to access a building is shown in the figure. Draw a suitable diagram using the given information and find the length of the ramp.

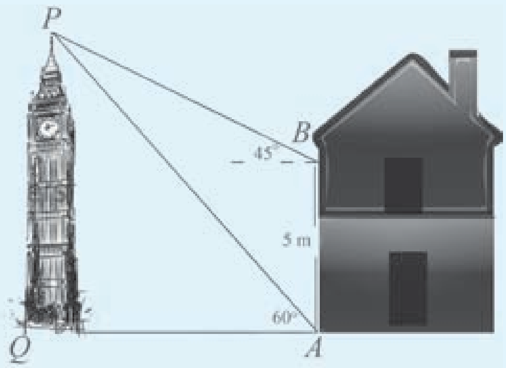


5. The mathematics teacher Mr. Weerasekera asked Chanuka to find the height of an inaccessible mango tree. Chanuka finds, using his clinometer, that the angle of elevation of the highest point P of the tree from a point A on the horizontal ground is 30° and that the angle of elevation of the same point P from a point B which is 10 m closer to the tree than A is 40°. The points A, B and the mango tree are in the same vertical plane. Find, the height of the mango tree by drawing a suitable scale diagram. (Ignore Chanuka's height).

6. Mr. Pieris observes that the angle of elevation to the top of a coconut tree from the upper floor balcony of his house is 40°. The distance from his house to the coconut tree is 6 m. Find the minimum length of a picking pole he could pluck coconuts with from the upper floor balcony.

7. The prefect Sithira was assigned the task of making arrangements for hoisting the national flag at the Independence Day celebration. Sithira needed to find the height of the flag pole. Form the upper floor of a building 10 m from the flagpole, using his clinometer, he observed the top of the flagpole with an angle of elevation of 20° and the bottom of the flagpole with an angle of depression of 50°. Draw a scale diagram using this information and find the height of the flag pole to the nearest metre.

8. The top, *P*, of a clock tower situated on a horizontal ground has an angle of elevation of 60° from a point *A* at the brink of a building. The angle of elevation of *P* from a point *B* in the building which is 5m directly above the point *A*, is 45° (see figure on the right). Using a suitable scale diagram, find the height of the clock tower and the distance from *A* to the foot *Q* of the clock tower.



9. An observer who stands on a horizontal ground 3m away from a bell tower, observes the top of a bell tower with an angle of elevation of 60° and the bottom of the clock tower with an angle of depression of 25°. Using a suitable scale diagram, find the height of the clock tower and the height of the observer.

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மடக்கைகள்

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	මධ්‍යන්‍ය අන්තරය / இடை வித்தியாசங்கள் / Mean Differences 1	2	3	4	5	6	7	8	9
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25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
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30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
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35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
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மடக்கைகள்

LOGARITHMS

											මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
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83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9