Chapter 2: Matrix Algebra

Trần Hoà Phú

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OUR GOAL

- Matrices
- Special matrices
- Operations on matrices:
    - Addition
    - Difference
    - Transposition
    - Scalar multiplication
    - Matrix multiplication
- Inverse of a square matrix
- Matrices and linear systems of equations
- Matrices and linear transformations

Definition

- An mxn matrix is *rectangular array* of numbers

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \\ \vdots \\ a \end{bmatrix}}\right\} m \text{ rows}$$

- (m x n): size of the matrix *m by n*
- A = [a_{ij}] // a_{ij} is called (i, j)-entry

Matrices - Examples

2×3 matrix $A = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 9 & 3 \end{pmatrix}$

$A(1,1) = 2$
$A(1,2) = 4$
$A(1,3) = -1$
$A(2,1) = 1$
$A(2,2) = 9$
$A(2,3) = 3$

Exercise

Given a matrix B as below

$$B = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 9 & 3 \\ 0 & 1 & 2 \\ -1 & 5 & 10 \end{pmatrix}$$

1) Find the size of matrix B.

2) Find $B(2,3), B(4,3), B(3,1)$.

Exercise

Give an example

a) 1×5 matrix

b) 4×1 matrix

c) 3×3 matrix

d) 4×2 matrix

Two matrices are called equal if

- They have the same size(cùng cỡ)
- Corresponding entries are equal (các phần tử tương ứng bằng nhau)

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Example 1: Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, discuss the possibility that $A = B, B = C, A = C$.

Solution ▶ $A = B$ is impossible because A and B are of different sizes: A is 2×2 whereas B is 2×3. Similarly, $B = C$ is impossible. But $A = C$ is possible provided that corresponding entries are equal: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ means $a = 1$, $b = 0$, $c = -1$, and $d = 2$.

Special matrices

- **Zero** matrix 0_{mxn}

$$0_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Main diagonal** of a matrix

$$\begin{bmatrix} 3 & -1 & 7 \\ 0 & 2 & 3 \\ -2 & 4 & -1 \end{bmatrix}, \begin{bmatrix} -4 & 1 & 0 \\ -2 & 3 & 5 \end{bmatrix}$$

Exercise What are $0_{3\times 3}, 0_{4\times 4}$?

Identity matrices

Identity matrix: square matrix $[a_{ij}]$ where $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise

What are I_4, I_5?

Triangular matrices, diagonal matrices

Upper triangular matrix $\begin{bmatrix} 3 & 13 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Lower triangular matrix $\begin{bmatrix} 3 & 0 & 0 \\ 11 & -1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$

Diagonal matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Transpose of a matrix

- The **transpose** of an mxn matrix $[a_{ij}]$ is an nxm matrix $[a_{ji}]$
- Notation: A^T // the transpose of A
- Example

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 0 \end{bmatrix}$$

Then,

$$A^T = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ -1 & 0 \end{bmatrix}$$

Exercise

Find transpose of following matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -5 & 7 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -4 & 3 \\ 5 & 8 & 9 \\ -10 & 11 & 22 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Symmetric matrices

Ma trn i xng

- Square matrix [a_{ij}] where $a_{ij} = a_{ji}$

$or\ A^T = A$

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & 3 & 7 \\ 5 & 7 & 4 \end{bmatrix}$$

$$A^T = A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & 3 & 7 \\ 5 & 7 & 4 \end{bmatrix}$$

- Addition $A+B=?$
- Scalar Multiplication $3.A=?$
- Matrix Multiplication $A.B=?$

Definition

If A and B are matrices of the same size, their sum $A + B$ is the matrix formed by adding corresponding entries.

Matrix Operations: Matrix Addition

Definition

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Example

$$\begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 4+1 & 8+0 \\ 3+5 & 7+2 \end{bmatrix}$$

Definition

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Example

$$\begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 4+1 & 8+0 \\ 3+5 & 7+2 \end{bmatrix}$$

Exercise

1. Find $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & -3 \\ 2 & 0 & 8 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 0 \\ 6 & 5 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

2. Find a, b, c if $\begin{pmatrix} a & b & c \end{pmatrix} + \begin{pmatrix} c & a & b \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \end{pmatrix}$

Properties

If A ,B and C are any matrices of the same size, then

- A+B=B+A (commutative law: giao hoán)
- A+(B+C)=(A+B)+C (associative law: kết hợp)

Exercise

Solve $\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} + X = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

Definition

If A is a matrix and k is any number, the scalar multiple kA is the matrix obtained from A by multiplying each entry of A by k.

Example

$$2 \cdot \begin{bmatrix} 10 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 10 & 2 \cdot 6 \\ 2 \cdot 4 & 2 \cdot 3 \end{bmatrix}$$

Exercise Find 3. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \end{pmatrix}$

Exercise

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Compute $5A, \quad (A+B)^T, \quad \frac{1}{2}B, \quad 3A-2B, \quad 2B^T - A^T$

- 1 hàng nhân 1 cột

Example

$$\begin{pmatrix} 1 & 3 \end{pmatrix} . \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 1.5 + 3.2 = 11$$

$$\begin{pmatrix} 1 & 3 & 4 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 1.2 + 3.(-1) + 4.3 = 11$$

Compute $\begin{pmatrix} 1 & 3 & 4 & -1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 & 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

Matrix multiplication

- $A_{m \times n} \cdot B_{n \times p} = C_{m \times p}$ //suitable size

- The entry c_{ij} = (row i of A).(column j of B)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix}$$

Matrix multiplication

- $A_{m \times n} \,.\, B_{n \times p} = C_{m \times p}$ //suitable size
- The entry c_{ij} = (row i of A).(column j of B)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1.1+2.1 & & & \\ & & & \\ & & & \end{pmatrix}$$

Matrix multiplication

- $A_{m \times n} . B_{n \times p} = C_{m \times p}$ //suitable size
- The entry c_{ij} = (row i of A).(column j of B)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 & & \\ & & & \\ & & & \end{pmatrix}$$

Matrix multiplication

- $A_{m \times n} \cdot B_{n \times p} = C_{m \times p}$ //suitable size
- The entry c_{ij} = (row i of A).(column j of B)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 & \\ & & & \\ & & & \end{pmatrix}$$

Matrix multiplication

- $A_{m \times n} . B_{n \times p} = C_{m \times p}$ //suitable size

- The entry c_{ij} = (row i of A).(column j of B)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ \boxed{-2} & \boxed{0} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 & \boxed{2} \\ 1 & 2 & 1 & \boxed{0} \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 & 2 \\ -1 & -2 & -1 & 0 \\ -2 & 0 & 2 & \boxed{-4} \end{pmatrix}$$

Exercise 2.3.1 Compute the following matrix products.

a. $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 5 & -7 \\ 9 & 7 \end{bmatrix}$

- $A(B+C) = AB + AC \quad$ //distributive law
- $A(BC) = (AB)C \quad$ //associative law
- $AI = A, IA = A$ //A,I square matrix + same size
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^T)^T = A, \quad (kA)^T = kA^T$

Note

- In general, $AB \neq BA \rightarrow$ Not Commutative
- $AB = 0 \nRightarrow A = 0$ or $B = 0$
- $AB = AC \nRightarrow B = C$

Exercise

a. $\left(A+3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$

b. $\left(3A^T+2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$

c. $\left(2A-3\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}\right)^T = 3A^T+\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$

d. $\left(2A^T-5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A-9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

Exercise

1. Let $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}, B=\begin{pmatrix}-1 & 1\\ 3 & 2\end{pmatrix}$ and $C=\begin{pmatrix}1 & 3 & -4\\ -1 & 2 & 1\end{pmatrix}$. Compute the matrix

a. $2A-B^T$ b. AB c. BA d. AC

e. CC^T f. C^TC g. A^3 h. B^2A^T

Definition If A is a square matrix, a matrix B is called an inverse of A if and only if $AB = I$ and $BA = I$ $(B = A^{-1})$
A matrix A that has an inverse is called invertible matrix.

Example Show that $B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ is an inverse of $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Solution Compute AB and BA

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Inversion Algorithm

The Inversion algorithm:

$$[\mathbf{A} \mid I_n] \to \ldots \to [I_n \mid \mathbf{A}^{-1}]$$

For example,

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\begin{array}{ccc}1 & -1 & 2\end{array}}^{\mathbf{A}}} & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-r_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-2r_2 + r_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right) \xrightarrow[3r_3 + r_2]{-2r_3 + r_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -4 & -2 \\ 0 & 1 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & \multicolumn{3}{c}{\underbrace{\begin{array}{ccc}0 & 2 & 1\end{array}}_{\mathbf{A}^{-1}}} \end{array}\right)$$

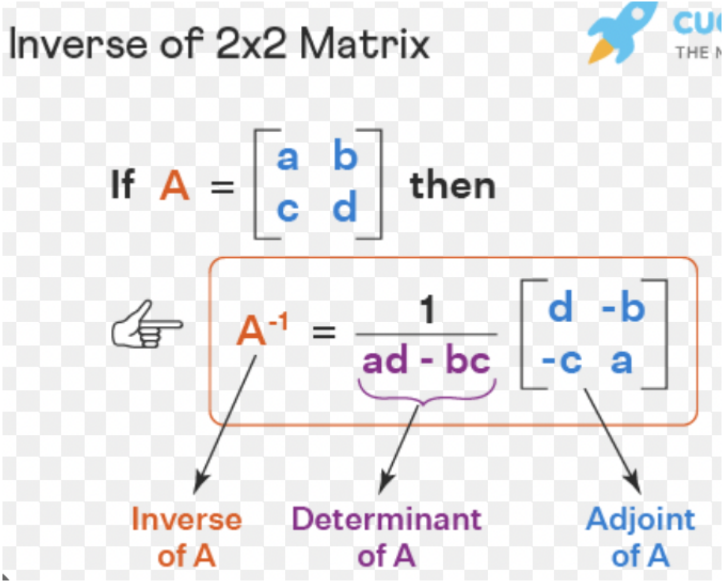
Exercise

Find the inverse of each of the following matrices

$$A = \begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{pmatrix}$$


Inverse of 2x2 Matrix
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then
$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
Inverse of A
Determinant of A
Adjoint of A

Matrix and system of linear equations

- Consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, these equations become the single matrix equation

$$AX = B$$

Example

Solve the linear system

$$\begin{cases} -2x + y = -1 \\ 3x - 2y = 5 \end{cases}$$

Đưa hệ về dạng ma trận

$$\begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$AX = B$

$$X = A^{-1}B = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ -7 \end{pmatrix}$$

Thus $x = -3, y = -7$.

Exercise

Write the system of linear equations in matrix form and then solve them.

$$\begin{cases} 2x - y = 4 \\ 3x + 2y = -4 \end{cases}$$

$$\begin{cases} 2x + 3y + z = 10 \\ 2x - 3y - 3z = 22 \\ 4x - 2y + 3z = -2 \end{cases}$$

Properties

Theorem 4

- $(A^{-1})^{-1}=A$
- $(AB)^{-1}=B^{-1}A^{-1}$
- $(A^T)^{-1}=(A^{-1})^T$
- $(A_1A_2\ldots A_k)^{-1}=A_k^{-1}\ldots A_2^{-1}A_1^{-1}$
- $(A^k)^{-1}=(A^{-1})^k$
- $(aA)^{-1}=A^{-1}/a$
- $I^{-1}=I$

Example

- Find A if

$$\left(A^T - 2I\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Matrix and linear transformation

- Example of a transformation

T(x, y) = (x, -y)

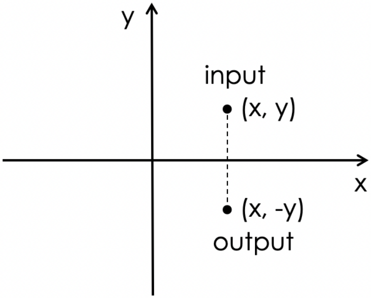
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

input

output

Matrix of
The transformation

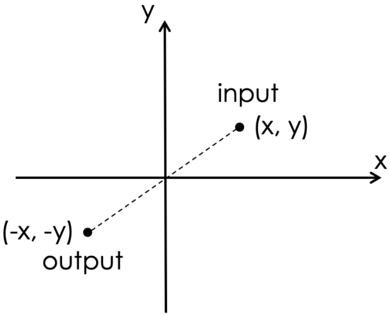


Matrix and linear transformation

- Example of a transformation

$S(x, y) = ?$

Find the matrix of S?



Suppose T is a linear transformation given by the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$$

Find T(1, 2, -3).

$$T(1, 2, -3) = T\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$$

Exercise

11. Let $T: R^2 \to R^2$ be a linear transformation, and assume that $T(1,2) = (-1,1)$ and $T(0,3) = (-3,3)$

a. Compute $T(11,-5)$

b. Compute $T(1,11)$

c. Find the matrix of T

d. Compute $T^{-1}(2,3)$

12. Let $T: R^2 \to R^2$ be a linear transformation such that the matrix of T is $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

Find $T(3,-2)$

The composition of transformations

Given T(x, y) = (x, y-x)

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y - x \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

And S(x, y) = (x-y, y)

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Find the composite transformation

(T∘S)(x, y) defined by

(T∘S)(x, y) = T(S(x, y))

Matrix of T ∘ S:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$\Rightarrow$ $(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y \end{pmatrix}$

Theorem

If the matrix of T is A, then the matrix of T^{-1} is A^{-1}

Example. Given T(x, y) = (x – y, -x + 2y), find T^{-1}, the inverse of T.

Solution.

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y \end{pmatrix}$ has the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

➔ T^{-1} has the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

➔ $T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + y \end{pmatrix}$

Note that $(T\,?\,T^{-1})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$