Chapter 5: The Vector Space R^n

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Contents

n-Vectors

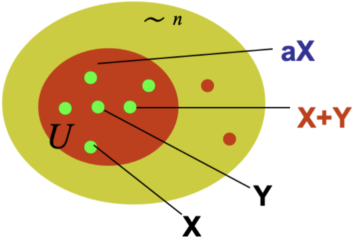
- (x_1, x_2) // vector in R^2
- (x_1, x_2, x_3) // vector in R^3
- (x_1, x_2, x_3, x_4) // vector in R^4
- $(x_1, x_2, \ldots, x_n)$ // vector in R^n
- A **vector** $(x_1, x_2, \ldots, x_n)$ in R^n is also called a **point** in R^n.
- $(0, 0, \ldots, 0)$: the zero vector in R^n

Subspace of R^n

(Không gian con của $\mathbb{R}^n$)

Definition Let $\emptyset \neq U$ be a subset of $\mathbb{R}^n$. U is called a **subspace of R^n** if

- vector $\mathbf{0} = (0, .., 0) \in \mathbf{U}$
- $\forall \mathbf{X}, \mathbf{Y} \in \mathbf{U} \Rightarrow \mathbf{X} + \mathbf{Y} \in \mathbf{U}$
- $\forall \mathbf{X} \in \mathbf{U}, \mathbf{a} \in \mathbf{R} \Rightarrow \mathbf{aX} \in \mathbf{U}$



Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{x}) | \mathbf{x} \in \mathbb{R}\}$ **is a subspace of** $\mathbb{R}^2$

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- vector $0 = (0, 0) \in U$
- $X, Y \in U$. Prove that $X + Y \in U$
  $X, Y \in U \Rightarrow \exists x_0, y_0 \in R : X = (x_0, x_0), Y = (y_0, y_0)$
  $X + Y = (x_0, x_0) + (y_0, y_0) = (x_0 + y_0, x_0 + y_0) \in U$
- $X \in U, a \in \mathbb{R}$. Prove that $aX \in U$.
  $X \in U \Rightarrow \exists x_0 : X = (x_0, x_0)$. Thus
  $aX = a(x_0, x_0) = (ax_0, ax_0) \in U$

Example

$\mathbf{U} = \{(\mathbf{t}, \mathbf{t}, \mathbf{2t}) | \mathbf{t} \in \mathbb{R}\}$ **is a subspace of** $\mathbb{R}^3$

Example

$\mathbf{U = \{(t, t, 2t) | t \in \mathbb{R}\}}$ **is a subspace of** $\mathbb{R}^3$

- vector $0 = (0, 0, 0) \in U$
- $X, Y \in U$. Prove that $X + Y \in U$
  $X = (x, x, 2x)$, $Y = (y, y, 2y)$. Then
  $X + Y = (x, x, 2x) + (y, y, 2y) = (x + y, x + y, 2(x + y)) \in U$
- $X \in U, a \in \mathbb{R}$. Prove that $aX \in U$
  $X = (x, x, 2x)$. Then $aX = a(x, x, 2x) = (ax, ax, 2ax) \in U$

Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) | \mathbf{x} + \mathbf{2y} - \mathbf{z} = \mathbf{0}\}$ **is a subspace of** $\mathbb{R}^3$

Example

$\mathbf{U = \{(x, y, z) | x + 2y - z = 0\}}$ is a subspace of $\mathbb{R}^3$

- vector $0 = (0, 0, 0) \in U$
- $X, Y \in U$. Prove that $X + Y \in U$

  $X, Y \in U \Rightarrow X = (x_1, y_1, z_1)$ with $x_1 + 2y_1 - z_1 = 0$ and
  $Y = (x_2, y_2, z_2)$ with $x_2 + 2y_2 - z_2 = 0$.
  $X + Y = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
  with $x_1 + x_2 + 2(y_1 + y_2) + z_1 + z_2 = 0$
- $X \in U, a \in \mathbb{R}$. Prove that $aX \in U$

  $X = (x_1, y_1, z_1)$ with $x_1 + 2y_1 - z_1 = 0$. Then
  $aX = a(x_1, y_1, z_1) = (ax_1, ay_1, az_1)$ with
  $ax_1 + 2ay_1 - az_1 = a(x_1 + 2y_1 - z_1) = 0$

Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{5x}, \mathbf{1}) | \mathbf{x} \in \mathbb{R}\}$ **is NOT a subspace of** $\mathbb{R}^3$

vector $0 = (0, 0, 0) \in U$?

Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y} - \mathbf{1}) | \mathbf{x}, \mathbf{y} \in \mathbb{R}\}$ **is NOT a subspace of** $\mathbb{R}^3$

vector $0 = (0, 0, 0) \in U$?

Exercise

Determine whether U is a subspace of R^3.

(i) $U=\{[0\ 1\ s]^T : s \in R\}$

(ii) $U=\{[0\ a\ b]^T : a,b \in R\}$

(iii) $U=\{[a\ b\ a+1]^T : a,b \in R\}$

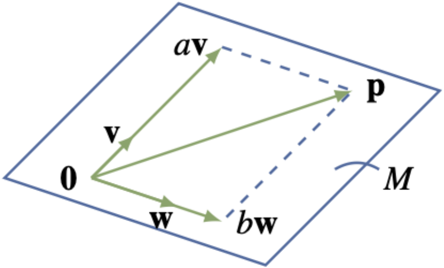
Spanning sets
(Tập sinh)

Let v, w be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. How to describe the plane M through origin containing these vectors ?

First way. The plane M has normal $n = v \times w$ and through origin so it consists of all vectors p: $\mathbf{n}.\mathbf{p} = \mathbf{0}$

Second way.

Let v, w be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. How to describe the plane M through origin containing these vectors ?



By a diagram, vector p is in M if and only if $p = av + bw$ for certain real numbers a, b.

$$\mathbf{M} = \{\mathbf{av} + \mathbf{bw} | \mathbf{a}, \mathbf{b} \in \mathbb{R}\} =: \mathbf{span}\{\mathbf{v}, \mathbf{w}\}$$

Spanning sets

Definition

- $\mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \{\mathbf{a}_1\mathbf{v}_1 + \mathbf{a}_2\mathbf{v}_2 | \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}\}$
- $\mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\mathbf{a}_1\mathbf{v}_1 + \mathbf{a}_2\mathbf{v}_2 + \mathbf{a}_3\mathbf{v}_3 | \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}\}$
- $\mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} = \{\mathbf{a}_1\mathbf{v}_1 + \mathbf{a}_2\mathbf{v}_2 + ... + \mathbf{a}_k\mathbf{v}_k | \mathbf{a}_1, ..., \mathbf{a}_k \in \mathbb{R}\}$

Example

Given $V = \text{span}\{(-1, 2, 1), (3, -5, -1)\}$.

a. $(-1, 1, 1) \in \mathbf{V}$?

b. Find all m such that $(-2, 1, m) \in \mathbf{V}$

Solution

a. Find x, y such that

$$\begin{aligned}(-1,1,1) &= x(-1,2,1) + y(3,-5,-1)\\ &= (-x,2x,x) + (3y,-5y,-y)\\ &= (-x+3y, 2x-5y, x-y)\end{aligned}$$

It follows that $\begin{cases} -x+3y=-1\\ 2x-5y=1\\ x-y=1\end{cases}$ (inconsistent)

Thus $(-1,1,1) \notin V$

b. Find x, y such that

$$\begin{aligned}(-2, 1, m) &= x(-1, 2, 1) + y(3, -5, -1)\\ &= (-x, 2x, x) + (3y, -5y, -y)\\ &= (-x + 3y, 2x - 5y, x - y)\end{aligned}$$

It follows that $\begin{cases} -x + 3y = -2 \\ 2x - 5y = 1 \\ x - y = m \end{cases} \Leftrightarrow \begin{pmatrix} -1 & 3 & -2 \\ 2 & -5 & 1 \\ 1 & -1 & m \end{pmatrix} \Leftrightarrow$

$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & m+4 \end{pmatrix} \Leftrightarrow m + 4 = 0 \Leftrightarrow m = -4$

Exercise

Let $\mathrm{U} = \mathrm{span}\{(1,1,2),(-1,2,1)\} \subset \mathbb{R}^3$ and $x = (m,-1,1)$

$x \in \mathrm{U}$ if and only if $m =$

A.-1 B.1 C.2 D.-2

Exercise

Let $U = \text{span}\{(1,1,2,1),(0,1,1,-2)\}$.
Find all values of t such that $(1,t,3,4) \in U$

A. there is no such t

B. -2

C. All nonzero numbers

D. None of the other choices is correct

E. All number different from -1

Exercise

Let
$x = (-1, -2 - 2), u = (0, 1, 4), v = (-1, 1, 2), w = (3, 1, 2) \in \mathbb{R}^3$.
Find real numbers a, b, c such that $x = au + bv + cw$

Exercise

Write v as a linear combination of u and w, if possible, where $u = (1, 2), w = (1, -1)$.

a. $v = (0, 1)$

b. $v = (2, 3)$

c. $v = (1, 4)$

Linear Independence and Linear Dependence (Độc lập tuyến tính và phụ thuộc tuyến tính)

Definition

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is called **linearly independent** if the system

$$\mathbf{t}_1\mathbf{v}_1 + \mathbf{t}_2\mathbf{v}_2 + \ldots + \mathbf{t}_k\mathbf{v}_k = \mathbf{0}$$

has only trivial Solution

$$\mathbf{t}_1 = \mathbf{t}_2 = \ldots = \mathbf{t}_k = \mathbf{0}$$

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is called **linearly dependent** if it is not linearly independent.

Example

Prove that $\{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ **is linearly independent**

Example

Prove that $\{(\mathbf{1}, \mathbf{2}, \mathbf{3}), (\mathbf{0}, \mathbf{1}, \mathbf{2}), (-\mathbf{2}, \mathbf{0}, \mathbf{1})\}$ **is linearly independent**

Solution

Given real numbers t_1, t_2, t_3 such that

$$t_1(1,2,3) + t_2(0,1,2) + t_3(-2,0,1) = \underline{(0,0,0)}$$

$$\Leftrightarrow (t_1, 2t_1, 3t_1) + (0, t_2, 2t_2) + (-2t_3, 0, t_3) = (0,0,0)$$

$$\Leftrightarrow (t_1 - 2t_3, 2t_1 + t_2, 3t_1 + 2t_2 + t_3) = (0,0,0)$$

homo...(pt thuan nhat)

$$\Leftrightarrow \begin{cases} t_1 - 2t_3 = 0 \\ 2t_1 + t_2 = 0 \\ 3t_1 + 2t_2 + t_3 = 0 \end{cases}, \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0 \Leftrightarrow \begin{cases} t_1 = 0 \\ t_2 = 0 \\ t_3 = 0 \end{cases}$$

=> det != 0

det = - 1

Theorem

Let $\{c_1, c_2, \dots, c_n\}$ *denotes the columns of A. Then*

- *if* $|\mathbf{A}| \neq 0$ ***or rank(A)=n*** *then* $\{c_1, c_2, \dots, c_n\}$ *are* ***linearly independent***
- *if* $|\mathbf{A}| = 0$ ***or rank(A)*** $< n$ *then* $\{c_1, c_2, \dots, c_n\}$ *are* ***linearly dependent***

Exercise

For what value of a is the set of vectors

$$S = \{(1,1,1),(2,0,4),(2,a,2)\} \text{ linearly dependent?}$$

A. -4 B. -2 C. 0 D. 2

Example

Find all $x \in \mathbb{R}$ such that $\{(1, 1, 2), (-2, \mathbf{x}, 1), (2, -1, 1)\}$ is a linearly independent set.

Example

Find all $x \in \mathbb{R}$ such that $\{(1,1,2),(-2,x,1),(2,-1,1)\}$ is a linearly independent set.

Solution

We solve

$$\begin{vmatrix} 1 & -2 & 2 \\ 1 & x & -1 \\ 2 & 1 & 1 \end{vmatrix} \neq 0 \Leftrightarrow 9-3x \neq 0 \Leftrightarrow x \neq 3$$

Example

Prove that $\{(1,2,3),(-2,0,1)\}$ is linearly independent.

Exercise

3. Determine whether the set S is linearly independent or linearly dependent

a. $S = \{(-1,2),(3,1),(2,1)\}$

b. $S = \{(-1,2,3),(1,3,5)\}$

c. $S = \{(1,-2,2),(2,3,5),(3,1,7)\}$

d. $S = \{(-1,2,1),(2,4,0),(3,1,1)\}$

e. $S = \{(1,-2,2,1),(1,2,3,5),(-1,3,1,7)\}$

Exercise

4. For which values of k is each set linearly independent?

a. $S = \{(-1,2,1),(k,4,0),(3,1,1)\}$

b. $S = \{(-1,k,1),(1,1,0),(2,-1,1)\}$

c. $S = \{(k,1,1),(1,k,1),(1,1,k)\}$

d. $S = \{(1,2,1,0),(-2,1,1,-1),(-1,3,2,k)\}$

Basis (Cơ sở)

Definition If U is a subspace of $\mathbb{R}^n$, a set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\mathbf{m}\}$ vectors in U is called basis of U if it satisfies the following two conditions:

- $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\mathbf{m}\}$ is linearly independent
- $U = \text{span}\,\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\mathbf{m}\}$

Dimension of U (dim U) = number of vectors in basis of U

Example

Prove that $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$

Example

Prove that $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$

Solution

- verify $\{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent? Yes,
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$
- verify $\mathbb{R}^3 = \text{span}\ \{(1,0,0),(0,1,0),(0,0,1)\}$. Take $v = (\alpha, \beta, \gamma) \in \mathbb{R}^3$, then
$$\begin{aligned} v =& (\alpha, \beta, \gamma) \\ &= (\alpha, 0, 0) + (0, \beta, 0) + (0, 0, \gamma) \\ &= \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) \in \text{span}\{(1,0,0),(0,1,0),(0 \end{aligned}$$

Example

- $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2 \Rightarrow \dim \mathbb{R}^2 = 2$
- $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3 \Rightarrow \dim \mathbb{R}^3 = 3$
- $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ is a basis of $\mathbb{R}^4 \Rightarrow \dim \mathbb{R}^4 = 4$
- $\dim \mathbb{R}^n = n$

Example

Prove that $\{(1,1,1),(1,2,1),(2,3,1)\}$ is a basis of $\mathbb{R}^3$.

Example

Prove that $\{(\mathbf{1},\mathbf{1},\mathbf{1}),(\mathbf{1},\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{3},\mathbf{1})\}$ is a basis of $\mathbb{R}^3$.

Solution

- $\{(1,1,1),(1,2,1),(2,3,1)\}$ is linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 3 \neq 0$$

- $\mathbb{R}^3 = \text{span}\{(1,1,1),(1,2,1),(2,3,1)\}$.
  Take $u = (\alpha, \beta, \gamma) \in \mathbb{R}^3$, then find x, y, z such that

$$(\alpha, \beta, \gamma) = x(1,1,1) + y(1,2,1) + z(2,3,1)$$

$$. \Leftrightarrow \begin{cases} x + y + 2z = \alpha \\ x + 2y + 3z = \beta \\ x + y + z = \gamma \end{cases}$$

This system has unique solution (x, y, z) because

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \cancel{3} \neq 0$$

-1

Example

Let $\mathbf{W} = \{(\mathbf{r}, \mathbf{s}, \mathbf{r}) | \mathbf{s}, \mathbf{r} \in \mathbb{R}\}$. Find a basis and calculate dimW.

Example

Let W $= \{(\mathbf{r},\mathbf{s},\mathbf{r}) | \mathbf{s},\mathbf{r} \in \mathbb{R}\}$. Find a basis and calculate dimW.

Solution

$(r,s,r) = (r,0,r) + (0,s,0) = r(1,0,1) + s(0,1,0)$

Thus, $(r,s,r) \in$ span $\{(1,0,1),(0,1,0)\}$

Then $W =$ span $\{(1,0,1),(0,1,0)\}$

Moreover, $(1,0,1)$ and $(0,1,0)$ are linearly independent (why?).

Thus $\{(1,0,1),(0,1,0)\}$ is a basis of W and dim W $= 2$

Example

Let $W = \{(x, y, z) | x + y + z = 0, x - y = 0\}$. Find a basis and calculate dim W.

Example

Let $\mathbf{W = \{(x, y, z) | x + y + z = 0, x - y = 0\}}$. Find a basis and calculate dim W.

Solution

$$\begin{cases} x + y + z = 0 \\ x - y = 0 \end{cases} \Leftrightarrow \begin{cases} x + y + z = 0 \\ -2y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x = t \\ y = t \\ z = -2t \end{cases}$$

Thus $(x, y, z) = (t, t, -2t) = t(1, 1, -2)$

$\Rightarrow W = \text{span } \{(1, 1, -2)\}$.

Moreover $(1, 1, -2)$ is linearly independent.

In conclusion, $\{(1, 1, -2)\}$ is a basis of W and dim W $= 1$

Exercise

Let $U = \{(x, y, z)|2x - y + z = 0\}$ be a subspace of $\mathbb{R}^3$. Which of the following statements are true?

1 mt sinh bi 2 vector => choose i

i) $U = \text{span } \{(1, 0, -2), (0, 1, 1)\}$

ii) $U = \text{span } \{(1, 2, 0)\}$

A. (i) only

B (ii) only

C. Both (i) and (ii)

D. None of the other choices is correct

Column space and Row space
(Không gian cột và không gian hàng)

Let A be a m $\times$ n matrix, we define

- **The column space of A, col A, is the subspace of $\mathbb{R}^m$ spanned by columns of A**
- **The row space of A, row A, is the subspace of $\mathbb{R}^m$ spanned by rows of A**

Example Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

col A$=$ span $\{(1,4),(2,5),(3,6)\}$

row A $=$ span $\{(1,2,3),(4,5,6)\}$

Theorem

- dim(col(A))=dim(row(A))= rank of A
- nonzero rows of row-echelon of A are a basis of row A

c s hàng A là các hàng khác 0 ca A

- columns consisting leading 1s of row-echelon of A are a basis of col A

Example

Find bases and dim of

U =span $\{(1,2,2,-1),(3,6,5,0),(1,2,1,2)\}$

Example

Find bases and dim of

U =span $\{(1,2,2,-1),(3,6,5,0),(1,2,1,2)\}$

Solution

$$\begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{(1,2,2,-1),(0,0,-1,3)\}$ is a basis of U and dim U $= 2$

Exercise

What is the dimension of the subspace of $\mathbb{R}^3$ spanned by

$$\{(1,2,-1),(1,-2,1),(-3,2,-1),(2,0,0)\}?$$

a)0 b)1 c)2 d)3 e)4

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \text{rank} = 2$$

Exercise

What is the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$\{(1, 1, 0, 9), (1, 1, 0, -1), (0, 0, 1, 7), (0, 0, 1, 0)\}?$$

a) 1 b)2 c)3 d)4

$$\begin{pmatrix} 1 & 1 & 0 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \text{rank} = 3$$

Exercise

Let $U = span\{(2,1,1),(1,-1,0),(3,0,1)\} \subset \mathbb{R}^3$.

Find the dimension of U

A. 1 B. 2 C. 3 D. $\{(2,1,1),(1,-1,0)\}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \text{Rank} = 2$$

Exercise

$$\begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 1/4 \end{pmatrix}$$

Let $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 5 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix}$. Find $\dim(\operatorname{col} A)$ $= \operatorname{rank} A$

$= 3$

A. 3 B. 1 C. 2 D. 4

Null Space
(Không gian nghiệm)

Definition **Let A be an $m \times n$ matrix, null space**

$$\mathbf{null(A)} := \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} | \mathbf{Ax} = \mathbf{0}\}$$

Theorem

dim(null(A)) = number of variables (n) - rank of A

(Số chiều không gian nghiệm của phương trình $Ax = 0$ thì bằng số biến - số hạng của A)

Example

Given $A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix}$. Find bases of null(A) and its dimension.

cơ sở

Solution

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \text{null(A)}, \text{ then } \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \to$$

$$\begin{cases} x_1 = 2s + t \\ x_2 = s \\ x_3 = -2t \\ x_4 = t \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2s + t \\ s \\ -2t \\ t \end{pmatrix} = s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Therefore, dim null(A) $=2$ and bases of A is $\{(2, 1, 0, 0), (1, 0, -2, 1)\}$

Exercise

Find a basis and the dimension of the solution space of the homonegeous system of linear equations.

= NULL spac

$$\begin{cases} -x + y + z = 0 \\ 3x - y = 0 \\ 2x - 4y - 5z = 0 \end{cases}$$

dim

rank = 2

Solution

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x - y - z = 0 \\ 2y + 3z = 0 \end{cases}$$

$$\begin{cases} x = -t/2 \\ y = -3t/2 \\ z = t \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1/2 \\ -3/2 \\ 1 \end{pmatrix}$$

A basis is $\{(-1/2, -3/2, 1)\}$ and dimension is 1.

Exercise

Let $U = \{(a, b, c, d) | a + 2d = 3b + c\}$ be a subspace in $\mathbb{R}^4$. Find the dimension of U.

$A.0 \quad B.1 \quad C.2 \quad D.3 \quad E.4$

$\Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \end{array}\right) \to R = 1$

$\to \dim = 4 - 1 = 3$

Exercise

Let A be a 4×7 matrix and rank(A)=1. Find the dimension of null space of A.

⟹ 7 biến

A.4 B.5 C.6 D.7

$7 - 1 = 6$

co 4 bien . rank = 2
=> dim = 2 (D OR E)

Exercise

A basis for the solution space of the system $\begin{cases} u - 2x + 3y + 4z = 0 \\ -2u + 4x - 5y - 6z = 0 \end{cases}$ is:

a) $\{(0,0,0,0)\}$

b) $\{(2,1,0,0)\}$

c) $\{(1,2,0,0)\}$

d) $\{(2,1,0,0),(1,-3,-4,1)\}$

e) $\{(2,1,0,0),(2,0,-2,1)\}$

f) $\{(2,0,-2,1)\}$

Exercise

Which one of the following is a basis for the subspace of $\mathbb{R}^3$ defined by $G = \{(x, y, z) : 2x - y + 3z = 0\}$? $3 - 1 = 2\ \dim(\text{basis})$

a) $(1, 2, 0)$ and $(0, 3, 1)$

b) $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$

c) $(1, 2, 0)$

d) $(1, \cancel{0}, 0)$ and $(1, 2, 0)$

e) $(3, 0, -2)$

f) $(-3, 0, 2)$ and $(1, \cancel{0}, 0)$