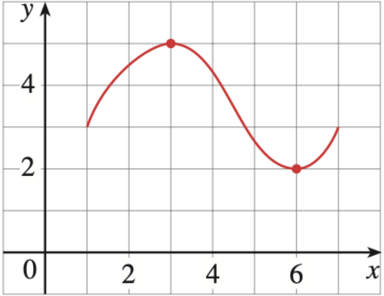
Chapter 4: Application of Derivative

Trần Hoà Phú

Ngày 11 tháng 3 năm 2023

Maximum and Minimum Values



- The highest point in the graph is $(3, 5)$.
- The largest value of f is $f(3) = 5$.
- We say that $f(3) = 5$ is the absolute maximum.
- The smallest value of f is $f(6) = 2$.
- We say that $f(6) = 2$ is the absolute minimum.

Absolute Maximum and Absolute Minimum Value (Giá trị cực đại tuyệt đối và cực tiểu tuyệt đối)

Definition

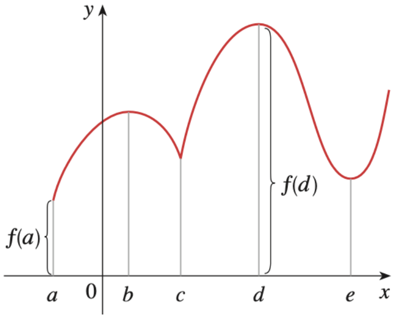
Let c be a number in the domain D of a function f. Then $f(c)$ is the

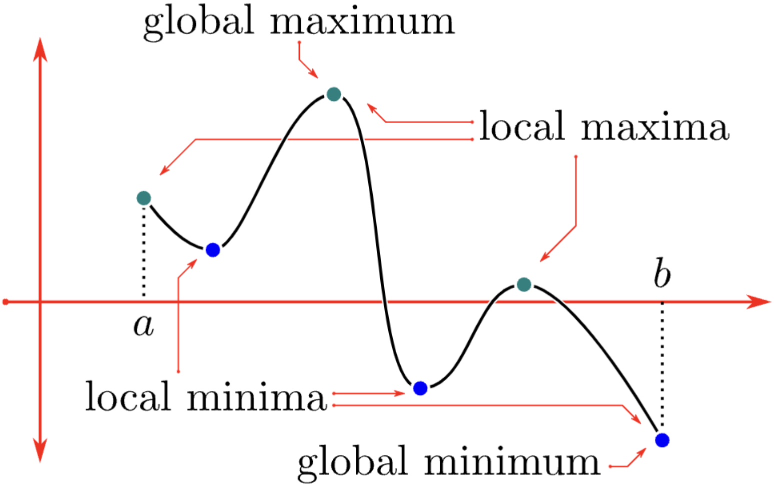
- absolute maximum value of f on D if $\mathbf{f(c) \geq f(x)\ \forall x \in D}$
- absolute minimum value of f on D if $\mathbf{f(c) \leq f(x)\ \forall x \in D}$

Local Maximum and Local Minimum Value
(Giá trị cực đại địa phương và giá trị cực tiểu địa phương)

Definition The number $f(c)$ is a

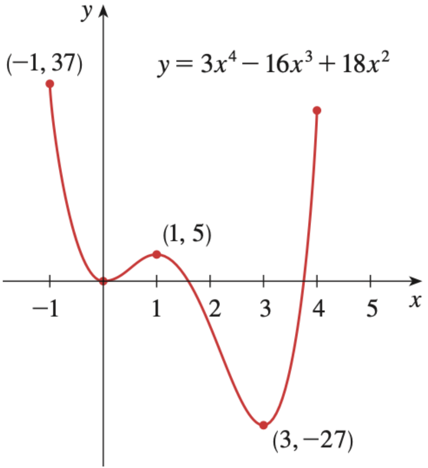
- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c




global maximum
local maxima
b
a
local minima
global minimum

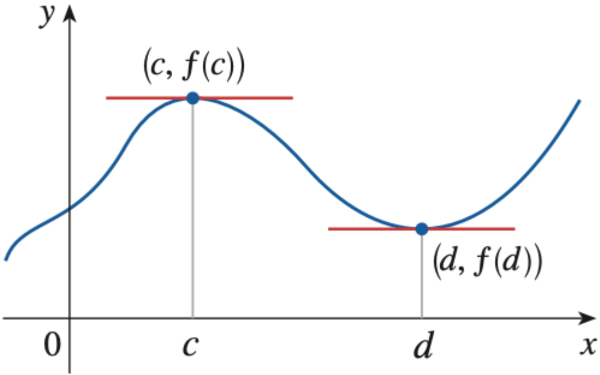
Exercise

Find absolute maximum(minimum) and local maximum (minimum).

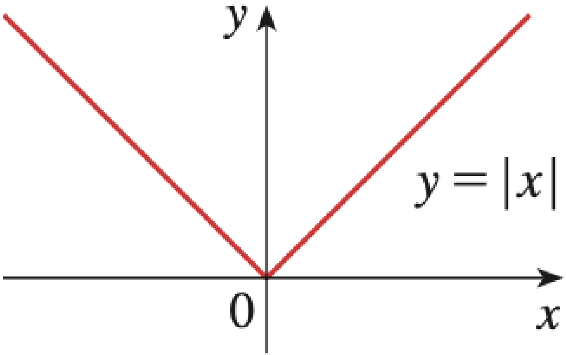


Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $\mathbf{f'(c) = 0}$



Let $\mathbf{f(x) = |x|, f(0) = 0}$ is a minimum value, but $f'(0)$ does not exists.



Critical numbers
(Điểm tới hạn)

Definition **A critical number of a function f is a number c in the domain of f such that one of the following hods.**

- $\mathbf{f'(c) = 0}$
- $\mathbf{f'(c)}$ **does not exist.**

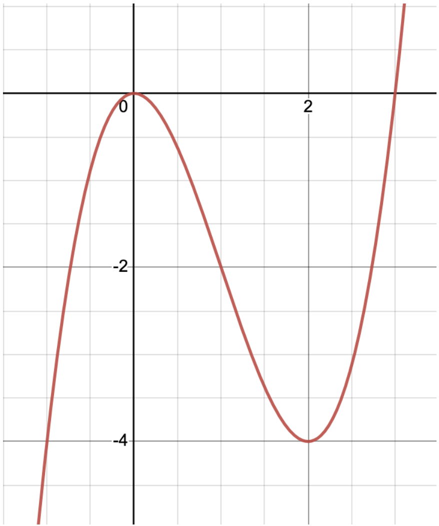
Example Find critical numbers of $\mathbf{f(x) = x^3 - 3x^2}$.

Solution

$f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \Leftrightarrow x = 0$ or $x = 2$

f has two critical numbers 0 and 2.

Graph of $y = x^3 - 3x^2$



Exercise

Find critical numbers of

$$g(x) = \sqrt[3]{x^2 - x}$$

A.1 B.0 C.0;1 D.0; 1; 1/2 E.1/2

Closed Interval Method

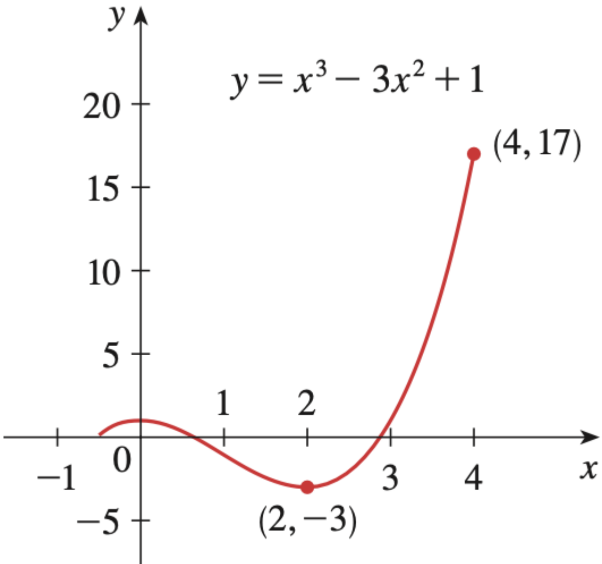
To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$

1. Find the values of f at the critical numbers of f in $[a, b]$
2. Find the values of f at the endpoints of the interval
3. The largest value from 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

Exercise

Find the absolute minimum and maximum values of the

$$f(x) = x^3 - 3x^2 + 1 \quad \frac{-1}{2} \le x \le 4$$


y
$y = x^3 - 3x^2 + 1$
20
15
10
5
−5
1
2
3
4
−1
0
x
(4, 17)
(2, −3)

Exercise

Find the absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 3x + 1, \quad \text{on} \quad [0, 2]$$

A. Absolute maximum: 2, absolute minimum: 1

B. Absolute maximum: 3, absolute minimum: 1

C. Absolute maximum: 3, absolute minimum: 0

D. Absolute maximum: 3, absolute minimum: 2

Example

Find two positive numbers whose product is 24 and whose the sum of 2 times the first and 3 times the second is a minimum.

A. 12,2   B. 8,3   C.4,6   D.6,4

Exercise

Find an equation of the line through the point $(4, 5)$ that cuts off the least area from the first quadrant.

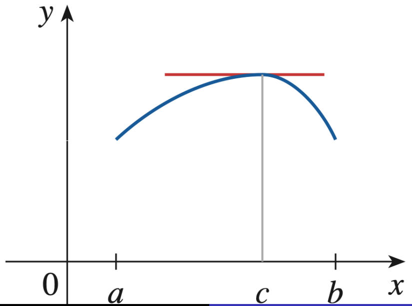
A. $\dfrac{x}{8} - \dfrac{y}{10} = 0$ B. $\dfrac{x}{8} - \dfrac{y}{10} = 1$ C. $\dfrac{x}{8} - \dfrac{y}{10} = -1$

Rolle's Theorem

Let f be a function that satisfies:

- f is **continuous** on the closed interval $[a, b]$
- f is **differentiable** on the open interval (a, b)
- $\mathbf{f(a) = f(b)}$

Then there is a number $\mathbf{c \in (a, b)}$ such that $\mathbf{f'(c) = 0}$



The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

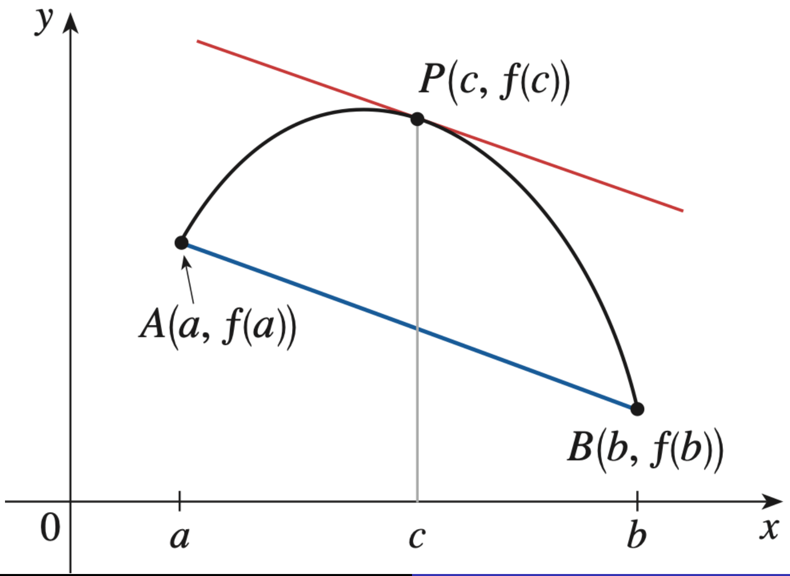
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$


y
P(c, f(c))
A(a, f(a))
B(b, f(b))
0
a
c
b
x

Example

Suppose that $f(0) = -3$ and $f'(x) \leq 5 \quad \forall x \in \mathbb{R}$.
How large can $f(2)$ possibly be?

A.5 B.6 C.7 D.8

Exercise

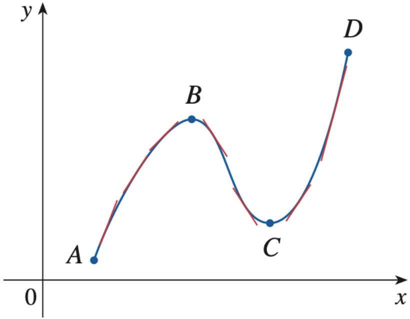
Suppose $f(1) = 3$ and $7 \le f'(x) \le 10$ for all x.

How small can $f(5)$ possible be?

A. 21 B. 43 C. 31 D. None of the

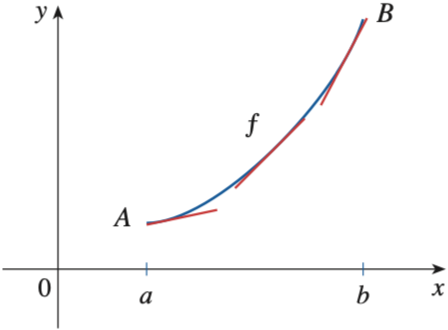
Increasing/Decreasing Test

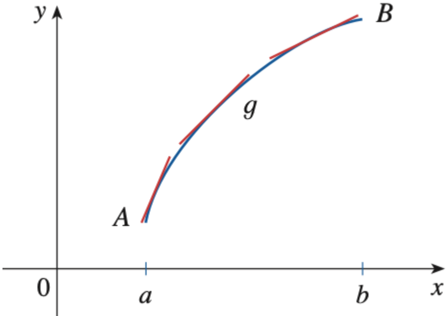
- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval



Concave Upward/Downward

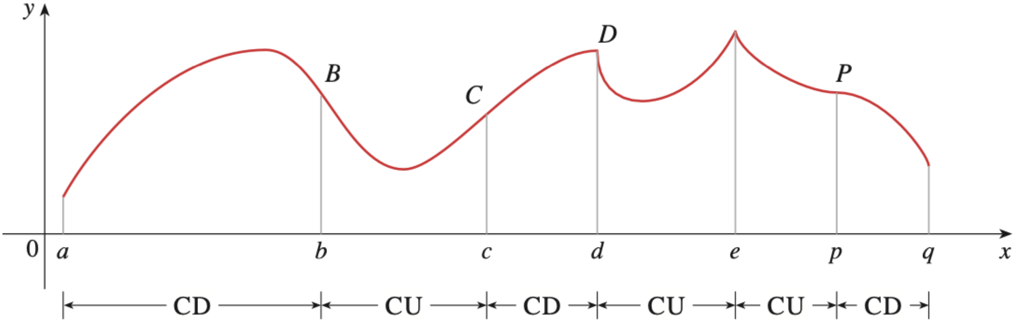
- The curve lies above the tangents and f is called concave upward on (a, b)
- The curve lies below the tangents and f is called concave downward on (a, b)





Concavity Test

- If $f''(x) > 0\ \forall x \in I$, then the graph of f is **concave upward** on I
- If $f''(x) < 0\ \forall x \in I$, then the graph of f is **concave downward** on I



Exercise

Determine where the function

$$f(x) = x^3 + 3x^2 - x - 24$$

is concave up and where it is concave down.

(i) Concave up on $(-\infty, \infty)$

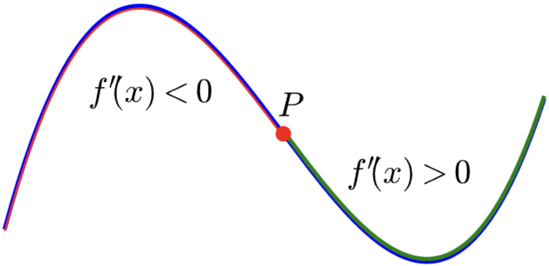
(ii) Concave down on $(-1, \infty)$ and concave up on $(-\infty, -1)$

(iii) Concave up on $(-1, \infty)$ and concave down on $(-\infty, -1)$

(iv) Concave down $(-\infty, \infty)$

Inflection point
(Điểm uốn)

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the **curve changes from concave upward to concave downward or vice versa.**

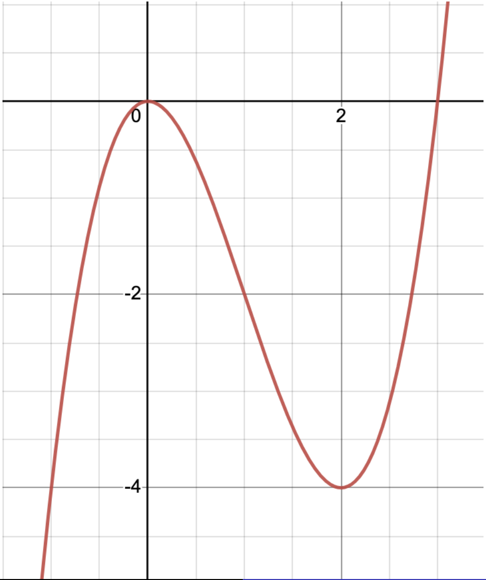


Method to find inflection points

- **Find** x_0 **such that** $\mathbf{f''(x_0) = 0}$
- **Check** x **near** x_0, $f''(x)$ **changes its sign.**

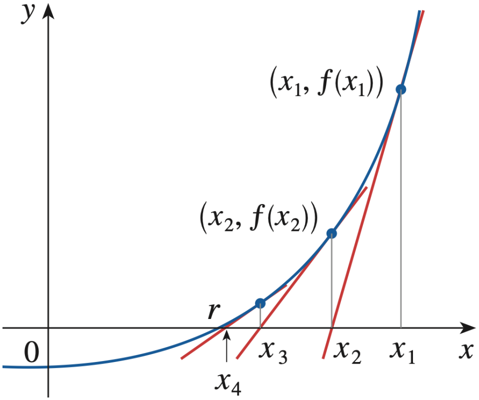
Example **Find inflection points of the curve** $y = x^4 - 4x^3$

Graph of $y = x^4 - 4x^3$



Newton Methods

Newton method is an algorithm to **approximate the solutions of functions**



$$\mathbf{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Proof

The tangent line at $(x_1, f(x_1))$ is

$$\mathbf{y = f'(x_1)(x - x_1) + f(x_1)}$$

Since $(x_2, 0)$ is on this tangent line, then

$$\mathbf{0 = f'(x_1)(x_2 - x_1) + f(x_1)}$$

Thus

$$\mathbf{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

Exercise

Starting with $x_1 = 2$, find the third approximation x_3 to the solution of the equation $x^3 - 2x - 5 = 0$.

≈ 2.095

Exercise

Use Newton's method with the initial approximation $x_1 = 1$ to find x_3, the third approximation to the root of the equation

$$x^5 - 10 = 0$$

Round your answer to 4 decimal places.

A. 2.4341 B. 2.8000 C. 2.2725 D. 1.9952

Antiderivatives
(Nguyên hàm)

<u>**Definition**</u> **A function F is called an antiderivative of f on an interval I if $\mathbf{F'(x) = f(x)}$ for all x in I.**

Notation

$$\int \mathbf{f(x)dx = F(x) + C}$$

Example

$$\int x dx = \frac{x^2}{2} + C$$

$$\int \cos x dx = \sin x + C$$

Table of Antidifferentiation Formulas

$\int 0 \mathrm{d}x = C$	$\int a^x \mathrm{d}x = \dfrac{a^x}{\ln a} + ,\ (a > 0, a \neq 1)$
$\int \mathrm{d}x = x + C$	$\int \cos x \mathrm{d}x = \sin x + C$
$\int x^{\alpha} \mathrm{d}x = \dfrac{1}{\alpha + 1} x^{\alpha+1} + C\ (\alpha \neq -1)$	$\int \sin x \mathrm{d}x = -\cos x + C$
$\int \dfrac{1}{x} \mathrm{d}x = \ln\lvert x\rvert + C$	$\int \dfrac{1}{\cos^2 x} \mathrm{d}x = \tan x + C$
$\int e^x \mathrm{d}x = e^x + C$	$\int \dfrac{1}{\sin^2 x} \mathrm{d}x = -\cot x + C$

Exercise

Find $f(x)$ if $f'(x) = x\sqrt{x}$ if $f(1) = 3$