Chapter 5: Integral

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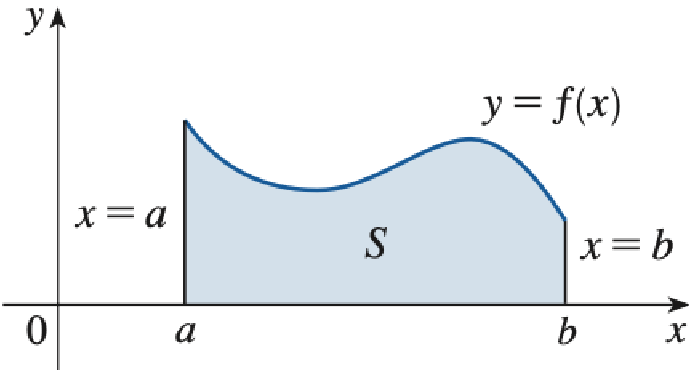
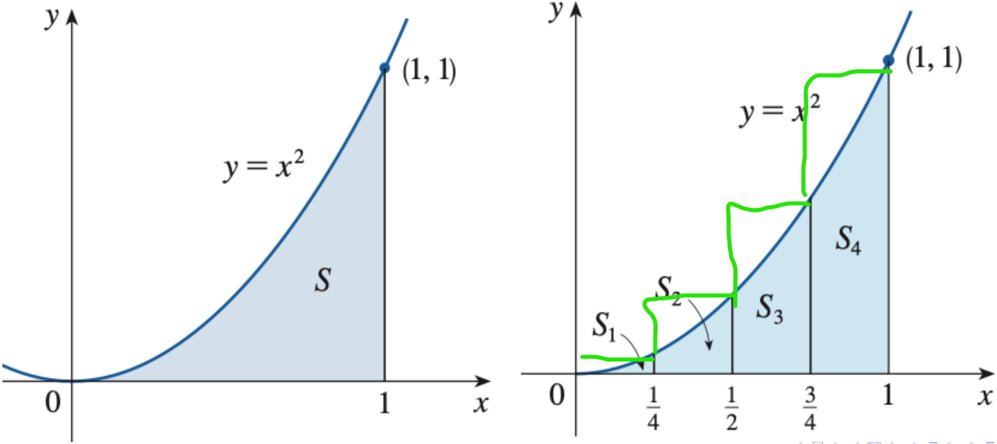
Area Problem

Problem Find the area of the region S that lies under the curve $y = f(x)$ from $x = a$ to $x = b$.

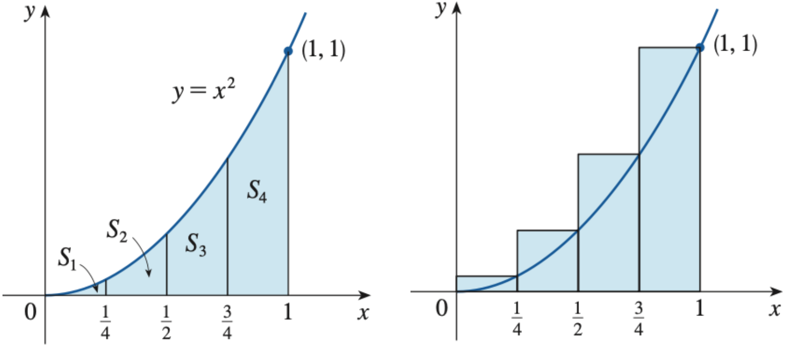


Example

Find the area S lies under the $y = x^2$ from $x = 0$ to $x = 1$.



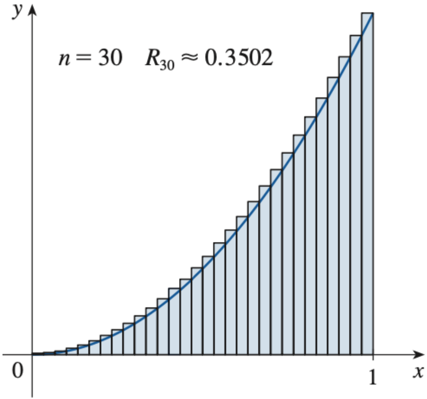
Divide S into four strips $\mathbf{S_1, S_2, S_3, S_4}$ by drawing $\mathbf{x = 1/4, x = 1/2, x = 3/4}$.

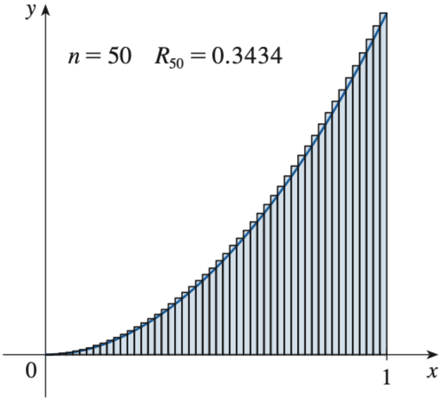


$$R_4 = \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{1}{2}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) + \frac{1}{4}f(1) = \frac{1}{4}\left(\frac{1}{4^2} + \frac{1}{2^2} + \frac{3^2}{4^2} + 1^2\right) =$$

0.4685

The more intervals we divide, the better approximations we get.





Problem How many intervals should we divide to have exact approximation?

$$\mathbf{S = \lim_{n\to\infty} R_n}$$

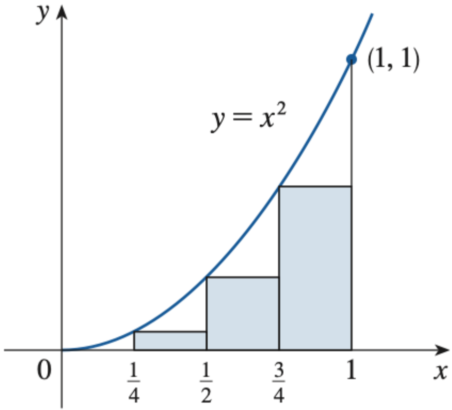
$$R_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \ldots + \frac{1}{n} f\left(\frac{n}{n}\right)$$

$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n} . \frac{1}{n^2}(1^2 + 2^2 + \ldots + n^2)$$

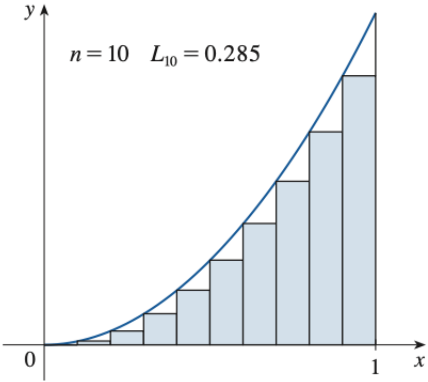
$$= \frac{1}{n^3} . \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

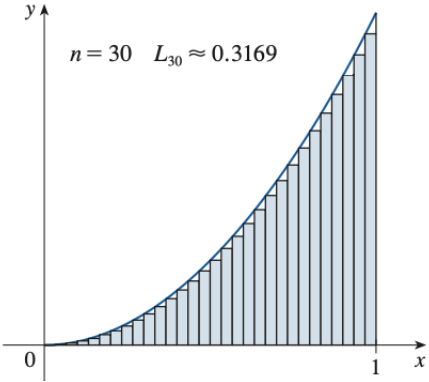
Thus $S = \lim_{n\to\infty} R_n = 1/3$



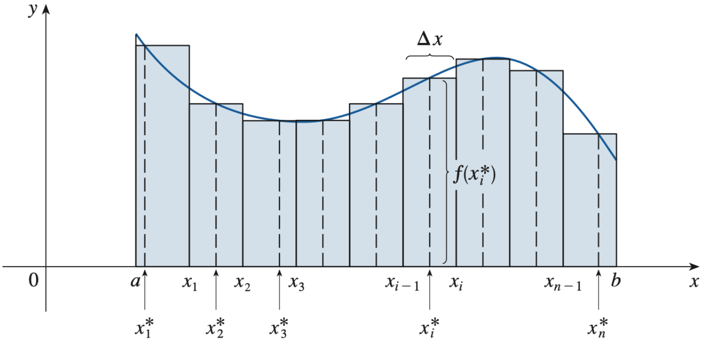
$$L_4 = \left(\frac{1}{4}f(0) + \frac{1}{4}f(\frac{1}{4}) + \frac{1}{4}f(\frac{2}{4}) + \frac{1}{4}f(\frac{3}{4})\right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{4^2} + \frac{2^2}{4^2} + \frac{3^2}{4^2}\right) = 0.21875$$


y
$n = 10 \quad L_{10} = 0.285$
0
1
x


y
$n = 30 \quad L_{30} \approx 0.3169$
0
1
x

- **Sample points**



$$\Delta x = \frac{b-a}{n}$$

$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definite integral

Definition **Definite integral of f from a to b is**

$$\int_a^b f(x)dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x}_{\text{Riemann Sum}} \quad \text{(provided this limit exists)}$$

If it does exist, we say f is integrable on $[a, b]$.

Exercise

Estimate the area under the graph of

$$f(x) = 25 - x^2$$

on $[0, 5]$ using 5 rectangles and right endpoints

A.50 B.60 C.70 D.55

Exercise

Express the limit as a definite integral over $[0, 1]$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \cos^2(2\pi x_i^*)\Delta x$$

$A.\ \int_0^1 \cos^2(2\pi)dx$ $\quad B.\ \int_0^1 \cos^2(\frac{2\pi}{x})dx$

$C.\ \int_{-1}^1 \cos^2(2\pi x)dx$ $\quad D.\ \int_0^1 \cos^2(2\pi x)dx$

Exercise

Use the Right-endpoint rule with n=4 to estimate the value of the integral

$$\int_1^3 f(x)dx$$

x	1	1.5	2	2.5	3
f(x)	0.31	0.5	0.36	1.35	2.04

A. 2.145 B. 1.620 C. 4.290 D. 3.240

Exercise

Find the Riemann sum for

$$f(x) = 3x^2 - 5, 0 \le x \le 2,$$

with four equal subintervals, taking the sample points to be left endpoints.

A. 9.5 B. -9.5 C. 4.75 D. -4.75

Exercise

Estimate the area under the graph $f(x) = x + 1/x$ over $[1, 9]$ using 4 rectangles and left endpoints.

A. 49.57 B. 42.19 C.40 D. 35.35

Properties of the Integral

1. $\displaystyle\int_a^b c\,dx = c(b-a)$
2. $\displaystyle\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. $\int_a^b cf(x)dxc \int_a^b f(x)dx =,$ c is a constant
4. $\displaystyle\int_a^b (f(x)-g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
5. If $m \le f(x) \le M\ \forall a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

$$\int_0^{\pi/4} \cos 2x dx = \frac{1}{2}\sin 2x\Big|_{x=0}^{\pi/4} = \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0 = \frac{1}{2}$$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

:displacement of the object during the time period from t_1 to t_2.

- If we want to calculate the distance the object travels during a time interval, then the distance is

$$\int_{t_1}^{t_2} |v(t)|dt = \text{total distance traveled}$$

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$.

a. Find the displacement of the particle during the time period $1 \leq t \leq 4$.

b. Find the distance traveled during this time period.

Solution

a. Displacement is

$$\int_1^4 v(t)dt = \int_0^1 (t^2 - t - 6) = \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t \right) \Bigg|_{t=1}^{t=4} = -9/2$$

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leqslant 0$ on the interval $[1, 3]$ and $v(t) \geqslant 0$ on $[3, 4]$. Thus, from Equation 3, the distance traveled is

$$\begin{aligned}
\int_1^4 |v(t)|\, dt &= \int_1^3 [-v(t)]\, dt + \int_3^4 v(t)\, dt \\
&= \int_1^3 (-t^2 + t + 6)\, dt + \int_3^4 (t^2 - t - 6)\, dt \\
&= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 \\
&= \frac{61}{6} \approx 10.17 \text{ m}
\end{aligned}$$

■

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Theorem **If f is continuous on** $[a,b]$ **then the function** g **defined by**

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on $[a,b]$ **and differentiable on** (a,b) **and** $g'(x) = f(x)$

Example

$$\left(\int_0^x \sqrt{t^2+1}dx\right)' = x^2+1$$

$$\left(\int_1^x \sin(2t)dt\right)' = \sin 2x$$

Exercise

Find $\frac{dy}{dx}$ for $y = \int_1^x \frac{1}{\sqrt{16-t^2}} dt$

(i) $\dfrac{1}{\sqrt{16-x^3}}$

(ii) $\dfrac{1}{\sqrt{16-x}}$

(iii) $\dfrac{1}{\sqrt{16-x^2}}$

(iv) $\dfrac{x}{\sqrt{16-t^2}}$

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f(u(x))u'(x)$$

Example

$$\left(\int_1^{x^2}(2t-1)dt\right)' = (2x^2-1)(x^2)' = 2x(2x^2-1)$$

$$\left(\int_0^{2-x}\sin t dt\right)' = \sin(2-x)(2-x)' = -\sin(2-x)$$

Exercise

Find $$\frac{d}{dx}\int_{3}^{1+x^2} \ln t\,dt$$
$2x\ln(1+x^2)$
$2x/(1+x^2)$
$\ln(1+x^2)-\ln 3$
None of the others.

Exercise

Suppose $g(x) = \int_1^{x^2} \sin(t-1)dt$ Find g'(x).	
a	$g'(x)=\sin(x-1)$
b	$g'(x)=\sin(x^2-1)$
c	$g'(x)=\cos(x-1)$
d	$g'(x)=2x\cos(x^2-1)$
e	$g'(x)=2x\sin(x^2-1)$

Exercise

Find $\dfrac{dy}{dx}$ for

$$y = \int_1^{\sqrt{x}} t dt$$

A. x B. $x-1$ C. $1/2$ D.1 E. $1/x$

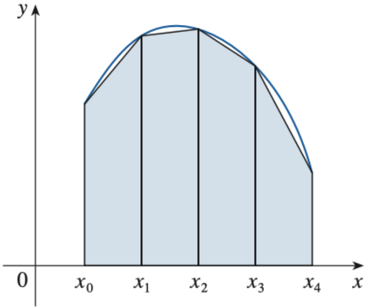
Average value of f on $[a, b] := \frac{1}{b-a}\int_a^b f(x)dx$

Find the average value of the function $y=x^2 -2x$ on the interval [0,3].
0
3/2
-1/2
1
None of the others.

Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{\Delta}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = \dfrac{b-a}{n}, \quad x_i = a + i\Delta x$



Example

Use the Trapezoidal Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x} dx$

Solution

$$\int_1^2 \frac{1}{x} dx = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$\approx 0.695635$$