Chapter 5: The Vector Space R^n

Trần Hoà Phú

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Contents

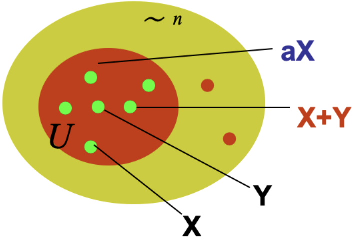
n-Vectors

- (x_1, x_2) // vector in R^2
- (x_1, x_2, x_3) // vector in R^3
- (x_1, x_2, x_3, x_4) // vector in R^4
- $(x_1, x_2, \ldots, x_n)$ // vector in R^n
- A **vector** $(x_1, x_2, \ldots, x_n)$ in R^n is also called a **point** in R^n.
- $(0, 0, \ldots, 0)$: the zero vector in R^n

Subspace of R^n
(Không gian con của $\mathbb{R}^n$)

Definition **Let** $\emptyset \neq U$ **be a subset of** $\mathbb{R}^n$. U **is called a subspace of** R^n **if**

- **vector** $\mathbf{0} = (0, .., 0) \in \mathbf{U}$
- $\forall \mathbf{X}, \mathbf{Y} \in \mathbf{U} \Rightarrow \mathbf{X} + \mathbf{Y} \in \mathbf{U}$
- $\forall \mathbf{X} \in \mathbf{U}, \mathbf{a} \in \mathbf{R} \Rightarrow \mathbf{aX} \in \mathbf{U}$



Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{x}) | \mathbf{x} \in \mathbb{R}\}$ **is a subspace of** $\mathbb{R}^2$

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- vector $0 = (0, 0) \in U$
- $X, Y \in U$. Prove that $X + Y \in U$
  $X, Y \in U \Rightarrow \exists x_0, y_0 \in R : X = (x_0, x_0), Y = (y_0, y_0)$
  $X + Y = (x_0, x_0) + (y_0, y_0) = (x_0 + y_0, x_0 + y_0) \in U$
- $X \in U, a \in \mathbb{R}$. Prove that $aX \in U$.
  $X \in U \Rightarrow \exists x_0 : X = (x_0, x_0)$. Thus
  $aX = a(x_0, x_0) = (ax_0, ax_0) \in U$

Example

$\mathbf{U} = \{(\mathbf{t}, \mathbf{t}, \mathbf{2t}) | \mathbf{t} \in \mathbb{R}\}$ **is a subspace of** $\mathbb{R}^3$

Example

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- vector $0 = (0, 0, 0) \in U$
- $X, Y \in U$. Prove that $X + Y \in U$
  $X = (x, x, 2x)$, $Y = (y, y, 2y)$. Then
  $X + Y = (x, x, 2x) + (y, y, 2y) = (x + y, x + y, 2(x + y)) \in U$
- $X \in U, a \in \mathbb{R}$. Prove that $aX \in U$
  $X = (x, x, 2x)$. Then $aX = a(x, x, 2x) = (ax, ax, 2ax) \in U$

Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) | \mathbf{x} + \mathbf{2y} - \mathbf{z} = \mathbf{0}\}$ **is a subspace of** $\mathbb{R}^3$

Example

$\mathbf{U = \{(x, y, z) | x + 2y - z = 0\}}$ is a subspace of $\mathbb{R}^3$

- vector $0 = (0, 0, 0) \in U$
- $X, Y \in U$. Prove that $X + Y \in U$

  $X, Y \in U \Rightarrow X = (x_1, y_1, z_1)$ with $x_1 + 2y_1 - z_1 = 0$ and
  $Y = (x_2, y_2, z_2)$ with $x_2 + 2y_2 - z_2 = 0$.
  $X + Y = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
  with $x_1 + x_2 + 2(y_1 + y_2) + z_1 + z_2 = 0$
- $X \in U, a \in \mathbb{R}$. Prove that $aX \in U$

  $X = (x_1, y_1, z_1)$ with $x_1 + 2y_1 - z_1 = 0$. Then
  $aX = a(x_1, y_1, z_1) = (ax_1, ay_1, az_1)$ with
  $ax_1 + 2ay_1 - az_1 = a(x_1 + 2y_1 - z_1) = 0$

Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{5x}, \mathbf{1}) | \mathbf{x} \in \mathbb{R}\}$ **is NOT a subspace of** $\mathbb{R}^3$

vector $0 = (0, 0, 0) \in U$?

Example

$\mathbf{U} = \{(\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y} - \mathbf{1}) | \mathbf{x}, \mathbf{y} \in \mathbb{R}\}$ **is NOT a subspace of** $\mathbb{R}^3$

vector $0 = (0, 0, 0) \in U$?

Exercise

Determine whether U is a subspace of R^3.

(i) $U=\{[0\ 1\ s]^T : s \in R\}$

(ii) $U=\{[0\ a\ b]^T : a,b \in R\}$

(iii) $U=\{[a\ b\ a+1]^T : a,b \in R\}$

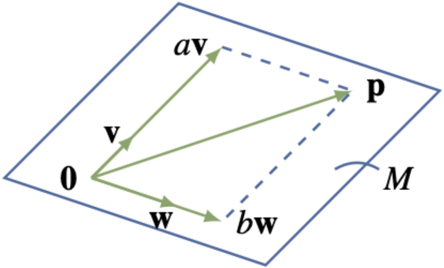
Spanning sets
(Tập sinh)

Let v, w be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. How to describe the plane M through origin containing these vectors ?

First way. The plane M has normal $n = v \times w$ and through origin so it consists of all vectors p: $\mathbf{n.p} = \mathbf{0}$

Second way.

Let v, w be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. How to describe the plane M through origin containing these vectors ?



By a diagram, vector p is in M if and only if $p = av + bw$ for certain real numbers a, b.

$$\mathbf{M = \{av + bw | a, b \in \mathbb{R}\} =: span\{v, w\}}$$

Spanning sets

Definition

- $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \{\mathbf{a}_1\mathbf{v}_1 + \mathbf{a}_2\mathbf{v}_2 | \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}\}$
- $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\mathbf{a}_1\mathbf{v}_1 + \mathbf{a}_2\mathbf{v}_2 + \mathbf{a}_3\mathbf{v}_3 | \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}\}$
- $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} = \{\mathbf{a}_1\mathbf{v}_1 + \mathbf{a}_2\mathbf{v}_2 + ... + \mathbf{a}_k\mathbf{v}_k | \mathbf{a}_1, ..., \mathbf{a}_k \in \mathbb{R}\}$

Example

Given $V = \text{span}\{(-1, 2, 1), (3, -5, -1)\}$.

a. $(-1, 1, 1) \in \mathbf{V}$?

b. Find all m such that $(-2, 1, m) \in \mathbf{V}$

Solution

a. Find x, y such that

$$\begin{aligned}(-1,1,1) &= x(-1,2,1) + y(3,-5,-1)\\ &= (-x,2x,x) + (3y,-5y,-y)\\ &= (-x+3y, 2x-5y, x-y)\end{aligned}$$

It follows that $\begin{cases} -x+3y=-1 \\ 2x-5y=1 \\ x-y=1 \end{cases}$ (inconsistent)

Thus $(-1,1,1) \notin V$

b. Find x, y such that

$$\begin{aligned}(-2,1,m) &= x(-1,2,1)+y(3,-5,-1)\\ &= (-x,2x,x)+(3y,-5y,-y)\\ &= (-x+3y,2x-5y,x-y)\end{aligned}$$

It follows that $\begin{cases}-x+3y=-2\\ 2x-5y=1\\ x-y=m\end{cases} \Leftrightarrow \begin{pmatrix}-1 & 3 & -2\\ 2 & -5 & 1\\ 1 & -1 & m\end{pmatrix} \Leftrightarrow$

$\begin{pmatrix}1 & -3 & 2\\ 0 & 1 & -3\\ 0 & 0 & m+4\end{pmatrix} \Leftrightarrow m+4=0 \Leftrightarrow m=-4$

Exercise

Let $U = \text{span}\{(1,1,2),(-1,2,1)\} \subset \mathbb{R}^3$ and $x = (m,-1,1)$

$x \in U$ if and only if $m =$

A.-1 B.1 C.2 D.-2

Exercise

Let $U = \text{span}\{(1,1,2,1),(0,1,1,-2)\}$.
Find all values of t such that $(1,t,3,4) \in U$

A. there is no such t

B. -2

C. All nonzero numbers

D. None of the other choices is correct

E. All number different from -1

Exercise

Let
$x = (-1, -2 - 2), u = (0, 1, 4), v = (-1, 1, 2), w = (3, 1, 2) \in \mathbb{R}^3$.
Find real numbers a, b, c such that $x = au + bv + cw$

Exercise

Write v as a linear combination of u and w, if possible, where $u = (1, 2), w = (1, -1)$.

a. $v = (0, 1)$

b. $v = (2, 3)$

c. $v = (1, 4)$

Linear Independence and Linear Dependence (Độc lập tuyến tính và phụ thuộc tuyến tính)

Definition

- A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is called **linearly independent** if the system

$$\mathbf{t_1 v_1} + \mathbf{t_2 v_2} + \ldots + \mathbf{t_k v_k} = \mathbf{0}$$

  has only trivial Solution

$$\mathbf{t_1} = \mathbf{t_2} = \ldots = \mathbf{t_k} = \mathbf{0}$$

- A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is called **linearly dependent** if it is not linearly independent.

Example

Prove that $\{(\mathbf{1}, \mathbf{2}, \mathbf{3}), (\mathbf{0}, \mathbf{1}, \mathbf{2}), (-\mathbf{2}, \mathbf{0}, \mathbf{1})\}$ **is linearly independent**

Example

Prove that $\{(\mathbf{1},\mathbf{2},\mathbf{3}),(\mathbf{0},\mathbf{1},\mathbf{2}),(-\mathbf{2},\mathbf{0},\mathbf{1})\}$ **is linearly independent**

Solution

Given real numbers t_1, t_2, t_3 such that

$$t_1(1,2,3)+t_2(0,1,2)+t_3(-2,0,1)=(0,0,0)$$
$$\Leftrightarrow (t_1,2t_1,3t_1)+(0,t_2,2t_2)+(-2t_3,0,t_3)=(0,0,0)$$
$$\Leftrightarrow (t_1-2t_3,2t_1+t_2,3t_1+2t_2+t_3)=(0,0,0)$$

$$\Leftrightarrow \begin{cases} t_1-2t_3=0 \\ 2t_1+t_2=0 \\ 3t_1+2t_2+t_3=0 \end{cases}, \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0 \Leftrightarrow \begin{cases} t_1=0 \\ t_2=0 \\ t_3=0 \end{cases}$$

Theorem

Let $\{c_1, c_2, ..., c_n\}$ denotes the columns of A. Then

- *if* $|\mathbf{A}| \neq 0$ ***or rank(A)=n*** *then* $\{c_1, c_2, ..., c_n\}$ *are* ***linearly independent***
- *if* $|\mathbf{A}| = 0$ ***or rank(A)*** $<$ $\boldsymbol{n}$ *then* $\{c_1, c_2, ..., c_n\}$ *are* ***linearly dependent***

Exercise

For what value of a is the set of vectors

$$S = \{(1,1,1),(2,0,4),(2,a,2)\} \text{ linearly dependent?}$$

A. -4 B. -2 C. 0 D. 2

Example

Find all $x \in \mathbb{R}$ **such that** $\{(1,1,2),(-2,x,1),(2,-1,1)\}$ **is a linearly independent set.**

Example

Find all $x \in \mathbb{R}$ such that $\{(1,1,2),(-2,x,1),(2,-1,1)\}$ is a linearly independent set.

Solution

We solve

$$\begin{vmatrix} 1 & -2 & 2 \\ 1 & x & -1 \\ 2 & 1 & 1 \end{vmatrix} \neq 0 \Leftrightarrow 9 - 3x \neq 0 \Leftrightarrow x \neq 3$$

Example

Prove that $\{(1,2,3),(-2,0,1)\}$ is linearly independent.

Exercise

3. Determine whether the set S is linearly independent or linearly dependent

a. $S = \{(-1,2),(3,1),(2,1)\}$

b. $S = \{(-1,2,3),(1,3,5)\}$

c. $S = \{(1,-2,2),(2,3,5),(3,1,7)\}$

d. $S = \{(-1,2,1),(2,4,0),(3,1,1)\}$

e. $S = \{(1,-2,2,1),(1,2,3,5),(-1,3,1,7)\}$

Exercise

4. For which values of k is each set linearly independent?

a. $S = \{(-1,2,1),(k,4,0),(3,1,1)\}$

b. $S = \{(-1,k,1),(1,1,0),(2,-1,1)\}$

c. $S = \{(k,1,1),(1,k,1),(1,1,k)\}$

d. $S = \{(1,2,1,0),(-2,1,1,-1),(-1,3,2,k)\}$

Basis (Cơ sở)

Definition If U is a subspace of $\mathbb{R}^n$, a set $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ vectors in U is called **basis** of U if it satisfies the following two conditions:

- $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ is linearly independent
- $U = \text{span}\,\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$

Dimension of U (dim U) = number of vectors in basis of U

Example

Prove that $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$

Example

Prove that $\{(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1},\mathbf{0}),(\mathbf{0},\mathbf{0},\mathbf{1})\}$ is a basis of $\mathbb{R}^3$

Solution

- verify $\{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent? Yes, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$
- verify $\mathbb{R}^3 = \text{span}\,\{(1,0,0),(0,1,0),(0,0,1)\}$. Take $v = (\alpha,\beta,\gamma) \in \mathbb{R}^3$, then

$$\begin{aligned} v =&(\alpha,\beta,\gamma) \\ &= (\alpha,0,0)+(0,\beta,0)+(0,0,\gamma) \\ &= \alpha(1,0,0)+\beta(0,1,0)+\gamma(0,0,1) \in \text{span}\{(1,0,0),(0,1,0),(0,\end{aligned}$$

Example

- $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2 \Rightarrow \dim \mathbb{R}^2 = 2$
- $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3 \Rightarrow \dim \mathbb{R}^3 = 3$
- $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ is a basis of $\mathbb{R}^4 \Rightarrow \dim \mathbb{R}^4 = 4$
- $\dim \mathbb{R}^n = n$

Example

Prove that $\{(1, 1, 1), (1, 2, 1), (2, 3, 1)\}$ is a basis of $\mathbb{R}^3$.

Example

Prove that $\{(\mathbf{1},\mathbf{1},\mathbf{1}),(\mathbf{1},\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{3},\mathbf{1})\}$ is a basis of $\mathbb{R}^3$.

Solution

- $\{(1,1,1),(1,2,1),(2,3,1)\}$ is linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 3 \neq 0$$

- $\mathbb{R}^3 = \text{span}\{(1,1,1),(1,2,1),(2,3,1)\}$.
  Take $u = (\alpha, \beta, \gamma) \in \mathbb{R}^3$, then find x, y, z such that
  $$(\alpha, \beta, \gamma) = x(1,1,1) + y(1,2,1) + z(2,3,1)$$
  $$. \Leftrightarrow \begin{cases} x + y + 2z = \alpha \\ x + 2y + 3z = \beta \\ x + y + z = \gamma \end{cases}$$
  This system has unique solution (x, y, z) because
  $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 3 \neq 0$$

Example

Let $\mathbf{W} = \{(\mathbf{r}, \mathbf{s}, \mathbf{r}) | \mathbf{s}, \mathbf{r} \in \mathbb{R}\}$**. Find a basis and calculate dimW.**

Example

Let W $= \{(\mathbf{r}, \mathbf{s}, \mathbf{r}) | \mathbf{s}, \mathbf{r} \in \mathbb{R}\}$. Find a basis and calculate dimW.

Solution

$(r, s, r) = (r, 0, r) + (0, s, 0) = r(1, 0, 1) + s(0, 1, 0)$

Thus, $(r, s, r) \in$ span $\{(1, 0, 1), (0, 1, 0)\}$

Then $W =$ span $\{(1, 0, 1), (0, 1, 0)\}$

Moreover, $(1, 0, 1)$ and $(0, 1, 0)$ are linearly independent (why?).

Thus $\{(1, 0, 1), (0, 1, 0)\}$ is a basis of W and dim W $= 2$

Example

Let W $= \{(x, y, z) | x + y + z = 0, x - y = 0\}$**. Find a basis and calculate dim W.**

Example

Let $\mathbf{W} = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) | \mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}, \mathbf{x} - \mathbf{y} = \mathbf{0}\}$. Find a basis and calculate dim W.

Solution

$$\begin{cases} x + y + z = 0 \\ x - y = 0 \end{cases} \Leftrightarrow \begin{cases} x + y + z = 0 \\ -2y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x = t \\ y = t \\ z = -2t \end{cases}$$

Thus $(x, y, z) = (t, t, -2t) = t(1, 1, -2)$

$\Rightarrow W = \text{span}\ \{(1, 1, -2)\}$.

Moreover $(1, 1, -2)$ is linearly independent.

In conclusion, $\{(1, 1, -2)\}$ is a basis of W and dim W $= 1$

Exercise

Let $U = \{(x, y, z) | 2x - y + z = 0\}$ be a subspace of $\mathbb{R}^3$. Which of the following statements are true?

i) $U =$ span $\{(1, 0, -2), (0, 1, 1)\}$

ii) $U =$ span $\{(1, 2, 0)\}$

A. (i) only

B (ii) only

C. Both (i) and (ii)

D. None of the other choices is correct

Column space and Row space
(Không gian cột và không gian hàng)

Let A be a m $\times$ n matrix, we define

- **The column space of A, col A, is the subspace of $\mathbb{R}^m$ spanned by columns of A**
- **The row space of A, row A, is the subspace of $\mathbb{R}^m$ spanned by rows of A**

Example Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

col A$=$ span $\{(1,4),(2,5),(3,6)\}$

row A $=$ span $\{(1,2,3),(4,5,6)\}$

Theorem

- dim(col(A))=dim(row(A))= rank of A
- nonzero rows of row-echelon of A are a basis of row A
- columns consisting leading 1s of row-echelon of A are a basis of col A

Example

Find bases and dim of

$U = \text{span}\{(1, 2, 2, -1), (3, 6, 5, 0), (1, 2, 1, 2)\}$

Example

Find bases and dim of

U =span $\{(1,2,2,-1),(3,6,5,0),(1,2,1,2)\}$

Solution

$$\begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{(1,2,2,-1),(0,0,-1,3)\}$ is a basis of U and dim U $=2$

Exercise

What is the dimension of the subspace of $\mathbb{R}^3$ spanned by

$$\{(1,2,-1),(1,-2,1),(-3,2,-1),(2,0,0)\}?$$

a)0 b)1 c)2 d)3 e)4

Exercise

What is the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$\{(1,1,0,9),(1,1,0,-1),(0,0,1,7),(0,0,1,0)\}?$$

a) 1 b)2 c)3 d)4

Exercise

Let $U = span\{(2,1,1),(1,-1,0),(3,0,1)\} \subset \mathbb{R}^3$.

Find the dimension of U

A. 1 B. 2 C. 3 D. $\{(2,1,1),(1,-1,0)\}$

Exercise

Let $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 5 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix}$. Find $\dim(\operatorname{col} A)$

A. 3 B. 1 C. 2 D. 4

Null Space
(Không gian nghiệm)

Definition **Let** A **be an** $m \times n$ **matrix, null space**

$$\mathbf{null(A)} := \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} | \mathbf{Ax} = \mathbf{0}\}$$

Theorem

dim(null(A)) = number of variables (n) - rank of A

(Số chiều không gian nghiệm của phương trình $Ax = 0$ thì bằng số biến - số hạng của A)

Example

Given $A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix}$. Find bases of null(A) and its dimension.

Solution

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \text{null(A), then } \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \to$$

$$\begin{cases} x_1 = 2s + t \\ x_2 = s \\ x_3 = -2t \\ x_4 = t \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2s + t \\ s \\ -2t \\ t \end{pmatrix} = s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Therefore, dim null(A) $=2$ and bases of A is $\{(2, 1, 0, 0), (1, 0, -2, 1)\}$

Exercise

Find a basis and the dimension of the solution space of the homonegeous system of linear equations.

$$\begin{cases} -x + y + z = 0 \\ 3x - y = 0 \\ 2x - 4y - 5z = 0 \end{cases}$$

Solution

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x - y - z = 0 \\ 2y + 3z = 0 \end{cases}$$

$$\begin{cases} x = -t/2 \\ y = -3t/2 \\ z = t \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1/2 \\ -3/2 \\ 1 \end{pmatrix}$$

A basis is $\{(-1/2, -3/2, 1)\}$ and dimension is 1.

Exercise

Let $U = \{(a, b, c, d) | a + 2d = 3b + c\}$ be a subspace in $\mathbb{R}^4$. Find the dimension of U.

$A.0 \quad B.1 \quad C.2 \quad D.3 \quad E.4$

Exercise

Let A be a 4×7 matrix and rank(A)=1. Find the dimension of null space of A.

A.4 B.5 C.6 D.7

Exercise

A basis for the $\boxed{\text{solution space}}$ of the system $\begin{matrix} u - 2x + 3y + 4z = 0 \\ -2u + 4x - 5y - 6z = 0 \end{matrix}$ is:

a) $\{(0, 0, 0, 0)\}$

b) $\{(2, 1, 0, 0)\}$

c) $\{(1, 2, 0, 0)\}$

d) $\{(2, 1, 0, 0), (1, -3, -4, 1)\}$

e) $\{(2, 1, 0, 0), (2, 0, -2, 1)\}$

f) $\{(2, 0, -2, 1)\}$

Exercise

Which one of the following is a basis for the subspace of $\mathbb{R}^3$ defined by $G = \{(x, y, z) : 2x - y + 3z = 0\}$?

a) $(1, 2, 0)$ and $(0, 3, 1)$

b) $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$

c) $(1, 2, 0)$

d) $(1, 0, 0)$ and $(1, 2, 0)$

e) $(3, 0, -2)$

f) $(-3, 0, 2)$ and $(1, 0, 0)$

Dot Product, Length and Distance in $\mathbb{R}^n$

Definition

- **Dot Product** let $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, we define

$$\mathbf{u.v = u_1v_1 + u_2v_2 + \ldots + u_nv_n}$$

- **Length** Let $x = (x_1, x_2, \ldots, x_n)$, then

$$\|\mathbf{x}\| := \mathbf{x.x} = \sqrt{\mathbf{x_1^2 + x_2^2 + \ldots + x_n^2}}$$

- **Distance** d(x,y) is denoted as the distance between two vectors x and y

$$\mathbf{d(x, y)} = \|\mathbf{x - y}\|$$

Exercise

Let $x = (1, 2, 3, 1)$ and $y = (-1, 2, 0, 2)$.
Find $x.y$, $\|x\|$, $\|y\|$, $\|x + y\|$ and $d(x, y)$.

Solution

$x.y = 1.(-1) + 2.2 + 3.0 + 1.2 = 5$

$\|x\| = \sqrt{1^2 + 2^2 + 3^2 + 1^2} = \sqrt{15}$

$\|y\| = \sqrt{1^2 + (-2)^2 + 0^2 + 2^2} = 3$

$\|x + y\| = \|(0, 4, 3, 3)\| = \sqrt{0^2 + 4^2 + 3^2 + 3^2}$

$d(x, y) = \|x - y\| = \|(2, 0, 3, -1)\| = \sqrt{2^2 + 0^2 + 3^2 + (-1)^2} =$
$\sqrt{14}$

Orthogonal Set
(Tập trực giao)

Definition **A set of vectors** $\{u_1, u_2, ..., u_m\}$ **in** $\mathbb{R}^n$ **is called an orthogonal set if** $\mathbf{u_j.u_k} = \mathbf{0}$ **for all** $j \neq k$ **and** $u_i \neq 0$ **for all i**

Example $\{(1,1);(-1,1)\}$ is an orthogonal set in $\mathbb{R}^2$

Example $\{(1,1,1),(2,2,-4),(1,-1,0)\}$ is called an orthogonal set in $\mathbb{R}^3$

Orthonormal set
(Tập trực chuẩn)

Definition A set of vectors $\{u_1, u_2, \ldots, u_m\}$ in $\mathbb{R}^n$ is called an orthonormal set if

- it is orthogonal: $\mathbf{u_j.u_k = 0 \quad \forall j \neq k}$
- each u_i is a unit vectors: $\|\mathbf{u_i}\| = \mathbf{1} \quad \forall \mathbf{1 \leq i \leq m}$

Example

$\{(1,0,0),(0,1,0),(0,0,1)\}$ is an orthonormal set in $\mathbb{R}^3$

Example $\{(1,2,3),(-1,-1,1)\}$ is an orthogonal set but not orthonormal.

Exercise

Let {X,Y} be an orthogonal set with ||X||=||Y||=1. Which of the following sets are orthogonal?

(i) {X+Y, X-Y}

(ii) {X+2Y, X-Y}

A. Only (i) is orthogonal

B. Not enough information

C. None of (i) and (ii) is orthogonal

D. Only (ii) is orthogonal

E. Both (i) and (ii) are orthogonal