Chapter 3: Derivative

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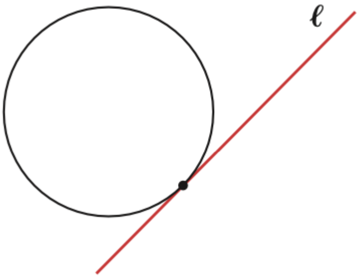
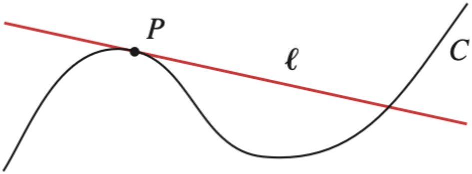
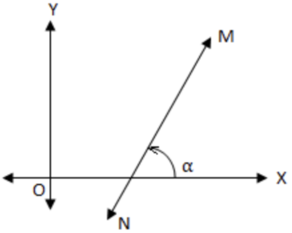
Tangent Lines
(Đường thẳng tiếp tuyến)

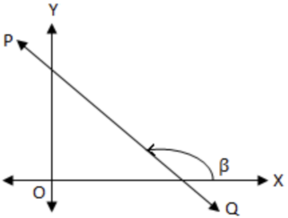




Slope of a line
(Hệ số góc của 1 đường thẳng)

Definition **Slope of a line** $= \tan\theta$, where θ is angel between the line and x-axis ($0 \leq \theta < \pi$).

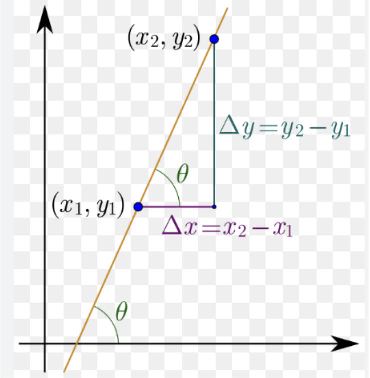




- **Slope > 0:** graph of a line rises from left to right
- **Slope < 0:** graph of a line falls from left to right
- **Slope $=0$:** graph of a line is horizontal

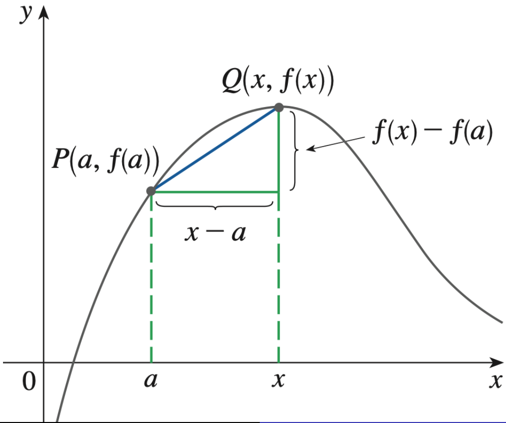
Theorem

- **Equation of the line through P(a,b) and has slope** m**:** $\mathbf{y = m(x - a) + b}$
- **Slope of a line through** $P(x_1, y_1)$ **and** $Q(x_2, y_2)$ $= \dfrac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}}$



Tangent Problems

Given a graph $y = f(x)$ and $P(a, f(a))$ lies on the graph. How to write equation of the tangent line of the graph at P.

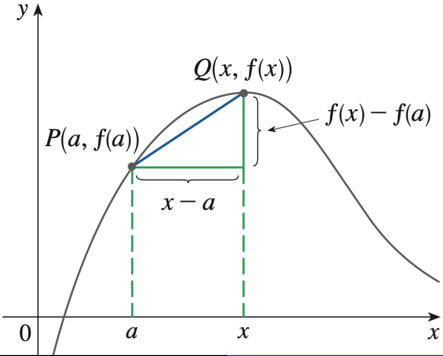


Solution

Slope of line PQ: $m_{PQ} = \dfrac{f(x) - f(a)}{x - a}$.

Slope of tangent line: $\lim\limits_{Q \to P} m_{PQ} = m = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

The equation of tangent line is $y = m(x - a) + f(a)$



Theorem

The tangent line to the curve $y = f(x)$ at the point $P(a, b)$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The equation of the tangent line is

$$y = m(x - a) + b$$

Example

Find an equation of the tangent line to the parabola $y = x^2$ at $(1, 1)$.

Solution

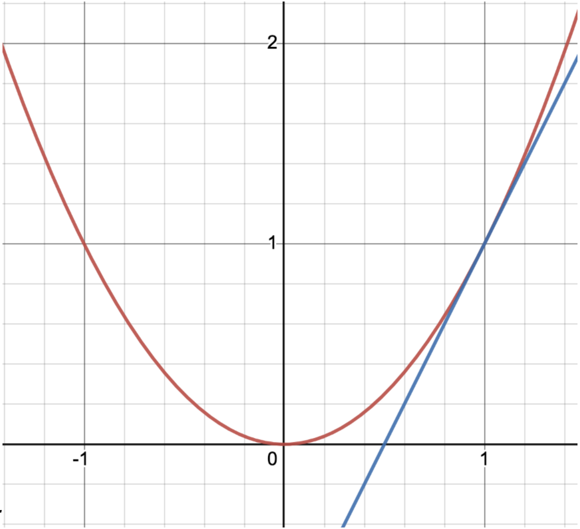
Slope of the tangent line:

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} (x+1) = 2$$

The equation of the tangent line is

$$y = 2(x-1) + 1 = 2x - 1$$

$$y = x^2, \quad y = 2x - 1$$



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Derivative as function

Notation:

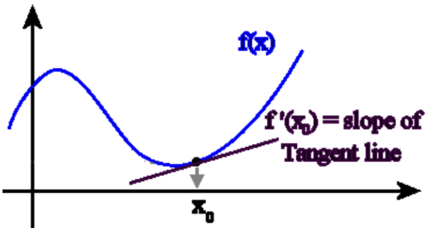
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Write $x = a + h$ and see that $x \to a \Leftrightarrow h \to 0$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Replacing a by variable x,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- $\mathbf{f'(x_0) > 0}$: **đồ thị đi lên trong khoảng lân cận điểm** x_0 **(nhìn từ trái sang phải)**
- $\mathbf{f'(x_0) < 0}$: **đồ thị đi xuống trong khoảng lân cận điểm** x_0 **(nhìn từ trái sang phải)**

Exercise

Determine where the function is increasing and where it is decreasing

$$f(x) = x^3 - 5x^4$$

$f' > 0$ $f' < 0$

A. Increasing on $(-\infty, 3/20)$, decreasing on $(3/20, \infty)$

B. Decreasing on $(-\infty, 3/20)$, increasing on $(3/20, \infty)$

C. Decreasing on $(-\infty, \cancel{-}3/20)$ and $(0, 3/20)$, increasing on $(-3/20, 0)$ and $(3/20, \infty)$

D. Increasing on $(-\infty, \cancel{-}3/20)$, decreasing on $(-3/20, \infty)$

Example

Find $f'(x)$ if

a. $f(x) = 2$

b. $f(x) = x^2$

Solution

a.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2-2}{h} = 0$$

b.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{h^2 + 2xh}{h} = 2x$$

Theorem: **The equation of tangent line to the curve $\mathbf{y = f(x)}$ at $\mathbf{P(a, b)}$: $\mathbf{y = f'(a)(x - a) + b}$**

Example Find an equation of the tangent line of the curve $y = x^2$ at the point $(-2, 4)$.

Solution

$f(x) = x^2$

We have $f'(x) = 2x \Rightarrow f'(-2) = -4$.

The equation of tangent line is

$$y = f'(-2)(x + 2) + 4 = -4(x + 2) + 4 = -4x - 4$$

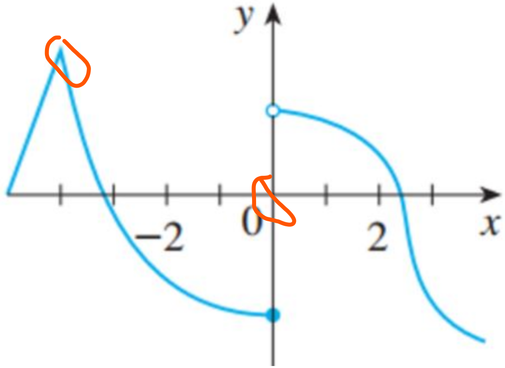
Derivative

Definition

- A function f is differentiable at a if $f'(a)$ exists.
- A function is differentiable on an open interval D if it is differentiable at every point in the interval D.

Exercise

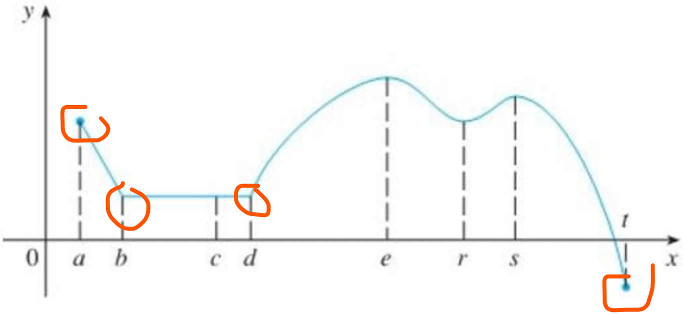
State the numbers at which f(x) is not differentiable.



A.-4 B.0 C.-4;0 D.-2

Exercise

State the numbers at which f(x) is not differentiable.



A. a,b　B. a,b,d,t　C. a,d,r　D. a,t

Differentiation formulas

- $C' = 0$. Ex: $1' = 0, (-5)' = 0, \ldots$
- $(x^\alpha)' = \alpha x^{\alpha-1}, \alpha \in \mathbb{R}$. Ex: $(x^2)' = 2x, (x^3)' = 3x^2, \ldots$
- $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}, \quad \left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$
- $(e^x)' = e^x, \quad (a^x)' = a^x \ln a$
- $(\ln x)' = \dfrac{1}{x}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x, \quad (\cot x)' = -\dfrac{1}{\sin^2 x} = -(1 + \cot^2 x)$

Differentiation Rules

- $(\mathbf{f}+\mathbf{g})' = \mathbf{f}' + \mathbf{g}'$.
  $(\sin \mathbf{x} + \mathbf{x}^2)' = (\sin \mathbf{x})' + (\mathbf{x}^2)' = \cos \mathbf{x} + \mathbf{2x}$
- $(\mathbf{f}-\mathbf{g})' = \mathbf{f}' - \mathbf{g}'$
  $(\mathbf{x}^3 - \mathbf{x}^2)' = (\mathbf{x}^3)' - (\mathbf{x}^2)' = \mathbf{3x}^2 - \mathbf{2x}$
- $(\mathbf{f}.\mathbf{g})' = \mathbf{f}'\mathbf{g} + \mathbf{f}\mathbf{g}'$
  $(x^2 \sin x)' = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x$
- $\left(\dfrac{\mathbf{f}}{\mathbf{g}}\right)' = \dfrac{\mathbf{f}'\mathbf{g} - \mathbf{f}\mathbf{g}'}{\mathbf{g}^2}$
  $$\left(\frac{x}{\sin x}\right)' = \frac{x' \sin x - x(\sin x)'}{(\sin^2 x)} = \frac{\sin x - x \cos x}{\sin^2 x}$$

Theorem: **The equation of tangent line to the curve** $\mathbf{y = f(x)}$ **at** $\mathbf{P(a, b)}$: $\mathbf{y = f'(a)(x - a) + b}$

Example Find an equation of the tangent line of the curve $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

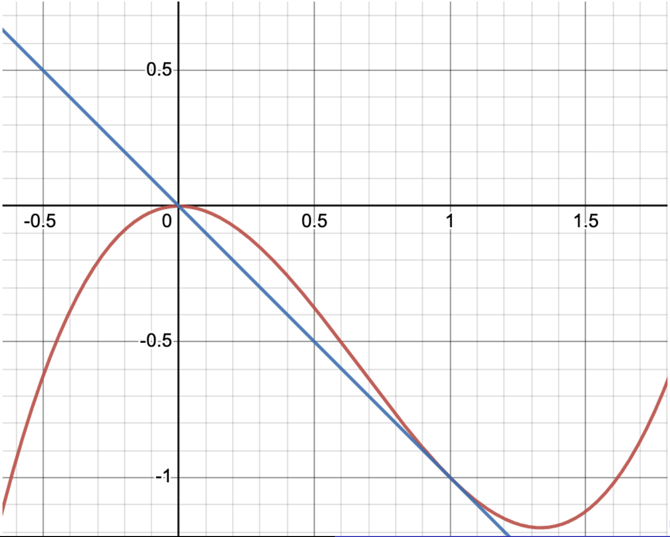
$f(x) = x^3 - 2x^2$

We have $f'(x) = 3x^3 - 4x \Rightarrow f'(1) = -1$.

The equation of tangent line is

$$y = f'(1)(x - 1) - 1 = -1(x - 1) - 1 = -x$$

$$y = x^3 - 2x^2, \quad y = -x.$$



Exercise

$$\text{Find } y'(2) \text{ for } y = \frac{x^3}{x-1}$$

A. 4 B. 2 C.-2 D.-4

Solution

$$y(x) = \frac{x^3}{x-1}$$

$$y'(x) = \frac{(x^3)'(x-1) - (x-1)'x^3}{(x-1)^2}$$

$$y'(x) = \frac{3x^2(x-1) - x^3}{(x-1)^2}$$

$$y'(x) = \frac{2x^3 - 3x^2}{(x-1)^2}$$

$$y'(2) = \frac{16-12}{1} = 4$$

Chain Rule: $(f \circ g)'(x) = f'(g(x))g'(x)$

Example

$$(\sin 4x)' = (\cos 4x)(4x)' = 4\cos 4x$$
$$\cos(x^2+2x)' = -\sin(x^2+2x)(x^2+2x)' = -(2x+2)\sin(x^2+2x)$$
$$(\sqrt{4+3x})' = \frac{1}{2\sqrt{4+3x}}(4+3x)' = 3\sqrt{1}2\sqrt{4+3x}$$
$$(\ln\ 4x)' = \frac{1}{4x}(4x)' = \frac{1}{x}$$

Exercise

Let f(x)=g(sin3x). Find f' in terms of g'.	
A	3cos3xg'(x)
B	3cos3xg'(sin3x)
C	cos3xg'(sin3x)

Exercise

Suppose h(x)=f(g(x)) and f(2)=3, g(2)=1, g'(2)=-1, f'(2)=2, f'(1)=5. Find h'(2).	
A	1
B	2
C	5
D	4
E	-5

Exercise

Let $h(x) = \sin(f(x))$.
Given that $f(0) = \pi$ and $f'(0) = 2$. Find $h'(0)$.

A.-2 B.2 C.-1 D.1

Implicit Differentiation

Example Let $x^2 + y^2 = 25$. Find $\dfrac{dy}{dx}$.

Solution

$$x^2 + y^2(x) = 25$$

$$\left(x^2 + y^2(x)\right)' = 25'$$

$$\left(2x + 2y(x)y'(x)\right) = 0$$

$$y(x)y'(x) = -x$$

$$y'(x) = \frac{x}{y(x)}$$

Exercise

Find $\dfrac{dy}{dx}$ by implicit differentiation

$$9xy + 2y - 2 = 0$$

A. $\dfrac{-9y(x+1)}{2}$ B. $\dfrac{-9y}{9xy+2}$

C. $\dfrac{-9(x+y)}{2}$ D. $\dfrac{-9y}{9x+2}$

Exercise

Find $\dfrac{dy}{dx}$ by implicit differentiation

$$\cos(xy) + 2y = 3$$

A. $\dfrac{y\sin(xy)}{2 + x\sin(xy)}$ B. $\dfrac{y\sin(xy)}{2 - x\sin(xy)}$

C. $\dfrac{\cancel{3 + y\sin(xy)}}{2 - x\sin(xy)}$ D. $\dfrac{3 + y\sin(xy)}{2 + x\sin(xy)}$

Exercise

Calculate y' from the equation

$$xy^3 + x^5y = 2x + 4y$$

A. $y' = \dfrac{y^3 + 5x^4y - 2}{4 - x^5 - 3xy^2}$

B. $y' = \dfrac{2y^3 + 5x^4y + 2}{4 - x^5 - 3xy^2}$

C. $y' = \dfrac{3y^2 + 5x^4y - 2x}{4 - x^5 - 3xy^2}$

D. $y' = \dfrac{y^3 + 5x^4y - 2x}{4 - x^5 - 3xy^2}$

Velocity Problem

Investigate the example of a falling ball.

- Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground.
- Find the velocity of the ball after 5 seconds.

Solution

Galileo's Law: distance fallen after t seconds is $4.9t^2$ (m).

Let $s(t)$ be distance fallen after t seconds then: $s(t) = 4.9t^2$

Average velocity from 5s to 5.1s:

$$\frac{\text{Distance from 5s to 5.1s}}{\text{time elapsed from 5.s to 5.1}} = \frac{s(5.1) - s(5)}{5.1 - 5.0} = 49.49m/s$$

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

Thus, (instantaneous) velocity after 5s is $v = 49m/s$

Actually, (instantaneous) velocity after 5s is $s'(5) = (9.8) \times 5 = 49$

Derivative of position function = velocity function

A particle moves along a straight line with displacement given by $s(t) = t^2 - 8t + 18$. What is the instantaneous velocity when $t = 4$?

A.8 B.4 C.2 D.0

Exercise

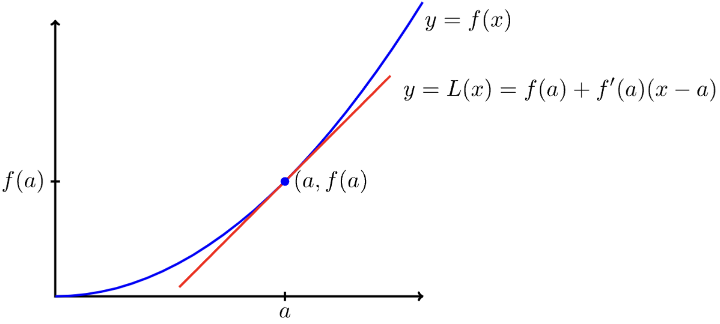
The position in meter of a particle after t seconds is modeled by the function:

$$f(t) = 2te^{t}, \text{ where } t \geq 0 .$$

At what rate, in meters per second, is the position of the particle changing at $t = 3$?

A. $8e^3$ B. $6e^3$ C. $2e^3$ D. $2 + e^3$

Linear Approximations



$f(x) \approx L(x)$ for x near a

su tuyen tinh hoa cua f(x) tai x=a

- $\mathbf{f(a) + f'(a)(x - a) \approx f(x)}$, when $\mathbf{x \approx a}$
- $\mathbf{L(x) = f(a) + f'(a)(x - a)}$ **is called the linearization of f(x) at x=a**

Approximate $\sqrt{5}$

Let $f(x) = \sqrt{x}$ and need to approximate $f(5)$.

$$\mathbf{f(a) + f'(a)(x - a) \approx f(x)}, \text{when} \quad \mathbf{a \approx x}$$

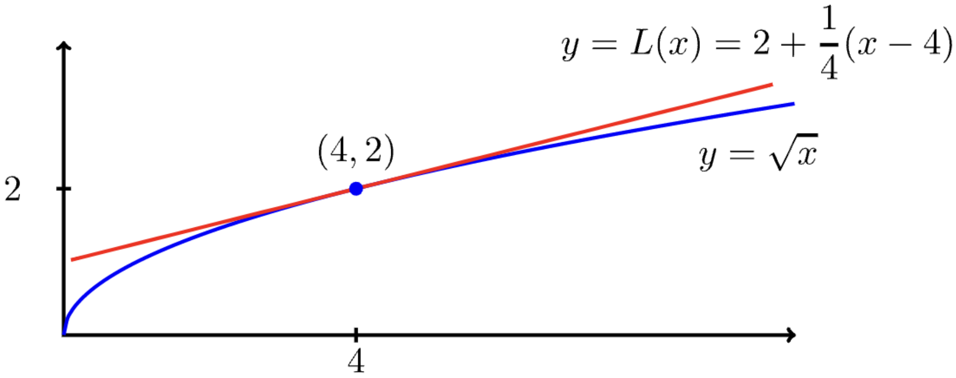
Replace x=5, $f(a) + f'(a)(5 - a) \approx f(5)$, when $a \approx 5$

$$\mathbf{\sqrt{a} + \frac{1}{2\sqrt{a}}(5 - a) \approx \sqrt{5}}, \text{when} \quad \mathbf{a \approx 5}$$

We should choose a such that a is near 5 and $\sqrt{a}$ is easy to compute. Choosing $a = 4$,

$$\mathbf{2 + \frac{1}{2.2}(5 - 4) = 2.25 \approx \sqrt{5}}$$

Linear Approximation



$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4) \text{ for } x \text{ near } 4$$

Approximate $e^{-0.1}$

Let $f(x) = e^x$ and we need to approximate $f(-0.1)$

$$\mathbf{f(a) + f'(a)(x - a) \approx f(x)}, \text{when} \quad \mathbf{a \approx x}$$

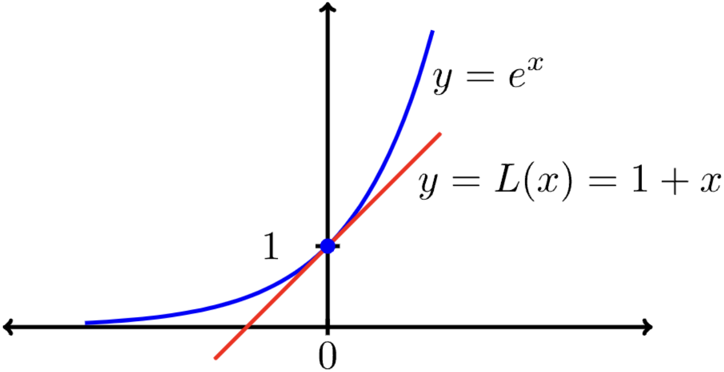
Replace $x = -0.1$, $f(a) + f'(a)(-0.1 - a) \approx f(-0.1)$, when $a \approx -0.1$

$$\mathbf{e^a + e^a(-0.1 - a) \approx e^{-0.1}}, \text{when} \quad \mathbf{a \approx -0.1}$$

We should choose a such that a is near -0.1 and e^a is easy to compute. Choosing a=0

$$\mathbf{e^0 + e^0(-0.1 - 0) = 0.9 \approx e^{-0.1}}$$

Linear Approximation



$$e^x \approx 1 + x \text{ for } x \text{ near } 0$$

Exercise

Find the linear approximation for

f'(2)= 2, f(2)=3
f(a)+f'(a)(x-a) voi a = 2

$$f(x) = \sqrt{x^3+1} \; at \; x = 2$$

A. 3x-2 B. 3x+2 C. 2x-1 D. 2x+1

Exercise

Find a linear approximation for

$$f(x) = e^{2x} \quad \text{at} \quad x = 1$$

A. $2e^2x - e^2$ B. $2e^2x + e^2$

C. $e^2x - e^2$ D. $e^2x + e^2$

Related Rates

Problem

Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ cm^3/s$
How fast is the radius of the balloon increasing when the diameter is 50 cm.

Solution

volume : the tich
rate of change of volume : van toc cua the tich
radius : ban kinh

$V(t)$: volume of the balloon at time t

$\Rightarrow V'(t)$: rate of change of volume at time t

$r(t)$: radius of the balloon at time t.

rate of change of radius : vtoc ban kinh

$\Rightarrow r'(t)$: rate of change of radius of the balloon at time t.

Relation between volume and radius of balloon

$$\mathbf{V} = \frac{4}{3}\pi \mathbf{r}^3$$

It means that, at any time t: $\mathbf{V(t)} = \dfrac{4}{3}\pi \mathbf{r^3(t)}$

Differentiating both sides: $\mathbf{V'(t)} = 4\pi \mathbf{r^2(t) r'(t)} \Rightarrow \mathbf{r'(t)} = \dfrac{\mathbf{V'(t)}}{4\pi \mathbf{r^2(t)}}$

Replacing $r(t) = \cancel{50}$: $\mathbf{r'(t)} = \dfrac{\mathbf{V'(t)}}{4\pi \cancel{50}^2} = \cancel{\dfrac{100}{4\pi 50^2}} \approx \cancel{0.0127}$ cm/s

25 25 1/25π

Exercise

If $4y^2 + 9z^2 = 36$ and $\dfrac{dz}{dt} = 4$. Find

$$\frac{dy}{dt} \quad \text{when} \quad z = \sqrt{2}$$

A. $\pm 3/\sqrt{2}$ B. $\pm 2/\sqrt{3}$ C. ± 2 D. ± 6

Exercise

If $x^2 + y^2 = 4x$ and $dy/dt = 6$. Find

dx/dt at the point $(1, \sqrt{3})$

A. $6\sqrt{3}$ B. $-6\sqrt{3}$ C. $12\sqrt{3}$ D. $-12\sqrt{3}$