Chapter 2: Limits and Continuity

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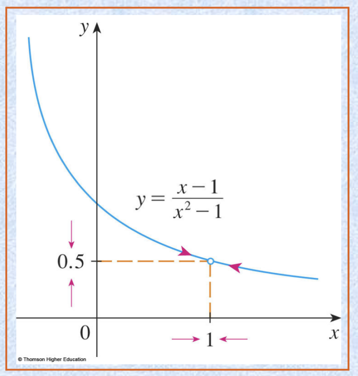
Contents

The limit of a function
(Giới hạn của một hàm số)

Definition We write $\lim_{x \to a} f(x) = L$

Heuristically, "x is close to a", then "$f(x)$ is close to L"



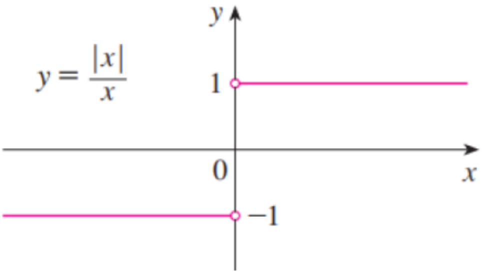
$\lim_{x \to 1} \frac{x-1}{x^2-1} = 1/2$,

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

1	0.5	1	0.5

Example

Prove that $\lim_{x \to 0} \frac{|x|}{x}$ **does not exist.**



Solution

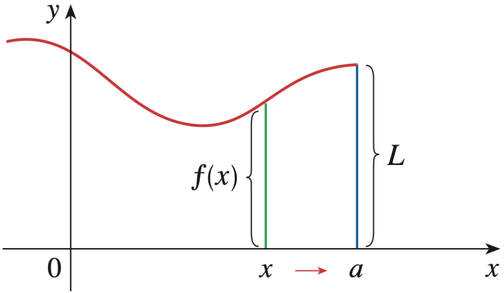
- $x \to 0$ from the right side, $f(x)$ is 1
- $x \to 0$ from the left side, $f(x)$ is -1

Limit from the left

(Giới hạn bên trái)

Definition We write $\lim\limits_{x \to a^-} f(x) = L$

Heuristically, "x is close to a, x<a" then "f(x) is close to L"

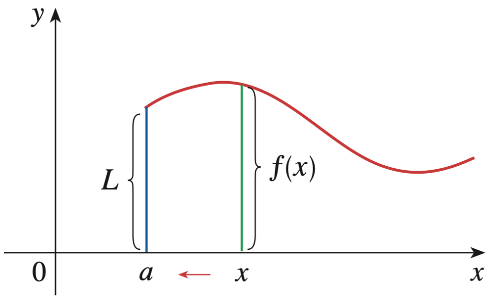


(a) $\lim\limits_{x \to a^-} f(x) = L$

Limit from the left
(Giới hạn bên phải)

Definition We write $\lim\limits_{x \to a^+} f(x) = L$

Heuristically, "x is close to a, x>a" then " f(x) is close to L"



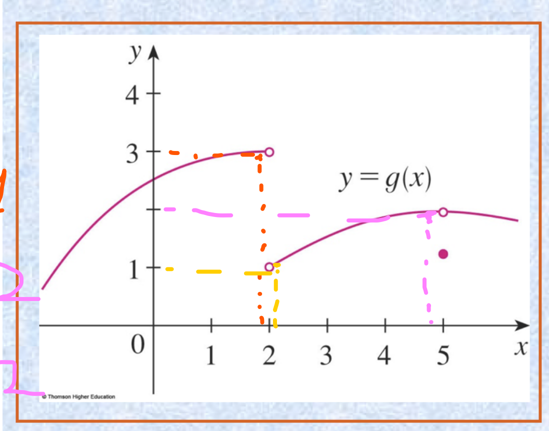
(b) $\lim\limits_{x \to a^+} f(x) = L$

Exercise

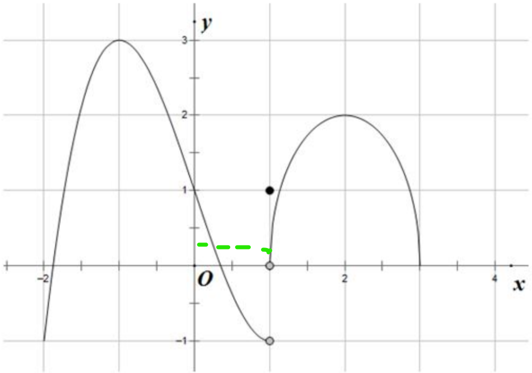
$\lim_{x\to 2^-} g(x) = 3$ $\lim_{x\to 2^+} g(x) = 1$

$\lim_{x\to 2} g(x)$ 0 tồn tại $\lim_{x\to 5^-} g(x) = 2$

$\lim_{x\to 5^+} g(x) = 2$ $\lim_{x\to 5} g(x) = 2$



The graph of a function f is shown.



Find the limit: $\lim_{x \to 1^+} f(x)$

A. 0 B. 1 C. -1 D. Does not exist

Exercise

$$\lim_{x\to 0^+} f(x),\ \lim_{x\to 0^-} f(x),\ \lim_{x\to 0} f(x),\ \lim_{x\to 1} f(x),\ \lim_{x\to 4} f(x)$$

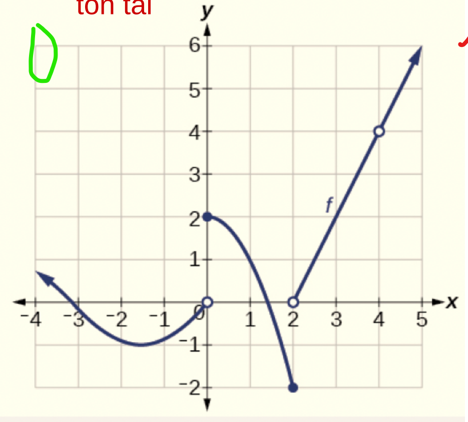
= 2

= 0

vì -> 0- khác 0+ nên ko ton tai

= 1

= 4



Limit Laws

Theorem Suppose c is a constant and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} c = c$

Example

Compute $\lim\limits_{x\to 2}(x^2 - 4x + 5)$, $\quad \lim\limits_{x\to 2}\dfrac{x^2 - 2x}{x - 2}$

Solution

$$\lim_{x\to 2}(x^2 - 4x + 5) = 2^2 - 4.2 + 5 = 1$$

$$\lim_{x\to 2}\frac{x^2 - 2x}{x - 2} = \lim_{x\to 2}\frac{x(x - 2)}{(x - 2)} = \lim_{x\to 2} x = 2$$

the x=2 thay dang 0/0
=> bien doi, rut gon

Exercise

Find

$$\lim_{x\to 3} \frac{\sqrt{x+13}-10}{x+3} = \frac{\sqrt{3+13}-10}{}$$

A. -1   B. 1/2   C. 1   D. 2   E. 4

Exercise

$\frac{\infty}{\infty}$

Find

$$\lim_{x\to\infty} \frac{x-1}{(x-2)\sqrt{x}}$$

A.positive infinity B. negative infinity C.1
D.0 E.None

Exercise

$\infty - \infty$

Find

$$\lim_{x\to\infty}[\sqrt{x+4}-\sqrt{x}]$$

A.1 B. 4 C.0 D.2 E.infinity

Exercise

Evaluate the limit, if it exists

$$\lim_{x \to 1^+} \frac{|x^2 - 4x + 3|}{x - 1}$$

A. 2 B. -2 C. 3 D. -3 E. 0

Exercise

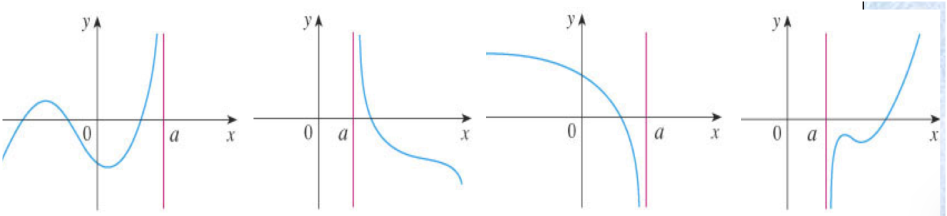
$$\text{Find } \lim_{x\to\infty} \frac{\sqrt{x^2-2x+3}-\sqrt{x+1}}{x-2}$$

A. 0 B. 1 C. -1 D. None

Vertical Asymptotes
(Tiệm cận đứng)

Definition $\mathbf{x = a}$ is called the **vertical asymptote** of $f(x)$ if we have one the following conditions

$$\lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^+} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = -\infty$$



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

Example

Find the vertical asympototes of $f(x) = \dfrac{2x}{x^2 - 5x + 6}$

Solution

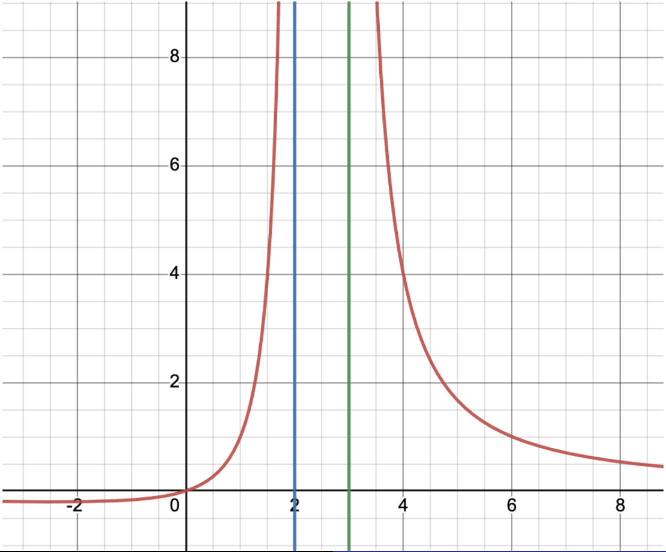
$$f(x) = \frac{2x}{x^2 - 5x + 6} = \frac{2x}{(x-2)(x-3)}$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{2x}{(x-2)(x-3)} = -\infty \qquad \frac{4}{0^+(-1)}$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} \frac{2x}{(x-2)(x-3)} = +\infty \qquad \frac{6}{1 , (0^+)}$$

$x = 2, x = 3$ are vertical symptotes of f.

Graph of $f(x) = \dfrac{2x}{x^2 - 5x + 6}$

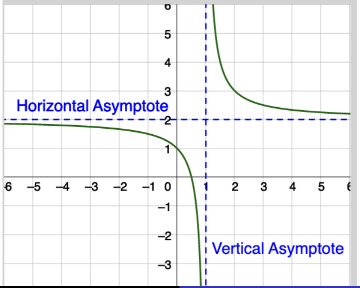


Horizontal Asymptotes
(Tiệm cận ngang)

Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

Example $y = 2$ is a horizontal asymptote



Example

Let $f(x) = 5 - \dfrac{1}{x}$. Determine the horizontal asymptote for f

Solution

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left(5 - \frac{1}{x}\right) = 5$$

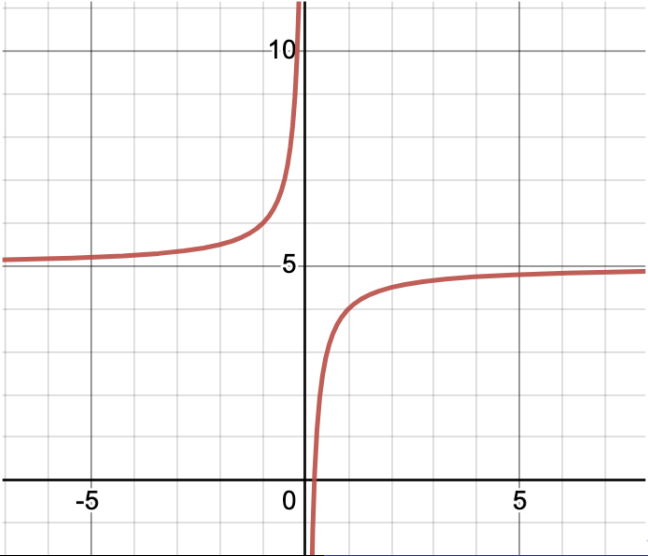
0

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(5 - \frac{1}{x}\right) = 5$$

0

Thus, f has only one horizontal asymptote $y = 5$

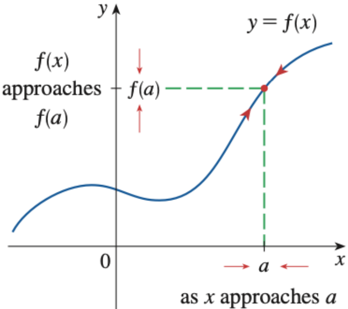
The graph of function $f(x) = 5 - \frac{1}{x}$ is given below.



Continuity
(Sự liên tục)

Definition

- A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$
- A function f is continuous on an interval if it is continuous at every point in the interval.

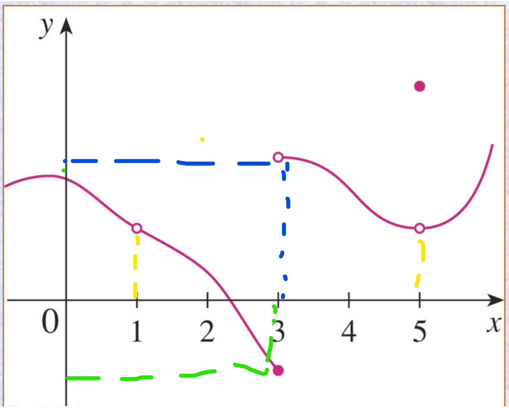


Definition f is discontinuous at a if f is not continuous at a

Geometrically, f is discontinuous at a if x is close to a, but f(x) is not close to f(a).

x gan a nhng f(x) ko gan f(a)

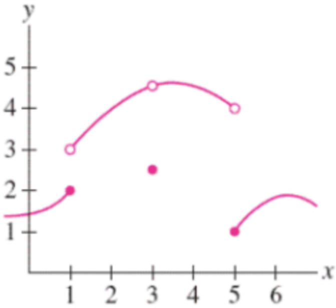
Example Which points is f continuous?



Definition

- A function f is **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$
- A function f is **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$

Example Is f continuous from the left or right at 1,3,5?



Theorem

- A function f is continuous at a if and only if f is continuous from the left and right at a
- $\lim_{x \to a} f(x) = f(a) \Leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

Exercise

Find all points of discontinuity of the function

$$f(x) = \begin{cases} 2x+1, & x < -1 \\ 2, & -1 \le x < 2 \\ x^2-2, & x \ge 2 \end{cases}$$

A.2 B.-1 C. 2;-1 D. None of them

Exercise

Find the numbers at which the function

$$f(x) = \begin{cases} \sin x, & x \leq 0 \\ 2x, & 0 < x < 3 \\ 3 - x^2, & x > 3 \end{cases}$$

is discontinuous

A.0 B.3 C.0;3 D. None of them

Exercise

Find the constant m that makes f continuous on $\mathbb{R}$

$$\mathbf{f(x)} = \begin{cases} mx^2 + 2x, x < 2 \\ x^3 - mx, x \geq 2 \end{cases}$$

Solution f is continuous on $\mathbb{R} \setminus \{2\}$. We need to find m such that f is also continuous at 2. f is continuous at 2 if

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

We compute

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (mx^2 + 2x) = 4m + 4$$

and

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^3 - mx) = 8 - 2m$$

Thus f continuous at 2 if

$$4m + 4 = 8 - 2m \Leftrightarrow m = 2/3$$

Theorem

If f, g are continuous at a and c is a constant, then the following functions are also continuous at a:

- $\mathbf{f+g}$
- $\mathbf{f-g}$
- **cf**
- **fg**
- $\dfrac{\mathbf{f}}{\mathbf{g}}$ **if** $\mathbf{g(a) \neq 0}$

Theorem

The following types of functions are continuous in their domains: mien xac dinh

- **Polynomials:** $2x+1, x^2-3x, x^5-2x^3, \ldots$ lien tuc tren R
- **Rational functions:** $\frac{x}{x-2}, \frac{x^3-x}{x^4}, \ldots$ lien tuc tren R\{2}
- **Root functions:** $\sqrt{x}, \sqrt[3]{x}, \sqrt[4]{x}, \ldots$ lien tuc tren (0-> vc)
- **Trigonometric functions:** $\sin x, \cos x, \tan x, \cot x$

sin cos lt trên R lt trên cos,sin \{0}

Exercise

12. Determine where the function is continuous

$$a. f(x) = \frac{2x^2 + x - 1}{x - 2}$$

tìm liên tuc tren dau

R\{2}

$$b. f(x) = \frac{x - 9}{\sqrt{4x^2 + 4x + 1}}$$

R\{1/2}

$$c. f(x) = ln(2x + 5)$$

lnA có nghia khi A>0
=> (-5/2 -> vc)

Exercise

3. On what intervals is the function $f(x) = \dfrac{\sqrt{4-x^2}}{x}$ continuous ?

A. $[-2,2]$ B. $(-2,2)$ C. $[-2,0)\cup(0,2]$ D. $(-\infty,-2]\cup[2,+\infty)$

Solution

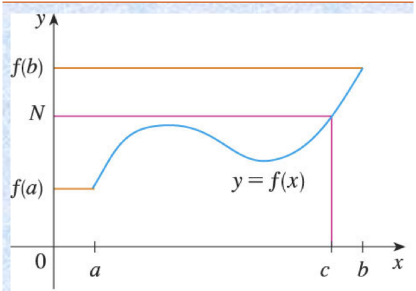
We need to determine the domain of f

$$\begin{cases} 4 - x^2 \geq 0 \\ x \neq 0 \end{cases} \Leftrightarrow \begin{cases} -2 \leq x \leq 2 \\ x \neq 0 \end{cases}$$

. Thus $D = [-2, 2] \setminus \{0\} \to C$

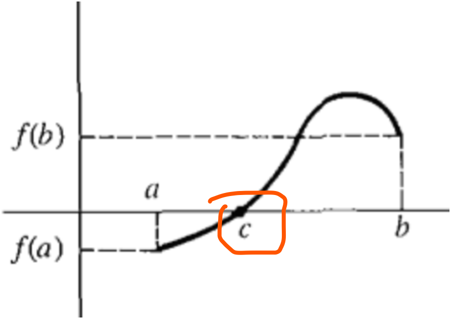
Intermediate Value Theorem
(Định lý giá trị trung bình)

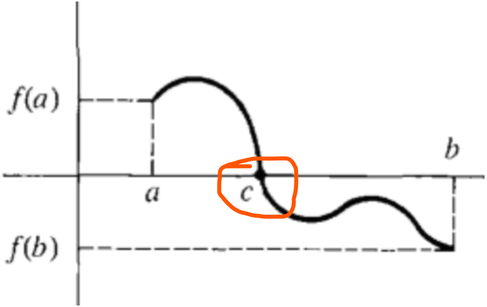
Theorem **Let** f **be continuous on the closed interval** $[a, b]$ **and let** N **be any number between** $f(a)$ **and** $f(b)$**, where** $f(a) \neq f(b)$**. Then, there exists a real number** $c \in (a, b)$ **such that** $f(c) = N$



Theorem Suppose the real function f is continuous on the closed interval [a, b] with $f(a).f(b) < 0$. Then

$$f(x) = 0 \text{ has a solution in } (a, b) \text{ or } \exists c \in (a, b) : f(c) = 0$$





Example

Prove that $\mathbf{x^2 = \cos x}$ has a solution in $(\mathbf{0}, \pi)$

Solution

$\mathbf{x^2} = \cos \mathbf{x} \Leftrightarrow \boxed{\cos x - x^2} = 0 \Leftrightarrow f(x) = 0$ with $f(x) = \cos x - x^2$

$f(0) = 0^3 - \cos 0 = 1, \quad f(\pi) = \cos \pi - \pi^2 = -1 - \pi^2$

Thus $f(0)f(\pi) < 0$ and $f(x) = 0$ has a solution in $(0, \pi)$.