Chapter 5: Integral

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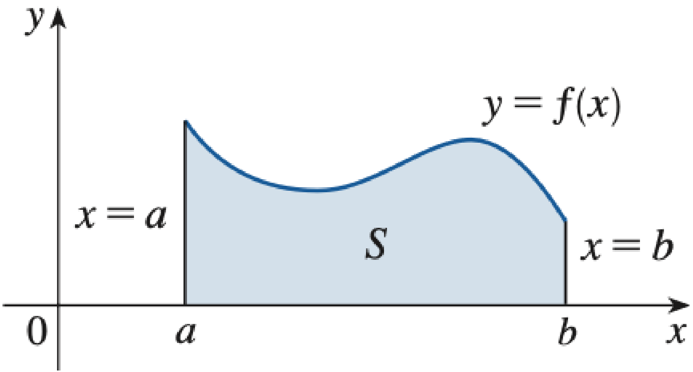
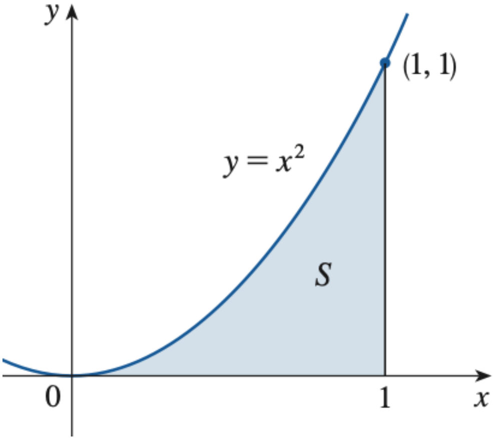
Area Problem

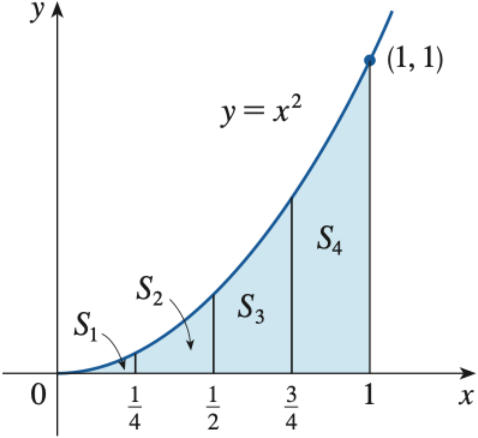
Problem Find the area of the region S that lies under the curve $y = f(x)$ from $x = a$ to $x = b$.



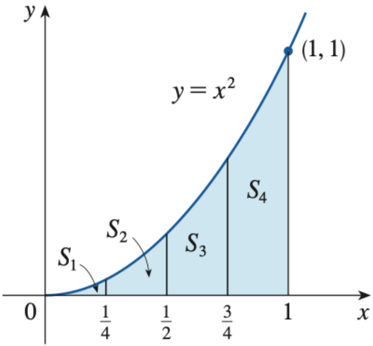
Example

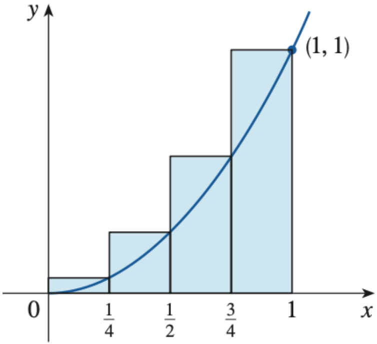
Find the area S lies under the $y = x^2$ from $x = 0$ to $x = 1$.





Divide S into four strips $\mathbf{S_1, S_2, S_3, S_4}$ by drawing $\mathbf{x = 1/4, x = 1/2, x = 3/4}$.

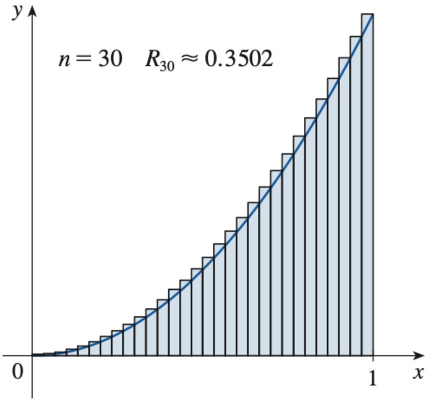


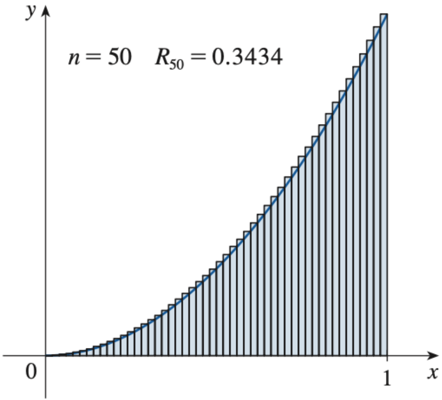


$$R_4 = \frac{1}{4}f(\frac{1}{4}) + \frac{1}{4}f(\frac{1}{2}) + \frac{1}{4}f(\frac{3}{4}) + \frac{1}{4}f(1) = \frac{1}{4}\left(\frac{1}{4^2} + \frac{1}{2^2} + \frac{3^2}{4^2} + 1^2\right) =$$

0.4685

The more intervals we divide, the better approximations we get.



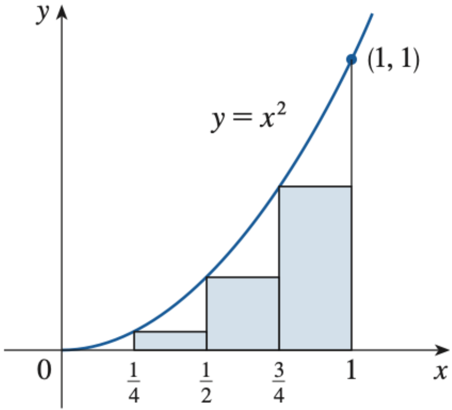


Problem **How many intervals should we divide to have exact approximation?**

$$\mathbf{S = \lim_{n\to\infty} R_n}$$

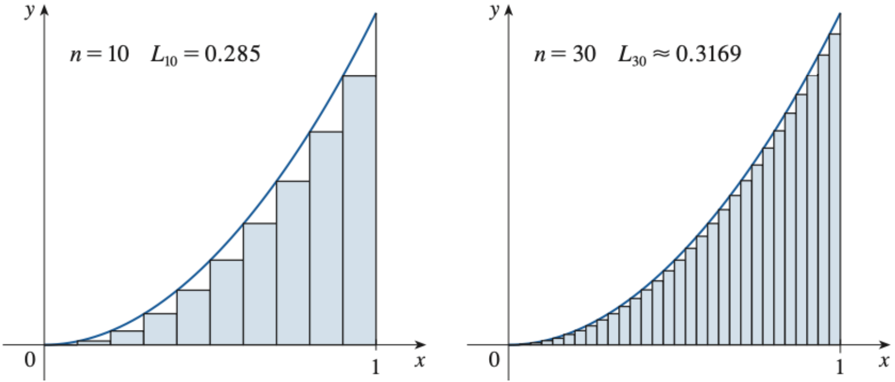
$$\begin{aligned}
R_n &= \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \ldots + \frac{1}{n} f\left(\frac{n}{n}\right) \\
&= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\
&= \frac{1}{n}.\frac{1}{n^2}(1^2 + 2^2 + \ldots + n^2) \\
&= \frac{1}{n^3}.\frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)
\end{aligned}$$

Thus $S = \lim_{n\to\infty} R_n = 1/3$

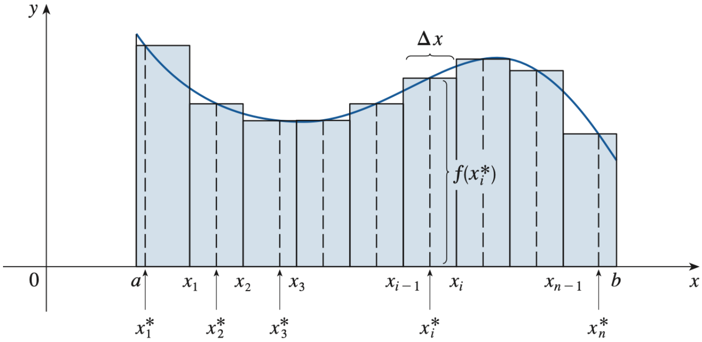


$$L_4 = \left(\frac{1}{4}f(0) + \frac{1}{4}f(\frac{1}{4}) + \frac{1}{4}f(\frac{2}{4}) + \frac{1}{4}f(\frac{3}{4})\right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{4^2} + \frac{2^2}{4^2} + \frac{3^2}{4^2}\right) = 0.21875$$


y
$n = 10 \quad L_{10} = 0.285$
0
1
x
y
$n = 30 \quad L_{30} \approx 0.3169$
0
1
x

- **Sample points**



$$\Delta x = \frac{b-a}{n}$$

$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definite integral

Definition **Definite integral of f from a to b is**

$$\int_a^b f(x)dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x}_{\text{Riemann Sum}} \quad \text{(provided this limit exists)}$$

If it does exist, we say f is integrable on $[a, b]$.

Note

$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$$

Exercise

Estimate the area under the graph of

$$f(x) = 25 - x^2$$

on $[0, 5]$ using 5 rectangles and right endpoints

A.50 B.60 C.70 D.55

Exercise

Express the limit as a definite integral over $[0, 1]$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \cos^2(2\pi x_i^*)\Delta x$$

$A.\ \int_0^1 \cos^2(2\pi)dx$ $\quad B.\ \int_0^1 \cos^2(\frac{2\pi}{x})dx$

$C.\ \int_{-1}^1 \cos^2(2\pi x)dx$ $\quad D.\ \int_0^1 \cos^2(2\pi x)dx$

Exercise

Use the Right-endpoint rule with n=4 to estimate the value of the integral

$$\int_1^3 f(x)dx$$

x	1	1.5	2	2.5	3
f(x)	0.31	0.5	0.36	1.35	2.04

A. 2.145 B. 1.620 C. 4.290 D. 3.240

Exercise

Find the Riemann sum for

$$f(x) = 3x^2 - 5, 0 \leq x \leq 2,$$

with four equal subintervals, taking the sample points to be left endpoints.

A.9.5 B. -9.5 C. 4.75 D. -4.75

Exercise

Estimate the area under the graph $f(x) = x + 1/x$ over $[1, 9]$ using 4 rectangles and left endpoints.

A. 49.57 B. 42.19 C.40 D. 35.35

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$
2. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
5. If $m \leq f(x) \leq M\ \forall a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Exercise

Let

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 2 \\ x^2, & x \geq 2 \end{cases}$$

Find $\int_0^3 f(x)dx$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$$\int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_{x=0}^{x=1} = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

$$\int_0^{\pi/4} \cos 2x dx = \left.\frac{1}{2}\sin 2x\right|_{x=0}^{\pi/4} = \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0 = \frac{1}{2}$$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

:displacement of the object during the time period from t_1 to t_2.

- If we want to calculate the distance the object travels during a time interval, then the distance is

$$\int_{t_1}^{t_2} |v(t)|dt = \text{total distance traveled}$$

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$.

a. Find the displacement of the particle during the time period $1 \leq t \leq 4$.

b. Find the distance traveled during this time period.

Solution

a. Displacement is

$$\int_1^4 v(t)dt = \int_0^1 (t^2 - t - 6) = \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Bigg|_{t=1}^{t=4} = -9/2$$

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leqslant 0$ on the interval $[1, 3]$ and $v(t) \geqslant 0$ on $[3, 4]$. Thus, from Equation 3, the distance traveled is

$$\begin{aligned}
\int_1^4 |v(t)|\,dt &= \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt \\
&= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt \\
&= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\
&= \frac{61}{6} \approx 10.17 \text{ m}
\end{aligned}$$

Exercise

A particle moves along a line so that its velocity at time t is

$$v(t) = 3t^2 - 2t - 5 \text{ (measured in meters per second)}$$

Find the distance of the particle during the time $2 \leq t \leq 5$

A.78 B.89 C.81 D.87

Exercise

A particle moves along a line so that its velocity at time t is

$v(t) = 6t^2 - 2t - 3$ (measured in meters per second).

Find the displacement of the particle during the time $0 \le t \le 7$.

A. 616 B. 321 C. 661 D. 116

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Theorem **If f is continuous on $[a, b]$ then the function g defined by**

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$

Example

$$\left(\int_0^x \sqrt{t^2+1}dt\right)' = x^2 + 1$$

$$\left(\int_1^x \sin(2t)dt\right)' = \sin 2x$$

Exercise

Find $\dfrac{dy}{dx}$ for

$$y = \int_1^x \frac{1}{\sqrt{16 - t^2}} dt$$

A. $\dfrac{1}{\sqrt{16 - x^3}}$ B. $\dfrac{1}{\sqrt{16 - x}}$

$C.\dfrac{1}{\sqrt{16 - x^2}}$ $D.\dfrac{x}{\sqrt{16 - x^2}}$

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f(u(x))u'(x)$$

Example

$$\left(\int_1^{x^2}(2t-1)dt\right)' = (2x^2-1)(x^2)' = 2x(2x^2-1)$$

$$\left(\int_0^{2-x}\sin t dt\right)' = \sin(2-x)(2-x)' = -\sin(2-x)$$

Exercise

Find

$$\frac{d}{dx}\int_{3}^{1+x^2} \ln t dt$$

A. $2x\ln(1+x^2)$

B. $2x/(1+x^2)$

C. $\ln(1+x^2) - \ln 3$

D. None of the others

Exercise

Suppose

$$g(x) = \int_1^{x^2} \sin(t-1)dt$$

Find $g'(x)$

$A. g'(x) = \sin(x-1)$

$B. g'(x) = \sin(x^2-1)$

$C. g'(x) = 2x\sin(x^2-1)$

$D. g'(x) = 2x\cos(x^2-1)$

Exercise

Find $\dfrac{dy}{dx}$ for

$$y = \int_1^{\sqrt{x}} t dt$$

A. x B. $x - 1$ C. $1/2$ D.1 E. $1/x$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f(u(x))u'(x) - f(v(x))v'(x)$$

Example

Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_{2x}^{x^2} t^2 dt$

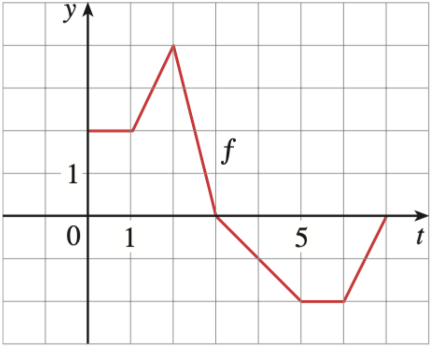
Solution

$y'(x) = (x^2)^2(x^2)' - (2x)^2(2x)' = 2x^5 - 8x^2$

Exercise

Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

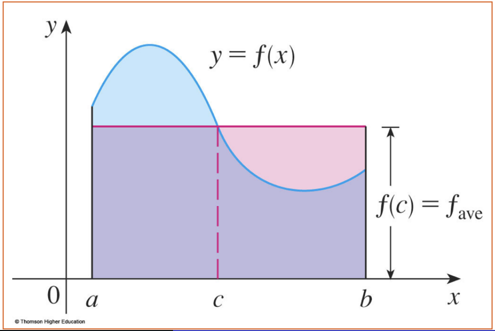
(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.



MEAN VALUE THEOREM

The geometric interpretation of the Mean Value Theorem for Integrals is as follows.

- For ‘positive’ functions *f*, there is a number *c* such that the rectangle with base [*a*, *b*] and height *f*(*c*) has the same area as the region under the graph of *f* from *a* to *b*.



MEAN VALUE THEOREM

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

that is, $$\boxed{f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$$\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

Average value of f on $[a, b] := \frac{1}{b-a}\int_a^b f(x)dx$

Find the average value of the function $y=x^2 -2x$ on the interval [0,3].
0
3/2
-1/2
1
None of the others.

Exercise

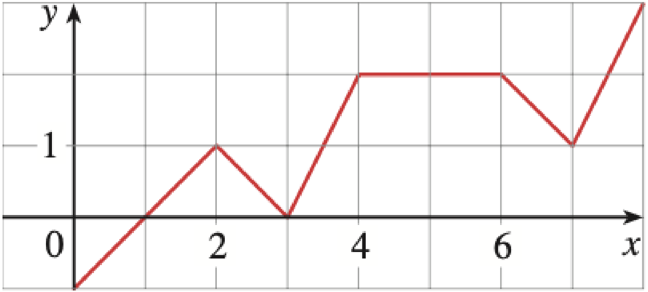
In a certain city the temperature (in °F) t hours after 9 AM was modeled by the function

$$T(t) = 50 + 14 \sin \frac{\pi t}{12}$$

Find the average temperature during the period from 9 AM to 9 PM.

Exercise

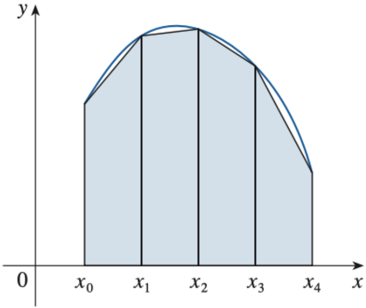
Find the average value of f on $[0, 8]$.



Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{\Delta}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = \dfrac{b-a}{n}, \quad x_i = a + i\Delta x$



Example

Use the Trapezoidal Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x} dx$

Solution

$$\int_1^2 \frac{1}{x} dx = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$\approx 0.695635$$

$$\int_1^2 \frac{1}{x} dx = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= 0.1\left(\frac{1}{\ } + \frac{2}{\ } + \frac{2}{\ } + \frac{2}{\ } + \frac{2}{\ } + \frac{1}{\ }\right)$$

Exercise

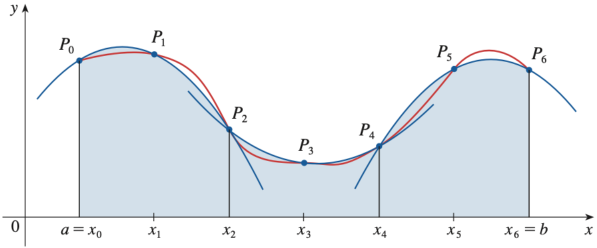
Use the Trapezoidal Rule with $n = 5$ steps to approximate the integral

$$\int_1^6 f(x)dx$$

x	1	2	3	4	5	6
f(x)	3.2	1.6	2.4	3.8	4.4	1.3

A. 15.55 B. 31.1 C. 28.90 D. 14.45

Simpson Rule



$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \\ + 4f(x_3) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = \dfrac{b-a}{n}$.

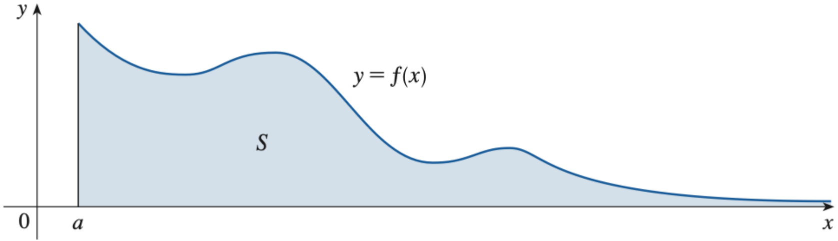
Exercise

Using Simpson's Rule with n=8 to approximate $\int_0^8 f(x)dx$

x	0	1	2	3	4	5	6	7	8
f(x)	3	1	2	5	3	8	7	6	2

A. 36.3 B. 30 C. 40 D. 28 E. 34.5

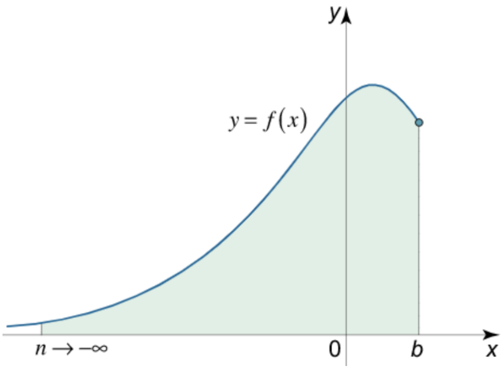
Improper Integral of type 1: infinite intervals



If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty \mathbf{f(x)dx} = \lim_{t\to\infty} \int_a^t \mathbf{f(x)dx}$$

provided this limit exists.



If $\int_t^b dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} f(x)dx$$

provided this limit exists.

Example

Find $\displaystyle\int_1^\infty \frac{1}{x}dx$ and $\displaystyle\int_1^\infty \frac{1}{x^2}dx$.

Solution

$$\begin{aligned}
\int_1^\infty \frac{1}{x}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x}dx \\
&= \lim_{t\to\infty} \ln|x| \Big|_{x=1}^{x=t} \\
&= \lim_{t\to\infty} (\ln t - \ln 1) \\
&= \lim_{t\to\infty} \ln t = \infty
\end{aligned}$$

Example

Find $\int_1^{\infty} \frac{1}{x^2} dx$

Solution

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2} dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2} dx \\ &= \lim_{t\to\infty} -\frac{1}{x}\bigg|_{x=1}^{x=t} \\ &= \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) \\ &= 1\end{aligned}$$

Convergence and Divergence of Improper Integral

Definition The improper integrals $\int_a^\infty f(x)dx$ and $\int_\infty^b f(x)dx$ are called

- **convergent** if the corresponding limit exists (a finite number)
- **divergent** if the corresponding limit does not exist

Example

$\int_1^\infty \frac{1}{x}dx$ is divergent, $\int_1^\infty \frac{1}{x^2}dx$ is convergent.

Example

Prove that $\int_1^\infty \frac{1}{x^p}dx$ is convergent for all $p > 1$ and disvergent for all $p \leq 1$

Solution

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}dx = \lim_{t\to\infty}\frac{1}{1-p}x^{1-p}\Big|_{x=1}^{x=t} = \lim_{t\to\infty}\frac{1}{1-p}(t^{1-p}-1)$$

If $p > 1$: $\lim_{t\to\infty}\frac{1}{1-p}(t^{1-p}-1) = \frac{1}{p-1}$

If $p < 1$: $\lim_{t\to\infty}\frac{1}{1-p}(t^{1-p}-1) = \infty$

Example

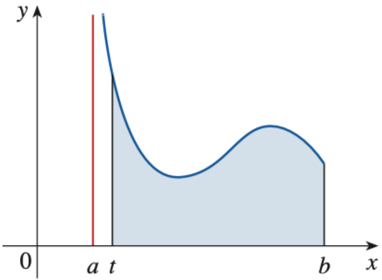
Find $\int_0^{\infty} e^{-x} dx$

Solution

$$\int_0^{\infty} e^{-x} = \lim_{t\to\infty} \int_0^t e^{-x} dx = \lim_{t\to\infty} -e^{-x}\Big|_{x=0}^{x=t}$$

$$= \lim_{t\to\infty} (-e^{-t} + 1) = 1$$

Improper Integral of Type 2

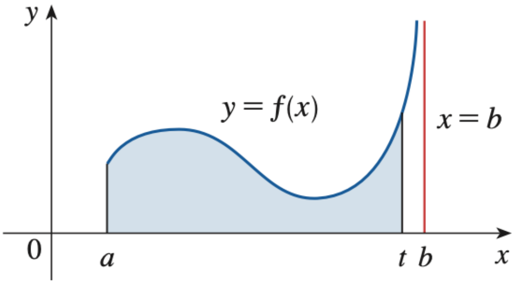


If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

If this limit exists (as a finite number)

Improper Integral of Type 2



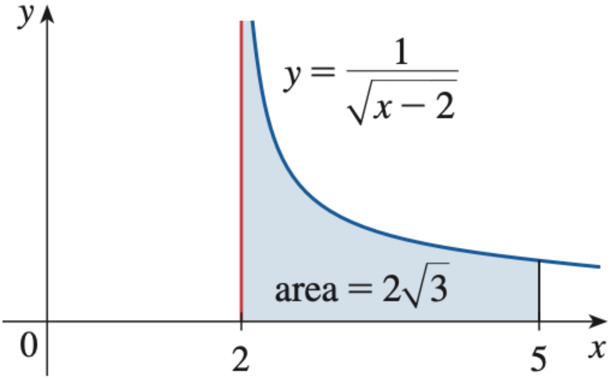
If f is continuous on $[a, , b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

If this limit exists (as a finite number)

Example

Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.



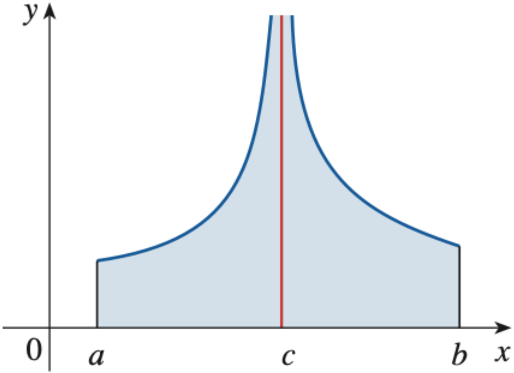
y
$y = \frac{1}{\sqrt{x-2}}$
area $= 2\sqrt{3}$
0
2
5
x

Exercise

Find $\int_0^1 \frac{1}{\sqrt{x}} dx$

Definition If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



Exercise

Evaluate $\int_0^3 \frac{1}{x-1} dx$.

Definition Improper integral $\int_a^b f(x)dx$ is called

- **convergent** if the corresponding limit exists
- **divergent** if this limit does not exist

Exercise

Which of the following improper integrals converge?

$$i) \quad \int_0^1 \frac{dx}{(1-x)^{3/2}} \qquad ii) \quad \int_0^1 \frac{dx}{x^{1/2}}$$

A.i B. ii C. Both D. None

Exercise

Determine whether the improper integrals converge or diverge

$$I = \int_0^1 \frac{1}{\sqrt{x}} dx, \qquad J = \int_1^{\infty} \frac{1}{x^2} dx$$

A. I diverges, J converges

B. I converges, J diverges

C. Both diverge

D. Both converge

Exercise

Evaluate

$$\int_0^3 \frac{1}{\sqrt{3-x}}dx$$

A. diverges B. $2\sqrt{3}$ C. $-2\sqrt{3}$ D. $4\sqrt{3}$

The Substitution Rule

Theorem If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Exercise

Find

1. $\int xe^{x^2}dx$

2. $\int \sqrt{2x+1}dx$

3. $\int \dfrac{x}{\sqrt{1-4x^2}}dx$

4. $\int \sin^2 x \cos x dx$

Theorem If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Exercise

Find

1. $\int_1^3 (2x-1)^3 dx$
2. $\int_3^5 \frac{1}{(x-2)^2}$
3. $\int_1^e \frac{\ln x}{x} dx$

Integration by Parts

To compute $\int f(x)g(x)dx$, we put

$$\begin{cases} \mathbf{u = f(x) \Rightarrow du = f'(x)dx} \\ \mathbf{v = \int g(x)dx} \end{cases}$$

Then

$$\int \mathbf{f(x)g(x)dx = uv - \int vdu}$$

Exercise

Find

1. $\int x \sin x dx$

2. $\int \ln x \; dx$

Exercise

Find $\int x^3 e^{x^2} dx$

i) $e^{x^2} - x^2 + C$
ii) $e^{x^2}(x^2 - 1)/2 + C$
iii) $e^{x^2}(x^2 - 1) + C$
iv) $e^x - x^3 + C$

Exercise

Evaluate $\int \frac{(lnx)^3}{x} dx$

i. $\frac{1}{4x}(ln\ x)^4 + C$

ii.$4(ln\ x)^4 + C$

iii.$1/2(ln\ x)^2 + C$

iv.$1/4(ln\ x)^4 + C$

Integration by Parts 2

To compute $\int_a^b f(x)g(x)dx$, we put

$$\begin{cases} \mathbf{u = f(x)} \Rightarrow du = f'(x)dx \\ \mathbf{v = \int g(x)dx} \end{cases}$$

Then

$$\mathbf{\int_a^b f(x)g(x)dx = uv\Big|_a^b - \int_a^b vdu}$$

Exercise

Find

1. $\int_0^{\pi/2} x \sin x dx$
2. $\int_0^1 xe^{2x} dx$
3. $\int_1^e x \ln x dx$