Chapter 3: Application of Derivative

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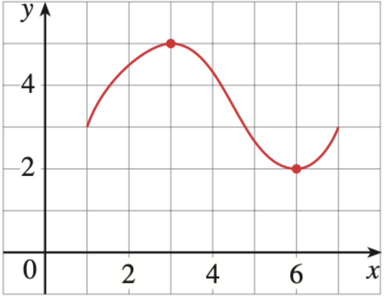
Ngày 8 tháng 3 năm 2023

Maximum and Minimum Values



- The highest point in the graph is $(3, 5)$.
- The largest value of f is $f(3) = 5$.
- We say that $f(3) = 5$ is the absolute maximum.
- The smallest value of f is $f(6) = 2$.
- We say that $f(6) = 2$ is the absolute minimum.

Absolute Maximum and Absolute Minimum Value (Giá trị cực đại tuyệt đối và cực tiểu tuyệt đối)

Definition

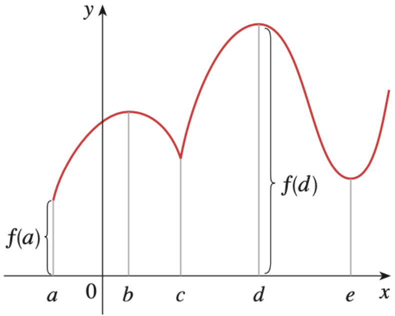
Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $\mathbf{f(c) \geq f(x)\ \forall x \in D}$
- **absolute minimum** value of f on D if $\mathbf{f(c) \leq f(x)\ \forall x \in D}$

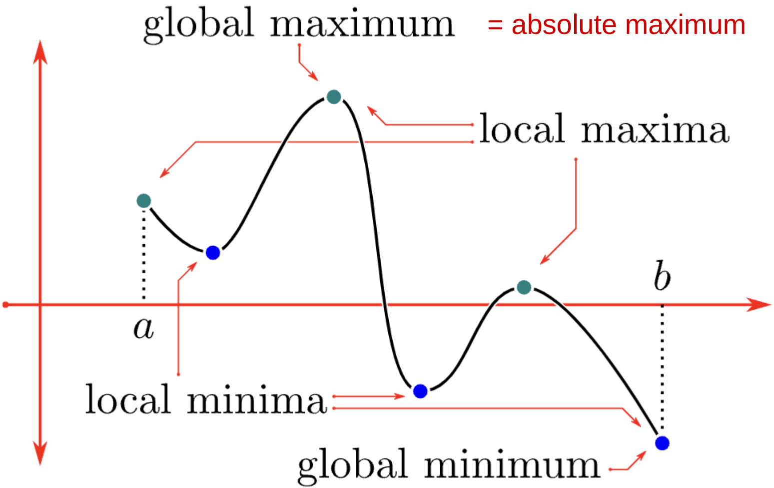
Local Maximum and Local Minimum Value
(Giá trị cực đại địa phương và giá trị cực tiểu địa phương)

Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c



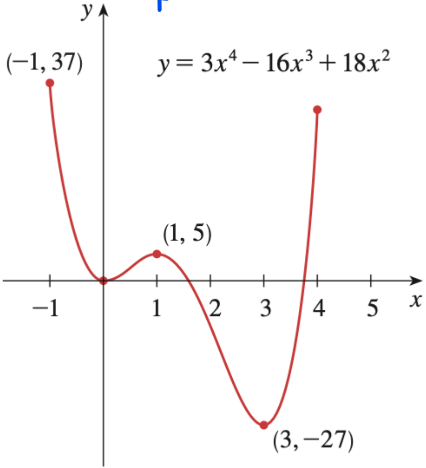
dia phuong giong nhu
la " diem " cd,ct


global maximum
= absolute maximum
local maxima
b
a
local minima
global minimum

Exercise

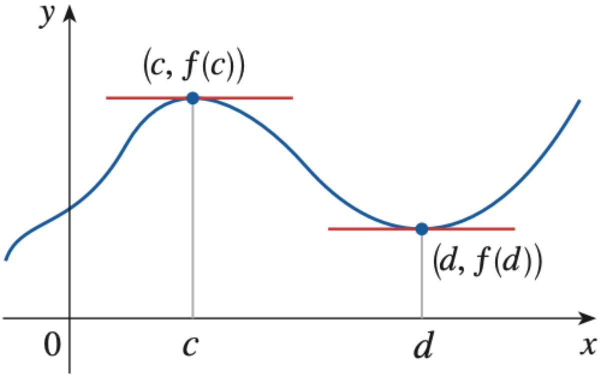
Find absolute maximum(minimum) and local maximum (minimum).

37   −27   5   0

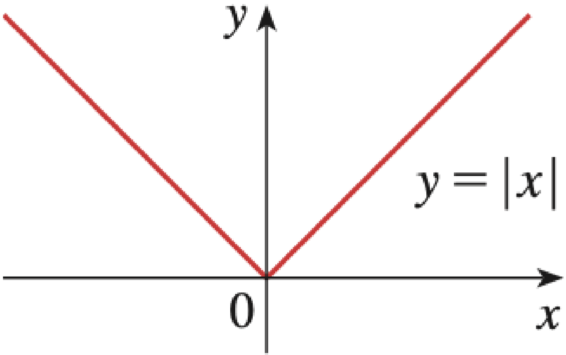


Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $\mathbf{f'(c) = 0}$



Let $f(x) = |x|, f(0) = 0$ is a minimum value, but $f'(0)$ does not exists.



Critical numbers

giai pt f'(x) = 0

(Điểm tới hạn)

Definition **A critical number of a function f is a number c in the domain of f such that one of the following hods.**

- $\mathbf{f'(c) = 0}$
- **$\mathbf{f'(c)}$ does not exist.**

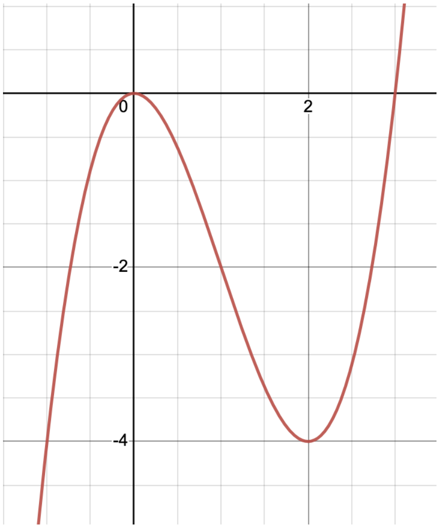
Example Find critical numbers of $\mathbf{f(x) = x^3 - 3x^2}$.

Solution

$f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \Leftrightarrow x = 0$ or $x = 2$

f has two critical numbers 0 and 2.

Graph of $y = x^3 - 3x^2$



Exercise

Find critical numbers of

$$g(x) = \sqrt[3]{x^2 - x}$$

A.1 B.0 C.0;1 D.0; 1; 1/2 E.1/2

Closed Interval Method

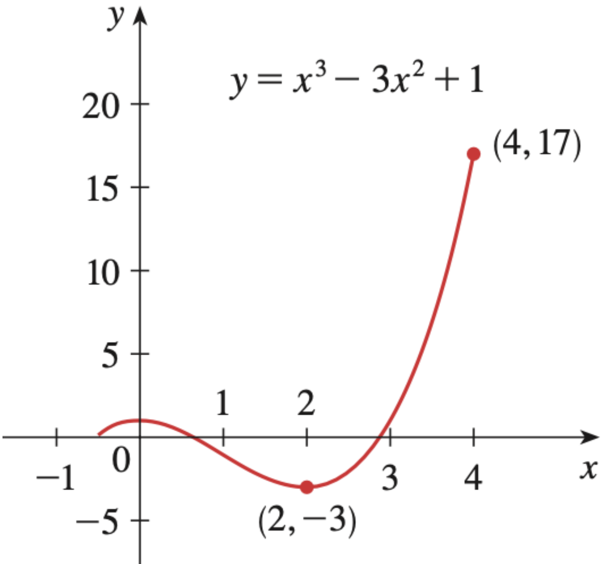
To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$

1. **Find the values of f at the critical numbers of f in $[a, b]$**
2. **Find the values of f at the endpoints of the interval** 2 dau mut
3. **The largest value from 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.**

Exercise

Find the absolute minimum and maximum values of the

$$f(x) = x^3 - 3x^2 + 1 \quad \frac{-1}{2} \leq x \leq 4$$


y
$y = x^3 - 3x^2 + 1$
20
15
10
5
−5
(4, 17)
1
2
0
−1
3
4
x
(2, −3)

Exercise

Find the absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 3x + 1, \quad \text{on} \quad [0,2]$$

A. Absolute maximum: 2, absolute minimum: 1

B. Absolute maximum: 3, absolute minimum: 1

C. Absolute maximum: 3, absolute minimum: 0

D. Absolute maximum: 3, absolute minimum: 2

Example

$a \cdot b = 24$

Find two positive numbers whose product is 24 and whose the sum of 2 times the first and 3 times the second is a minimum.

$2a + 3b$ min

A. 12,2   B. 8,3   C.4,6   D.6,4

t o : thay a,b vô thy min thì chn

Exercise

Find an equation of the line through the point $(4, 5)$ that cuts off the least area from the first quadrant.

ct góc phn t th 1 to din tich nh nhat

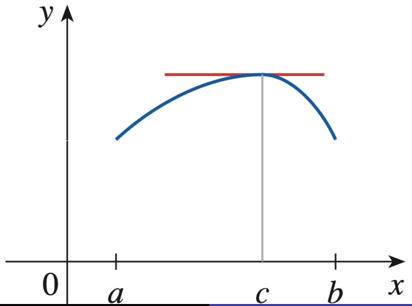
A. $\dfrac{x}{8} - \dfrac{y}{10} = 0$ B. $\dfrac{x}{8} - \dfrac{y}{10} = 1$ C. $\dfrac{x}{8} - \dfrac{y}{10} = -1$

Rolle's Theorem

Let f be a function that satisfies:

- **f is continuous on the closed interval $[a, b]$**
- **f is differentiable on the open interval (a, b)**
- **$\mathbf{f(a) = f(b)}$**

Then there is a number $\mathbf{c \in (a, b)}$ such that $\mathbf{f'(c) = 0}$



nh lý giá tr trung bình

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

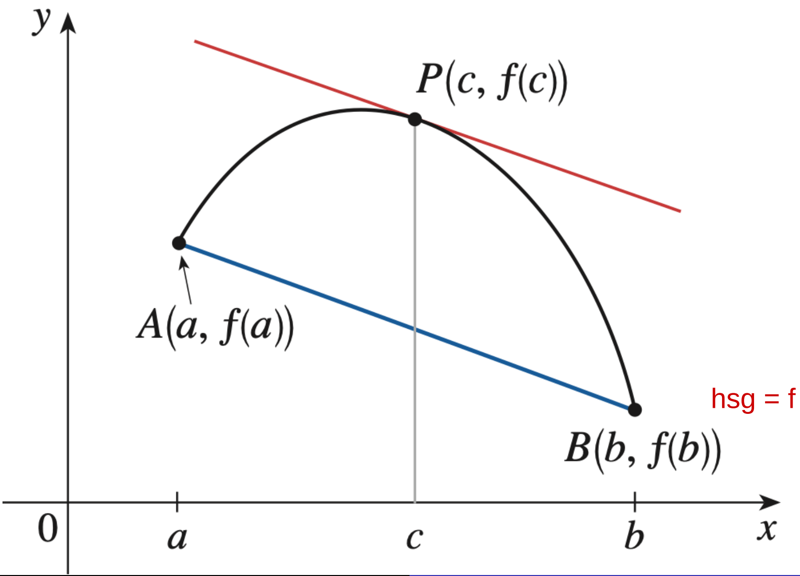
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$


y
P(c, f(c))
A(a, f(a))
hsg = f
B(b, f(b))
0
a
c
b
x

Example

Suppose that $f(0) = -3$ and $f'(x) \leq 5 \quad \forall x \in \mathbb{R}$.
How large can $f(2)$ possibly be?

A.5 B.6 C.7 D.8

Exercise

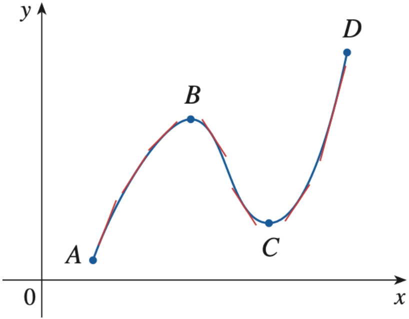
Suppose $f(1) = 3$ and $7 \le f'(x) \le 10$ for all x.

How small can $f(5)$ possible be?

A. 21 B. 43 C. 31 D. None of the

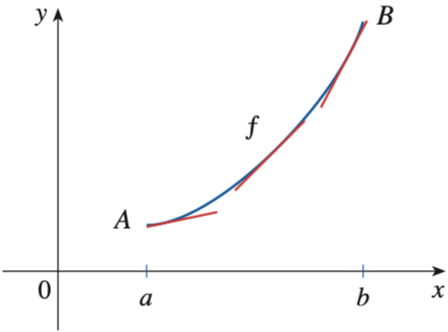
Increasing/Decreasing Test

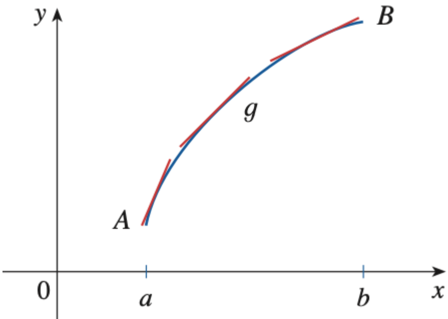
- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval



Concave Upward/Downward

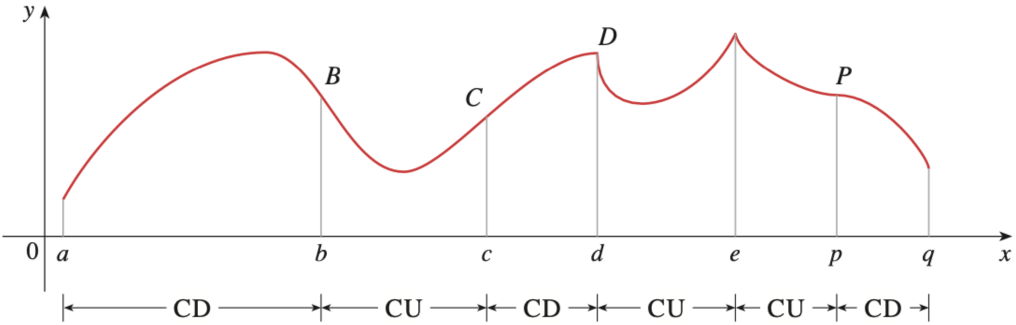
- **The curve lies above the tangents and f is called concave upward on (a, b)**
- **The curve lies below the tangents and f is called concave downward on (a, b)**





Concavity Test

- If $f''(x) > 0 \; \forall x \in I$, then the graph of f is **concave upward** on I
- If $f''(x) < 0 \; \forall x \in I$, then the graph of f is **concave downward** on I



Exercise

Determine where the function

$$f(x) = x^3 + 3x^2 - x - 24$$

is concave up and where it is concave down.

(i) Concave up on $(-\infty, \infty)$

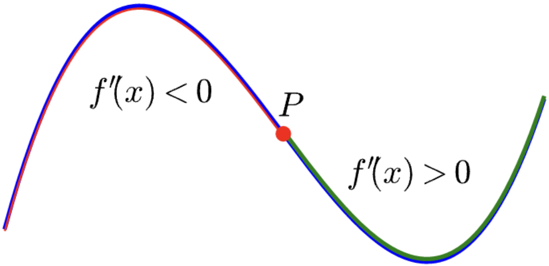
(ii) Concave down on $(-1, \infty)$ and concave up on $(-\infty, -1)$

(iii) Concave up on $(-1, \infty)$ and concave down on $(-\infty, -1)$

(iv) Concave down $(-\infty, \infty)$

Inflection point
(Điểm uốn)

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the **curve changes from concave upward to concave downward or vice versa.**

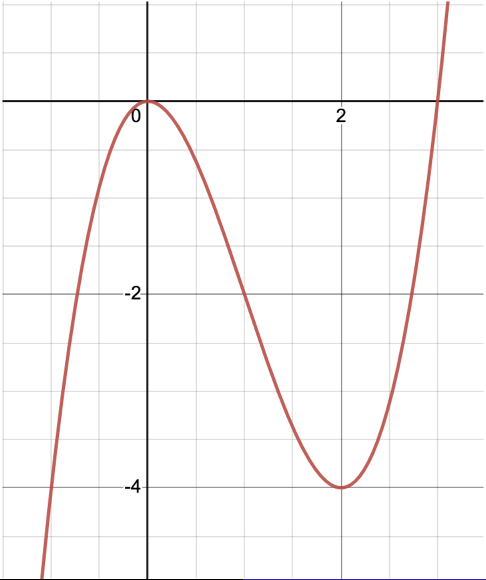


Method to find inflection points

- **Find x_0 such that $\mathbf{f''(x_0) = 0}$**
- **Check x near x_0, $f''(x)$ changes its sign.**

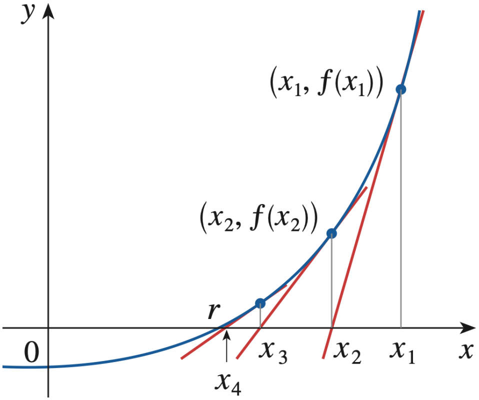
Example **Find inflection points of the curve $y = x^4 - 4x^3$**

Graph of $y = x^4 - 4x^3$



Newton Methods

Newton method is an algorithm to **approximate the solutions of functions**



$$\mathbf{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Proof

The tangent line at $(x_1, f(x_1))$ is

$$\mathbf{y = f'(x_1)(x - x_1) + f(x_1)}$$

Since $(x_2, 0)$ is on this tangent line, then

$$\mathbf{0 = f'(x_1)(x_2 - x_1) + f(x_1)}$$

Thus

$$\mathbf{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

Exercise

Starting with $x_1 = 2$, find the third approximation x_3 to the solution of the equation $x^3 - 2x - 5 = 0$.

Solution

Let $f(x) = x^3 - 2x - 5, \quad f'(x) = 3x^2 - 2$

$$\mathbf{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}}$$

With $n = 1$

$$\mathbf{x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} = 2.1}$$

With $n = 2$ $\quad \mathbf{x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} \approx 2.0946}$

Exercise

Use Newton's method with the initial approximation $x_1 = 1$ to find x_3, the third approximation to the root of the equation

$$x^5 - 10 = 0$$

Round your answer to 4 decimal places.

A. 2.4341 B. 2.8000 C. 2.2725 D. 1.9952

Antiderivatives
(Nguyên hàm)

Definition A function F is called an **antiderivative** of f on an interval **I** if $\mathbf{F'(x) = f(x)}$ for all x in **I**.

Notation

$$\int \mathbf{f(x)dx = F(x) + C}$$

Example

$$\int x dx = \frac{x^2}{2} + C$$

$$\int \cos x dx = \sin x + C$$

Table of Antidifferentiation Formulas

$\int 0\,dx = C$	$\int a^x dx = \dfrac{a^x}{\ln a} + ,\ (a > 0,\ a \neq 1)$
$\int dx = x + C$	$\int \cos x\,dx = \sin x + C$
$\int x^\alpha dx = \dfrac{1}{\alpha+1} x^{\alpha+1} + C\ (\alpha \neq -1)$	$\int \sin x\,dx = -\cos x + C$
$\int \dfrac{1}{x} dx = \ln\lvert x\rvert + C$	$\int \dfrac{1}{\cos^2 x} dx = \tan x + C$
$\int e^x dx = e^x + C$	$\int \dfrac{1}{\sin^2 x} dx = -\cot x + C$

Exercise

Find $f(x)$ if $f'(x) = x\sqrt{x}$ if $f(1) = 3$

Exercise

A particle moves along a line so that its velocity at time t is

$v(t) = 6t^2 - 2t - 3$ (measured in meters per second).

Find the displacement of the particle during the time $0 \le t \le 7$.

A. 616 B. 321 C. 661 D. 116

Solution

Let $s(t)$ be de displacement of the particle at time t. Then

$$\mathbf{s'(t) = v(t) = 6t^2 - 2t - 3}$$

Then

$$\mathbf{s(t) = 2t^3 - t^2 - 3t}$$

The displacement of the particle during the time $0 \leq t \leq 7$ is

$$\mathbf{s(7) - s(0) = 2.7^3 - 7^2 - 3.7 = 661}$$