Chapter 2: Basic Structures: Sets, Functions, Sequences, Sums and Matrices

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Sets

Definition

- A set is an unordered collection of objects, called elements or members of the set.
- We write $a \in A$: a is an element of the set A.
- The notation $a \notin A$:a is not an element of the set A.

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Example

The set V of all vowels in the English alphabet: $V = \{a, e, i, o, u\}$.
Elements of V are a, e, i, o, u.

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Example

The set V of all vowels in the English alphabet: $V = \{a, e, i, o, u\}$.
Elements of V are a, e, i, o, u.
We can write $a \in V, b \notin V, f \notin V$.

Example

The set of positive integers less than 100 can be denoted by $A = \{1, 2, 3, ..., 99\}$

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or $A = \{x | x \text{ is an positive integer less than } 100\}$

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Exercise

List the members of these sets

a) $A = \{2, a, b, c\}$

b) $B = \left\{\{2\}, 1, 3\right\}$

c) $C = \left\{\{1\}, \{\{1\}\}\right\}$

d) $\{x | \text{x is a real number such that } x^2 = 1\}$

e) $\{x | x \text{ is a positive integer less than } 12\}$

f) $\{x | x \text{ is the square of an integer and } x < 100\}$

Exercise

2. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in R \mid x \text{ is an integer greater than } 1\}$

b) $\{x \in R \mid x \text{ is the square of an integer}\}$

c) $\{2, \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

e) $\{\{2\}, \{2, \{2\}\}\}$

f) $\{\{\{2\}\}\}$

Equality of two sets

Definition

Two sets are equal if and only if they have the same element. Therefore, A and B are equal if and only if $\forall x(x \in A \longleftrightarrow x \in B)$ is T.

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Example

The sets $\{1, 3, 5\} = \{5, 3, 1\}$ are equal.

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Example

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Empty set (Tập rỗng)

Definition

The empty set $\emptyset$ is a set has no elements.

Be careful: $\emptyset \neq \{\emptyset\}$!!!

Subsets (Tập hợp con

Definition

The set A is a subset of B $(A \subseteq B)$ if and only if every element of A is also an element of B.
Therefore, $A \subseteq B$ if and only if $\forall x(x \in A \longrightarrow x \in B)$ is T.

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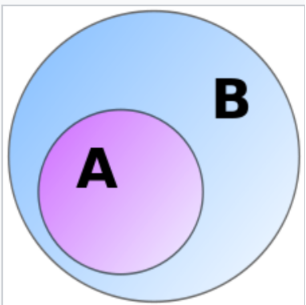

Example

$\{1,3,5\} \subseteq \{1,3,5,6,7\}$

Example

C: the set of all odd positive integers less than 10

D: the set of all positive integers less than 10.

Thus $C \subseteq D$.

Theorem

For every set S,

$$i)\ \emptyset \subseteq \mathbf{S}$$
$$ii)\ \mathbf{S} \subseteq \mathbf{S}$$

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Proof

i) To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \longrightarrow x \in S)$ is T.

We have "$x \in \emptyset$" is F because $\emptyset$ has no element. Thus "$x \in \emptyset \longrightarrow x \in S$" is T.

Therefore $\forall x(x \in \emptyset \longrightarrow x \in S)$ is True.

Exercise

List all the subsets of $A = \{1, 2\}$

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Solution

$\emptyset \subseteq \{1, 2\}$

$\{1\} \subseteq \{1, 2\}$

$\{2\} \subseteq \{1, 2\}$

$\{1, 2\} \subseteq \{1, 2\}$

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$\{1, 2\} \subseteq \{1, 2\}$

All subsets of A are: $\emptyset$, $\{1, 2\}$, $\{1\}$, $\{2\}$.

Exercise

List all the subsets of $B = \left\{ \emptyset, \{\emptyset\} \right\}$

Exercise

List all the subsets of $B = \left\{\emptyset, \{\emptyset\}\right\}$

Solution

All subsets of B are: $\emptyset, \left\{\emptyset, \{\emptyset\}\right\}, \{\emptyset\}, \left\{\{\emptyset\}\right\}$.

Exercise

3. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$ b) $\emptyset \in \{0\}$ c) $\{0\} \subset \emptyset$ d) $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$ f) $\{0\} \subset \{0\}$ g) $\{\emptyset\} \subseteq \{\emptyset\}$

Exercise

4. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ b) $\{x\} \subseteq \{x\}$ c) $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$ e) $\emptyset \subseteq \{x\}$ f) $\emptyset \in \{x\}$

The Size of a Set

Definition

Let S be a set.

- The cardinality of S, denoted by $|S|$, is the number of distinct elements in S.
- If the number of distinct elements is finite then S is called a finite set.

Example

Let $A = \{1, 2, 3, 4, 5\}$.Then $|A| = 5$.

Example

Let $B = \{1, 1, 2, 2, 3\}$.Then $|B| = 3$.

Example $|\emptyset| = 0$

Exercise

5. What is the cardinality of each of these sets?

a) $\{a\}$ b) $\{\{a\}\}$ c) $\{a, \{a\}\}$ d) $\{a, \{a\}, \{a, \{a\}\}\}$

6. What is the cardinality of each of these sets?

a) $\emptyset$ b) $\{\emptyset\}$ c) $\{\emptyset, \{\emptyset\}\}$ d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Definition

A set is said to be infinite (vô hạn) if the number of distinct elements is infinite.

Example The set of positive integers is infinite.

Example The set of rational numbers is infinite.

Power Sets

Definition

Given a set S, the power set of S is the set of all subsets of the set S.

The power set of S is denoted by $P(S)$.

Example

What is the power set of the set $\{1, 2\}$?

Solution

$$P(\{1,2\}) = \left\{\emptyset, \{1,2\}, \{1\}, \{2\}\right\}.$$

Exercise

Find $P(\{1, 2, 3\})$.

Theorem

Số các tập hợp con của tập A bằng $2^{\text{số phần tử của } A}$

$$|P(A)| = 2^{|A|}$$

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Example Let $C = \{1, 2, 2, a, a\}$

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Example Let $C = \{1, 2, 2, a, a\}$
Then $|P(C)| = 2^{|C|} = 2^3 = 8.$

Definition

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B = \{(a, b) | a \in A, b \in B\}$

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Example

Let $A = \{1, 2\}$ and $B = \{3, 4\}$.
Then $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Theorem

Số phần tử khác nhau của tập hợp $A \times B$ = *số phần tử khác nhau của tập* $A\times$ *số phần tử khác nhau của tập* B

$$|A \times B| = |A|.|B|$$

Example

$A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$

Theorem

Số phần tử khác nhau của tập hợp $A \times B$ = *số phần tử khác nhau của tập* $A\times$ *số phần tử khác nhau của tập* B

$$|A \times B| = |A|.|B|$$

Example

$A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$
$|A \times B| = |A| \times |B| = 3.4 = 12$
$|P(A \times B)| = 2^{|A\times B|} = 2^{|A|.|B|} = 2^{12}$

Definition

The Cartesian products of the sets $A_1, A_2, ..., A_n$ is the set of ordered $n-$ tuples $(a_1, a_2, ..., a_n)$, where $a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n$.

Example Given $A = \{0, 1\}, B = \{1, 2\}, C = \{2, 3\}$
$A \times B \times C$ consists of all ordered triples (a, b, c) where $a \in A, b \in B, c \in C$
$A \times B \times C = \{(0, 1, 2), (0, 1, 3), (0, 2, 2), (0, 2, 3)$
$(1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 3)\}$

Notation:

$A^2 = A \times A$

$A^3 = A \times A \times A$

Example

$A = \{1, 2\}$

$A^2 = \{(1, 1), (1, 2), (2, 2), (2, 1)\}$

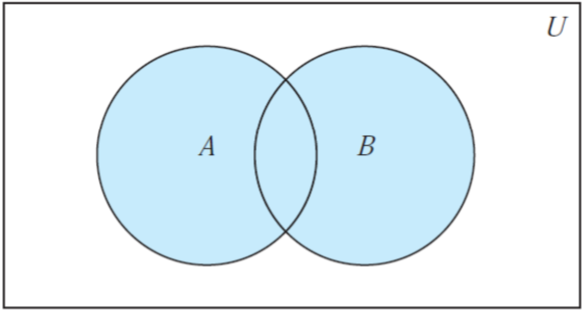
Exercise

Let $A = \{1, 2, 3, 2, a, 1, a\}$ and $B = \{a, b, c\}$

Find $|P(A^3)|$ and $|P(A \times B)|$

Definition

Let A and B be sets. The union of the sets A and B is
$A \cup B = \{x | x \in A \text{ or } x \in B\}$



$A \cup B$ is shaded.

Example

$A = \{1, 2\} \quad B = \{3, 4\}$.

$A \cup B =$

Example

$A = \{1, 2\} \qquad B = \{3, 4\}.$

$A \cup B = \{1, 2, 3, 4\}.$

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Example

$A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A \cup B =$

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Example

$A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$

Example

Let A be the set of all boy students in this class.

B be the set of all girl students in this class.

$A \cup B$:

Example

$A = \{1, 2\} \quad B = \{3, 4\}$.

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Example

$A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$

Example

Let A be the set of all boy students in this class.

B be the set of all girl students in this class.

$A \cup B$: set of all students in this class

Definition

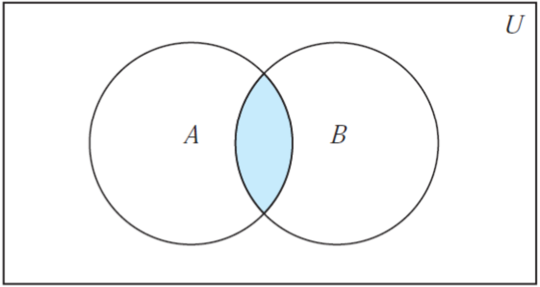
The intersection of A and B is

$A \cap B =$

Definition

The intersection of A and B is

$A \cap B = \{x | x \in A \text{ and } x \in B\}$



$A \cap B$ is shaded.

FIGURE 2 **Venn Diagram of the Intersection of A and B.**

Example

Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A \cap B =$

Example

Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A \cap B = \{2, 3\}$

Example

Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A \cap B = \{2, 3\}$

Example

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$

$A \cap B =$

Example

Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A \cap B = \{2, 3\}$

Example

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$

$A \cap B = \emptyset$

Example

Let A be the set of all computer science majors in in FPTU

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Let A be the set of all computer science majors in in FPTU

Let B be the set of all graphic design majors in FPTU

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$A \cap B$

Example

Let A be the set of all computer science majors in in FPTU

Let B be the set of all graphic design majors in FPTU

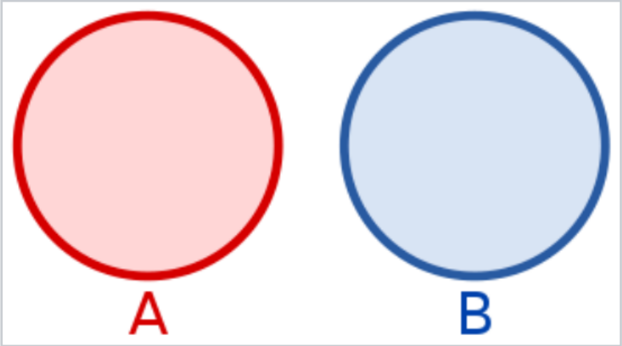
$A \cap B$

is the set of all student in FPTU who are joint majors in computer science and graphic design.

Disjoint (Rời nhau)

Definition

Two sets are called disjoint if their intersection is the empty set.



Two disjoint sets

Example Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$
A and B are disjoint?

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A and B are disjoint?
Solution
Yes! $A \cap B = \emptyset$

Example

Let A: "the set of all odd integer"
and B : "the set of all even integer".
A and B are disjoint ?

Example

Let A: "the set of all odd integer"

and B : "the set of all even integer".

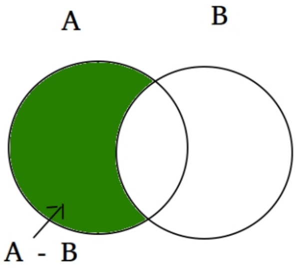
A and B are disjoint ?

Solution

Yes! $A \cap B = \emptyset$

Definition

The difference of A and B is
$A - B = \{x | x \in A \text{ and } x \notin B\}$



Example

$A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$

$A - B =$

Example

$A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$

$A - B = \{5\}$

Example

$A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$

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Example

A: the set of all positive integer less than 100

B: the set of all odd positive integer less than 100

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Example

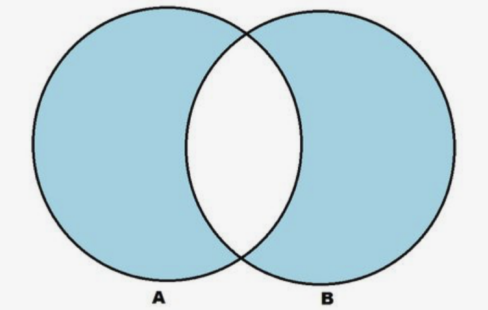
A: the set of all positive integer less than 100

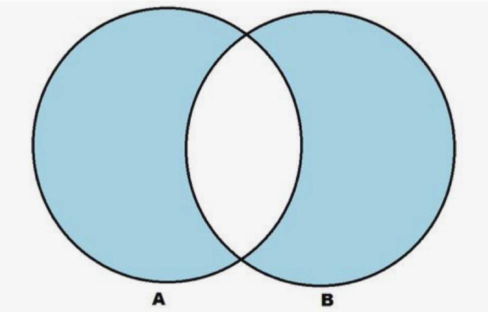
B: the set of all odd positive integer less than 100

$A - B$: The set of all even positive integer less than 100.

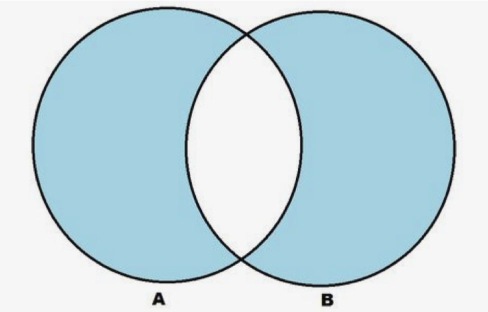
Definition

The symmetric difference of A and B, denoted $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.





Example 1: $A = \{1, 3, 5, 6, 7, 8\}$ and $B = \{2, 3, 4, 7, 9, 10\}$
$A \oplus B =$



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$A \oplus B = \{1, 2, 4, 5, 6, 9, 10\}$

Exercise

Let A be the set of students who live within one mile of school
Let B be the sets of students who walk to classes.
Describe the students in each of these sets
$A \cap B$, $A \cup B$, $A - B$, $B - A$.

Exercise

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$.
Find $A \cap B$, $\quad B \cup A$, $\quad A - B$, $\quad B - A$, $\quad A \oplus B$.

Definition

Let U be the universal set.

- The complement of the set A is the set $\overline{A} = U \setminus A$

Example Let A : the set of all positive integers and U : the set of all integer. What is $\overline{A}$?

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Solution

$\overline{A} = U \setminus A$: the set of all integers but not positive

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Solution

$\overline{A} = U \setminus A$: the set of all integers but not positive

or $\overline{A}$: the set of all nonpositive integers

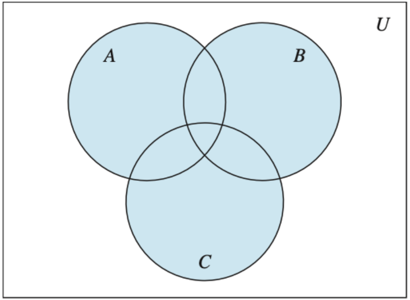
Set Idententes

TABLE 1 **Set Identities.**

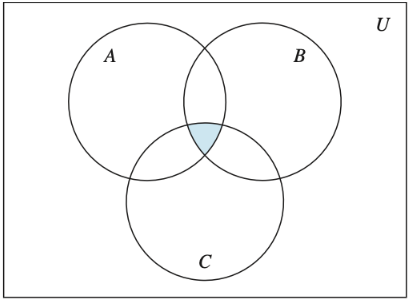
Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Generalization Unions and Intersections



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

Example

$A = \{0, 2, 4, 6, 8\}$
$B = \{0, 1, 2, 3, 4\}$
$C = \{0, 3, 6, 9\}$
What are $A \cap B \cap C$ and $A \cup B \cup C$?

Example

Let U be the universal set. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{2, 3, 5, 8\}$
The bit string that represents A is 0110100100

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$B = \{1, 2, 3, 4, 5\}$

Example

Let U be the universal set. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 3, 5, 8\}$

The bit string that represents A is 0110100100

$B = \{1, 2, 3, 4, 5\}$

The bit string that represents B is 1111100000

Exercise

Given a universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and
$A = \{1, 3, 5, 7, 9\}$
Find the bit strings which represent the set A and $U - A$.

Exercise

Given a universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
The bit string for $A = \{1, 3, 4, 5, 6, 9\}$ is 101111001
The bit string for $B = \{2, 3, 6, 7, 8\}$ is 011001110
Use bit strings to find the union and intersect of the set?

Exercise

U2-Q11

Suppose that the universal set is U = {a, b, c, d, e}.

Given the set represented by strings

A = "1 1 1 0 0"

B = "0 1 0 1 0"

List all elements in the set A - B.

A. $\{a, b, c\}$
B. $\{a, b, d\}$
C. $\{a, c\}$
D. $\{a, b\}$

Functions (Hàm số, Ánh xạ)

Definition

- Let A and B be nonempty sets.

A function f from A to B is an assignment such that each element of A is assigned to exactly one element of B.

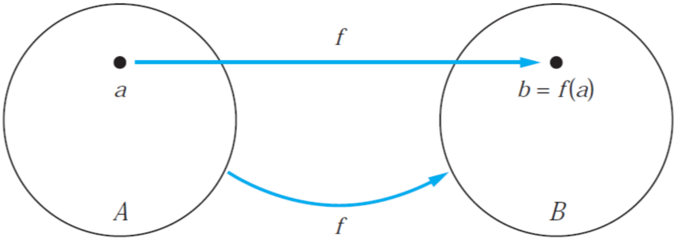
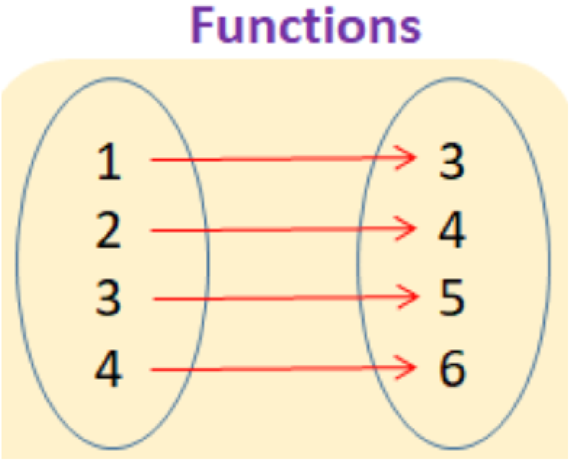
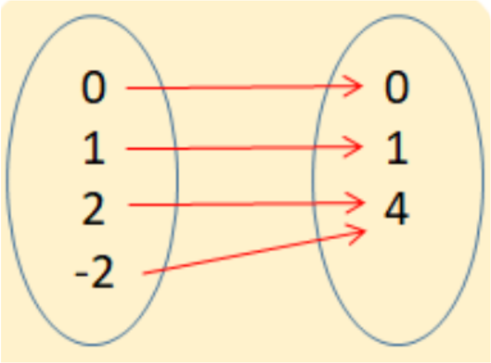


FIGURE 2 The Function f Maps A to B.

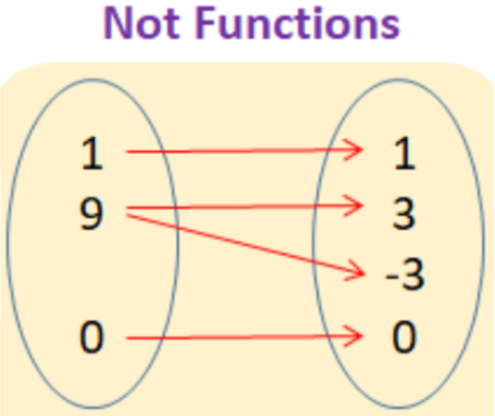
Example



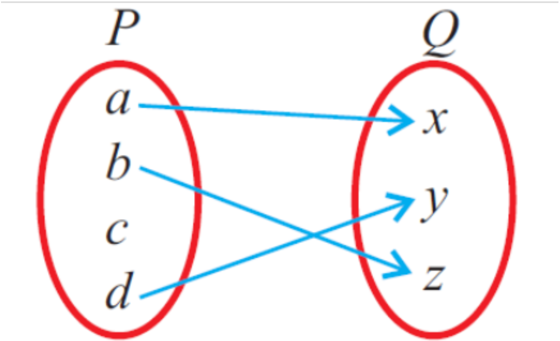
Example



Example



Example

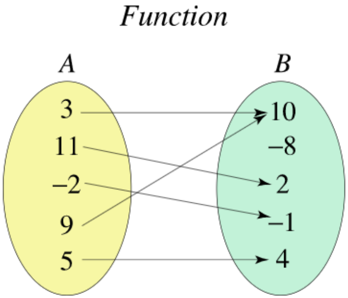


Definition

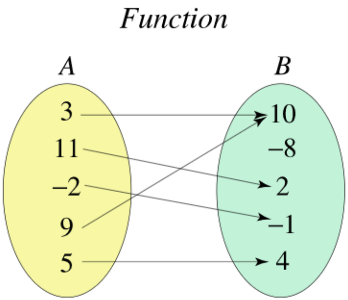
If f is a function from A to B, we say that

- A is the domain of f
- B is the codomain of f.
- If $f(a) = b$, we say that b is the image of a, and a is the preimage of b.
- The range, or image, of f is the set of all images of elements of A.

Example

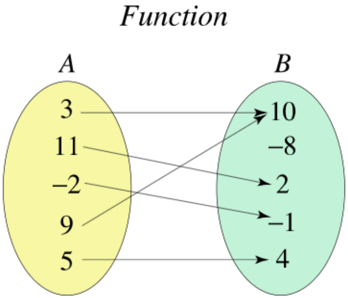


Example



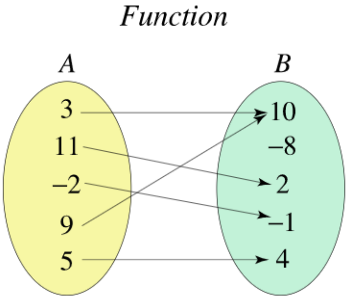
Let f be the function.
The domain of f is

Example



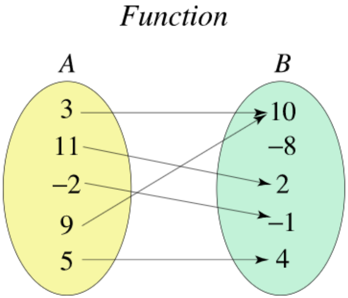
Let f be the function.
The domain of f is $\{3, 11, -2, 9, 5\}$
The codomain of f is

Example



Let f be the function.
The domain of f is $\{3, 11, -2, 9, 5\}$
The codomain of f is$\{10, -8, 2, -1, 4\}$
The images of f is

Example



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Using a formula to define a function

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f a function?

Using a formula to define a function

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f a function?

Solution

$\forall x (x \in \mathbb{Z} \to f(x) \in \mathbb{Z})$ is T?

Using a formula to define a function

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f a function?

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f is a function.

Example Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = \dfrac{x}{2}$. Is f a function?

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f is not a function.

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Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{x-1}$. Is f a function?

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Solution

- $\forall x(x \in \mathbb{R} \to f(x) \in \mathbb{R})$ is T?
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Example

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{x-1}$. Is f a function?

Solution

- $\forall x (x \in \mathbb{R} \to f(x) \in \mathbb{R})$ is T?
- $\forall x \left(x \in \mathbb{R} \to \dfrac{1}{x-1} \in \mathbb{R} \right)$ is T?

f is not a function

Exercise

What are functions?

$$f : \mathbb{R} \to \mathbb{R}\colon f(x) = \frac{2x}{(x-1)^2}$$

F : $x = 1$

$$g : \mathbb{R} \to \mathbb{Z}\colon g(x) = 2x$$

F

$x \in \mathbb{R} \to g(x) \notin \mathbb{Z}$

$$h : \mathbb{Z} \to \mathbb{R}\colon h(x) = \frac{1}{x - \sqrt{2}} + 2x + 7$$

Exercise

Determine whether if f is a function from Z to R if

$$\text{a) } f(n) = -n$$

$$\text{b) } f(n) = \sqrt{n^2+1}$$

$$\text{c) } f(n) = \frac{1}{n^2-4}$$

One-To-One functions (Hàm đơn ánh)

Definition

- Let f be a function from A to B.
- f is said to be one-to-one or an injunction if and only if for every $a, b \in A$ such that $a \neq b$ implies that $f(a) \neq f(b)$.

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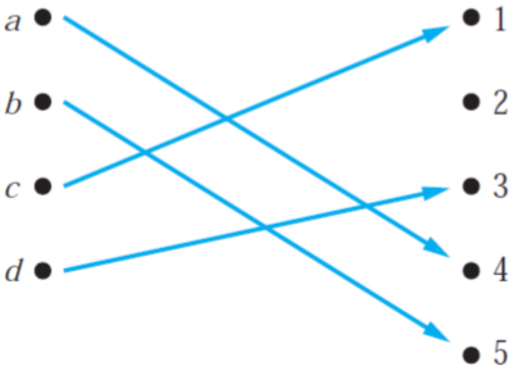
- f is said to be one-to-one or an injunction if and only if for every $a, b \in A$ such that $a \neq b$ implies that $f(a) \neq f(b)$.
- f is said to be one-to-one if and only if for every $a, b \in A$ such that $f(a) = f(b)$ implies that $a = b$.
- f is one-to-one from A to B if

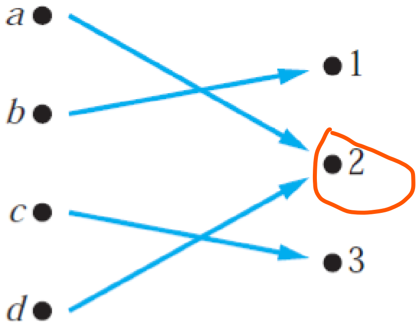
$$\forall a, b \in A\Big(f(a) = f(b) \rightarrow a = b\Big) \text{ is } T$$

Example Is the following function is one-to-one?



T

Example Is the following function is one-to-one?



F

Example Is $f(x) : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 5$ one-to-one?

$f(x)$

Example Is $f(x) : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 5$ one-to-one?

Solution

f is one-to-one if $\forall a, b \in \mathbb{Z}\Big(f(a) = f(b) \to a = b\Big)$ is T?

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T

Example Is $f(x):\mathbb{Z}\to\mathbb{Z},\ f(x)=2x+5$ one-to-one?

Solution

f is one-to-one if $\forall a,b\in\mathbb{Z}\Big(f(a)=f(b)\to a=b\Big)$ is T?

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Example Is $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ is one-to-one?

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Example Is $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ is one-to-one?

Solution

f is one-to-one if $\forall a, b \in \mathbb{Z}\Big(f(a) = f(b) \to a = b\Big)$ is T?

f is one-to-one if $\forall a, b \in \mathbb{Z}(a^2 = b^2 \to a = b)$ is T?

False

Example Is $f:\mathbb{Z}\to\mathbb{Z}$, $f(x)=x^2$ is one-to-one?

Solution

f is one-to-one if $\forall a,b\in\mathbb{Z}\Big(f(a)=f(b)\to a=b\Big)$ is T?

f is one-to-one if $\forall a,b\in\mathbb{Z}(a^2=b^2\to a=b)$ is T?

f is not one-to-one.

Onto Functions (Hàm toàn ánh)

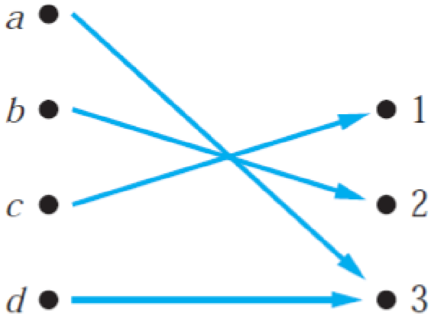
Definition

A function f from A to B is called onto, or a surjection if and only if for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$.

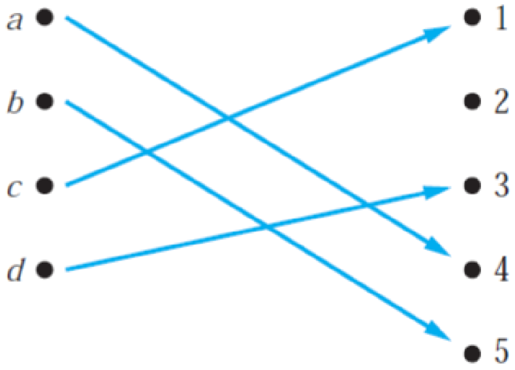
mọi ptử ở phải đều luôn có 1 thg trái gán qua

→ " mọi thg bên phải ko đứng một mình " ⇒ Onto

Example Is the following function onto?



Example The function below is onto?



Example Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f onto?

Example Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f onto?

Solution

f is onto

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $f(a) = b \quad (T?)$.

Example Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f onto?

Solution

f is onto

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $f(a) = b$ $\quad(T?)$.

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $a + 1 = b$ $\quad(T?)$.

Example Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f onto?

Solution

$f(a) = a + 1$

f is onto

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $f(a) = b$ (T?).

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $a + 1 = b$ (T?).

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $a = b - 1$ (T?).

mỗi số nguyên b, tồn tại số nguyên A sao cho $a = b - 1$

Example Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$. Is f onto?

Solution

f is onto

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $f(a) = b$ $\quad (T?)$.

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $a + 1 = b$ $\quad (T?)$.

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $a = b - 1$ $\quad (T?)$.

f is onto.

Example Let the function $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x + 2$. Is f onto?

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if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $3a + 2 = b$ $\quad (T?)$.

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $a = \dfrac{b-2}{3}$ $\quad (T?)$.

$b = 1$ $\rightarrow$ $a = -1/3$

$\Rightarrow$ False

Example Let the function $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x + 2$. Is f onto?

Solution

f is onto

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $f(a) = b \quad (T?)$.

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $3a + 2 = b \quad (T?)$.

if for every $b \in \mathbb{Z}$, there exists $a \in \mathbb{Z}$ such that $a = \dfrac{b-2}{3} \quad (T?)$.

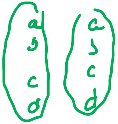
f is not onto.

Exercise

Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one (onto)

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

a)



đơn ánh

one to one

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

onto

toàn ánh

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

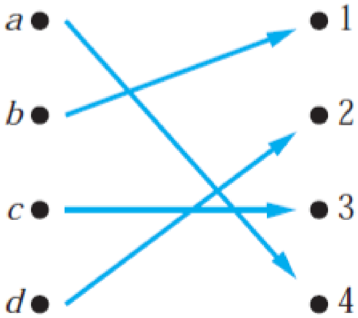
Bijective functions (Hàm song ánh)

Definition

The function f is a bijection if it is both one-to-one and onto. We also say that such a function is bijective.

mọi thg bên trái để gán duy nhất bên phải

Example The following function is bijective?

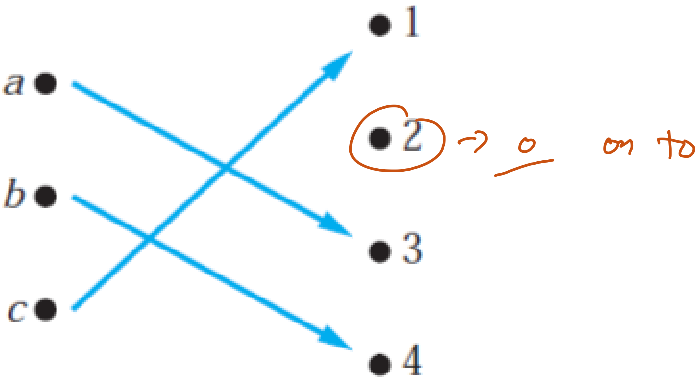


one-to one

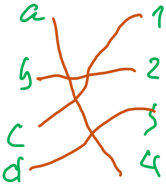
\+ $\rangle$ bijective

onto

Example The following function is bijective?



Example Let f be a function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1$ and $f(d) = 3$. Is f a bijection?



$\Rightarrow$ bijective

Example Let f be a function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1$ and $f(d) = 3$. Is f a bijection?

Solution

• f is one-to-one because no two values in the domain are assigned the same function value.

Example Let f be a function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1$ and $f(d) = 3$. Is f a bijection?

Solution

• f is one-to-one because no two values in the domain are assigned the same function value.

• f is onto because all four elements of the codomain are images of elements in the domain.

Hence f is bijective.

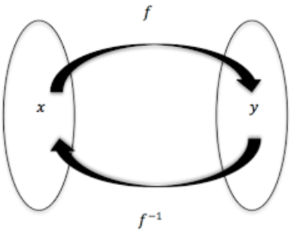
Inverse functions (Hàm số ngược)

Definition Let f be a bijective function from the set A to the set B. The inverse function of f (f^{-1}) is the function that assigns to an element $y \in B$ to the unique element $x \in A$ such that $f(x) = y$.

$f : \mathbb{R} \to \mathbb{R}$
$f(x) = 2x + 7$

1) hàm song ánh
2) $f^{-1}(y) = \ ?$ ←

$f(x) = 2x + 7 = y \to x = \frac{y-7}{2}.$



chỉ có hàm song ánh mới có hàm ngược

$a \longrightarrow 2$

$\to f^{-1} = 2 \to a.$

Method to find f^{-1} : $\mathbf{f(x) = y \Leftrightarrow f^{-1}(y) = x}$.

Example

phương pháp: giải x theo y.

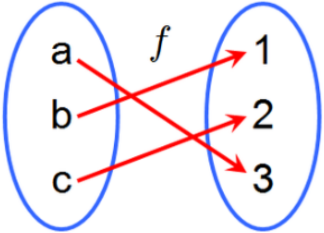
$f(x) = y$

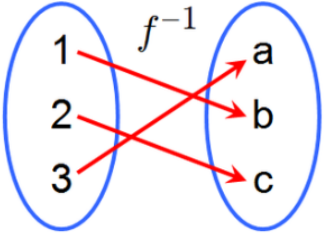
$\to \quad y = x + 1$

$\to \quad x = y - 1.$

$f^{-1}(y) = x = y - 1$

$\to \quad f^{-1}(x) = x - 1$





vd: $f: \mathbb{Z} \to \mathbb{Z}$

$f(x) = x + 1$

f'

$x \longrightarrow x+1$

$1 \longrightarrow 2$ $\qquad 2 \longrightarrow 1$

$2 \longrightarrow 3$ $\qquad 3 \longrightarrow 2$

$\ldots \qquad \ldots$

$10 \longrightarrow 11$ $\qquad 11 \longrightarrow 10$

$\boxed{y = f^{-1}}$

Example Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Is f invertible and if it is, what is its inverse?

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

1) Check song ánh ?

2) $f^{-1}(y) = ?$ $\sqrt{y}$

$f^{-1}(x) = \sqrt{x}$

$f(x) = x^2 = y$

$\left[\begin{array}{l} \boxed{x = +\sqrt{y}} \\ x = -\sqrt{y} \quad (\text{loại}) \end{array}\right.$

Example Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2, f(b) = 3$ and $f(c) = 1$. Is f invertible and if it is, what is its inverse?

Solution

f is invertible because f is bijective.

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Solution

f is invertible because f is bijective.

$f^{-1}(1) = c, f^{-1}(2) = a$ and $f^{-1}(3) = b$.

Example Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

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Solution

f is invertible because f is bijective (both one-to-one and onto).

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$f(x) = y \Leftrightarrow f^{-1}(y) = x$

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To find its inverse:

$\boxed{f(x) = y \Leftrightarrow f^{-1}(y) = x}$

Solve x: $f(x) = y$

$\Leftrightarrow x + 1 = y$

$\Leftrightarrow x = y - 1$.

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f is invertible because f is bijective (both one-to-one and onto).

To find its inverse:

$f(x) = y \Leftrightarrow f^{-1}(y) = x$

Solve x: $f(x) = y$

$\Leftrightarrow x + 1 = y$

$\Leftrightarrow x = y - 1$.

Thus $f^{-1}(y) = y - 1$.

Exercise

Find the inverse functions (if they exist) of the following functions?

1) Let $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = 2x + 3$.

2) Let $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 2x + 3$

Example Let f be a function from $\mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Is f invertible?

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Solution

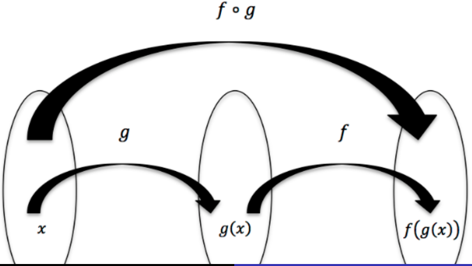
f is not one-to-one because $f(-2) = f(2)$. Hence f is not bijective. Thus f is not invertible.

Composition

Definition

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $x \in A$ by $f \circ g$, is denoted by

$$(f \circ g)(x) = f(g(x))$$

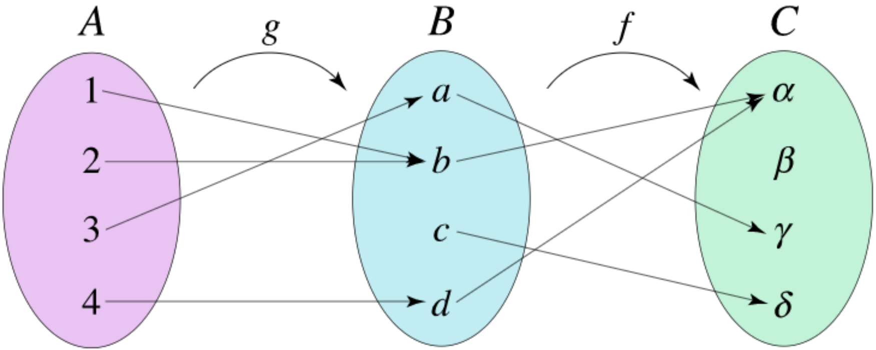


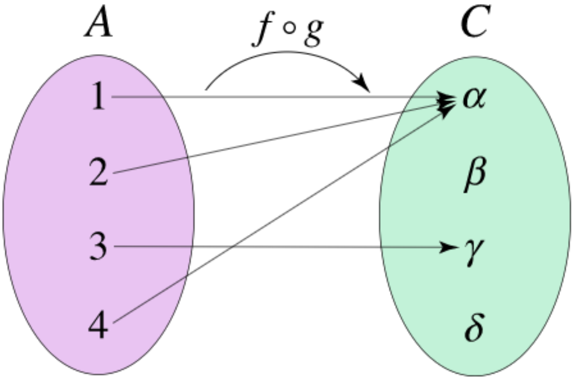
$g \quad f$

$2 \to 3 \to$

$\to (f \circ g) = 2 \to$

Example find $f \circ g$?





A
$f \circ g$
C
1
2
3
4
α
β
γ
δ

Example

Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be functions defined by

$$f(x) = 2x + 3, \quad g(x) = 3x + 2$$

What are $f \circ g, g \circ f, f \circ f, g \circ g$?

$(f \circ g)(x) = f(g(x))$; f hợp g = $g(\cancel{cx+s})$ = $g\,\cancel{(2x+3)}$

$(f \circ f)(x) = f(f(x)) = f(2x+3)$; $g \circ g = g(g(x))$

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$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7.$

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Exercise

Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$f(x) = \sin x \quad g(x) = 2x + 1$$

Find $f \circ g, g \circ f$.

Floor functions (hàm sàn)

Definition

• The floor function assigns to the real numbers x the largest integer that is less than or equal to x. (làm tròn xuống) $\left\lfloor \frac{19}{3} \right\rfloor = 6$

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Exercise

Find these values

a) $\lceil 1.1 \rceil$ b) $\lceil -0.1 \rceil$

c) $\lceil 4 \rceil$ d) $\lfloor 3.2 \rfloor$

e) $\lfloor -5.2 \rfloor$ f) $\left\lfloor \frac{1}{2} + \left\lceil \frac{2}{3} \right\rceil \right\rfloor$ $\lfloor \frac{1}{2} + 1 \rfloor = \lfloor 1,5 \rfloor = 1$

Sequences (Dãy số)

Định nghĩa Dãy số thực là một ánh xạ

$x_n = \left(\frac{1}{n}\right)_{n \geq 1}$

$$x : N \to R$$
$$n \to x(n) \equiv x_n$$

$\frac{1}{1}; \frac{1}{2}; \frac{1}{3}; \frac{1}{4} \ldots$

Ta dùng các kí hiệu sau để chỉ dãy số thực x: $\{x_n\}_{n \geq 0}$, $\{x_n\}_{n \geq 1}$

Ví dụ $\{x_n\}$ với $x_n = \frac{1}{n}$, $n \in N$ là 1 dãy số thực. Dãy này có 3 số hạng đầu là x_1, x_2, x_3, nghĩa là $1, 1/2, 1/3$

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Ví dụ $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ là 1 dãy số thực.

Example Consider the sequence $\{a_n\}$, where

$a_n = (n+1)^2, \quad n \in \mathbb{N}$

$\{(n+1)^2\}_{n \geq 0}$

$a_1 = (1+1)^2 = 4$

$a_2 = (2+1)^2 = 9$

$a_{10} = (10+1)^2 = 121$

$a_{1000} = 1001^2$

Exercise

Liệt kê 5 số hạng đầu tiên của dãy $\{x_n\}$ với $x_n = 2n + 1, n \in N$

Recurrence Relations

Example

Let $\{a_n\}$ be a sequence such that

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Recurrence Relations

Example

(truy hồi)

Let $\{a_n\}$ be a sequence such that

$$a_n = a_{n-1} + 1 \quad \forall n \geq 1, a_0 = 1 \tag{1}$$

$$a_1 = a_0 + 1 = 1 + 1 = 2$$
$$a_2 = a_1 + 1 = 2 + 1 = 3$$
$$a_3 = a_2 + 1 = 3 + 1 = 4$$

Example Let $\{a_n\}$ be a sequence such that

$$a_n = a_{n-1} + a_{n-2} \quad \forall n \geq 2, \quad a_0 = 1, a_1 = 1. \tag{2}$$

$a_2 = a_1 + a_0 = 1 + 1 = 2$
$a_3 = a_2 + a_1 = 2 + 1 = 3$
$a_4 = a_3 + a_2 = 3 + 2 = 5$

Exercise

Find the first five terms of the sequence defined by each of these recurrence relations and initial condition.

a) $a_n = 6a_{n-1} \; \forall n \geq 1, \quad a_0 = 2$

b) $a_n = a_{n-1}^2 \; \forall n \geq 1, a_0 = 2$

c) $a_n = a_{n-1}^2 + 2a_{n-2} \; \forall n \geq 2, \quad a_0 = 1, a_1 = 2$

b) $a_1 = a_0^2 = 4$

$a_2 = a_1^2 = 4^2 = 16$

$a_3 = a_2^2 = 16^2$

Exercise

Given a sequence $\{a_n\}$ satisfying the recurrence relation

$$a_0 = -1,\ a_n = a_{n-1} + 2^n \text{ for } n = 1, 2, \ldots$$

Find a_6.

$a)123 \quad b)125 \quad c)127 \quad d)121 \quad e)$None of them is correct

Geometric progression

$f(n) = ar^n \rightarrow a, ar, ar^2, ar^3, \ldots, ar^n$: cấp số nhân

công bội

số hạng đầu

Arithmetic progression

$f(n) = a + nd \rightarrow a, a+d, a+2d, \ldots, a+nd$: cấp số cộng

công sai

- a: initial term,
- r: common ratio, a real number
- d: common difference, real number

vd:

<u>Do yourself</u>

$b_n = (-1)^n$, $n>=0$ $\qquad$ $c_n = 2(5)^n$, $n>=0$

$t_n = 7-3n$, $n>=0$ $\qquad$ $a_n = -1 + 4n$, $n>=0$

Summation

Notation

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

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vd: $\sum_{j=1}^{10} j^2 = 1^2 + 2^2 + \ldots + 10^2$

Example

$$\sum_{j=1}^{100} \frac{1}{j} =$$

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Exercise

What are the values of these sums?

$$A = \sum_{k=1}^{5}(k+1) = 2 + 3 + 4 + 5 + 6 =$$

$$B = \sum_{j=0}^{4}(-2)^j = (-2)^0 + (-2)^1 + (-2)^3 + (-2)^4$$

$$C = \sum_{i=1}^{5} 3 = 15 \ (3 \times 5)$$

$$D = \sum_{j=0}^{3}(j-2)^2 = (-2)^2 + (-1)^2 + 0^2 + 1^2$$

$$\sum_{j=2}^{4} \sum_{i=1}^{3} ij$$

$$\sum_{j=2}^{4} (j + 2j + 3j) = 6j$$

Given $S = \{1, 3, 5, 7\}$

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$$\sum_{j \in S} j^2 = 1^2 + 3^2 + 5^2 + 7^2$$

$$\sum_{j \in S} (j + \frac{1}{j}) = ?$$

Double Summations

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij =$$

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Exercise Find the values of the following sums

$$\sum_{i=1}^{3}\sum_{j=0}^{2}(2i-3j)$$

$$\sum_{i=0}^{2}\sum_{j=0}^{3} i^2 j^3$$

Theorem

If a and r are real numbers and $r \neq 0, 1$. Then

$$\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$$

$= \underbrace{a + ar + ar^2 + \cdots + ar^n}$: tổng $(n+1)$ số hạng đầu tiên của 1 cấp số nhân.

Example

$$\sum_{j=0}^{8} 3.2^j = \frac{3.2^9 - 3}{2 - 1} = 3.2^9 - 3$$

Theorem

$1+2+\dots+n =$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad n \in \mathbb{N}$$

$1+2^2+\dots+n^2$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad n \in \mathbb{N}$$

Example

$500+501+\dots+1000 = \underbrace{(1+2+\dots+499)+(500+\dots+1000)}_{\frac{1000.1001}{2}} - (1+2+\dots+499)$

$- \frac{499.500}{2} \dots$

$$\sum_{k=1}^{50} k = \frac{50(50+1)}{2} = 1275$$

$$\sum_{k=1}^{30} k^2 = \frac{30.31.61}{6} = 9455$$