Chapter 10: Graph

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Graph

vertices

Edges

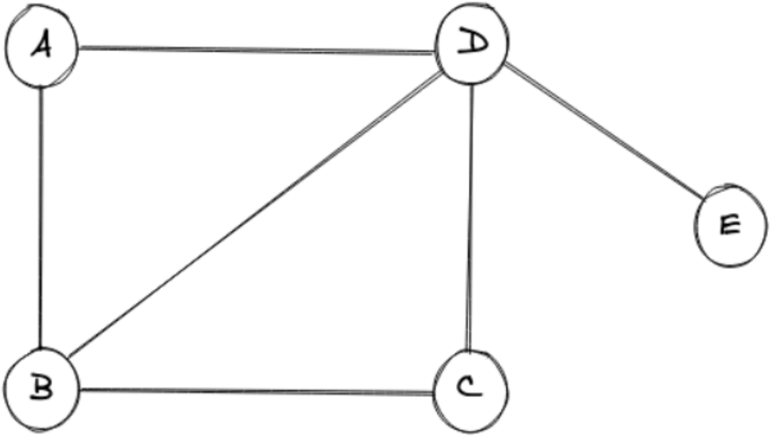
Definition

- A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges.

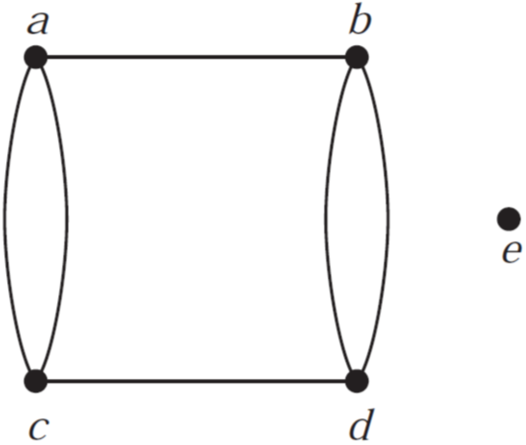
Remark

- The set of vertices V of a graph G may be infinite. có hạn
- **An infinite graph: infinite vertex set or an infinite number of edges.**
- **A finite graph: finite vertex set and a finite edge set.**

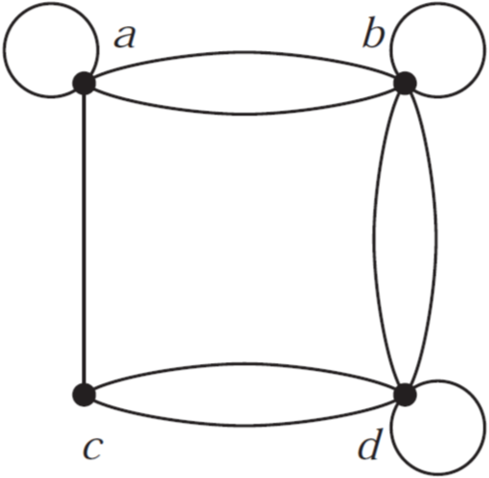
Example 1



Example 2



Example 3



Simple Graph

(Đồ thị đơn)

“ k có vòng lặp ”

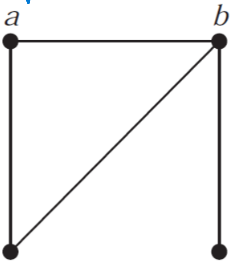
- **no loops**: each edge connects two different vertices (mỗi cạnh nối với 2 đỉnh khác nhau)

“ giữa 2 đỉnh k có nhiều hơn 1 cạnh ”

- **no multiple edges**: no two edges connect the same pair of vertices (không có cạnh nào nối với cùng 1 cặp đỉnh)

Example

Simple Graph >< Non-simple Graph:

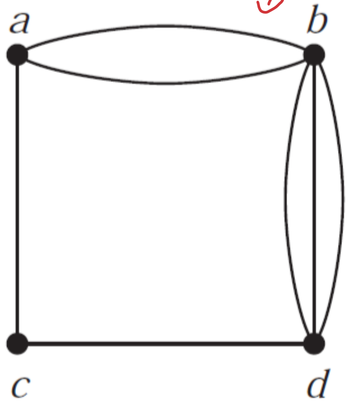


Multigraph

- no loops : o có vòng lặp
- multiple edges allowed giữa 2 đỉnh được nhiều hơn 1 cạnh.

Example

2 edges

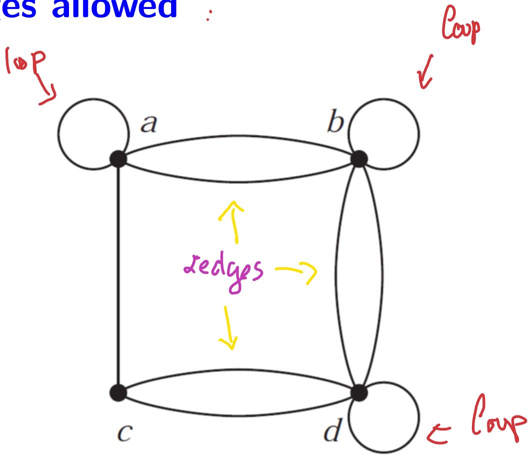


3 edges

Pseudographs

- loops allowed : có vòng lặp
- multiple edges allowed :

Example



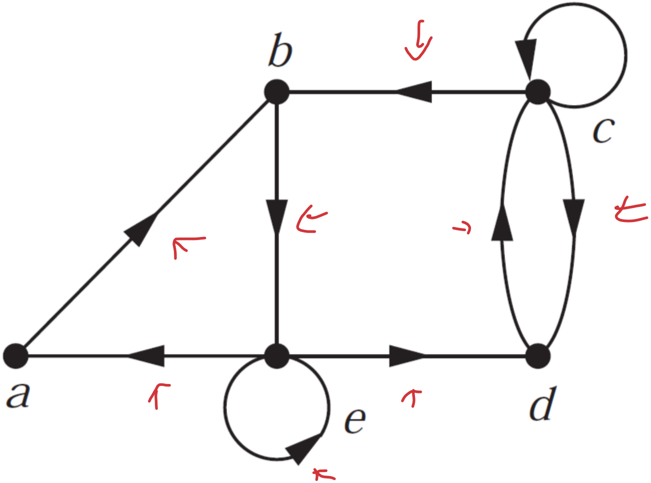
Directed Graph : Có cạnh + có hướng

(Đồ thị có hướng)

Definition

- Directed graph = graph + directed edge (cạnh có hướng)

Example



Graph Models

: v bắc cầu

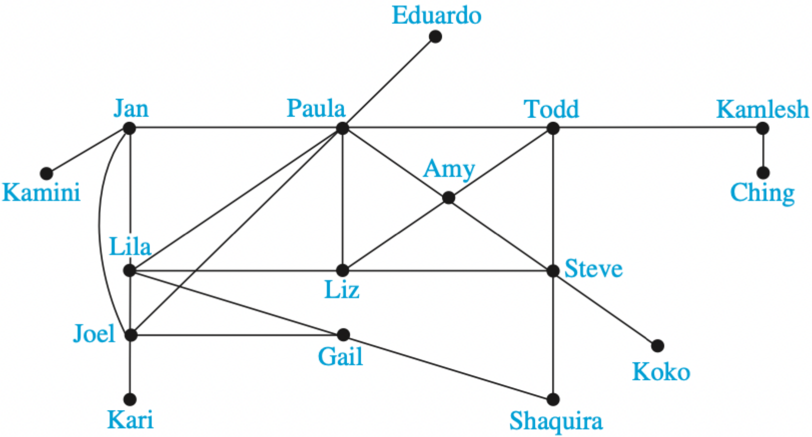


FIGURE 6 An Acquaintanceship Graph.

Graph Models

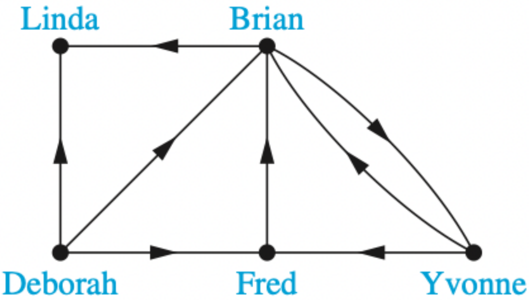


FIGURE 7 An Influence Graph.

Graph Models

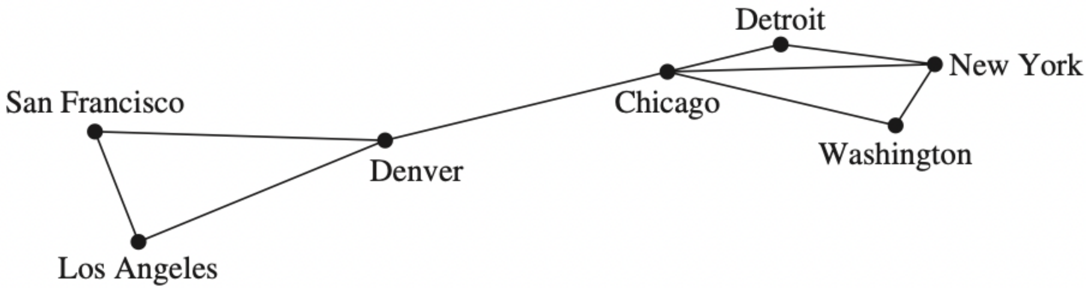


FIGURE 1 A Computer Network.

Graph Models

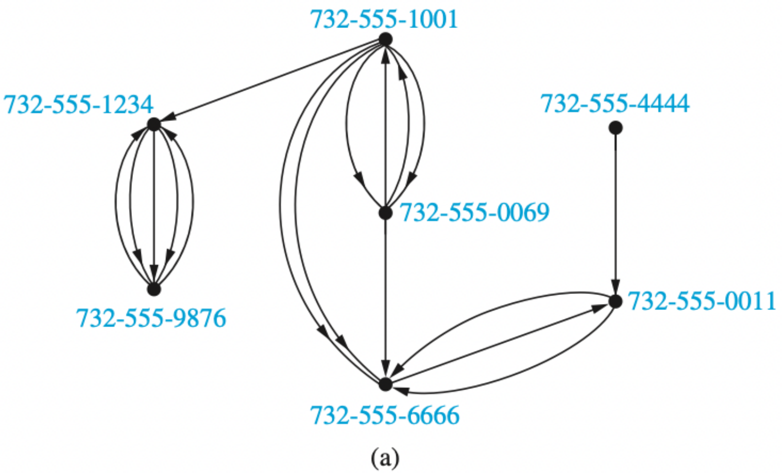
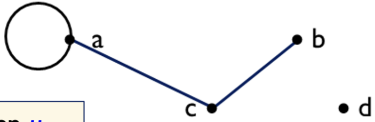


FIGURE 8 **A Call Graph.**

Basic *Terminology*



degree: bậc

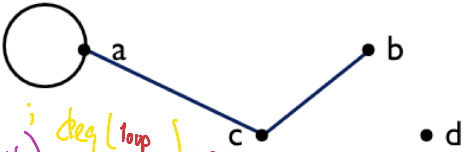
If an *edge* **{u, v}** exists, then *u* *and v* are called ***adjacent***: liền kề nhau

vertex	*adjacency* list : cạnh liền kề
a	a, c
b	c
c	a, b
d	

The *edge {u, v}* is called ***incident*** with u and v
gắn liền với

edge	*incident* vertices : đỉnh.
{a, a}	a
{a, c}	a, c
{c, b}	c, b

Basic Terminology



bậc = "số cạnh nối vs đỉnh đó" = deg(v)

deg(loop) = 2.

The **degree** of a vertex v:

= the number of edges *incident* with v, except that *a loop* at a vertex *contributes twice*.

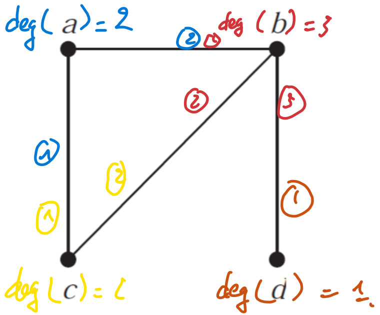
Notation: $\boxed{\text{deg(v)}}$

vertex	degree
a	deg(a) = 3 = 1 + 2. (1 loop)
b	deg(b) = 1
c	deg(c) = 2
d	deg(d) = 0

b is called **pendant** : đơn độc: khi chỉ có 1 cạnh nối vs đỉnh. b

d is called **isolated** : cô lập khi đỉnh d đứng một mình

Example



$\deg(a) = 2, \deg(b) = 3, \deg(c)=2, \deg(d) = 1.$

$N(a) = \{b, c\}, N(b) = \{a, c, d\}, N(c) = \{a, b\}$ and $N(d) = \{b\}$

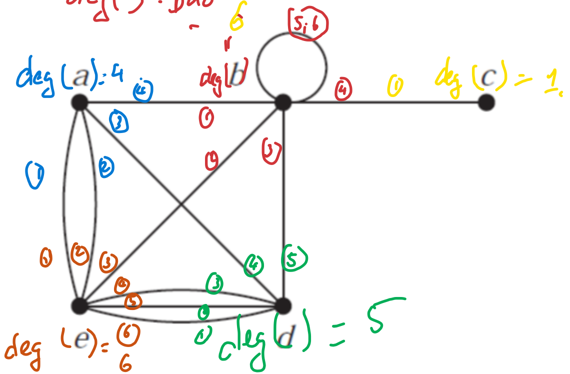
vertex

adjacency

Exercise

What are the degrees of the vertices in the graph H?

deg( ) : bậc



H

Handshaking Theorem (for undirected graph)

"The sum of the degrees is twice the number of edges"

2x

bậc

cạnh

$$\boxed{\sum_{v \in V} \deg(v) = 2|E|}$$

Exercise

Can a simple graph exist with 15 vertices each of degree five?

15 đỉnh
5 bậc

$\Rightarrow \deg(v) = 75$

(sai vì phải là số chẵn)

Exercise

How many edges does a graph have if its **degree sequence is** $5, 5, 4, 3, 1, 0$?

Draw a such graph.

$\deg(a) = 5$, $(d) = 3$
$(b) = 5$ $(e) = 1$ $\Rightarrow \sum \deg = 18$
$(c) = 4$ $(f) = 0$

$\sum \deg = 2 \times |E| = 18$

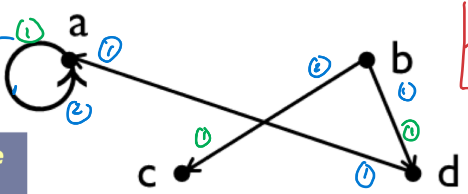
$\rightarrow |E| = 9$

Directed graphs -Basic Terminology

Bậc

từ đỉnh | đi vào chính nó | từ nó đi ra

Vertex	In-degree deg^-	Out-degree deg^+
a	2	1
b	0	2
c	1	0
d	1	1



0 tính loop

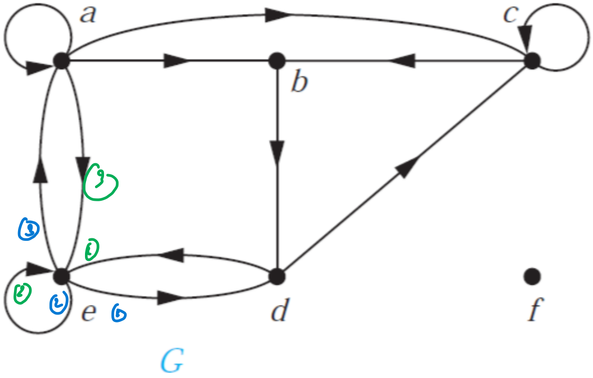
$b \longrightarrow c$

initial vertex   terminal vertex

(b, c)   ~~{b, c}~~   ~~(c, b)~~

$$\boxed{\sum deg^- = \sum deg^+} = 4 \textit{ directed edges}$$

Example 1 Find the in-degree and out-degree of each vertex in the graph G below?



$deg^-(e) = 3$

$deg^+(e) = 3$

$deg^-(a) = 2$, $deg^+(a) = 4$.

Theorem

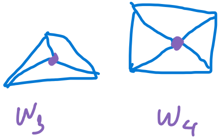
let $G = (V, E)$ *be a graph with directed edges. Then*

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

.

Some special simple graphs

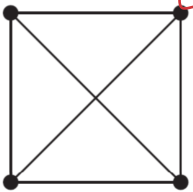
- **Complete graphs** K_n : có đỉnh n, n đỉnh luôn nối với nhau.
  Số cạnh $|E| = \frac{n(n-1)}{2}$
- **Cycles** C_n : có n đỉnh và n cạnh.
- **Wheels** W_n = C_n + 1 vertex (đỉnh), 2n edge.
- **n-cube** Q_n : 2 bit-string ≠ nhau có đúng 1 vị trí nối.



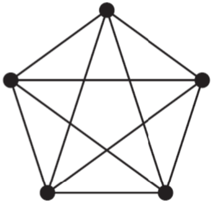
Complete graphs K_n

- **n vertices** : n đỉnh.
- **simple graph that contains exactly one edge between each pair of distinct vertices** : có đúng nhất 1 cạnh giữa các đỉnh.

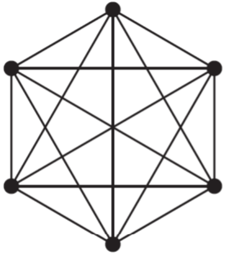
Số cạnh $= |E| = \dfrac{n(n-1)}{2}$



K_4 $|E| = \dfrac{4.3}{2} = 6.$



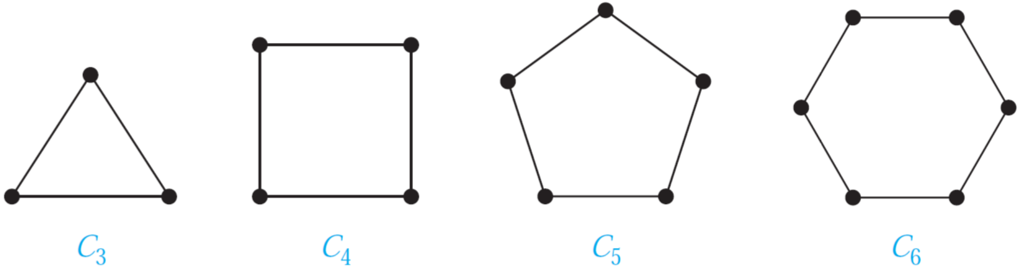
K_5 $|E| = \dfrac{5.4}{2} = 10$



K_6 $|E| = \dfrac{6.5}{2} = 15$

Cycles C_n

số đỉnh = số cạnh = n.

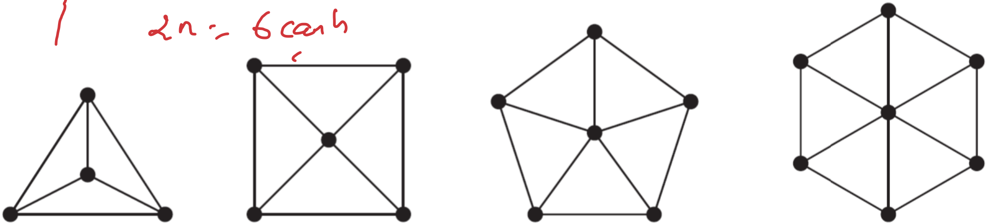


Wheels W_n

Wheels $W_n = C_n + 1$ additional vertex.

W_3 thì có $3+1 = 4$ đỉnh

$2n = 6$ cạnh



W_3 W_4 W_5 W_6

W_3: $3+1 = 4$ đỉnh, $3 \times 2 = 6$ cạnh

W_4: $4+1 = 5$ đỉnh, $4 \times 2 = 8$ cạnh

W_5: $5+1 = 6$ đỉnh, $5 \times 2 = 10$ cạnh

W_6: $6+1 = 7$ đỉnh, $6 \times 2 = 12$ cạnh

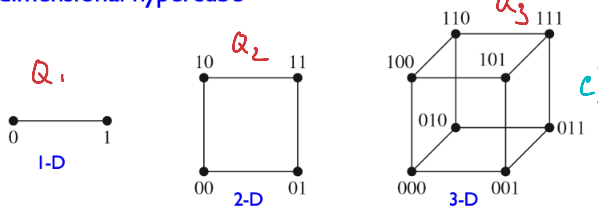
Q_n is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if two bit strings differ in exactly one bit position.

có 2^n đỉnh

$(n \times 2^{n-1})$ cạnh

n-cubes Q_n

n-dimensional hypercube



a) $2^1 = 2$ đỉnh; $1 . 2^{1-1} = 1$ cạnh

b) $2^2 = 4$ đỉnh; $2 . 2^{2-1} = 4$ cạnh

c) $2^3 = 8$ đỉnh; $3 . 2^{3-1} = 12$ cạnh

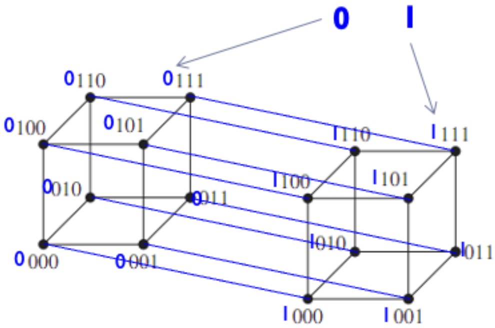
Graph	|V| = number of vertices	|E| = number of edges
Q_n	2^n	$n.2^{n-1}$

n-cube Q_n

- Construct Q_4 from two copies of Q_3

$2^4 = 16$ đỉnh

$4 \cdot 2^{4-1} = 32$ cạnh



Exercise

5. Draw these graphs.

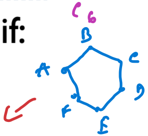
a) K_7 b) $K_{1,8}$ c) $K_{4,4}$

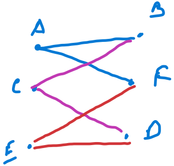
d) C_7 e) W_7 f) Q_4

Bipartite graphs (lưỡng phân): chia làm 2 phe nối nhau

- A simple graph G = (**V**, E) is called ***bipartite*** if:
  - **V** $= V1 \cup V2$, $V1 \cap V2 = \emptyset$
  - *no edge* connects two vertices in V1
  - *no edge* connects two vertices in V2

Phe 1: Chỉ nối với đứa khác





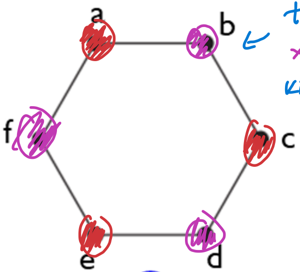
Phe 2: bị nối với được nối

- A simple graph $G = (V, E)$ is called ***bipartite*** if:
  - $V = V1 \cup V2, \qquad V1 \cap V2 = \varnothing$
  - *no edge* connects two vertices in $V1$
  - *no edge* connects two vertices in $V2$
- We call the pair $(V1,V2)$ a *bipartition* of V.

Ex. Study graph C_6 (cycle)

$V = \{ a, b, c, d, e, f \}$

⊕ Đồ thị xuất hiện 1 △ thì o tô được



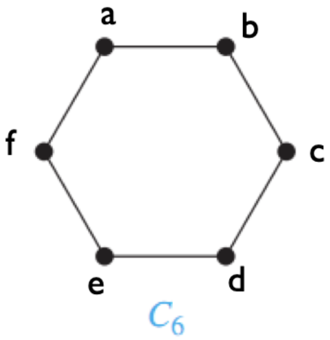
C_6

tô đỉnh bởi 2 màu xanh đỏ. nếu 2 đỉnh kề nhau thì màu ≠ nhau
⇓
lưỡng phân

Bipartite graphs

Ex.

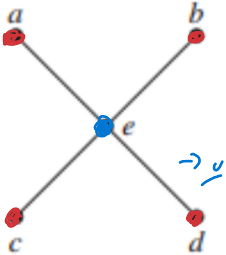
- Let **color** the vertices using *2 different colors*
- Two ***adjacent vertices*** must have ***different colors*** (e.g., **red** and **black**)



Exercise

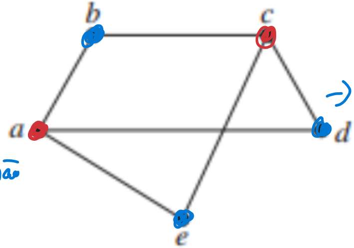
Determine whether the graph is bipartite

a)



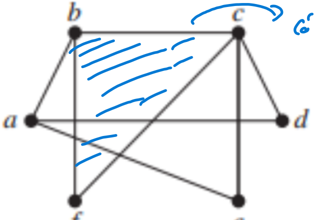
→ ø có Δ nào

b)



→ bipartite

c)

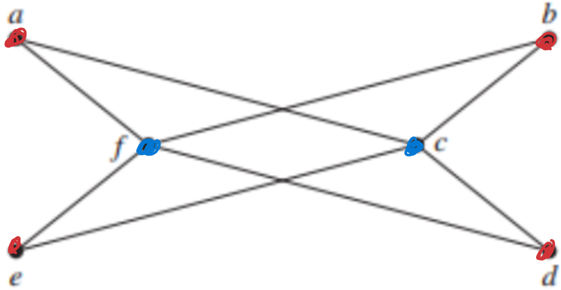


có 1 Δ ⇒ ø tô được

→ ø bipartite

Exercise

Determine whether the graph is bipartite



→ bipartite

Exercise

For which values of n are these graph bipartite?

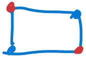
a) K_n b) C_n c) W_n d) Q_n

K_2 

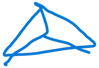
K_3, K_4 ×

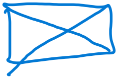
→ K_n $(n = 2)$

C_3 

C_4 

→ C_n ($n \in$ even) → bipartite

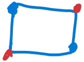
W_3 

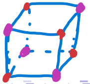
W_4 

W_n ⊄ lưỡng phân $\forall n$

→ Q_n lưỡng phân $\forall n$

Q_1  ✓

Q_2  ✓

Q_3:  ✓

Complete Bipartite Graphs $K_{m,n}$ → n đỉnh dưới.

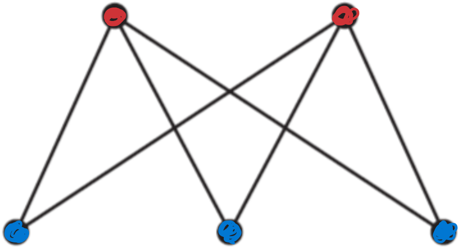
đồ thị lưỡng phân đầy đủ

m đỉnh trên

Example:

$K_{2,3}$ ←

Có 2×3 cạnh
2+3 đỉnh

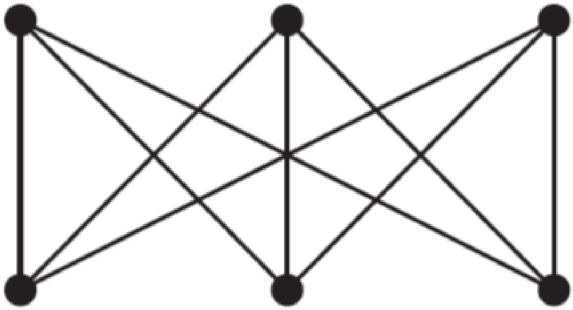


$K_{2,3}$

mọi đỉnh trên đều phải nối mọi đỉnh dưới

m × n cạnh

m + n đỉnh



$K_{3,3}$

Exercise

A complete bipartite graph with 15 edges has .... vertices. đỉnh

$K_{m,n}$

$m \times n = 15 \Rightarrow (m+n) = ?$

A.7 B.8 C.9 D.10 E.None of them

1	15
3	5
5	3

$\to m+n = 16$

$= 8$

Exercise

Fill in the blank: "The graph $K_{m,n}$... has ... edges"

cạnh

i) C_n, $\cancel{2n}$ n

ii) $K_{m,n}$, mn

iii) W_n, $2n\ \cancel{+1}$

iv) K_n, $\cancel{n^2}$ $C_n^2 = \frac{n(n-1)}{2}$

Representing Graphs : mô tả đồ thị

- Use adjacency lists : ds liền kề
- Use adjacency matrix :
- Incidence matrix :

3 ways

Adjacency List

Example

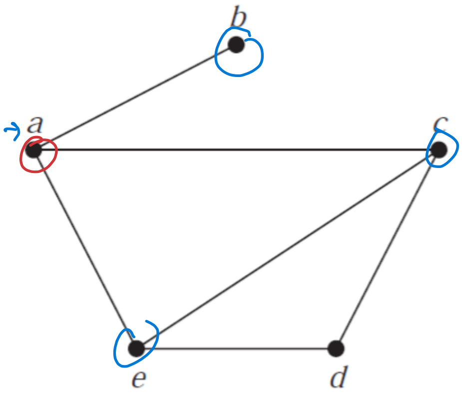


TABLE 1 An Adjacency List for a Simple Graph.

Vertex: đỉnh	*Adjacent Vertices*: các đỉnh kề
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

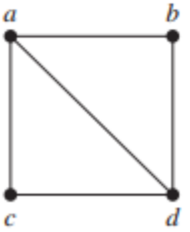
Exercise

1. use an adjacency list to represent the given graph.

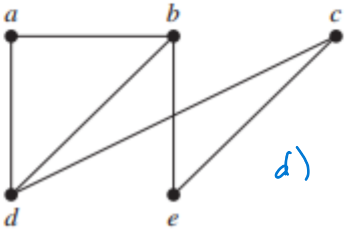
(c)

V	AV
a	a, b, c, d
b	~~b~~ d
c	b, a
d	d, c, b

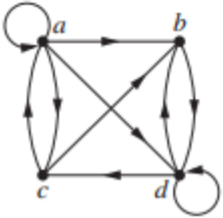
a)



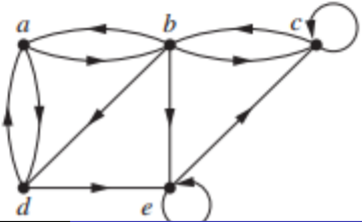
b)



c)

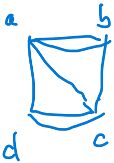


d)



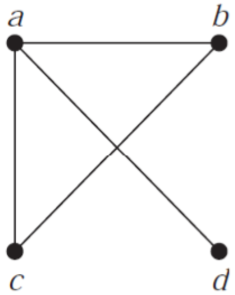
d) V AV

Adjacency Matrices



	a	b	c	d
a	0	2	1	1
b	2	0	1	0
c	1	1	0	2
d	1	0	2	0

Example 1



$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

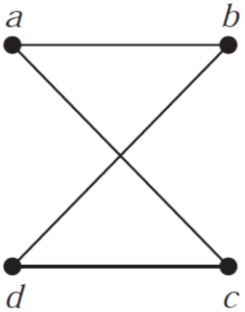
Exercise

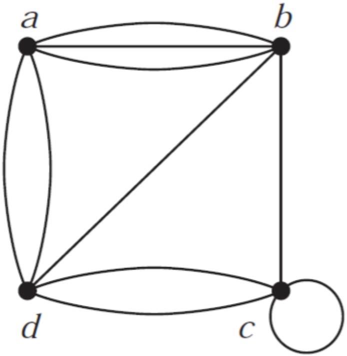
Draw the graphs below the adjacency matrix with respect to the ordering of vertices a, b, c, d.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 1 & 1 & 0 \end{array}$$

=> đối xứng

=>





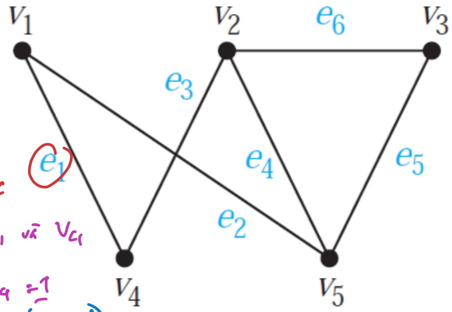
Exercise

Draw a graph with the given adjacency matrix

$$a)\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\quad b)\begin{pmatrix}1&2&1\\2&0&0\\0&2&2\end{pmatrix}\quad c)\begin{pmatrix}0&0&1&1\\0&0&1&0\\1&1&0&1\\1&1&1&0\end{pmatrix}$$

Incidence matrices : các cạnh và các đỉnh

Example 1



e1 nối v1 và v4

=> v1, v4 = 1

còn lại = 0

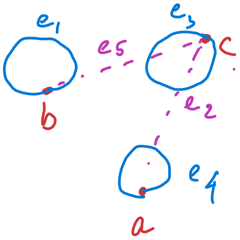
$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 1 & 1 & 0 & 1 \\
v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\
v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\
v_5 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

Exercise

Given a pseudo graph G with incident matrix

$$\begin{array}{c|c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

3 loops



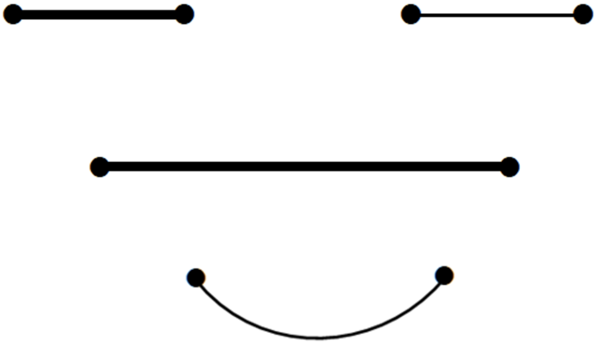
How many loops does G have?

A.5 B.1 C.0 E.3

Isomorphism

Đồng hình.

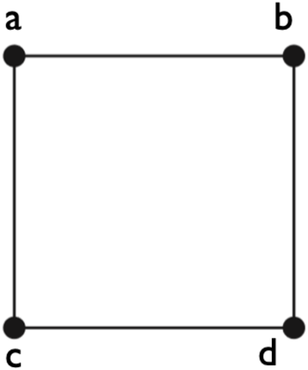
số đỉnh = nhau
số cạnh = nhau
dãy các bậc → (bậc: số các cạnh nối với đỉnh đó)



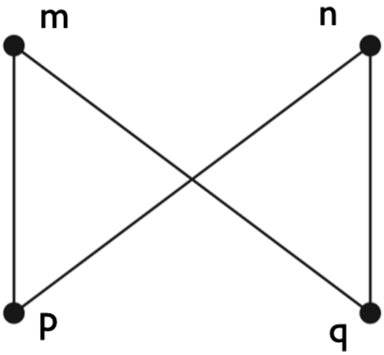
THE SAME

SAME?

số đỉnh = 4
cạnh = 4.
bậc = 2.



=



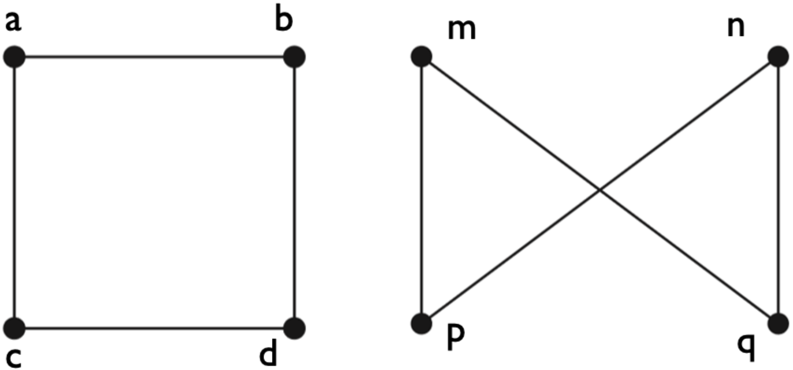
Isomophism of Graphs

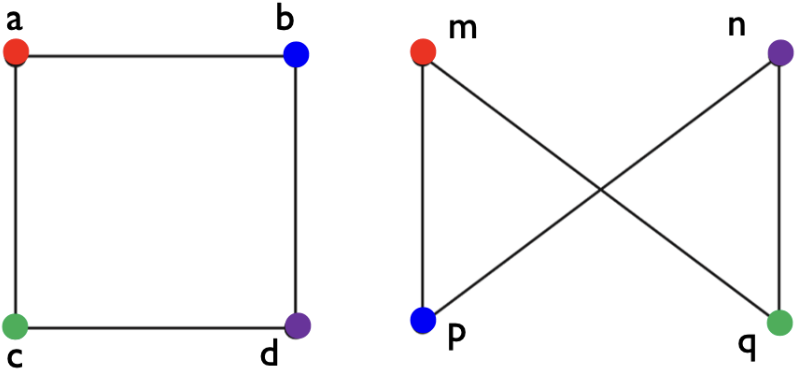
Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if $\exists f : V_1 \to V_2$ such that

- one-to-one
- onto
- a, b are adjacent in $V_1 \Leftrightarrow f(a), f(b)$ are also adjacent in V_2.

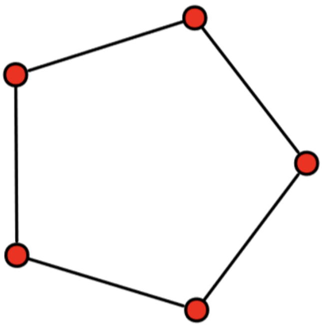
Example The graphs below are isomorphic?

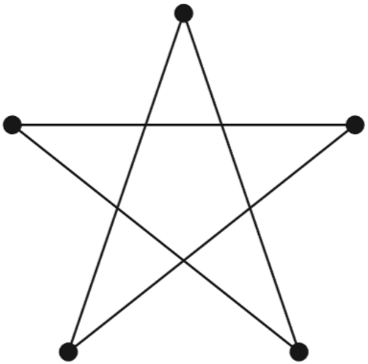




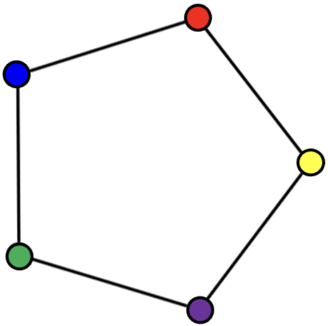
Setting $f(a) = m, f(b) = p, f(c) = q$ and $f(d) = n$ is a bijection between V and W. This correspondence preserves adjacency.

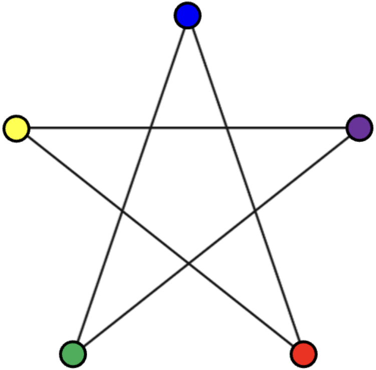
Isomorphic?



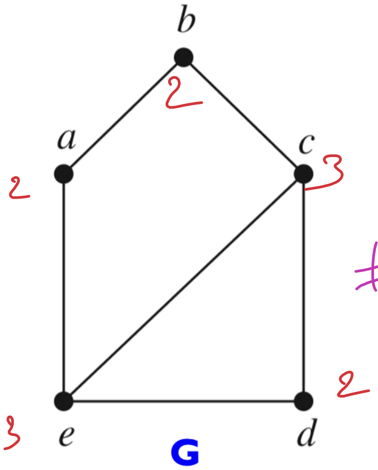


YES

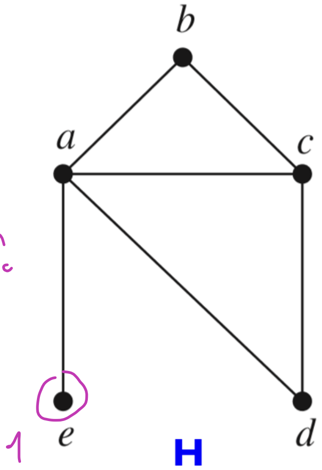




Isomorphic?



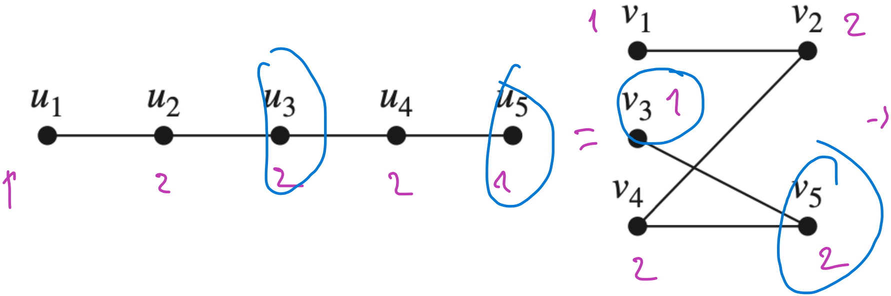
≠ bậc



⇒ ư đẳng hình

Exercise

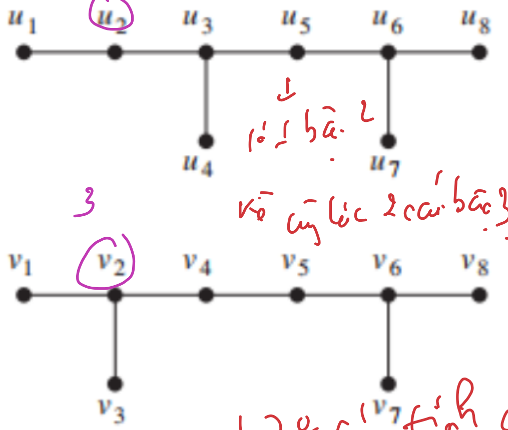
Determine whether the given pairs of graph is isomorphic?



→ cần tay, kéo thả ra

Exercise

Determine whether the given pairs of graph is isomorphic?



2

có 1 bậc 2

3

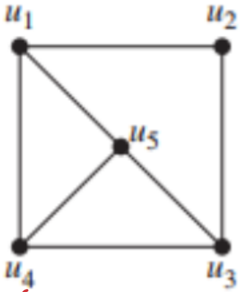
với cạnh lúc 2 cái bậc 3

0 đồng hình.

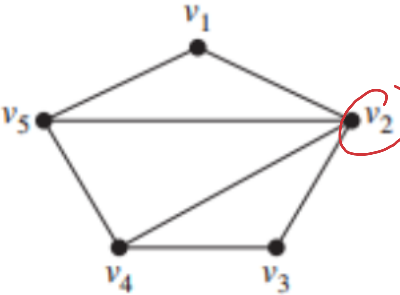
(*) 0 có tính chất như vậy

Exercise

Determine whether the given pairs of graph is isomorphic?



$\neq$



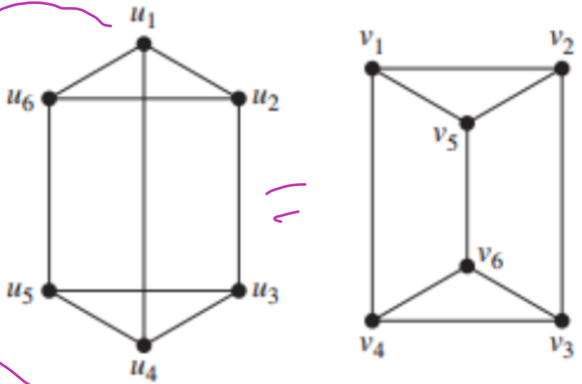
bậc 4

tối đa bậc 3

Exercise

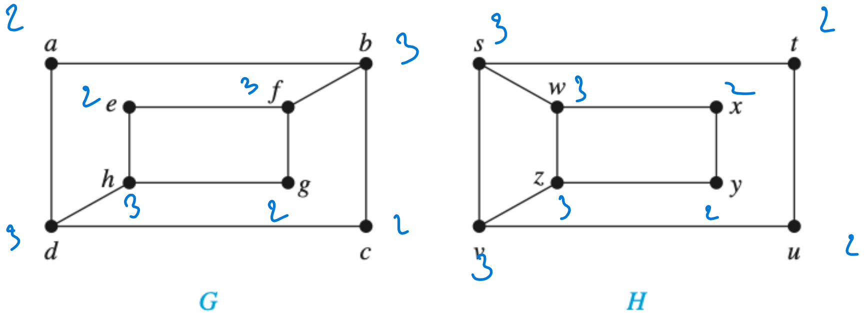
Determine whether the given pairs of graph is isomorphic?

Cầm 2, đầu kéo vô trong.



Exercise

Determine whether the given pairs of graph is isomorphic?



Exercise

Determine whether the given pairs of graph is isomorphic?

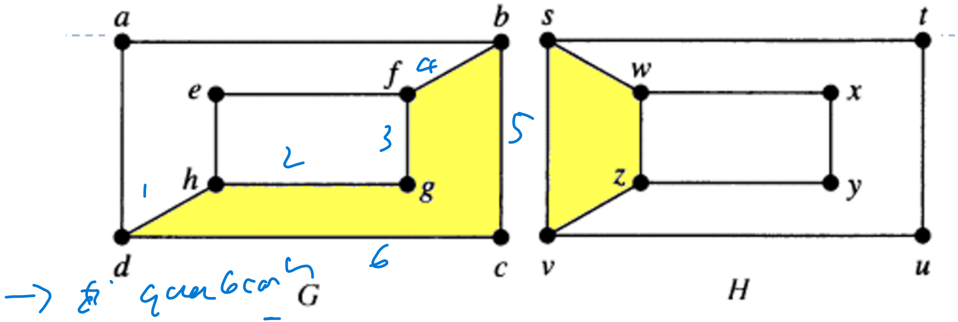
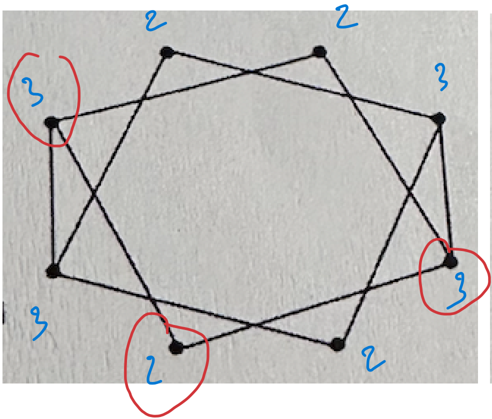
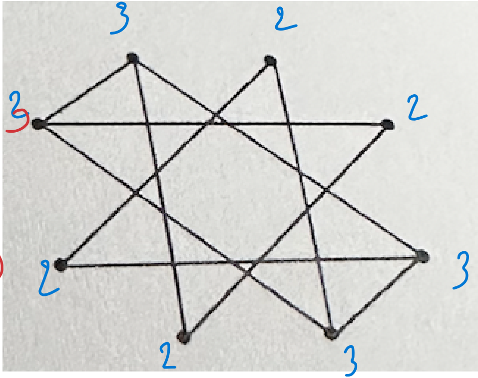


Figure 10: G and H are not isomorphic

Exercise





Exercise

Are these two graphs isomorphic? If not, what is the reason?

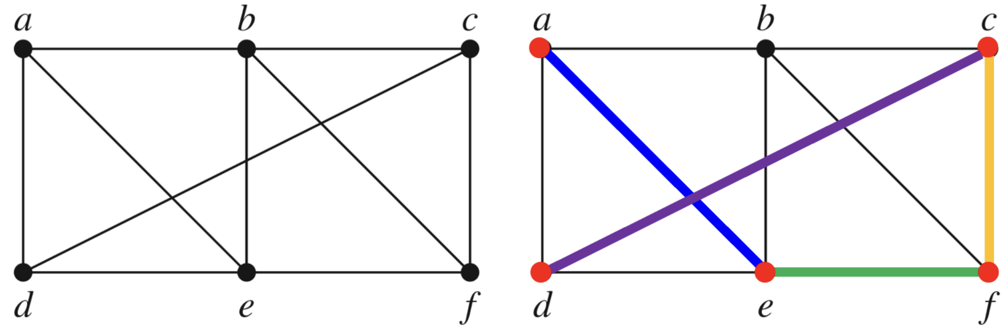
A. Yes, they are isomorphic

B. No they are not isomorphic because the vertices of degree 3 of the graph on the right form a circuit, and the graph on the left does not have that property

C. No they are not isomorphic because they do not have the same number of vertices of degree 3.

D. None of the other choices is correct

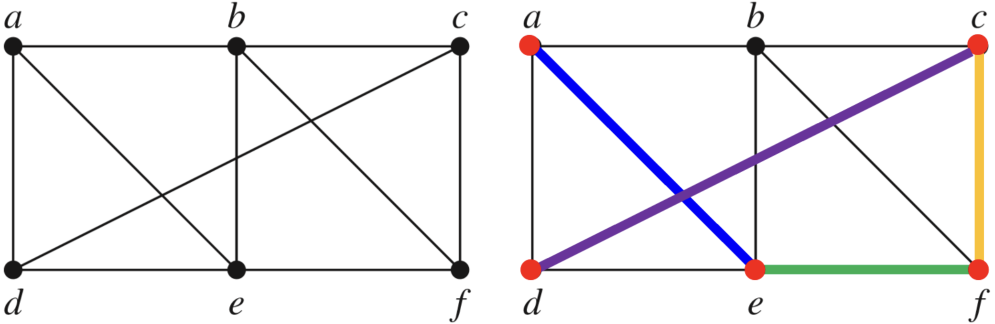
Path
(Đường đi)



a,e,f,c,d: đi qua 4 cạnh : length = 4 : có thể đi cùng cạnh nhiều lần a, b, d, b, e : 4.

Path

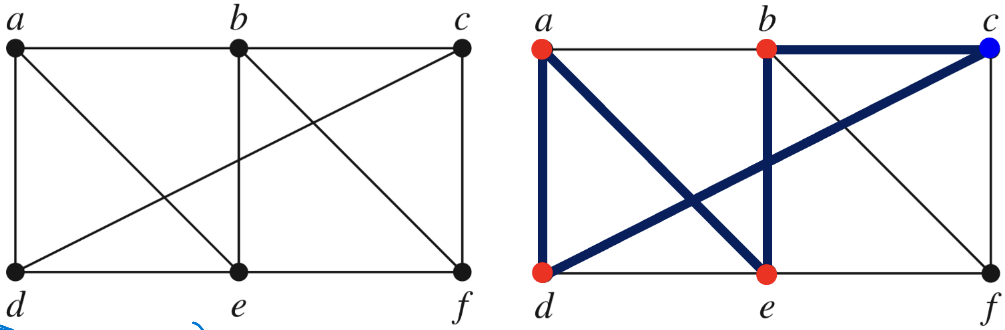
(Đường đi)



a,e,f,c,d: is a path of length 4

A path of length n from u to v is a sequence of n consecutive edges

Circuit
(Chu trình)



c, b, e, a, d, c: is a path and ends the same vertex.

A circuit is a path that starts and ends at the same vertex.

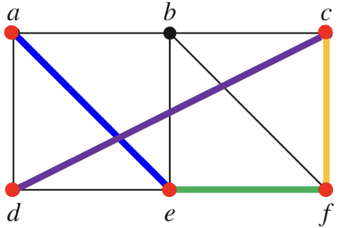
Simple paths/circuits
(Đường đi đơn, chu trình đơn)

Ko đi qua 1 cạnh quá 1 lần

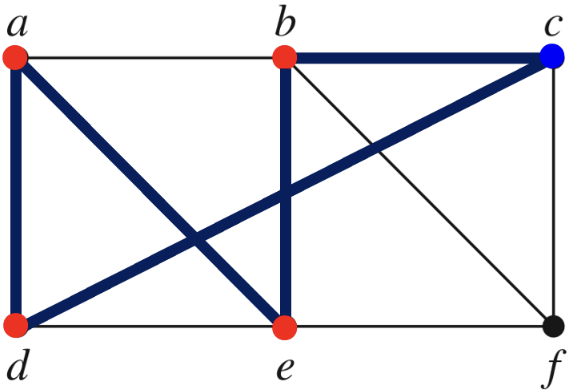
Definition

A path/circuit is simple if it does not contain the same edge more than once.

(Một đường đi (chu trình) gọi là đơn nếu đường đi(chu trình) đó không đi qua cạnh nào quá 1 lần)



a,e,f,c,d is a simple path



c, b, e, a, d, c is a simple circuit

Connectedness in Undirected Graphs

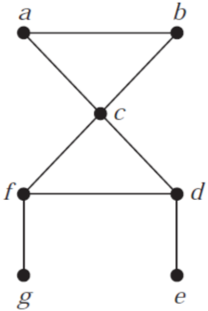
liên thông

Definition

• **An undirected graph is called connected (liên thông) if there is a path between every pair of distinct vertices of the graph.**

giữa 2 đỉnh bất kỳ luôn có 1 đường đi nối với nhau

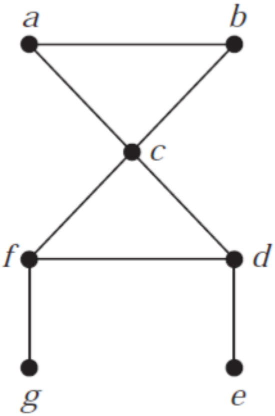
Example 1 Check the connectivity of the graph G_1.



ví dụ: giữa g và e

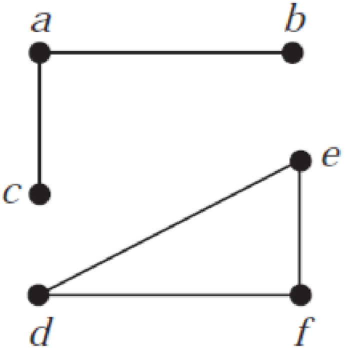
có g, f, d, e.

Example 1 Check the connectivity of the graph G_1.



G_1

Example 2 Check the connectivity of the graph G_2.



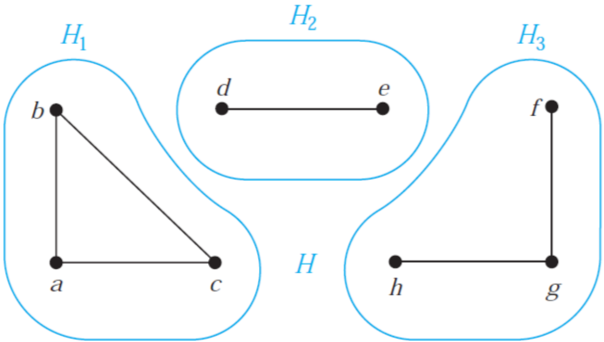
G_2

$\Rightarrow$ $\underline{\text{ko}}$ liên thông

Connected Components
(Thành phần liên thông)

Definition A connected component of a graph G is a maximal connected subgraph of G.

Example



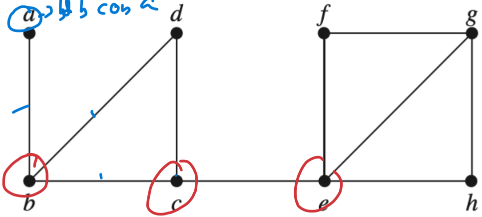
Cut Vertices
(Đỉnh cắt)

Definition Cut vertices are vertices if they are removed from a graph and all incident edges produces a subgraph with more connected component.

chừng nào bỏ đỉnh đi tạo thành liên thông mới gọi là đỉnh cắt

bỏ đỉnh và các cạnh liên quan

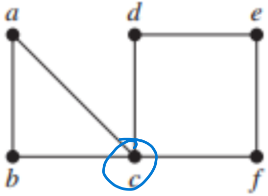
a → bỏ b còn a

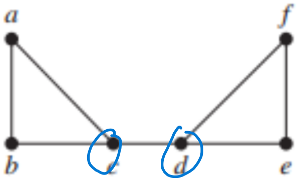


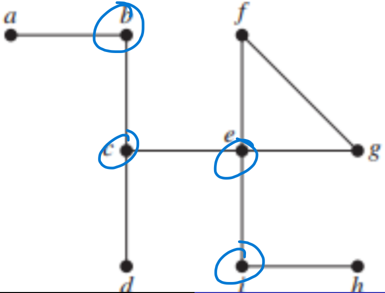
G_1

Exercise

Find all cut vertices of the graph below.



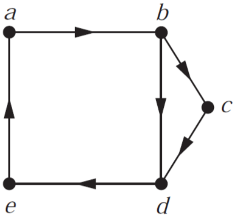




Strongly Connected : liên thông mạnh

Definition **A directed graph is strongly connected if there is a path from** a **to** b **and from** b **to** a **whenever** a **and** b **are vertices in the graph** : giữa 2 đỉnh bất kỳ luôn có đường đi chiều về

Example 1 Is the graph G below is strongly connected?

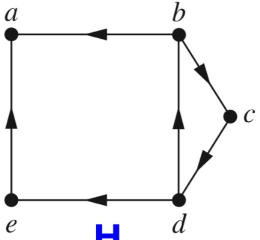


G

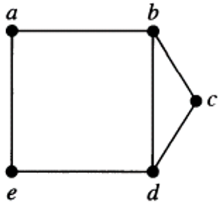
Weakly Connected

Definition **A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.**

Example The graph H below is weakly connected.



H



underlying undirected graph of H

nếu bỏ hướng
vẫn là liên thông

H: ***weakly connected*** ⬅ connected

Counting Paths Between Vertices

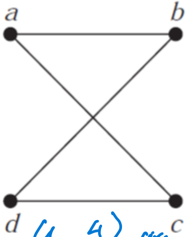
đếm số đường đi giữa các đỉnh (trả vấn đề)

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed).
The **number of different paths of length r from v_i to v_j $=$ (i, j)th entry of A^r**.

Example 15 How many paths of length four are there from a to d in the simple graph G below?

qua 4 cạnh



vị trí (1 4) của ma trận $A^4 = 8$

length 4, $a \to d$

→ thành lập ma trận liền kề:

$$\mathbf{A} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}. \quad \mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

$c \to c$, length 4: $(3,3)$ A^4

$b \to b$, length 2: $(2,2)$ A^2

length 4, $b \to c$: $(2,3)$: A^4: 8

The number of paths of length four from a to d is the (1,4)th entry A^4. It's 8.

Exercise

3. Find the number of paths of length n between two different vertices in K_4 if n is

a)2 b)3 c)4

4. Find the number of paths between c and d in the graph of length

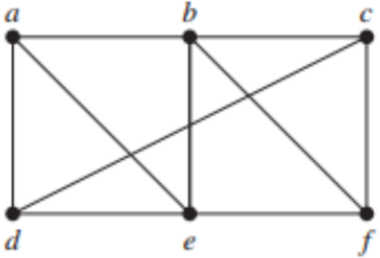
a) 2 b)3 c)4

lgth 2, $c \to d : (3,4) : A^2$.

hàng 3 × cột 4: $(0\ 1\ 0\ 1\ 0\ 1) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

lgth 3: $c \to d : (3,4) : A^3$

lgth 4: $c \to d\ (3,4) : A^4$



	a	b	c	d	e	f
a	0	1	0	1	1	0
b	1	0	1	0	1	1
c	0	1	0	1	0	1
d	1	0	1	0	1	0
e	1	1	0	1	0	1
f	0	1	1	0	1	0

Exercise

Let G be an undirected graph with 4 vertices A, B, C, D, whose adjacency matrix (in that order of vertices) is

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

How many paths of length 2 from B to D?

A.0 B.2 C.3 D.4

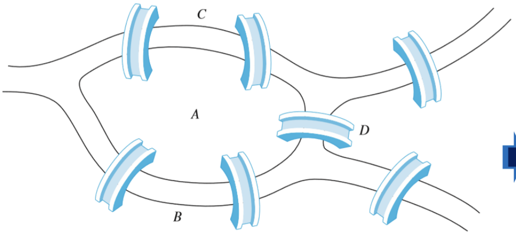
$(1\ 0\ 2\ 0)\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = (1 \times 1) + (2 \times 1) = 3$

$(2,4)$: A^2

→ cần hỏi A^2 phải bấm máy

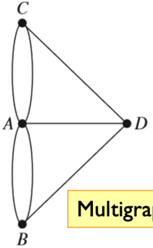
Euler and Hamilton paths - introduction

Can one travel across *all the bridges once* and *return* to the starting point?



The Seven Bridges of Königsberg.

LEONHARD **EULER**
(1707–1783)

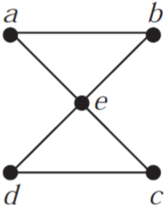


Multigraph Model

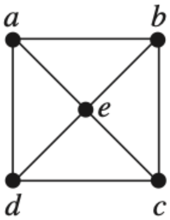
Euler circuit and Euler path
(Chu trình Euler và đường đi Euler)

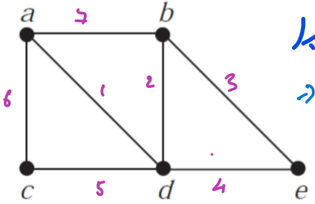
Definition

- An Euler circuit in a graph G is a simple circuit containing every edge of G.(đường đi khép kín và đi qua tất cả các cạnh của đồ thị đúng 1 lần)
- An Euler path in G is a simple path containing every edge of G.(đường đi (không nhất thiết khép kín) sao cho đi qua mọi cạnh của đồ thị đúng 1 lần)



G_1

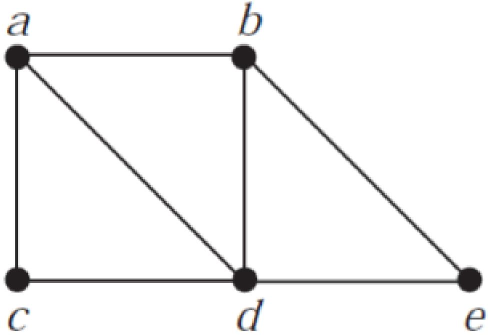




G_3

KV
→ Có chu trình Euler

Example 1 The graph G_3 below has no an Euler circuit, But it has an Euler path: a, c, d, e, b, d, a, b.



G_3

Euler circuit

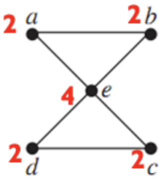
Theorem.

A *connected* multigraph, ≥ 2 vertices,

has an *Euler circuit* $\Leftrightarrow$ every vertex has **even degree**

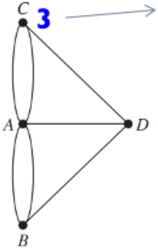
mọi đỉnh bậc chẵn

=> có chu trình Euler



G_1

OK



NOT OK

Euler path

Theorem.

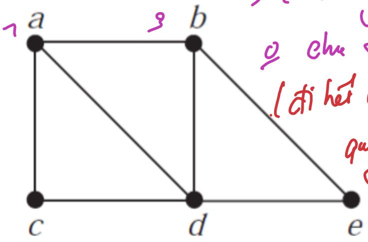
A connected multigraph has

an **Euler path** but *not an* Euler circuit

$\Leftrightarrow$

it has *exactly*

two vertices *of odd degree*



G_3

Có đúng 2 đỉnh có bậc lẻ

⇒ Có đường đi nhưng

o chu trình

(đi hết nhưng o

quay lại đ

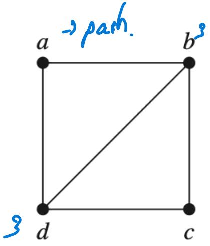
xuất phát)

Note that: an Euler circuit is also an Euler path

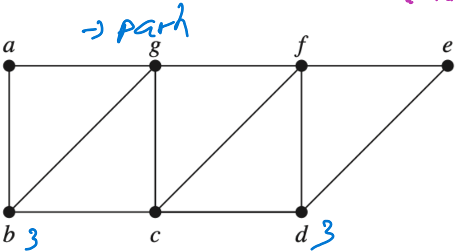
Exercise

Exist Euler circuit? Euler path?

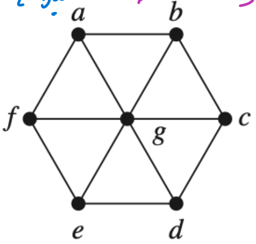
bậc lẻ > 2
có bậc chẵn ~~chẵn~~ 0 chẵn
0 có path + circuit.



G_1



G_2

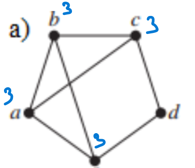


G_3

Exercise

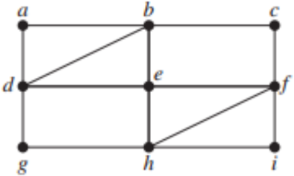
Exist Euler circuit? If yes, construct a such one.

a)



→ 0 path + 0 circuit

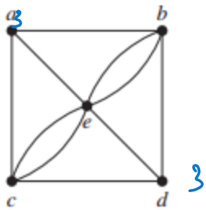
b)



→ all Even :

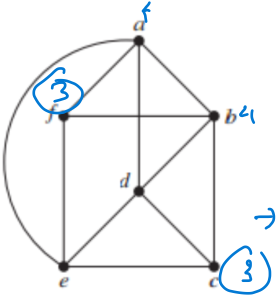
→ Circuit

c)



→ path

d)



→ path.

Exercise

3

For which values of n do these graphs have an Euler circuit?

a) K_n b) C_n c) W_n d) Q_n

(n lẻ) (∀n) (o có n) (n chẵn)

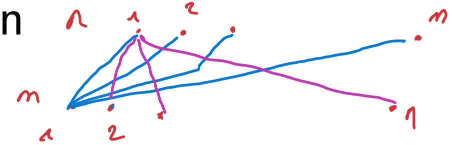
K (chéo bên trong)
C (chỉ có ngoài)
W (nối với • ở giữa)
Q. —

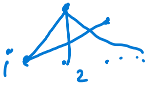
4

For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

a) Euler circuit { n = 2, n lẻ

b) Euler path : m lẻ





m lẻ

Hamilton Paths and Circuits

Definition

Hamilton path

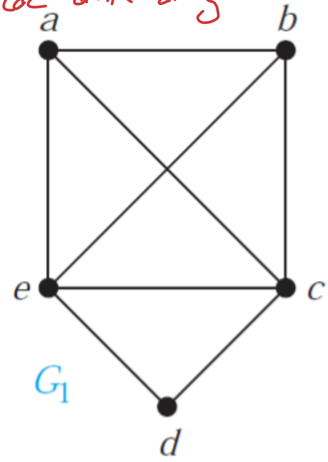
- simple path (đường đi không đi qua cạnh nào quá 1 lần)
- passes through every vertex exactly once (đi qua mỗi đỉnh đúng 1 lần)

Hamilton circuit

- simple circuit (đường đi khép kín không đi qua cạnh nào quá 1 lần)
- passes through every vertex exactly once (đi qua mỗi đỉnh đúng 1 lần)

Example 1 Exist a Hamilton circuit?

đi qua các đỉnh đúng 1 lần và quay lại

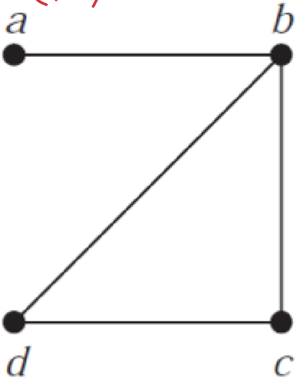


a → e → d → c → b → a

Example 2 Exist Hamilton circuit ? exist a Hamilton path? (đ. hết đỉnh)

(✗)

(✓) : v qua quá 1 lần



G_2

Hamilton circuits - sufficient conditions

Dirac's theorem.

G is a graph:
- *simple*
- n ($\geq$ 3) vertices
- $\forall v_i$, **$deg(v_i) \geq \frac{n}{2}$**

⇒ G has a ***Hamilton circuit***

nếu o thoả mãn ⟷ thì cũng chưa chắc nó o phải là chu trình Halmiton

Ore's theorem.

G is a graph:
- *simple*
- n ($\geq$ 3) vertices
- $\forall u, \forall v$, *non-adjacent*
  $deg(u) + deg(v) \geq n$

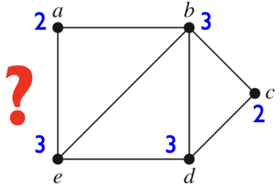
⇒ G has a ***Hamilton circuit***

Hamilton circuits - sufficient conditions

Dirac's theorem.

G is a graph:
- *simple*
- n ($\geq$ 3) vertices
- $\forall v_i$, $\mathbf{deg(v_i)} \geq \frac{n}{2}$

$\Rightarrow$ G has a ***Hamilton circuit***



Ore's theorem.

G is a graph:
- *simple*
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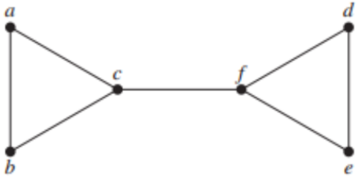
- $\forall u$, $\forall v$, *non-adjacent* $\mathbf{deg(u) + deg(v) \geq n}$

$\Rightarrow$ G has a ***Hamilton circuit***

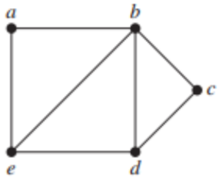
Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit.

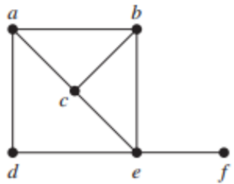
a)



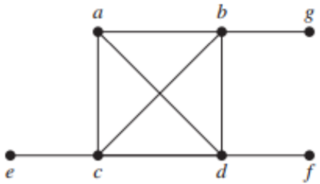
b)



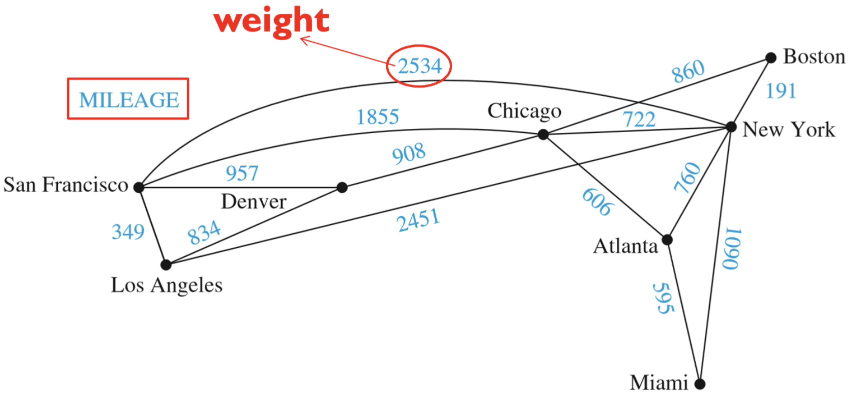
c)



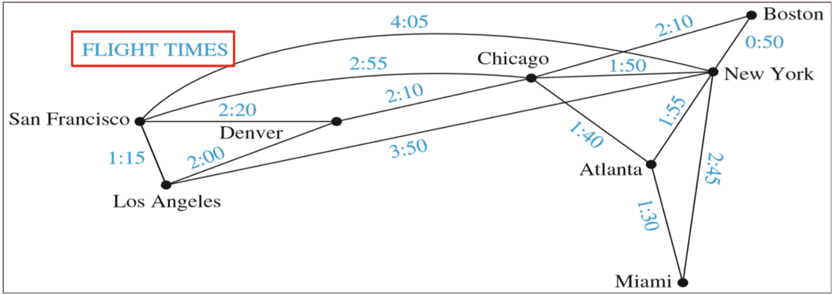
d)



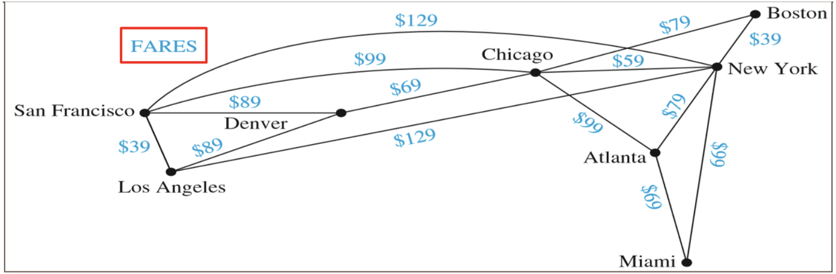
Shortest-path problems



Shortest-path problems



Shortest-path problems



Definition

- Weighted Graphs are graphs that have a number assigned to each edge.
- Length of a path in a weighted graph is the sum of the weights of the edges of this path.

Example

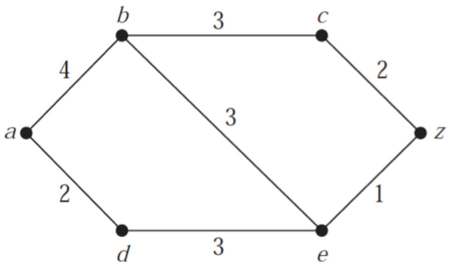
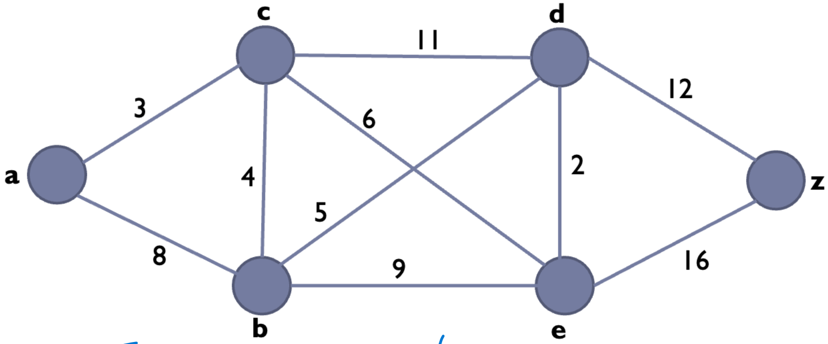


FIGURE 3 **A Weighted Simple Graph.**

The path a, b, e, z has length $= 4 + 3 + 1 = 8$.

Dijkstra's algorithm



vd: Đường đi ngắn nhất nối A -> Z, A -> D, A -> F ???

=> Dijkstra's Algorithm

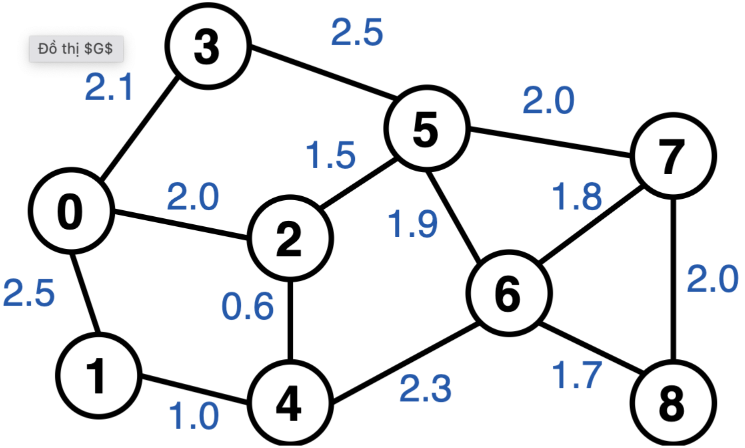
Dijkstra's Algorithm

Bước 1: Từ đỉnh gốc, khởi tạo khoảng cách tới chính nó là 0, khởi tạo khoảng cách nhỏ nhất ban đầu tới các đỉnh khác là $+\infty$ Ta được danh sách các khoảng cách tới các đỉnh.

Bước 2: Chọn đỉnh a có khoảng cách nhỏ nhất trong danh sách này và ghi nhận. Các lần sau sẽ không xét tới đỉnh này nữa.

Bước 3: Lần lượt xét các đỉnh kề b của đỉnh a. Nếu khoảng cách từ đỉnh gốc tới đỉnh b nhỏ hơn khoảng cách hiện tại đang được ghi nhận thì cập nhật giá trị và đỉnh kề a vào khoảng cách hiện tại của b.

Bước 4: Sau khi xét tất cả đỉnh kề b của đỉnh a. Lúc này ta được danh sách khoảng cách tới các điểm đã được cập nhật. Quay lại Bước 2 với danh sách này. Thuật toán kết thúc khi chọn được



Đồ thị G

0	1	2	3	4	5	6	7	8
0	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$
–	$(2.5, 0)$	$\mathbf{(2.0, 0)}$	$(2.1, 0)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$
–	$(2.5, 0)$	–	$\mathbf{(2.1, 0)}$	$(2.6, 2)$	$(3.5, 2)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$
–	$\mathbf{(2.5, 0)}$	–	–	$(2.6, 2)$	$(3.5, 2)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$
–	–	–	–	$\mathbf{(2.6, 2)}$	$(3.5, 2)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$
–	–	–	–	–	$\mathbf{(3.5, 2)}$	$(4.9, 4)$	$(\infty,-)$	$(\infty,-)$
–	–	–	–	–	–	$\mathbf{(4.9, 4)}$	$(5.5, 5)$	$(\infty,-)$
–	–	–	–	–	–	–	$\mathbf{(5.5, 5)}$	$(6.6, 6)$
–	–	–	–	–	–	–	–	$\mathbf{(6.6, 6)}$

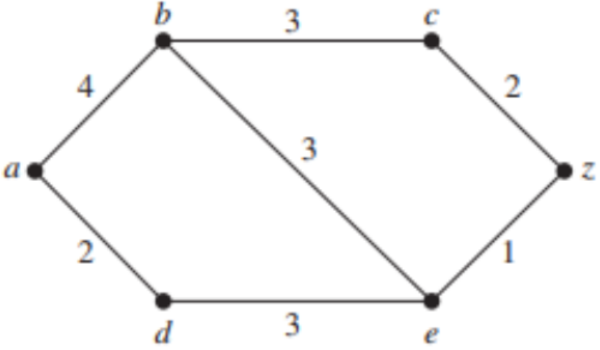
đi từ đỉnh 0 đến

đi từ đỉnh 2 → (4,5)

đi từ đỉnh 4 đến

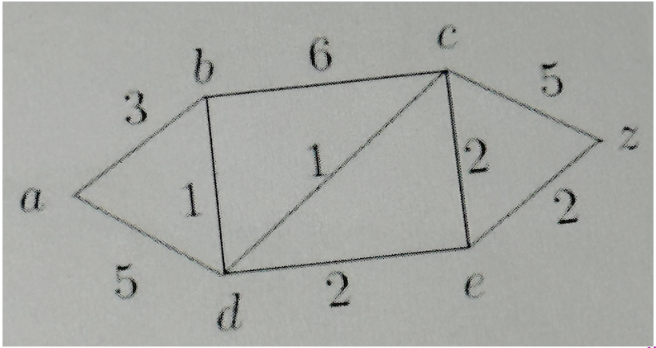
Exercise

1. What is the length of a shortest path between a and z in the weighted graph shown in the figure?



Exercise

Apply the Dijkstra algorithm to find the shortest path from A to Z. What are the first 4 vertices chosen? Để kiếm tìm đường đi ngắn 'I', nhưng thực chất là tìm các đỉnh gần I, II, III.



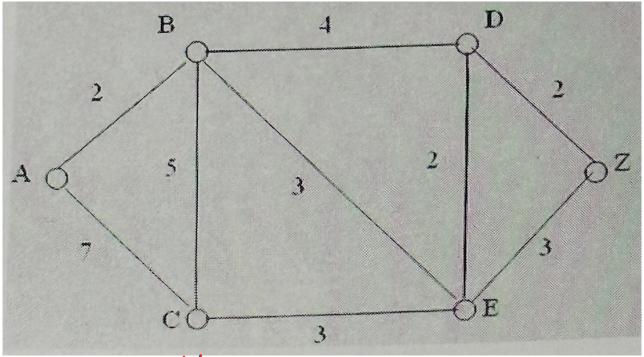
nhìn bảng

đỉnh b gần a nhất

A. a,d,b,c B. a,d,e,z D. a,b,c,z E. a,b,d,c

Exercise

Apply the Dijkstra algorithm to find the shortest path from A to Z. What are the first 4 vertices chosen?



chọn đỉnh gần I, II, III

A. A-B-C-D B. A-B-E-D ~~C. A-B-E-Z~~ D. A-B-D-Z

Exercise

2.Find a shortest path between a and z in the given weighted

Chọn đỉnh gần nhất, đỉnh gần nhì, ba . . .