Chapter 6: Counting

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Several examples of counting problems

- How many passwords consisting of 8 letters are there? (a letter can be chosen from $0, .., 9, a, \dots, z, A, \dots, Z$)
- How many bit strings of length 8 that do not have two consecutive 0s?
- How to determine the complexity of an recursive algorithm?

Basic Counting Principles

- The Product Rule
- The Sum Rule
- The Subtraction Rule

Quy tắc đếm

The Product Rule

2 giai đoạn

Suppose that a mission needs two stages to be completed.

The frist stage has n_1 ways to be done

The second stage has n_2 ways to be done

$n_1 \times n_2$

There are total $n_1 n_2$ ways to complete the mission.

Example 1

A new company with just two employees, An and Tuan, rent a floor of a building with 12 offices.

How many ways are there to assign different offices to these two employees?

Solution

Assigning the first person to an office: 12 ways

Assigning the second to other office: 11 ways

The answer is 132 ways.

Example 2

How many functions from a set A with 3 **elements** to a set B with 4 **elements**?

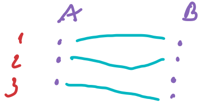
$\Rightarrow 4^3$ (b^a)

Solution

Assigning the 1st-element in A to an element in B: 4 ways
Assigning the 2nd-element in A to an element in B: 4 ways
Assigning the 3rd-element in A to an element in B: 4 ways
The answer is: $\boxed{4^3 = 64 \text{ ways}}$

Example 3

How many one-to-one functions are there from a set A with 3 **elements** to a set B with 4 **elements**?



one to one

Solution

1 thứ A gán duy I 1 thứ B

Assigning the $1^{st}-$ element in A to an element in B: 4 ways

Assigning the $2^{nd}-$element in A to an element in B: 3 ways

Assigning the $3^{rd}-$ element in A to an element in B: 2 ways

The answer is $2.3.4 = 24$ ways

Example 4 How many bijective functions from a set A with 3 **elements** to a set B with 3 **elements**?

Solution

Assigning the $1^{st}-$ element in A to an element in B: 3 ways

Assigning the $2^{nd}-$ element in A to an element in B: 2 ways

Assigning the $3^{rd}-$ element in A to an element in B: 1 ways

The answer is 6 ways

The Sum Rule

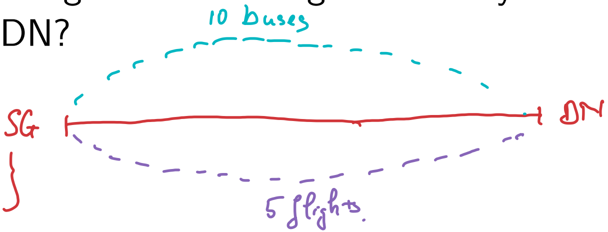
Suppose that a mission needs two solutions A or B to be completed.

- The solution A has n_1 ways to be done.
- The solution B has n_2 ways to be done.
- A và B are independent (cannot implement two solutions at the same time)

$n_1 + n_2$

There are total $n_1 + n_2$ ways to complete the mission.

Example 1 We can travel from HCM city to Da Nang city by bus or plane. There are 10 buses and 5 flights to Da Nang. How many ways to travel from HCM city to DN?



Solution

Traveling to DN by bus: 10 ways
Traveling to DN by plane: 5 ways
There are total 15 ways to travel from HCM city to Da Nang city.

The Subtraction Rule (Principle of Inclusion-Exclusion)

Suppose that a mission needs two solutions A and B to be completed.

- The solution A has n_1 ways to be done.
- The solution B has n_2 ways to be done.
- There are m ways to implement both at the same time

$(n_1 + n_2) - m$

There are $n_1 + n_2 - m$ ways to complete the mission.

$1 \le x \le N$: Có: $\left\lfloor \frac{N}{k} \right\rfloor$ $(1 \ldots 50 : k)$

$1 \le x \le 100$: $\vdots 2$ or $\vdots 3$ — $\left\lfloor \frac{100}{2} \right\rfloor = 50$, $\left\lfloor \frac{100}{3} \right\rfloor = 33$, $\vdots 2, \vdots 3$: $\left\lfloor \frac{100}{6} \right\rfloor = 16$ — $(50+33) - 16 = 67$

Example 1

How many bit strings of length eight either start with bit 1 or end with two bits 00?

Solution

Bit strings of length eight start with bit 1: 2^7 — 1xxxxxxx (7 bits)

Bit strings of length eight end with two bits 00 bit: 2^6 — xxxxxx00 (6 bits)

Bit strings of length eight start with bit 1 **and** end with two bits 00: 2^5 — $1\underbrace{xxxxx}_{2^5}00$

The answer is $2^7 + 2^6 - 2^5$

k° vượt quá 50

Example 2

How many positive integers not exceeding 50, which is divisible by 2 or 3?

Solution

Positive integers not exceeding 50 and they are divisible by 2: 25 $= \left\lfloor \frac{50}{2} \right\rfloor$ numbers

Positive integers not exceeding 50 and they are divisible by 3: 16 $= \left\lfloor \frac{50}{3} \right\rfloor$ numbers

Positive integers not exceeding 50 and they are divisible by both 2, 3: 8 numbers $\left\lfloor \frac{50}{2\times 3} \right\rfloor$

The answer is $\boxed{25+16-8} = 33$ numbers

Exercise $(1 \le x \le 999)$

How many positive integers less than 1000

a) are divisible by 7 $\left\lfloor \frac{999}{7} \right\rfloor = 142.$

$\underbrace{\le 999, \vdots 7, \not\vdots 11}$ $+ \le 999, \vdots 7, \vdots 11 = \le 999, \vdots 7$

$\left\lfloor \frac{999}{7 \times 11} \right\rfloor$ $\left\lfloor \frac{999}{7} \right\rfloor$

b) are divisible by 7 but not by 11?

c) are divisible by both 7 and 11? $\left\lfloor \frac{999}{7 \times 11} \right\rfloor$

d) are divisible by either 7 or 11? $\left\lfloor \frac{999}{7} \right\rfloor + \left\lfloor \frac{999}{11} \right\rfloor - \left\lfloor \frac{999}{7 \times 11} \right\rfloor$

e) are divisible by exactly one of 7 and 11? $\underbrace{\le 999, \vdots 7, \not\vdots 11} + \underbrace{\le 999, \vdots 11, \not\vdots 7}$

lấy $999 \vdots 7$ − $999 \vdots (7 \times 11)$ lấy $999 \vdots 11$ − $999 \vdots (7 \times 11)$

f) are divisible by neither 7 nor 11?

g) have distinct digits?

$999 - \le 999, \not\vdots 7$ and $\not\vdots 11$.

$= 999 - \le 999, \vdots 7$ OR $\vdots 11 = 999 - d$

Solution

a) $\leq 999, \vdots 7 = \lfloor \dfrac{999}{7} \rfloor$

b) $\leq 999, \vdots 7, NOT \vdots 111 = \lfloor \dfrac{1000}{7} \rfloor - \lfloor \dfrac{999}{77} \rfloor$

c) $\leq 999, \vdots 7, \vdots 11 = \lfloor \dfrac{999}{77} \rfloor$

d) $\leq 999, \vdots 7 OR \vdots 11 = \lfloor \dfrac{999}{7} \rfloor + \lfloor \dfrac{999}{11} \rfloor - \lfloor \dfrac{999}{77} \rfloor$

e)
$$\left(\leq 999, \vdots 7, NOT \vdots 11 \right) + \left(\leq 999, \vdots 11, NOT \vdots 7 \right) = \lfloor \frac{999}{7} \rfloor - \lfloor \frac{999}{77} \rfloor + \lfloor \frac{999}{11} \rfloor - \lfloor \frac{999}{77} \rfloor$$

f) $\leq 999, NOT \vdots 7, NOT \vdots 11 = 999 - \left(\leq 999, \vdots 7 OR \vdots\vdots 11 \right) = \ldots$

g) distinct digits $= 9 + 9.9 + 9.8 = \ldots$

Exercise

7 length

1. How many bit strings of length seven either begin with two 0s or end with three 1s?

$(2^5 + 2^4) - 2^2$

2. How many bit strings of length 10 either begin with three 0s or end with two 0s?

$(2^7 + 2^8) - 2^5$

3. Let B be the set $\{\underbrace{a, b, c}_{3}\}$. How many functions are there from $\underbrace{B^2}_{a=9}$ to $\underbrace{B}_{b}$?

$\Rightarrow b^a =$

Solution

1. $2^5 + 2^4 - 2^2$ 00**111

2. $2^7 + 2^8 - 2^5$ 000*****00

3. There are $3^2 = 9$ elements in B^2.
So there are total 3^9 functions from a set consisting of 9 elements to a set consisting of 3 elements.

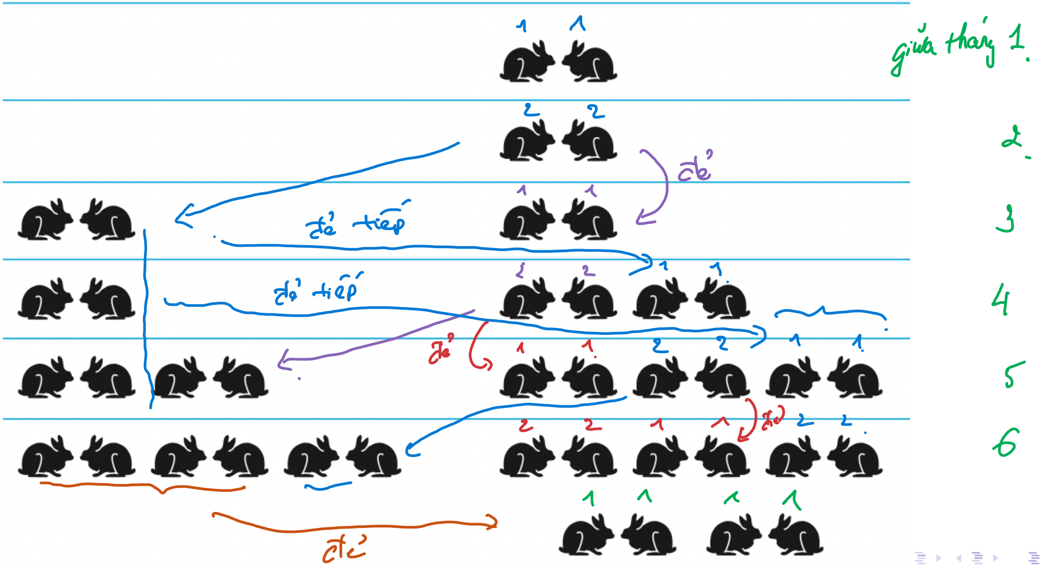
Advance Counting Techniques

- Applying recurrence relations for advance counting problems
- Applying Divide-and-Conquer Algorithms to determine complexity of a recursive algorithm.

Dãy số Fibonacci bắt nguồn từ bài toán cổ về việc sinh sản của các cặp thỏ. Bài toán đặt ra như sau:
1) Các con thỏ không bao giờ chết
2) Hai tháng sau khi ra đời, mỗi cặp thỏ mới sẽ sinh ra một cặp thỏ con (một đực, một cái)
3) Khi đã sinh con rồi thì cứ mỗi tháng tiếp theo chúng lại sinh được một cặp con mới
Giả sử từ đầu tháng 1 có một cặp mới ra đời thì đến giữa tháng thứ n sẽ có bao nhiêu cặp.

Reproducing pairs (at least two months old) ≥ 2 months

Young pairs (less than two months old) < 2 month.



Solution

Gọi a_n là số cặp thỏ giữa tháng thứ n. Ta có

$$\begin{cases} a_n &= a_{n-1} + a_{n-2}, \quad \forall n \geq 3 \\ a_1 &= 1, a_2 = 1 \end{cases}$$

Từ quan hệ trên, ta có thể tính

$$a_3 = a_1 + a_2 = 2, a_4 = a_3 + a_2 = 3, a_5 = a_4 + a_3 = 5, \ldots$$

Example

How many bit strings of length 8 that do not have two consecutive 0s?

2 số 0 liền kề

bit strings of length 1 that do not have two consecutive 0   1, 0 : 2

bit strings of length 2 that do not have two consecutive 0 (11) (10) (01) : 3

bit strings of length 3 that do not have two consecutive 0

(111) (110) (101) (010) (011) : 5

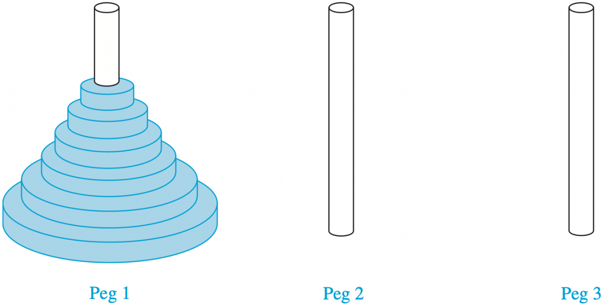
Solution

Let a_n be the number of bit strings of length n that do not have two consecutive 0s.

$$\begin{cases} a_n & = a_{n-1} + a_{n-2}, n \geq 3 \\ a_1 & = 2, a_2 = 3 \end{cases}$$

We have

$$a_3 = a_1 + a_2 = 5, a_4 = a_3 + a_2 = 8, a_5 = 13, a_6 = 21, a_7 = 34, a_8 = 55$$

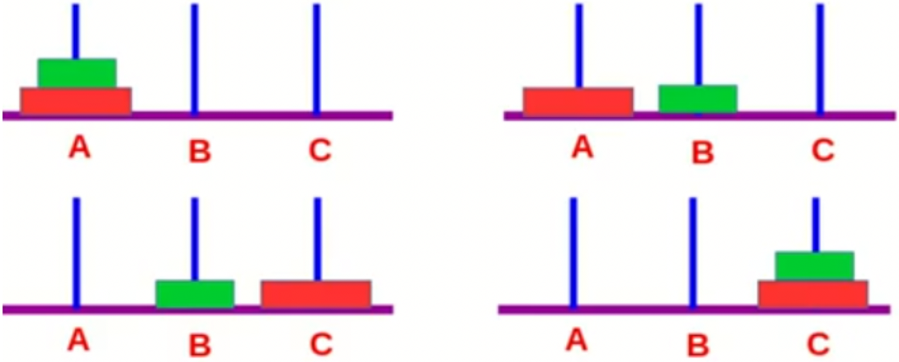


Question How can we move the disks to the third peg, one a time, following the rule: larger disks are never placed on top of smaller ones?

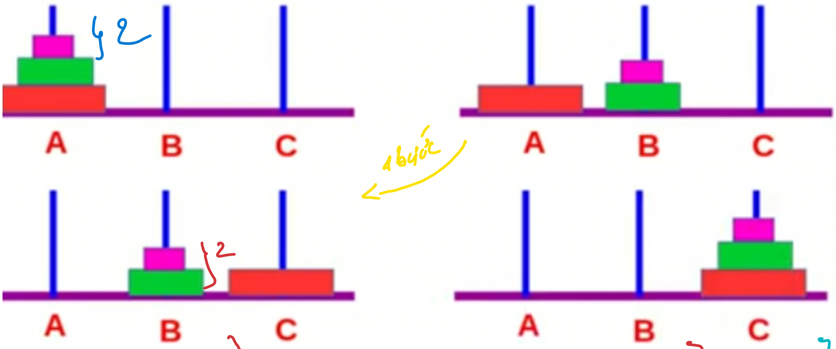
Let H_n denote the number of moves needed to solve the Tower of Hanoi puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$

- $n = 1$

$H_1 = 1$


A
B
C
A
B
C
A
B
C
A
B
C

3 bước.



2

1 bước

2

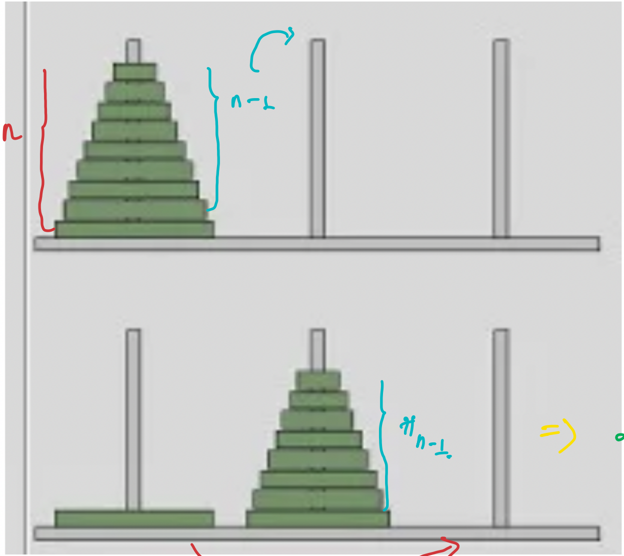
3 bước.

$\Rightarrow H_3 = 2H_2 + 1$

$H_4 = 2H_3 + 1$

$= 7$ bước.

$\boxed{H_n = 2H_{n-1} + 1}$



$\Rightarrow 2H_{n-1} + 1.$

+1 bước $\quad H_{n-1}.$

Solution Let h_n be number of moves needed to solve the problem with n disks. We have

$$\begin{cases} H_n = 2H_{n-1} + 1 \\ H_1 = 1 \end{cases}$$

From this recurrence relation, we can prove by induction that $H_n = 2^n - 1, \forall n \geq 1$. In fact, Let $P(n)$: "$H_n = 2^n - 1$".
basis step: $P(1)$: "$H_1 = 2^1 - 1$" (T)
inductive step: Suppose $P(k)$: "$H_k = 2^k - 1$" is True. We need to prove that $P(k+1)$: "$H_{k+1} = 2^{k+1} - 1$" also True.
We have

$$H_{k+1} = 2H_k - 1 = 2.(2^k - 1) - 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$$

"Cách tính độ phức tạp thuật toán đệ quy

"Chia để trị"

Divide-and-Conquer Algorithms and recurrence relations

- Divide: dividing a problem into one or more instances of the same problem of smaller size.
- Conquer: Using the solutions of the smaller problems to find a solution of the original problem, perhaps with some additional work

Divide-and-Conquer Recurrence Relations

- n : size of the original problem
- n/b : size of the sub-problem
- f(n) : number of operation required to solve the original problem.
- → f(n/b) : number of operation required to solve a sub-problem.
- g(n): overhead for additional work of the step **conquer.**
- Divide-and-conquer recurrence relation:

$$f(n) = af(n/b) + g(n)$$

A Recursive Binary Search Algorithm

procedure *binary search*$(i, j, x$: i, j, x integers, $1 \leq i \leq j \leq n)$
$m := \lfloor (i + j)/2 \rfloor$
if $x = a_m$ **then**
    return m
else if $(x < a_m$ and $i < m)$ **then**
    return *binary search*$(i, m - 1, x)$
else if $(x > a_m$ and $j > m)$ **then**
    return *binary search*$(m + 1, j, x)$
else return 0
{output is location of x in $a_1, a_2, \ldots, a_n$ if it appears; otherwise it is 0}

Recurrence Relations for Finding Maximum of a sequence

procedure max(i,j: integer ,a_i,a_{2+1},…,a_j: integers)
if i=j **then**
  begin
    max:= a_i
end
 else
   begin
    m= $\lfloor (i+j)/2 \rfloor$
    max1= max (i,m,a_i,a_{i+1},…,a_m)
    max2= max (m+1,j,a_{m+1},a_{m+2},…,a_j)
   if max1>max2 **then** max:= max1
   else max:=max2

$$f(n) = 2f(n/2) + 1$$

Theorem 1

- Let f be an **increasing function** that satisfies the recurrence relation f(n)= af(n/b) + c

whenever n is divisible by b, where $a \geq 1$, b is an integer and greater than 1, and c is a positive real number. **Then**

$$f(n)\ is \begin{cases} O(n^{\log_b a}) \text{ if } a>1 & \text{TH1} \\ O(\log n) \text{ if } a=1 & \text{TH2} \end{cases}$$

Furthermore, when $n=b^k$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2$$

Where C_1=f(1) + c/(a-1) and C_2= -c/(a-1)

Exercise

Suppose that f is an increasing function and satisfies

$$f(n) = \underbrace{5}_{a}f(n/\underbrace{2}_{b}) + \underbrace{3}_{c} \text{ if } n \vdots 2,\ f(1) = 7$$

Give a big-O estimate for $f(n)$.

TH1: $a > 1 \to O(n^{\log_b a})$

TH2 $a = 1$

Exercise

Suppose that f is an increasing function and satisfies

$$f(n) = f(n/3) + 4 \text{ if } n \vdots 3, f(1) = 3$$

$a=1$ $b=3$ $c=4$ → TH2 : $O(\log n)$

$(a=1)$

Give a big-O estimate for $f(n)$.

Theorem 2: Master Theorem

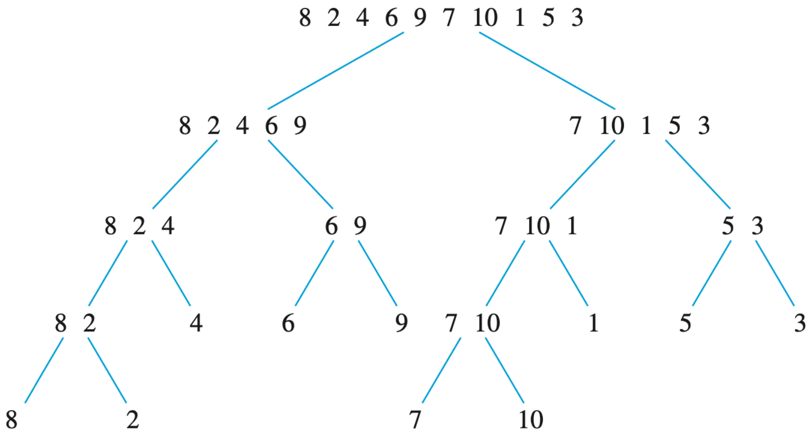
Let f be an **increasing function that satisfies the recurrence relation $f(n) = af(n/b) + cn^d$**

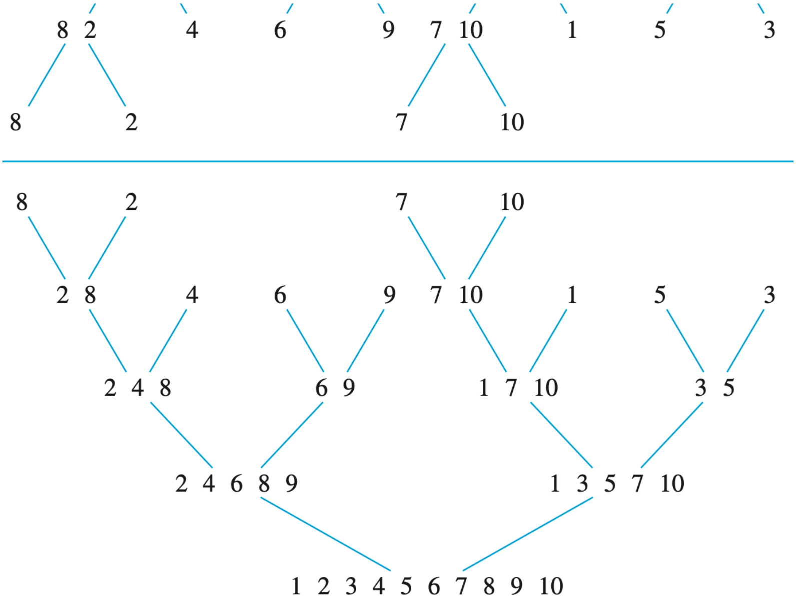
Whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

xd O(merge sort)

$$f(n) \text{ is } \begin{cases} O(n^d) \text{ if } a < b^d & \text{: TH1} \\ O(n^d \log n) \text{ if } a = b^d & \text{: TH2} \\ O(n^{\log_b a}) \text{ if } a > b^d & \text{: TH3} \end{cases}$$

Merge Sort

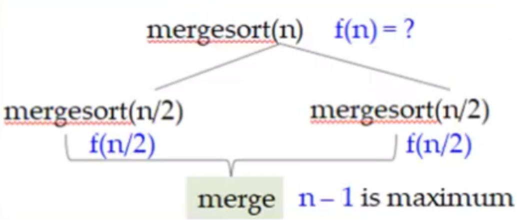




8 2 4 6 9 7 10 1 5 3
8 2 7 10
8 2 7 10
2 8 4 6 9 7 10 1 5 3
2 4 8 6 9 1 7 10 3 5
2 4 6 8 9 1 3 5 7 10
1 2 3 4 5 6 7 8 9 10

ALGORITHM 9 A Recursive Merge Sort.

procedure *mergesort*$(L = a_1, \ldots, a_n)$
if $n > 1$ **then**
    $m := \lfloor n/2 \rfloor$
    $L_1 := a_1, a_2, \ldots, a_m$
    $L_2 := a_{m+1}, a_{m+2}, \ldots, a_n$
    $L := merge(mergesort(L_1),\ mergesort(L_2))$
$\{L$ is now sorted into elements in nondecreasing order$\}$


mergesort(n)
f(n) = ?
mergesort(n/2)
mergesort(n/2)
f(n/2)
f(n/2)
merge
n – 1 is maximum

Exercise

Let f be an increasing function that satisfies

$$f(n) = 5f\left(\frac{n}{2}\right) + cn^2$$

$a = 5$, $b = 2$, $c = C$, $d = 2$.

Give a big-O estimate for $f(n)$

$\text{Vì } a = 5 > b^d = 4 \rightarrow O\left(n^{\log_b a}\right)$

$\text{TH3}:$ $= O\left(n^{\log_2 5}\right)$

Exercise

$f(3) = f(1) + 1 = 1 + 1 = 2$

2. Suppose that $f(n) = f(n/3) + 1$ when n is a positive integer divisible by 3, and $f(1) = 1$. Find

$f(9) = f(3) + 1 = 2 + 1$

a) f(3) b) f(27) c) f(729)

dãy $3^{mũ}$

3.Suppose that $f(n) = f(n/5) + 3n$ when n is a positive integer divisible by 5, and $f(1) = 4$. Find

$f(27) = f(3^3) + 1$

a) f(5) b) f(125) c) f(3125)

dãy $5^{mũ}$

$= f(3^2) + 1 + 1$

$= (f(3) + 1) + 1$

Exercise

In a weeding, how many ways to arrange 5 people in a row (including the bride and the groom), so that the bride stands somewhere on the right to the groom?

A.120 B.60 C.100 D.10

Exercise

A soccer tournament has 20 teams. Each team plays each other twice (first leg and second leg). How many games are there in the tournament?

A. 40 B.400 C.380 D.190

1 team → 19

× 2 lượt 2

38 × 20

→ × 20 team

Exercise

A yong pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 3 month old. After they are 3 month old they will produce 1 pair of rabbits each month. Let f_n be the number of pairs of rabbits after n months.

Find a recursive relation for f_n.

i) $f_n = f_{n-1} + f_{n-3}$

ii) $f_n = f_{n-1} + f_{n-2}$

iii) $f_n = f_{n-1} + f_{n-2} + f_{n-3}$

A.i B.ii C.iii D.None

Exercise

Suppose that the number of ways to solve an n-piece puzzle satisfies the recurrence relation

$$f_n = f_{n-1} + nf_{n-3}$$

with the initial conditions

$$f_1 = 1, f_2 = 2, f_3 = 3$$

How many ways are there to solve a 6-piece puzzle of this kind?

A.21 B.15 C.18 D.24 E.None

Exercise

How many bit strings of length 9 that do not have two consecutive 0s?

A. 34 B.55 C.89 D.144 E.None

Exercise

How many bit strings of length 4 that do have three consecutive 0s?

A.10 B.13 C.20 D.8

Exercise

How many positive integers not exceeding 200 and divisible by 5 or 7 ?

A.65 B.64 C.66 D.63 E. None

Exercise

How many positive integers less than 500 which are divisible by exactly one of 5 and 8?

A. 137 B. 138 C. 139 D. 149