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Class: SE 1871

Discrete Mathematics and Its Applications

Exercise Book

Trần Trọng Huỳnh - 2020

Chapter 1: The Foundations: Logic and Proofs

1.1 Propositional Logic

1. Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Boston is the capital of Massachusetts.

b) Miami is the capital of Florida.

c) 2 + 3 = 5.

d) 5 + 7 = 10.

e) $x + 2 = 11$.

f) Answer this question.

2. What is the negation of each of these propositions?

a) Mei has an MP3 player. → doesn't have

b) There is no pollution in New Jersey. → pollution

c) 2 + 1 = 3. → ≠

d) The summer in Maine is hot and sunny. → not + or

3. Let p and q be the propositions

p : I bought a lottery ticket this week.

q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$ not

b) $p \vee q$ or

c) $p \rightarrow q$ if p, q

d) $p \wedge q$ and

e) $p \leftrightarrow q$ if and only if

f) $\neg p \rightarrow \neg q$ if not, I won't

g) $\neg p \wedge \neg q$
not and won't

h) $\neg p \vee (p \wedge q)$
not or p and q

4. Let p and q be the propositions

p : It is below freezing.

q : It is snowing.

Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing and snowing.

b) It is below freezing but not snowing.

c) It is not below freezing and it is not snowing.

d) It is either snowing or below freezing (or both).

5. Let p and q be the propositions

p : You drive over 65 miles per hour.

q : You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).

a) You do not drive over 65 miles per hour.

b) You drive over 65 miles per hour, but you do not get a speeding ticket.

c) You will get a speeding ticket if you drive over 65 miles per hour.

d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.

e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.

f) You get a speeding ticket, but you do not drive over 65 miles per hour.

g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

6. Determine whether these biconditionals are true or false.

a) 2 + 2 = 4 if and only if 1 + 1 = 2. → True

T ⟷ T

b) 1 + 1 = 2 if and only if 2 + 3 = 4. → False

T ⟷ F

c) 1 + 1 = 3 if and only if monkeys can fly. → True

F ⟷ F

d) 0 > 1 if and only if 2 > 1. → False

F ⟷ T

7. Determine whether each of these conditional statements is true or false.

a) If 1 + 1 = 2, then 2 + 2 = 5. → False

T → F

b) If 1 + 1 = 3, then 2 + 2 = 4. → True

F → T

c) If 1 + 1 = 3, then 2 + 2 = 5. → True

F → F

d) If monkeys can fly, then 1 + 1 = 3. → True

F → F

8. Determine whether each of these conditional statements is true or false.

a) If 1 + 1 = 3, then unicorns exist. : True

F →

b) If 1 + 1 = 3, then dogs can fly. → True

F

c) If 1 + 1 = 2, then dogs can fly. : False

T → F

d) If 2 + 2 = 4, then 1 + 2 = 3. → True.

T → T

9. Write each of these statements in the form “if p, then q” p → q sufficient necessary.

a) It is necessary to wash the boss’s car to get promoted.
q p

b) Winds from the south imply a spring thaw.
p q

c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.
q p

d) Willy gets caught whenever he cheats.
q p

e) You can access the website only if you pay a subscription fee.
q p

f) Getting elected follows from knowing the right people.
q p

g) Carol gets seasick whenever she is on a boat.
q p

10. How many rows appear in a truth table for each of these compound propositions?

a) p → ¬p 2^1 b) (p ∨ ¬r) ∧ (q ∨ ¬s) 2^4 c) q ∨ p ∨ ¬s ∨ ¬r ∨ ¬t ∨ u 2^6

d) (p ∧ r ∧ t) ↔ (q ∧ t) 2^4 e) (q → ¬p) ∨ (¬p → ¬q) 2^2 f) (p ∨ ¬t) ∧ (p ∨ ¬s) 2^3

g) (p → r) ∨ (¬s → ¬t) ∨ (¬u → v) 2^6 h) (p ∧ r ∧ s) ∨ (q ∧ t) ∨ (r ∧ ¬t) 2^5

11. Construct a truth table for each of these compound propositions.

a) p ∧ ¬p b) p ∨ ¬p c) (p ∨ ¬q) → q d) (p ∨ q) → (p ∧ q)

e) (p → q) ↔ (¬q → ¬p) f) (p → q) → (q → p)

12. Construct a truth table for each of these compound propositions.

a) p → ¬p b) p ↔ ¬p c) p ⊕ (p ∨ q) d) (p ∧ q) → (p ∨ q)

e) (q → ¬p) ↔ (p ↔ q) f) (p ↔ q) ⊕ (p ↔ ¬q)

13. What is the value of x after each of these statements is encountered in a computer program, if x = 1 before the statement is reached?

a) if x + 2 = 3 then x := x + 1 = 2
T

b) if (x + 1 = 3) OR (2x + 2 = 3) then x := x + 1
F F = ~~x~~ = 1

c) if (2x + 3 = 5) AND (3x + 4 = 7) then x := x + 1
T T = 2

d) if (x + 1 = 2) XOR (x + 2 = 3) then x := x + 1
T ⊕ T → False x = 1

$+$ $\times$

14. Find the bitwise OR, bitwise AND, and bitwise XOR of each of these pairs of bit strings.

a) 101 1110, 010 0001

b) 1111 0000, 1010 1010

c) 00 0111 0001, 10 0100 1000

d) 11 1111 1111, 00 0000 0000

15. Evaluate each of these expressions.

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$

a) 1 1000 ∧ (0 1011 ∨ 1 1011)

b) (0 1111 ∧ 1 0101) ∨ 0 1000

c) (0 1010 ⊕ 1 1011) ⊕ 0 1000

d) (1 1011 ∨ 0 1010) ∧ (1 0001 ∨ 1 1011)

$(p \vee q) \wedge (p \vee r) \equiv p \vee (q \wedge r)$

1.2-Propositional Equivalences

always true

1. Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \wedge q) \rightarrow p$　　b) $p \rightarrow (p \vee q)$　　c) $\neg p \rightarrow (p \rightarrow q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$　　e) $\neg(p \rightarrow q) \rightarrow p$　　f) $\neg(p \rightarrow q) \rightarrow \neg q$

2. Show that each of these conditional statements is a tautology by using truth tables.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$　　b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

c) $[p \wedge (p \rightarrow q)] \rightarrow q$　　d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

3. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.

4. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

5. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology. : always True.

6. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent. : tương đương

7. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

8. Show that $(p \rightarrow q) \rightarrow (r \rightarrow s)$ and $(p \rightarrow r) \rightarrow (q \rightarrow s)$ are not logically equivalent.

9. Find the dual of each of these compound propositions.

a) $p \vee \neg q$　　b) $p \wedge (q \vee (r \wedge T))$　　c) $(p \wedge \neg q) \vee (q \wedge F)$　　0 ra thi

$\bar{p} \wedge q$　　$\bar{p} \vee (\bar{q} \wedge (\bar{r} \vee \bar{T}))$　　$(\bar{p} \vee q) \wedge (\bar{q} \vee \bar{F})$

1.4 Predicates and Quantifiers

1. Let P(x) denote the statement "$x \leq 4$". What are these truth values?

a) P(0) b) P(4) c) P(6)

2. Let P(x) be the statement "the word x contains the letter a." What are these truth values?

a) P (orange) b) P (lemon) c) P (true) d) P (false)

3. Let Q(x, y) denote the statement "x is the capital of y". What are these truth values?

a) Q(Denver, Colorado) b) Q(Detroit, Michigan)

c) Q(Massachusetts, Boston) d) Q(New York, New York)

4. State the value of x after the statement if P(x) then x := 1is executed, where P(x) is the statement "$x > 1$", if the value of x when this statement is reached is

a) $x = 0$. b) $x = 1$. c) $x = 2$.

5. Let P(x) be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$

6. Translate these statements into English, where C(x) is "x is a comedian" and F(x) is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$ b) $\forall x(C(x) \wedge F(x))$

c) $\exists x(C(x) \rightarrow F(x))$ d) $\exists x(C(x) \wedge F(x))$

7. Let P(x) be the statement "$x = x^2$". If the domain consists of the integers, what are these truth values?

a) P(0) b) P(1) c) P(2)

d) P(−1) e) $\exists x P(x)$ f) $\forall x P(x)$

8. Let Q(x) be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) Q(0) b) Q(−1) c) Q(1)

d) $\exists x Q(x)$ e) $\forall x Q(x)$ f) $\exists x \neg Q(x)$

9. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$ b) $\exists n(2n = 3n)$ c) $\exists n(n = -n)$ d) $\forall n(3n \leq 4n)$

10. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$ b) $\exists x(x^4 < x^2)$

c) $\forall x((-x)^2 = x^2)$ d) $\forall x(2x > x)$

11. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n(n^2 \geq 0)$ b) $\exists n(n^2 = 2)$ c) $\forall n(n^2 \geq n)$ d) $\exists n(n^2 < 0)$

12. Suppose that the domain of the propositional function P(x) consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists xP(x)$ b) $\forall xP(x)$ c) $\exists x\neg P(x)$

d) $\forall x\neg P(x)$ e) $\neg\exists xP(x)$ f) $\neg\forall xP(x)$

13. Suppose that the domain of the propositional function P(x) consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists xP(x)$ b) $\forall xP(x)$ c) $\neg\exists xP(x)$ d) $\neg\forall xP(x)$

1.5 Rules of Inference

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.

Socrates is human.

∴ Socrates is mortal.

2. Use rules of inference to show that the hypotheses “Randy works hard”, “If Randy works hard, then he is a dull boy”, and “If Randy is a dull boy, then he will not get the job” imply the conclusion “Randy will not get the job”.

3. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) “If I take the day off, it either rains or snows”. “I took Tuesday off or I took Thursday off”. “It was sunny on Tuesday”. “It did not snow on Thursday”.

b) “If I eat spicy foods, then I have strange dreams”. “I have strange dreams if there is thunder while I sleep”. “I did not have strange dreams”.

c) “I am either clever or lucky”. “I am not lucky”. “If I am lucky, then I will win the lottery”

d) “Every computer science major has a personal computer”. “Ralph does not have a personal computer”. “Ann has a personal computer”.

e) “What is good for corporations is good for the United States”. “What is good for the United States is good for you”. “What is good for corporations is for you to buy lots of stuff”.

f) “All rodents gnaw their food”. “Mice are rodents”. “Rabbits do not gnaw their food”. “Bats are not rodents.

4. Determine whether each of the following arguments is valid or not valid.

a) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.

b) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

c) No man is an island. Manhattan is an island. Therefore, Manhattan is not a man.

d) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.

e) If Mai knows French, Mai is smart. But Mai doesn't know French. So, she is not smart.

f) Lin can't go fishing if she doesn't have a bike. Last week, Lin went fishing with her friends. Therefore, she has got a bike.

Chapter 2: Basic Structures: Sets, Functions, Sequences, and Sums

2.1 Sets

1. List the members of these sets.

a) {x | x is a real number such that $x^2 = 1$}

b) {x | x is a positive integer less than 12}

c) {x | x is the square of an integer and x < 100}

d) {x | x is an integer such that $x^2 = 2$}

2. For each of the following sets, determine whether 2 is an element of that set.

a) {x ∈ R | x is an integer greater than 1} b) {x ∈ R | x is the square of an integer}

c) {2,{2}} d) {{2},{{2}}} e) {{2},{2,{2}}} f) {{{2}}}

3. Determine whether each of these statements is true or false.

a) 0 ∈ ∅ b) ∅ ∈ {0} c) {0} ⊂ ∅ d) ∅ ⊂ {0}

e) {0} ∈ {0} f) {0} ⊂ {0} g) {∅} ⊆ {∅}

4. Determine whether each of these statements is true or false.

a) x ∈ {x} b) {x} ⊆ {x} c) {x} ∈ {x}

d) {x} ∈ {{x}} e) ∅ ⊆ {x} f) ∅ ∈ {x}

5. What is the cardinality of each of these sets?

a) {a} b) {{a}} c) {a, {a}} d) {a, {a}, {a, {a}}}

6. What is the cardinality of each of these sets?

a) ∅ b) {∅} c) {∅, {∅}} d) {∅, {∅}, {∅, {∅}}}

7. Find the power set of each of these sets, where a and b are distinct elements.

a) {a} b) {a, b} c) {∅, {∅}}

8. How many elements does each of these sets have where a and b are distinct elements?

a) P({a, b, {a, b}}) b) P({∅, a, {a}, {{a}}}) c) P(P(∅))

9. Find A^2 and A^3 if

a) A = {1, 3} b) A = {1, a }

10. Let $A=\{1,2,3\}$ and $B=\{1,a\}$. What is the cardinality of each of these sets?

a) $A\times B$ b) A^2 c) $P(B)$ d) $P(B\times A)$ e) $A\cup B$

11. Find the truth set of each of these predicates where the domain is the set of integers.

a) P (x): $x^2 < 3$ b) Q(x): $x^2 > x$ c) R(x): $2x + 1 = 0$

2.2 Set operations

1. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$ b) $A \cap B$ c) $A - B$ d) $B - A$.

2. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$ b) $A \cap B$ c) $A - B$ d) $B - A$.

3. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

4. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$ b) $A \subseteq (A \cup B)$ c) $A - B \subseteq A$

d) $A \cap (B - A) = \emptyset$ e) $A \cup (B - A) = A \cup B$ f) $A \oplus B = (A \cup B) - (A \cap B)$.

5. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the i^{th} bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$ b) $\{1, 3, 6, 10\}$ c) $\{2, 3, 4, 7, 8, 9\}$

2.3 Functions

1. Why is f not a function from R to R if

a) f (x) = 1/x? b) f (x) = $\sqrt{x}$? c) f (x) = $\pm\sqrt{x^2+1}$?

2. Determine whether f is a function from Z to R if

a) f (n) = ±n b) $f(n)=\sqrt{n^2+1}$ c) $f(n)=\frac{1}{n^2-4}$

3. Find these values

a) $\lceil 1.1 \rceil$ b) $\lceil -0.1 \rceil$ c) $\lceil 4 \rceil$ d) $\lfloor 3.2 \rfloor$

e) $\lfloor -5.2 \rfloor$ e) $\lfloor 2 \rfloor$ e) $\left\lfloor \frac{1}{2}+\left\lceil \frac{2}{3} \right\rceil \right\rfloor$

4. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one (onto)

a) f (a) = b, f (b) = a, f (c) = c, f (d) = d

b) f (a) = b, f (b) = b, f (c) = d, f (d) = c

c) f (a) = d, f (b) = b, f (c) = c, f (d) = d

5. Determine whether each of these functions from Z to Z is one-to-one (onto)

a) f (n) = n − 1 b) f (n) = n^2 + 1 c) f (n) = n^3 d) $f(n)=\left\lceil \frac{n}{2} \right\rceil$

6. Determine whether f : Z × Z → Z is onto if

a) f (m, n) = 2m − n b) f (m, n) = $m^2 - n^2$ c) f (m, n) = m + n + 1

d) f (m, n) = |m| − |n| e) f (m, n) = $m^2 - 4$ f) f (m, n) = m + n

7. Determine whether each of these functions is a bijection from R to R.

a) f (x) = −3x + 4 b) f (x) = $-3x^2 + 7$ c) f (x) = (x + 1)/(x + 2) d) f (x) = $x^5 + 1$

8. Let S = {−1, 0, 2, 4, 7}. Find f (S) if

a) f (x) = 1 b) f (x) = 2x + 1 c) $f(x)=\left\lceil \frac{x}{5} \right\rceil$

9. Let f be the function from R to R defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$ b) $f^{-1}(\{x \mid 0 < x < 1\})$ c) $f^{-1}(\{x \mid x > 4\})$

2.4 Sequences and Summations

1. 1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2\ (-3)^n + 5n$.

a) a_0 b) a_1 c) a_4 d) a_5

2. What is the term a8 of the sequence {an} if an equals

a) $2n-1$? b) 7? c) $1 + (-1)^n$? d) $-(-2)^n$?

3. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$ b) $a_n = a_{n-1}^2, a_1 = 2$ c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

4. Find the solution to each of these recurrence relations and initial conditions.

a) $a_n = -a_{n-1}$, $a_0 = 5$ b) $a_n = a_{n-1} + 3$, $a_0 = 1$ c) $a_n = a_{n-1} - n$, $a_0 = 4$

d) $a_n = 2na_{n-1}$, $a_0 = 3$ e) $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 2$, $a_1 = -1$

5. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$ b) $\sum_{j}^{4}(j+2)^2$ c) $\sum_{i=0}^{2}\sum_{j=1}^{3}(2i-3j)$ d) $\sum_{i=1}^{3}\sum_{j=2}^{4}ij$

6. What are the values of these sums, where S = {1, 3, 5, 7}?

a) $\sum_{j\in S}\left(j+\frac{1}{j}\right)$ b) $\sum_{j\in S} j^2$ c) $\sum_{j\in S} 2$

7. What are the values of the following products?

a) $\prod_{i=0}^{10} i$ b) $\prod_{i=1}^{100}(-1)^i$ c) $\prod_{i=0}^{4} i!$

Chapter 3: The Fundamentals: Algorithms, the Integers, and Matrices

3.1 Algorithms

1. List all the steps used by Algorithm "max" to find the maximum of the list 1, 8, 12, 9, 11, 2, 14, 5, 10, 4.

2. Devise an algorithm that finds the sum of all the integers in a list.

3. List all the steps used to search for 9 in the sequence 1, 3, 4, 5, 6, 8, 9, 11 using

a) a linear search b) a binary search.

4. Describe an algorithm that inserts an integer x in the appropriate position into the list $a_1, a_2, \ldots, a_n$ of integers that are in increasing order.

5. Use the bubble sort to sort 3, 1, 5, 7, 4, showing the lists obtained at each step.

6. Consider the Linear search algorithm:

procedure linear search(x: integer, $a_1,a_2,\ldots,a_n$: distinct integers)

i := 1

while ($i \le n$ and $x \ne a_i$)

i := i + 1

if $i \le n$ then location := i

else location := 0

return location

Given the sequence a_n: 3, 1, 5, 7, 4, 6. How many comparisons required for searching x = 7?

3.2 The Growth of Functions

1. Determine whether each of these functions is O(x).

a) f (x) = 10 b) f (x) = 3x + 7 c) f (x) = x2 + x + 1 d) f (x) = 5 log x

2. Determine whether each of these functions is O(x2).

a) f (x) = 17x + 11 b) f (x) = x^2 + 1000 c) f (x) = x log x

d) $f\left(x\right)=\frac{x^4}{2}$ e) $f\left(x\right)=2^x$ f)) f (x) = $(x^3 + 2x)/(2x + 1)$

3. Find the least integer n such that f (x) is $O(x^n)$ for each of these functions.

a) f (x) = $2x^3 + x^2 \log x$ b) f (x) = $3x^3 + (\log x)^4$

c) f (x) = $(x^4 + x^2 + 1)/(x^3 + 1)$ d) f (x) = $(x^4 + 5 \log x)/(x^4 + 1)$

4. Determine whether x^3 is O(g(x)) for each of these functions g(x).

a) g(x) = x^2 b) g(x) = x^3 c) g(x) = $x^2 + x^3$

d) g(x) = $x^2 + x^4$ e) g(x) = 3^x f) g(x) = $x^3/2$

5. Arrange the functions $\sqrt{n}$, 1000 log n, n log n, 2n!, 2^n, 3^n, and $n^2/1,000,000$ in a list so that each function is big-O of the next function.

6. Give as good a big-O estimate as possible for each of these functions.

a) $(n^2 + 8)(n + 1)$ b) $(n \log n + n^2)(n^3 + 2)$ c) $(n! + 2^n)(n^3 + \log(n^2 + 1))$

3.3 Complexity of Algorithms

1. Consider the algorithm:

procedure *giaithuat*($a_1, a_2, \ldots, a_n$: *integers)*
count:= 0
for *i:= i* to *n* do
    if $a_i > 0$ then *count: = count + 1*
print*(count)*

Give the best big-O complexity for the algorithm above.

2. Consider the algorithm:

procedure GT(n : positive integer)
F:=1
for i:= 1 to n do
    F: = F * i
Print(F)

Give the best big-O complexity for the algorithm above.

3. Consider the algorithm:

procedure max(a ,a ,...,a : reals)
max:=a
for i:=2 to n
if max<a then max:=a

Give the best big-O complexity for the algorithm above.

3.4 The Integers and Division

1. Does 17 divide each of these numbers?

a) 68 b) 84 c) 357 d) 1001

2. What are the quotient and remainder when

a) 19 is divided by 7? b) −111 is divided by 11? c) 789 is divided by 23?

d) 1001 is divided by 13? e) 0 is divided by 19? f) 3 is divided by 5?

3. Suppose that a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \le c \le 12$ such that

a) $c \equiv 9a \pmod{13}$. b) $c \equiv 11b \pmod{13}$. c) $c \equiv a + b \pmod{13}$.

d) $c \equiv 2a + 3b \pmod{13}$. e) $c \equiv a^2 + b^2 \pmod{13}$. f) $c \equiv a^3 - b^3 \pmod{13}$.

4. Evaluate these quantities.

a) 13 mod 3 b) −97 mod 11 c) 155 mod 19 d) −221 mod 23

5. Find a **div** m and a **mod** m when

a) $a = -111$, $m = 99$. b) $a = -9999$, $m = 101$.

c) $a = 10299$, $m = 999$. d) $a = 123456$, $m = 1001$.

6. Decide whether each of these integers is congruent to 5 modulo 17.

a) 80 b) 103 c) −29 d) −122

7. Find each of these values.

a) $(992 \bmod 32)^3 \bmod 15$ b) $(34 \bmod 17)^2 \bmod 11$

c) $(193 \bmod 23)^2 \bmod 31$ d) $(893 \bmod 79)^4 \bmod 26$

8. Convert the decimal expansion of each of these integers to a binary expansion.

a) 23 b) 45 c) 241 d) 1025

9. Convert the binary expansion of each of these integers to a decimal expansion.

a) $(1\ 1011)_2$ b) $(10\ 1011\ 0101)_2$

c) $(11\ 1011\ 1110)_2$ d) $(111\ 1100\ 0001\ 1111)_2$

10. Convert the octal expansion of each of these integers to a binary expansion.

a) $(572)_8$ b) $(1604)_8$ c) $(423)_8$ d) $(2417)_8$

11. Convert each of the following expansions to **decimal expansion**.

a) $(1021)_3$ b) $(325)_7$ c) $(A3)_{12}$ d) $(401)_5$ e) $(12B7)_{13}$

12. Convert 69 to

a) a binary expansion b) a base 6 expansion c) a base 9 expansion

11. Suppose a ***mod*** 3 = 2 and b ***mod*** 6 = 4, find ab ***mod*** 3.

3.5 Primes and Greatest Common Divisors

1. Determine whether each of these integers is prime.

a) 21 b) 29 c) 71 d) 97

e) 111 f) 143 g) 93 h) 101

2. Find the prime factorization of each of these integers.

a) 39 b) 81 c) 101

d) 143 e) 289 f) 899

3. Find the prime factorization of 10!

4. Which positive integers less than 12 are relatively prime to 12?

5. Which positive integers less than 30 are relatively prime to 30?

6. Find these values of the Euler φ-function.

a) $\varphi(4)$ b) $\varphi(10)$ c) $\varphi(13)$

7. What are the greatest common divisors of these pairs of integers?

a) $37 \cdot 53 \cdot 73$, $211 \cdot 35 \cdot 59$ b) $11 \cdot 13 \cdot 17$, $29 \cdot 37 \cdot 55 \cdot 73$ c) 2331, 2317

d) $41 \cdot 43 \cdot 53$, $41 \cdot 43 \cdot 53$ e) $313 \cdot 517$, $212 \cdot 721$

3.6 Integers and Algorithms

1. Suppose ***pseudo-random numbers*** are produced by using: $x_{n+1} = (3x_n + 11) \bmod 13$. If $x_3 = 5$, find x_2 and x_4.

2. Suppose pseudo-random numbers are produced by using: $x_{n+1} = (2x_n + 7) \bmod 9$.

a) If $x_0 = 1$, find x_2 and x_3 b) If $x_3 = 3$, find x_2 and x_4.

3. Using the function $f(x) = (x + 10) \bmod 26$ to encrypt messages. Answer each of these questions.

a) Encrypt the message STOP b) Decrypt the message LEI

4. Which memory locations are assigned by the **hashing function** $h(k) = k \bmod 101$ to the records of insurance company customers with these Social Security Numbers?

a) 104578690 b) 432222187

5. Use the **Euclidean algorithm** to find

a) gcd(14, 28) b) gcd(8, 28) c) gcd(100, 101) d) gcd(28,35)

e) lcm(7, 28) f) lcm(12, 28) g) lcm(100, 101) h) lcm(28,35)

Chapter 4: Induction and Recursion

4.1 Mathematical Induction

1. Let P (n) be the statement that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for the positive integer n.

a) What is the statement P (1)?

b) Show that P (1) is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

2. Let P (n) be the statement that $1 + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{n^2} < 2 - \frac{1}{n}$, where n is an integer greater than 1.

a) What is the statement P (2)?

b) Show that P (2) is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

3. Prove the statement "6 divides n^3 - n for all integers $n \geq 0$".

4. Prove that $3^n < n!$ if n is an integer greater than 6.

5. Prove that $2^n > n^2$ if n is an integer greater than 4.

6. Prove that for every positive integer n, $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$.

7. Prove that $\ln n < \sum_{i=1}^{n} \frac{1}{i}$ whenever n is a positive integer.

4.2 Strong Induction

1. Let P (n) be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for $n \geq 8$.

a) Show that the statements P (8), P (9), and P (10) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 10$.

2. Let P (n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for $n \geq 18$.

a) Show statements P(18), P(19), P(20), and P(21) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

4.3 Recursive Definitions and Structural Induction

1. Find f(1), f(2), f(3), and f(4) if f(n) is defined recursively by f(0) = 1 and for n = 0, 1, . . .

a) $f(n + 1) = f(n) + 2$ b) $f(n + 1) = 3f(n)$

c) $f(n + 1) = 2f(n)$ d) $f(n + 1) = f(n)^2 + f(n) + 1$

2. Find f (2), f (3), f (4), and f (5) if f is defined recursively by f (0) = −1, f (1) = 2, and for n= 1, 2, . . .

a) $f(n + 1) = f(n) + 3f(n - 1)$ b) $f(n + 1) = f(n)^2 f(n - 1)$

c) $f(n + 1) = 3f(n)^2 - 4f(n - 1)^2$ d) $f(n + 1) = f(n - 1)/f(n)$

3. Find f (2), f (3), f (4), and f (5) if f is defined recursively by f (0) = f (1) = 1 and for n = 1, 2, . . .

a) $f(n + 1) = f(n) - f(n - 1)$ b) $f(n + 1) = f(n)f(n - 1)$

c) $f(n + 1) = f(n)2 + f(n - 1)^3$ d) $f(n + 1) = f(n)/f(n - 1)$

4. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for f(n) when n is a nonnegative integer and prove that your formula is valid.

a) $f(0) = 0$, $f(n) = 2f(n - 2)$ for $n \geq 1$

b) $f(0) = 1$, $f(n) = f(n - 1) - 1$ for $n \geq 1$

c) $f(0) = 2$, $f(1) = 3$, $f(n) = f(n - 1) - 1$ for $n \geq 2$

d) $f(0) = 1$, $f(1) = 2$, $f(n) = 2f(n - 2)$ for $n \geq 2$

e) $f(0) = 1$, $f(n) = 3f(n - 1)$ if n is odd and $n \geq 1$ and $f(n) = 9f(n - 2)$ if n is even and $n \geq 2$

5. Give a recursive definition of the sequence {an}, n = 1, 2, 3, . . . if

a) $a_n = 6n$ b) $a_n = 2n + 1$ c) $a_n = 10n$ d) $a_n = 5$

e) $a_n = 4n - 2$ f) $a_n = 1 + (-1)^n$ g) $a_n = n(n + 1)$ h) $a_n = n^2$

6. Let F be the function such that F(n) is the sum of the first n positive integers. Give a recursive definition of F(n).

7. Give a **recursive definition** of each of these sets.

a) A = {2, 5, 8, 11, 14, …} b) B = {…, -5, -1, 3, 7, 10, …}

c) C = {3, 12, 48, 192, 768, …} d) D = {1, 2, 4, 7, 11, 16, …}

8. The reversal of a string is the string consisting of the symbols of the string in reverse order. The reversal of the string w is denoted by w^R. Find the reversal of the following bit strings.

a) 0101 b) 1 1011 c) 1000 1001 0111

9. When does a string belong to the set A of bit strings defined recursively by

$\lambda \in A$, $0x1 \in A$ if $x \in A$, where λ is the empty string?

4.4 Recursive Algorithms

1. Give a recursive algorithm for computing nx whenever n is a positive integer and x is an integer, using just addition.

2. Consider an **recursive algorithm** to compute the n^{th} Fibonacci number:

procedure Fibo(n : positive integer)

if n = 1 return 1

else if n = 2 return 1

else return Fibo(n – 1) + Fibo(n – 2)

How many additions (+) are used to find Fibo(6) by the algorithm above?

3. Give a recursive algorithm for finding the sum of the first n odd positive integers.

4. Consider the following algorithm:

procedure tinh(a: real number; n: positive integer)

if n = 1 return a

else return a·tinh(a, n-1).

a) What is the output if inputs are: n = 4, a = 2.5? Explain your answer.

b) Show that the algorithm computes n·a using Mathematical Induction.

5. Consider the following algorithm:

procedure F($a_1, a_2, a_3, ..., a_n$: integers)

if n = 0 return 0

else return $a_n + F(a_1, a_2, a_3, ..., a_{n-1})$

Find

a) F(1,3,5) b) F(1,3,5) c) F(1,2,3,5,9)

Chapter 5-7: Counting

5.1 The Basics of Counting

1. How many different bit strings of length seven are there?

2. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

3. **Counting Functions**: How many functions are there from a set with m elements to a set with n elements?

4. **Counting One-to-One Functions**: How many one-to-one functions are there from a set with m elements to one with n elements?

5. **Counting bijection Functions**: How many one-to-one functions are there from a set with m elements to one with n elements?

6. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

7. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

8. How many bit strings of length four do not have two consecutive 1s?

9. How many bit strings of length ten both begin and end with a 1?

10. How many positive integers between 5 and 31

a) are divisible by 3? b) are divisible by 4? c) are divisible by 3 and by 4?

11. How many positive integers between 50 and 100

a) are divisible by 7? b) are divisible by 11? c) are divisible by both 7 and 11?

12. How many positive integers less than 1000

a) are divisible by 7?

b) are divisible by 7 but not by 11?

c) are divisible by both 7 and 11?

d) are divisible by either 7 or 11?

e) are divisible by exactly one of 7 and 11?

f) are divisible by neither 7 nor 11?

g) have distinct digits?

h) have distinct digits and are even?

13. How many positive integers between 100 and 999 inclusive

a) are divisible by 7?

b) are odd?

c) have the same three decimal digits?

d) are not divisible by 4?

e) are divisible by 3 or 4?

f) are not divisible by either 3 or 4?

g) are divisible by 3 but not by 4?

h) are divisible by 3 and 4?

14. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 b) 5 c) 6 d) 7

15. How many bit strings of length seven either begin with two 0s or end with three 1s?

16. How many bit strings of length 10 either begin with three 0s or end with two 0s?

17. How many bit strings of length 8 begin with 11 or end with 00?

18. Let B be the set {a, b, c}. How many functions are there from B^2 to B?

19. How may bit strings with length 10 that has exactly three 1s and end with 0?

20. A game consisting of flipping a coin ends when the player gets two heads in a row, two tails in a row or flips the coin four times. In how many ways can the game end?

7.1 Recurrence Relations

1. a) Find a recurrence relation for the number of permutations of a set with n elements.

b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

2. a) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain

three consecutive 0s?

3. What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

7.3 Divide-and-Conquer Algorithms and recurrence Relations

1. How many comparisons are needed for a binary search in a set of 64 elements?

2. Suppose that $f(n) = f(n/3) + 1$ when n is a positive integer divisible by 3, and $f(1) = 1$. Find

a) $f(3)$ b) $f(27)$ c) $f(729)$

3. Suppose that $f(n) = 2f(n/2) + 3$ when n is an even positive integer, and $f(1) = 5$. Find

a) $f(2)$ b) $f(8)$ c) $f(64)$ d) $f(1024)$.

4. Suppose that $f(n) = f(n/5) + 3n$ when n is a positive integer divisible by 5, and $f(1) = 4$. Find

a) $f(5)$ b) $f(125)$ c) $f(3125)$

Chapter 8: Relations

8.1 Relations and Their Properties

1. How many binary relations from A to B can be constructed?

2. How many relations are there on the set A?

3. How many relations are there on the set {1,2,3,4} that contain the pair (1,2) and (1,3)?

4. List the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3}, where (a, b) ∈ R if and only if

a) $a = b$ b) $a + b = 4$ c) $a > b$ d) $a \mid b$

e) $\gcd(a, b) = 1$ f) $\text{lcm}(a, b) = 2$

5. Let A = {1, 2, 3, 4, 5, 6}

a) List all the ordered pairs in the relation R = {(a, b) | a divides b} on A.

b) Display this relation graphically .

c) Display this relation in tabular form.

6. For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

c) {(2, 4), (4, 2)}

d) {(1, 2), (2, 3), (3, 4)}

e) {(1, 1), (2, 2), (3, 3), (4, 4)}

f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

7. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where (a, b) ∈ R if and only if

a) a is taller than b. b) a and b were born on the same day.

c) a has the same first name as b. d) a and b have a common grandparent

8. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x + y = 0$ b) $x = \pm y$ c) $x - y$ is a rational number

d) $xy \geq 0$ e) $xy = 0$ g) $xy > 1$.

9. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$ b) $xy \geq 1$ c) $x = y + 1$ or $x = y - 1$

d) $x \equiv y \pmod 7$ e) x is a multiple of y

Let R be a relation from a set A to a set B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** $\overline{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) / \in R\}$.

10. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find

a) $R_1 \cup R_2$ b) $R_1 \cap R_2$ c) $R_1 - R_2$

d) R^{-1} and $\overline{R}$ e) $R_1 \oplus R_2$

11. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$

12. a) List the 16 different relations on the set $\{0, 1\}$

b) Which of the the relations are

a) reflexive? b) irreflexive? c) symmetric?

d) antisymmetric? e) asymmetric? f) transitive?

13. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs (1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), and (5, 4). Find

a) R^2 b) R^3 c) R^4 d) R^5

8.2 n-ary Relations and Their Applications

TABLE 1 Students.			
Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

1. Which domains are primary keys for the n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?

2. Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the n-ary relation from Table 1, assuming that no n-tuples are ever added?

3. a) Find the records of computer science majors in the n-ary relation R shown in Table 1.

b) Find the records of students who have a grade point average above 3.5 in this database.

c) Find the records of computer science majors who have a GPA above 3.5.

4. What results when the projection $P_{1,3}$ is applied to the 4-tuples (2, 3, 0, 4), (Jane Doe, 234111001, Geography, 3.14), and (a_1, a_2, a_3, a_4)?

5. What relation results when the projection $P_{1,4}$ is applied to the relation in Table 1?

6. What is the table obtained when the projection P_2 is applied to the relation in Table 1?

7. What relation results when the join operator J_2 is used to combine the relation displayed in Tables 5 and 6?

TABLE 5 Teaching_assignments.		
Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.			
Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

8.3 Representing Relations

1. Suppose that A = {1, 2, 3} and B = {1, 2}. Let R be the relation from A to B containing (a, b) if a ∈ A, b ∈ B, and a > b. What is the matrix representing R?

2. Let A = {1, 2, 3} and B = {1, 2, 3, 4, 5}. Which ordered pairs are in the relation R represented by the matrix

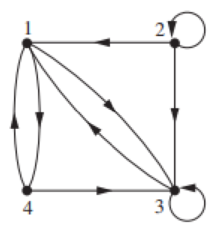
$$M_R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

3. Suppose that the relation R on a set is represented by the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Is R reflexive, symmetric, and/or antisymmetric?

4. Suppose that the relations R_1 and R_2 on a set *A* are represented by the matrices

$M_{R_1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $M_{R_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ .What are the matrices representing $R_1 \cup R_2$, $R_1 \cap R_2$ and $S \circ R$?

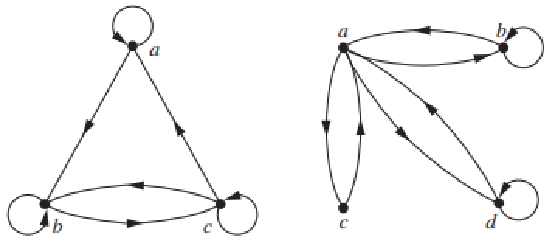
5.



a) What are the ordered pairs in the relation R represented by the directed graph shown in the Figure?

b) Determine whether the relations for the directed graphs shown in the Figure are reflexive, symmetric, antisymmetric, and/or transitive.

6.



(a) Directed graph of R (b) Directed graph of S

a) What are the ordered pairs in the relation R represented by the directed graph shown in the Figure?

b) Determine whether the relations for the directed graphs shown in the Figure are reflexive, symmetric, antisymmetric, and/or transitive.

7. Represent each of these relations on{1, 2, 3}with a matrix and a directed graphs

a) {(1, 1), (1, 2), (1, 3)}

b) {(1, 2), (2, 1), (2, 2), (3, 3)}

c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

d) {(1, 3), (3, 1)}

8. Represent each of these relations on{1, 2, 3}with a matrix and a directed graphs

a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}

c) {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}

d) {(2, 4), (3, 1), (3, 2), (3, 4)}

9. List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

10. Let R_1 and R_2 be relations on a set A represented by the matrices

$$M_{R_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } M_{R_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

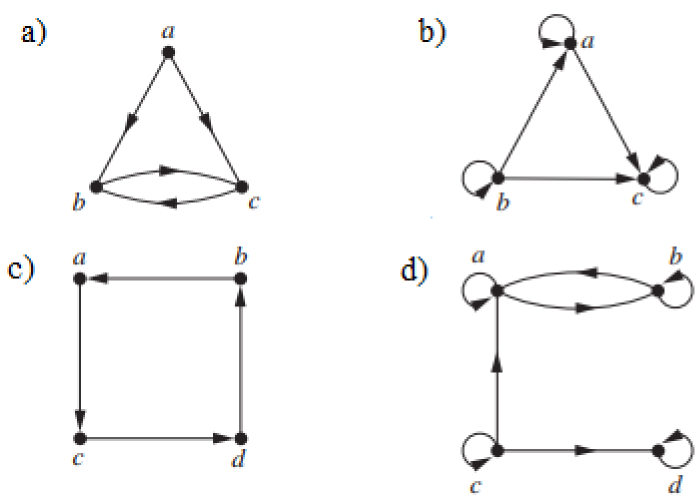
Find the matrices that represent

a) $R_1 \cup R_2$ b) $R_1 \cap R_2$ c) $R_2 \circ R_1$ d) $R_1 \oplus R_2$

e) R_2^2 f) R_1^3 g) R_1^2 h) R_1^4

11. Draw the directed graph that represents the relation {(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)}

12. List the ordered pairs in the relations represented by the directed graphs



8.5 Equivalence Relations

1. Let R be the relation on the set of real numbers such that aRb if and only if a − b is an integer. Is R an equivalence relation?

2. Let R be the relation on the set of integers such that aRb if and only if a = b or a = −b. What is the equivalence class of an integer for the equivalence relation R?

3. What are the equivalence classes of 0 and 1 for congruence modulo 4?

4. List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of S = {1, 2, 3, 4, 5, 6}.

5. What are the sets in the partition of the integers arising from congruence modulo 4?

6. Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

c) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

7. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) {(a, b) | a and b are the same age}

b) {(a, b) | a and b have the same parents}

c) {(a, b) | a and b share a common parent}

d) {(a, b) | a and b have met}

8. Which of these relations on the set of all functions from Z to Z are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) {(f, g) | f (1) = g(1)}

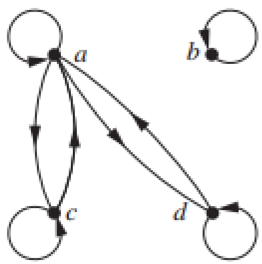
b) {(f, g) | f (0) = g(0) or f (1) = g(1)}

c) {(f, g) | f (x) − g(x) = 1 for all x ∈ Z}

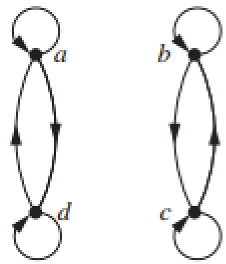
d) {(f, g) | for some C ∈ Z, for all x ∈ Z, f (x) − g(x) = C}

9. Determine whether the relation with the directed graph shown is an equivalence relation

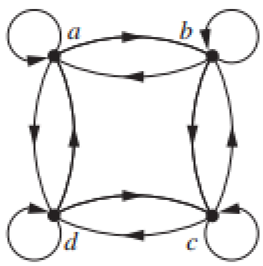
a)



b)



c)



10. Determine whether the relations represented by these zero–one matrices are equivalence relations

a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

11. Which of these collections of subsets are partitions of {1, 2, 3, 4, 5, 6}?

a) {1, 2},{2, 3, 4},{4, 5, 6} b) {1}, {2, 3, 6}, {4}, {5}

c) {2, 4, 6}, {1, 3, 5} d) {1, 4, 5}, {2, 6}

12. Let R={(1,1), (2,2), (3,3), (1,2), (2,1), (4,4), (4,5), (5,4), (5,5)} be a equivalence relation on the set {1,2,3,4,5}. How many equivalence classes with respect to R?

13. List the ordered pairs in the equivalence relations produced by these partitions of {a, b, c, d, e, f, g}.

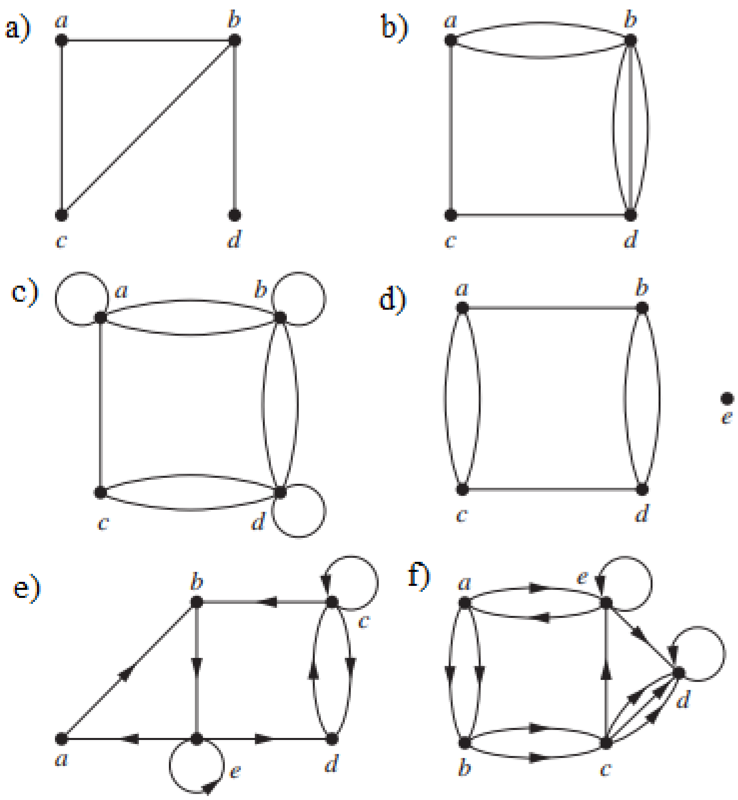
a) {a, b}, {c, d}, {e, f, g} b) {a}, {b}, {c, d}, {e, f }, {g}

c) {a, b, c, d}, {e, f, g} d) {a, c, e, g}, {b, d}, {f }

Chapter 9: Graphs

9.1 Graphs and Graph Models

1. Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops



2. Construct a precedence graph for the following program:

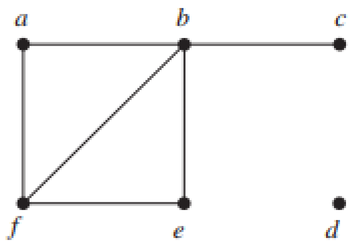
S_1: x := 0 S_2: x := x + 1 S_3: y := 2

S_4: z := y S_5: x := x + 2 S_6: y := x + z S_7: z := 4

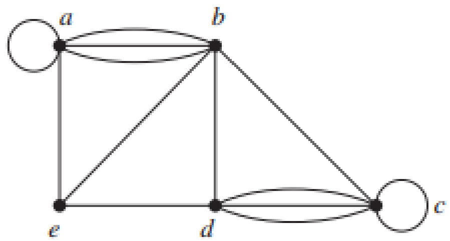
9.2 Graph Terminology and Special Types of Graphs

1. find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

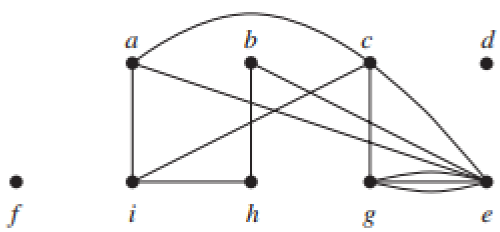
a)



b)



c)

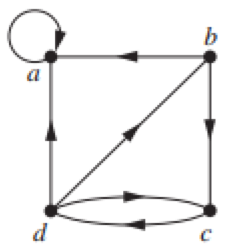


2. Find the sum of the degrees of the vertices of each graph in Exercises 1 and verify that it equals twice the number of edges in the graph.

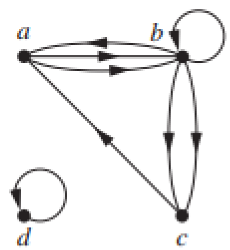
3. Can a simple graph exist with 15 vertices each of degree five?

4. determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph

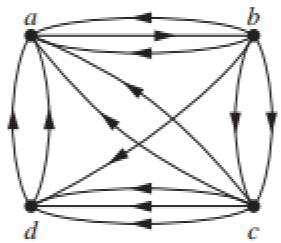
a)



b)



c)



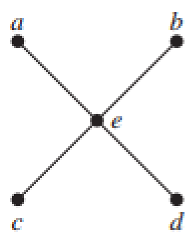
5. Draw these graphs.

a) K_7 b) $K_{1,8}$ c) $K_{4,4}$

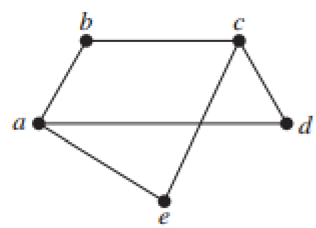
d) C_7 e) W_7 f) Q_4

6. Determine whether the graph is bipartite

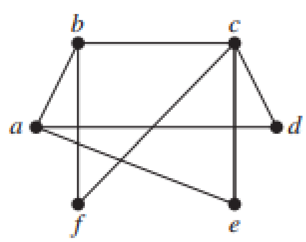
a)



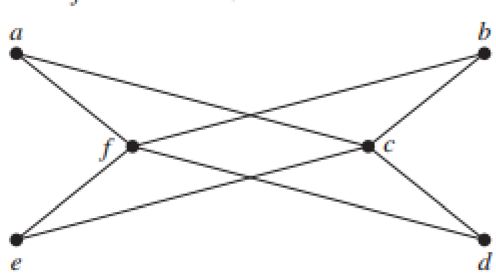
b)



c)



d)



7. For which values of n are these graphs bipartite?

a) K_n b) C_n c) W_n d) Q_n

8. How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

9. Find the degree sequences for each of the graphs in Exercises 6.

10. Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4 d) $K_{2,3}$ e) Q_3

11. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

12. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

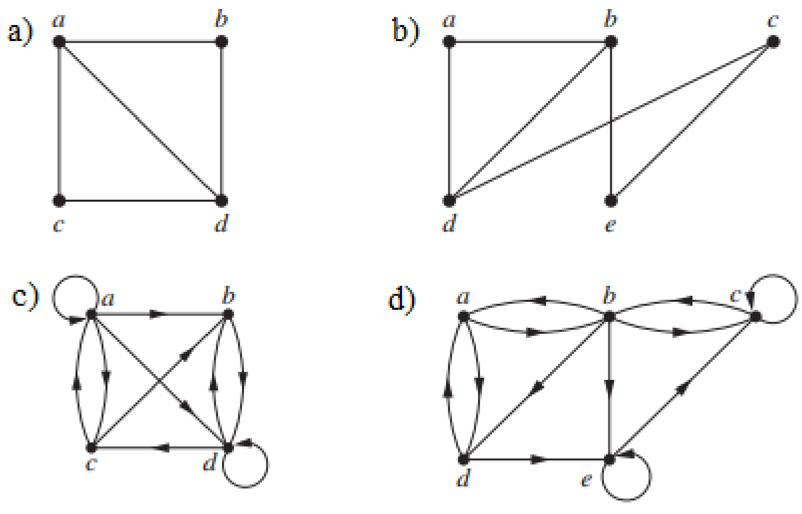
13. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 b) 6, 5, 4, 3, 2, 1 c) 4, 4, 3, 2, 1

d) 3, 3, 3, 2, 2, 2 e) 3, 3, 2, 2, 2, 2 f) 3, 2, 2, 1, 0

9.3 Representing Graphs and Graph Isomorphism

1. use an adjacency list to represent the given graph.



2. Represent the graph in Exercise 1 with an adjacency matrix.

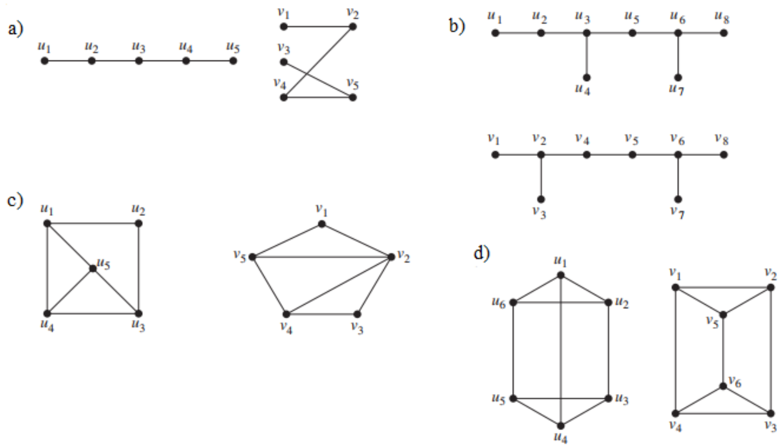
3. Represent each of these graphs with an adjacency matrix.

a) K_4 b) $K_{1,4}$ c) $K_{2,3}$ d) C_4 e) W_4 f) Q_3

4. Draw a graph with the given adjacency matrix

a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

5. Determine whether the given pair of graphs is isomorphic



6. Are the simple graphs with the following adjacency matrices isomorphic?

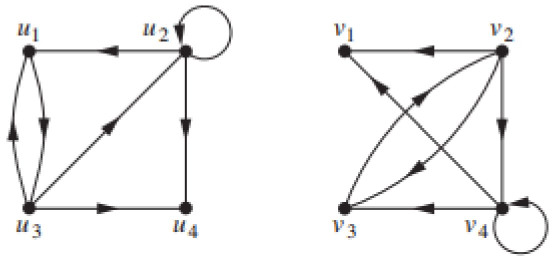
a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

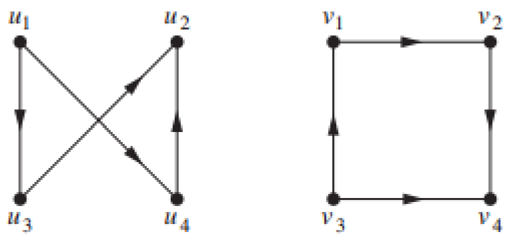
c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

7. Determine whether the given pair of directed graphs are isomorphic.

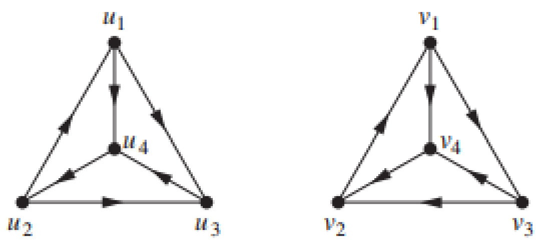
a)



b)



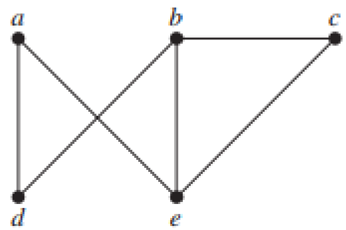
c)



9.4 Connectivity

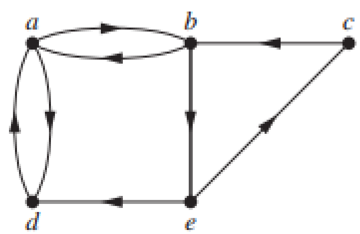
1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b b) a, e, a, d, b, c, a c) e, b, a, d, b, e d) c, b, d, a, e, c



2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b b) a, d, a, d, a c) a, d, b, e, a d) a, b, e, c, b, d, a

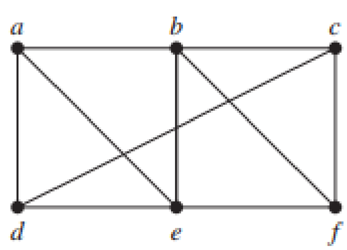


3. Find the number of paths of length n between two different vertices in K_4 if n is

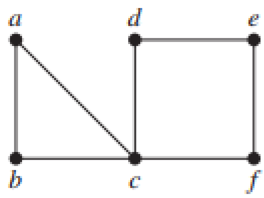
a) 2 b) 3 c) 4 d) 5

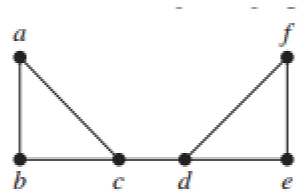
4. Find the number of paths between c and d in the graph of length

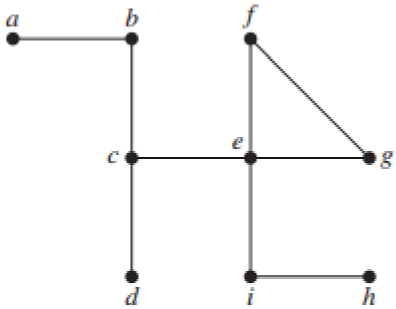
a) 2 b) 3 c) 4 d) 5 e) 6 f) 7



5. Find all the cut vertices of the given graph

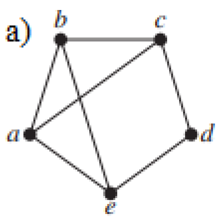
a) 

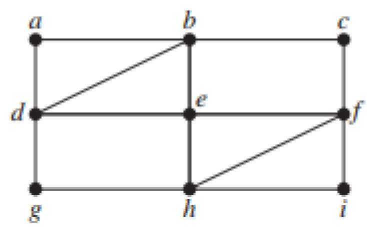
b) 

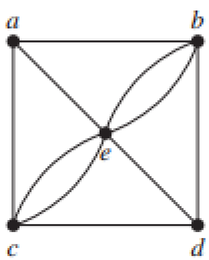
c) 

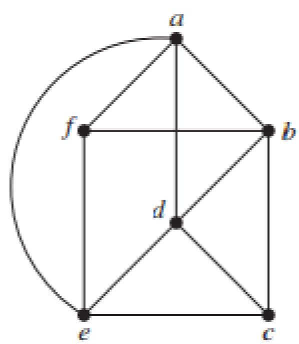
9.5 Euler and Hamilton Paths

1. Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists.

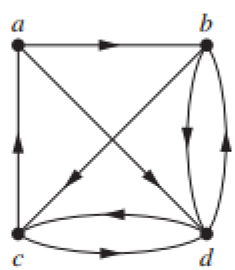
a)


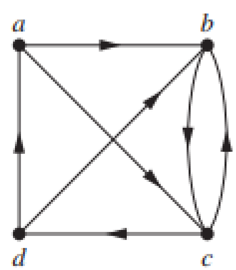
b)


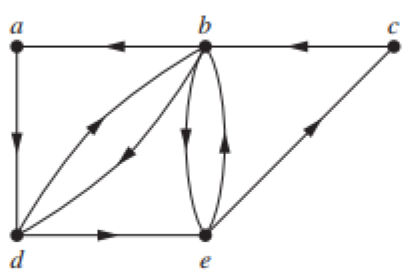
c)


d)


2. Determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists.

a)


b)


c)


3. For which values of n do these graphs have an Euler circuit?

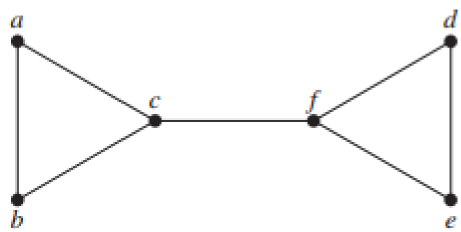
a) K_n b) C_n c) W_n d) Q_n

4. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

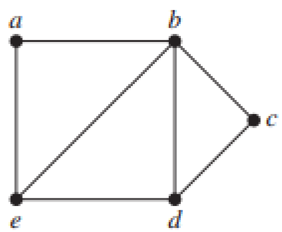
a) Euler circuit? b) Euler path?

5. Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit.

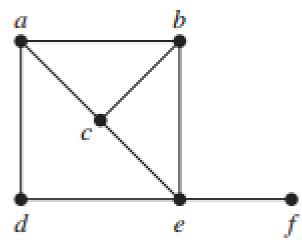
a)



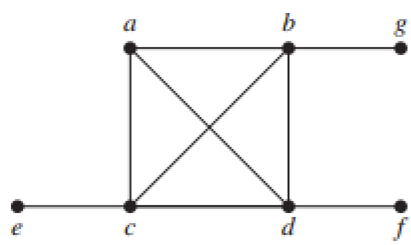
b)



c)

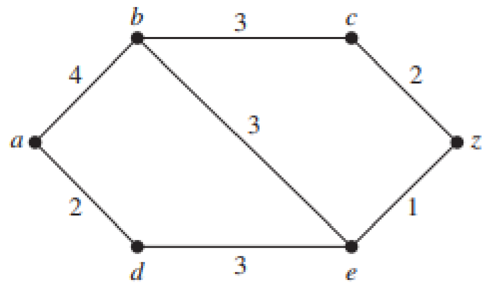


d)



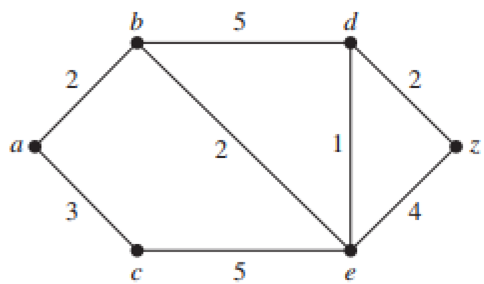
9.6 Shortest-Path Problems

1. What is the length of a shortest path between a and z in the weighted graph shown in the Figure?

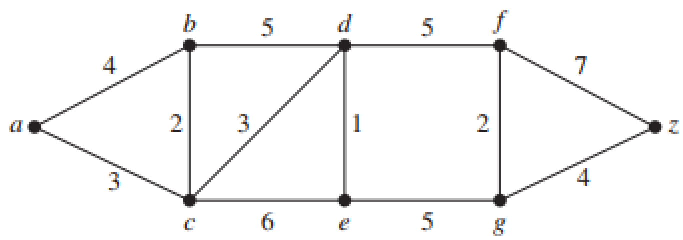


2. Find a shortest path between a and z in the given weighted graph.

a)



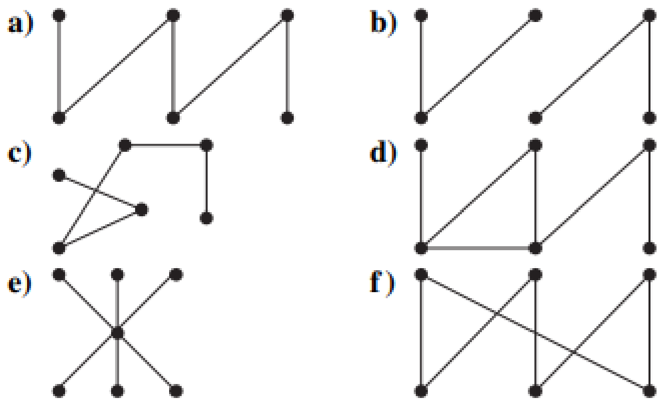
b)



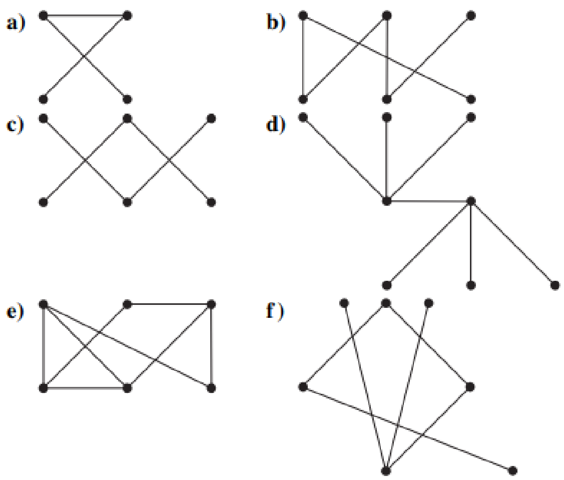
Chapter 10: Trees

10.1 Introduction to Trees

1. Which of these graphs are trees?



2. Which of these graphs are trees?



3. Construct a complete binary tree of height 4 and a complete 3-ary tree of height 3.

4. Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

5. How many edges does a tree with 10,000 vertices have?

6. How many vertices does a full 5-ary tree with 100 internal vertices have?

7. How many edges does a full binary tree with 1000 internal vertices have?

8. How many leaves does a full 3-ary tree with 100 vertices have?

10.2 Applications of Trees

1. Form a binary search tree for the words *mathematics, physics, geography, zoology, meteorology, geology, psychology* and *chemistry* (using alphabetical order).

2. Use Huffman coding to encode the following symbols with the frequencies listed: A: 0.08, B: 0.10, C: 0.12, D: 0.15, E: 0.20, F: 0.35. What is the average number of bits used to encode a character?

3. Build a binary search tree for the words *banana, peach, apple, pear, coconut, mango* and *papaya* using alphabetical order.

4. Build a binary search tree for the words *oenology, phrenology, campanology, ornithology, ichthyology, limnology, alchemy* and *astrology* using alphabetical order.

5. How many comparisons are needed to locate or to add each of these words in the search tree for Exercise 3, starting fresh each time?

a) pear b) banana c) kumquat d) orange

6. How many comparisons are needed to locate or to add each of the words in the search tree for Exercise 4, starting fresh each time?

a) palmistry b) etymology c) paleontology d) glaciology

7. Which of these codes are prefix codes?

a) a: 11, e: 00, t: 10, s: 01

b) a: 0, e: 1, t: 01, s: 001

c) a: 101, e: 11, t: 001, s: 011, n: 010

d) a: 010, e: 11, t: 011, s: 1011, n: 1001, i: 10101

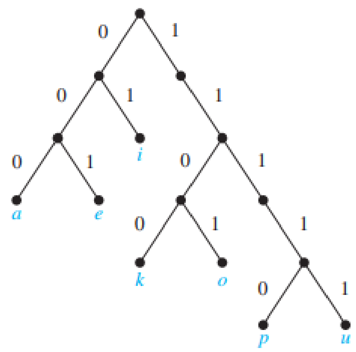
8. Construct the binary tree with prefix codes representing these coding schemes.

a) a: 11, e: 0, t: 101, s: 100

b) a: 1, e: 01, t: 001, s: 0001, n: 00001

c) a: 1010, e: 0, t: 11, s: 1011, n: 1001, i: 10001

9. What are the codes for a, e, i, k, o, p, and u if the coding scheme is represented by this tree?



10. Given the coding scheme a: 001, b: 0001, e: 1, r: 0000, s: 0100, t: 011, x: 01010, find the word represented by

a) 01110100011 b) 0001110000

c) 0100101010 d) 01100101010

11. Use Huffman coding to encode these symbols with given frequencies: a: 0.20, b: 0.10, c: 0.15, d: 0.25, e: 0.30. What is the average number of bits required to encode a character?

12. Use Huffman coding to encode these symbols with given frequencies: A: 0.10, B: 0.25, C: 0.05, D: 0.15, E: 0.30, F: 0.07, G: 0.08. What is the average number of bits required to encode a symbol?

13. Construct two different Huffman codes for these symbols and frequencies: t: 0.2, u: 0.3, v: 0.2, w: 0.3.

10.3 Tree Traversal

1. What is the ordered rooted tree that represents the expression $((x + y)\uparrow 2) + ((x - 4)/3)$?

2. What is the prefix form for $((x + y) \uparrow 2) + ((x - 4)/3)$?

3. What is the value of the prefix expression $+ - * 2\ 3\ 5/\uparrow 2\ 3\ 4$?

4. What is the postfix form of the expression $((x + y) \uparrow 2) + ((x - 4)/3)$?

5. What is the value of the postfix expression $7\ 2\ 3 * - 4 \uparrow 9\ 3/+$?

6. a) Represent the expression $((x + 2) \uparrow 3) * (y -(3 + x)) - 5$ using a binary tree.

Write this expression in

b) prefix notation c) postfix notation d) infix notation

7. a) Represent the expressions $(x + xy) + (x/y)$ and $x + ((xy + x)/y)$ using binary trees.

Write these expressions in

b) prefix notation c) postfix notation d) infix notation

8. a) Represent the compound propositions ¬(p ∧ q) ↔ (¬p ∨ ¬q) and (¬p ∧ (q ↔ ¬p)) ∨ ¬q using ordered rooted trees.

Write these expressions in

b) prefix notation c) postfix notation d) infix notation

9. a) Represent $(A \cap B) - (A \cup (B - A))$ using an ordered rooted tree.

Write this expression in

b) prefix notation c) postfix notation d) infix notation

10. Draw the ordered rooted tree corresponding to each of these arithmetic expressions written in prefix notation. Then write each expression using infix notation.

a) $+ * + - 5\ 3\ 2\ 1\ 4$ b) $\uparrow + 2\ 3 - 5\ 1$ c) $* / 9\ 3 + * 2\ 4 - 7\ 6$

11. What is the value of each of these prefix expressions?

a) $- * 2 / 8\ 4\ 3$ b) $\uparrow - * 3\ 3 * 4\ 2\ 5$

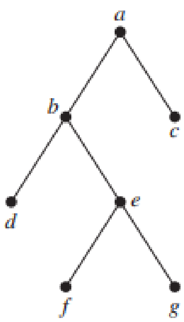
c) $+ - \uparrow 3\ 2 \uparrow 2\ 3 / 6 - 4\ 2$ d) $* + 3 + 3 \uparrow 3 + 3\ 3\ 3$

12. What is the value of each of these postfix expressions?

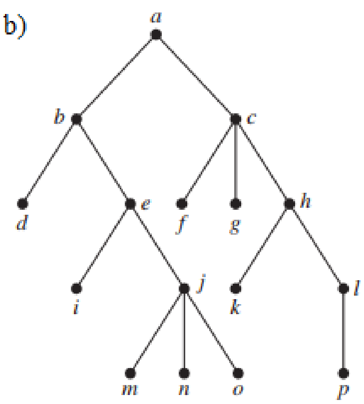
a) 5 2 1 − − 3 1 4 ++ ∗ b) 9 3 / 5 + 7 2 − ∗ c) 3 2 ∗ 2 ↑ 5 3 − 8 4 / ∗ −

13. Determine the order in which a preorder traversal visits the vertices of the given ordered rooted tree.

a)



b)



14. In which order are the vertices of the ordered rooted tree in Exercise 13 visited using an inorder traversal?

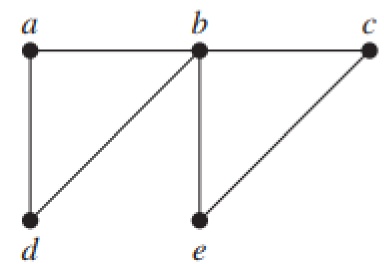
15. In which order are the vertices of the ordered rooted tree in Exercise 13 visited using a postorder traversal?

10.4 Spanning Trees

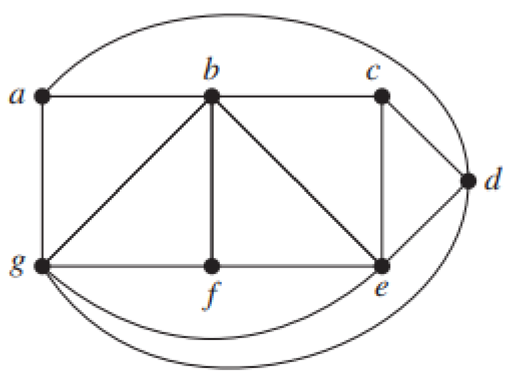
1. How many edges must be removed from a connected graph with n vertices and m edges to produce a spanning tree?

2. Find a spanning tree for the graph shown by removing edges in simple circuits.

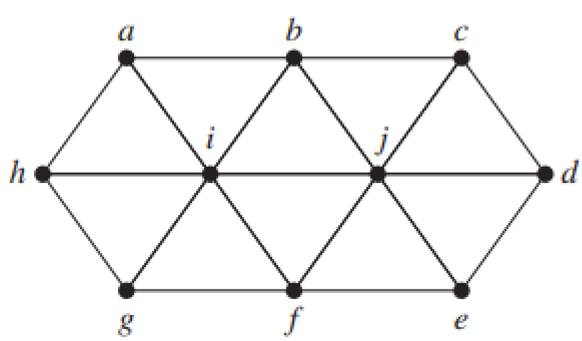
a.



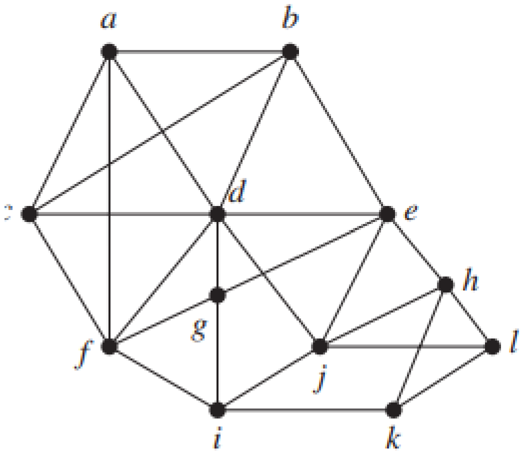
b.



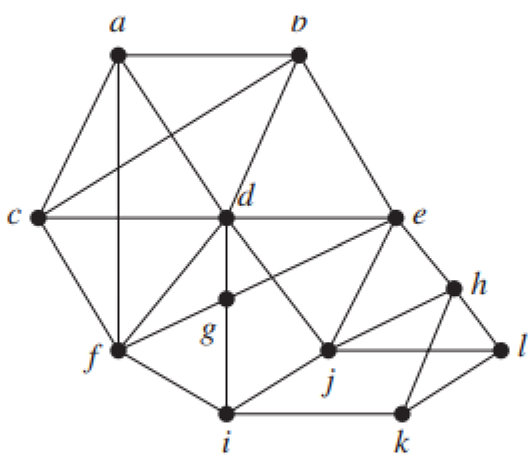
c.



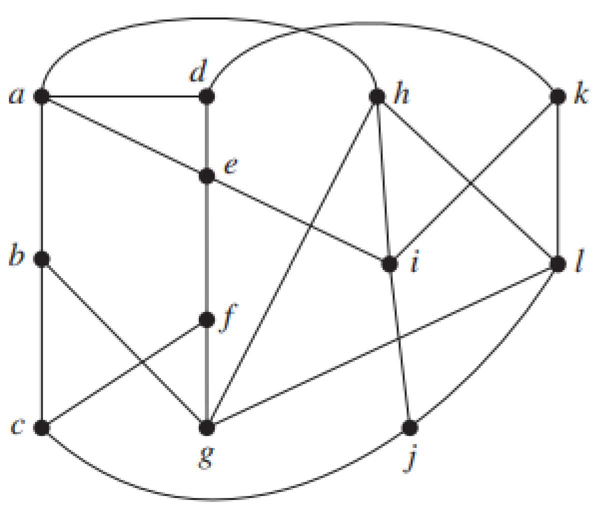
d.



e.



f.

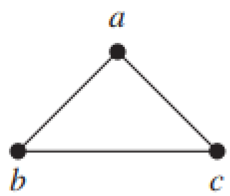


3. Find a spanning tree for each of these graphs.

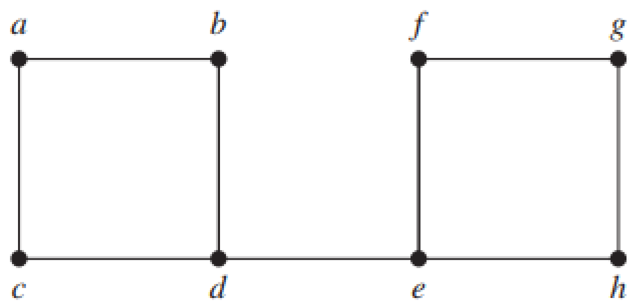
a) K_5 **b)** $K_{4,4}$ **c)** $K_{1,6}$
d) Q_3 **e)** C_5 **f)** W_5

4. Draw all the spanning trees of the given simple graphs.

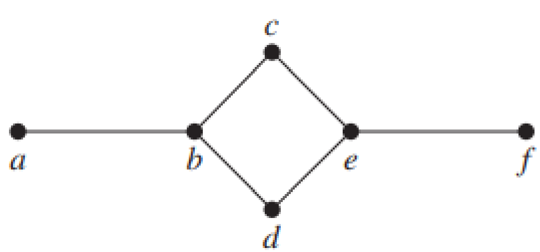
a.



b.

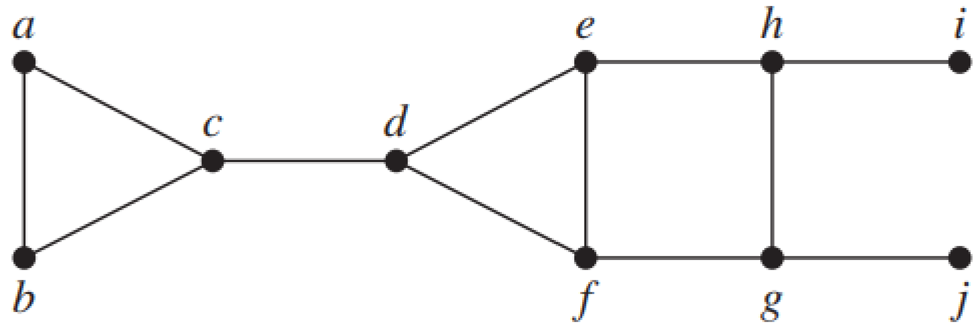


c.

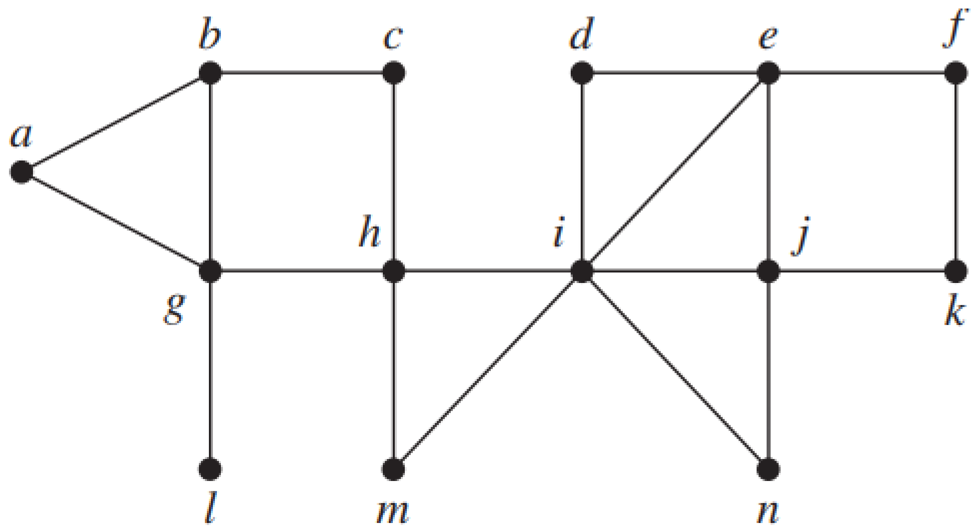


5. Use depth-first search and breadth-first search to produce a spanning tree for the given simple graph. Choose a as the root of this spanning tree and assume that the vertices are ordered alphabetically

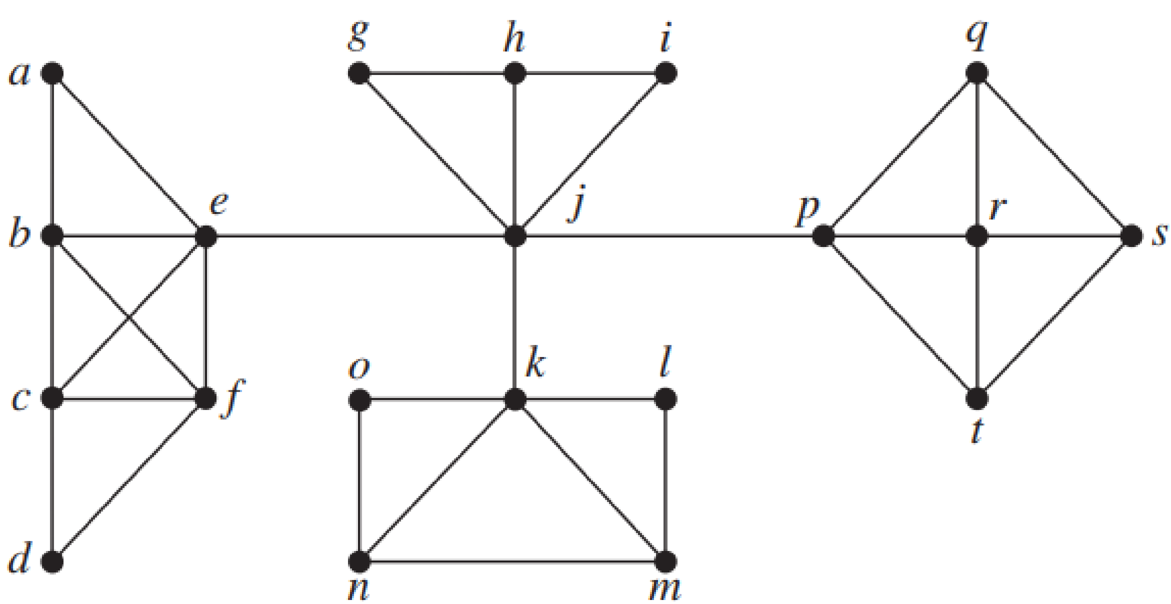
a.



b.



c.



6. Use depth-first search and breadth-first search to find a spanning tree of each of these graphs.

a) W_6 ,starting at the vertex of degree 6 b) K_5

c) $K_{3,4}$, starting at a vertex of degree 3 d) Q_3

7. Use backtracking to solve the n-queens problem for these values of n.

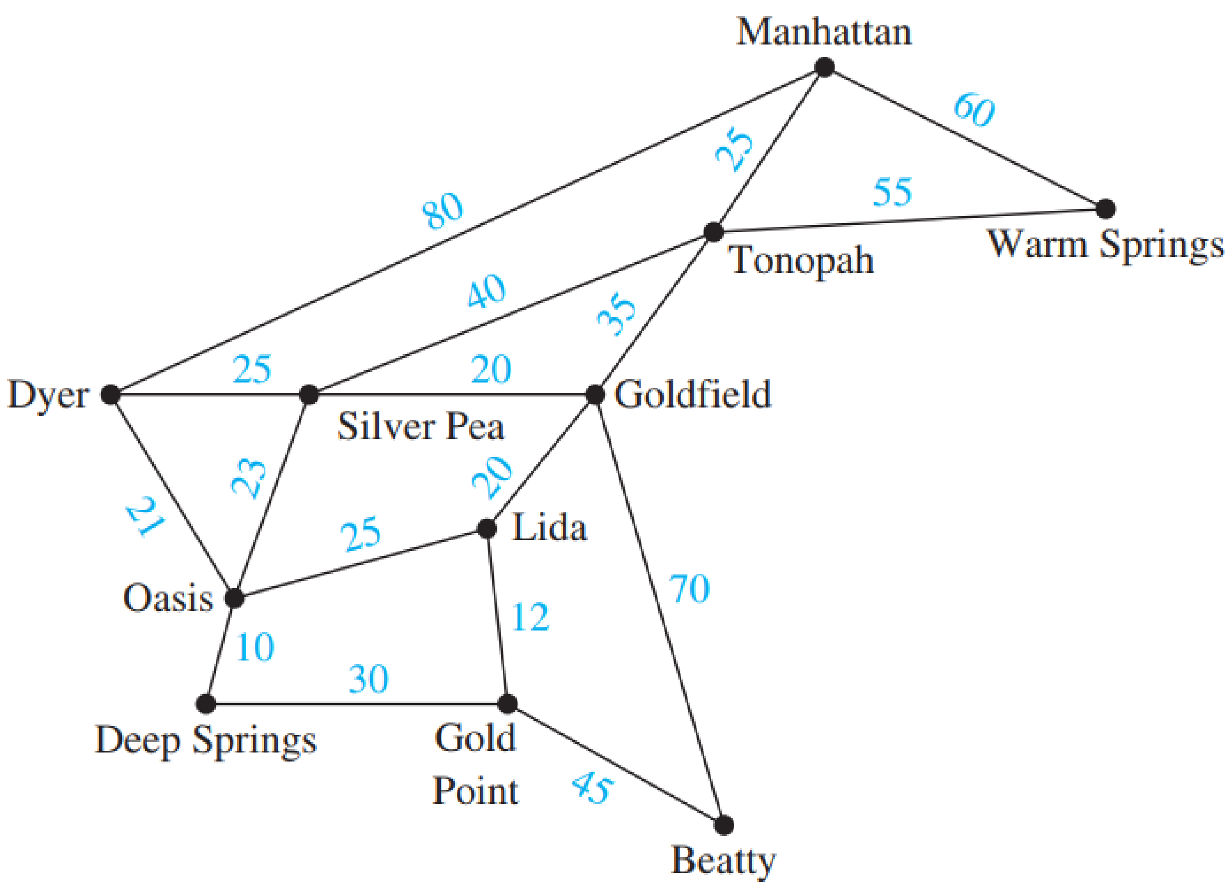
a) n = 3 b) n = 5 c) n = 6

8. Use backtracking to find a subset, if it exists, of the set {27, 24, 19, 14, 11, 8} with sum

a) 20 b) 41 c) 60

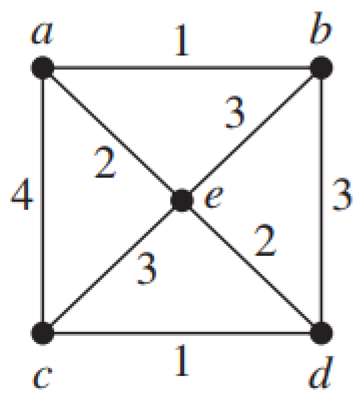
10.4 Minimum Spanning Trees

1. The roads represented by this graph are all unpaved. The lengths of the roads between pairs of towns are represented by edge weights. Which roads should be paved so that there is a path of paved roads between each pair of towns so that a minimum road length is paved? (Note: These towns are in Nevada.)

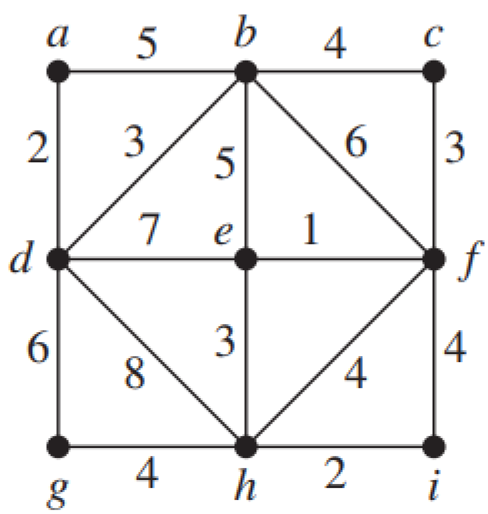


2. Use Prim's algorithm and Kruskal's algorithm to find a minimum spanning tree for the given weighted graph

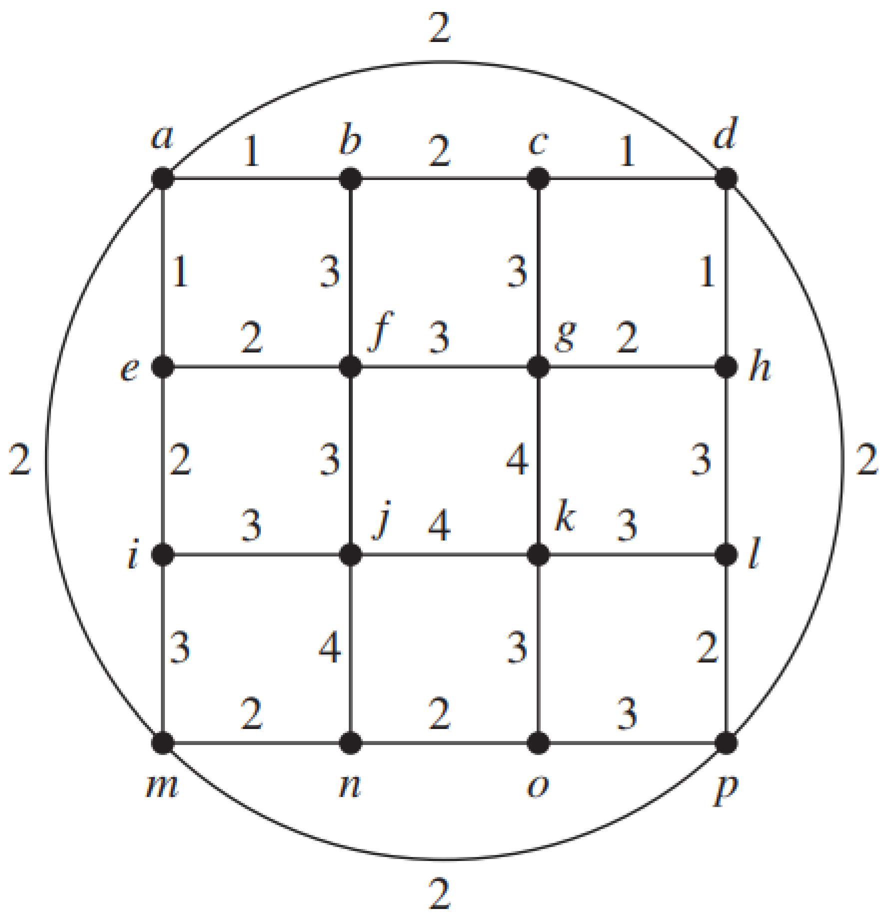
a.



b.



c.


2
a
1
b
2
c
1
d
1
3
3
1
e
2
f
3
g
2
h
2
2
3
4
3
2
i
3
j
4
k
3
l
3
4
3
2
2
2
3
m
n
o
p
2