A Paper Review on Aircraft Control using LQR and LQG Controllers with Optimal Estimation-Kalman Filter Design

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Abstract

This paper explores the development of an LQG controller for the control of a fixed wing aircraft's longitudinal and lateral flight dynamics. This uses the LQG as a robust controller while taking into account the varying dynamic parameters of the aircraft and the various disturbances of the system. The simulations have been run on MATLAB and Simulink and have been compared against the original paper, of which this is a review. This paper also briefly covers the design of the Kalman filter for achieving relatively noise free full state feedback. This allows the system to take advantage of multiple sensors to get a more accurate measurement of the state. In an actual aircraft, these could be the GPS, gyro, RADAR and IMU readings for the measurement of position and velocity. This paper also briefly covers aircraft dynamics and the various factors affecting them, such as the control surfaces.

1 Introduction

This paper is a review of the paper on Aircraft control using LQR and LQG controllers with optimal estimation-Kalman filter design [1]. Modern aircraft are complicated systems with many non-linear aspects to their dynamics. Standard controllers like the PID controller come up short when trying to control the various modes of operation. This is without considering the obvious measurement and process noise present in the aircraft. Secondly, the performance of any control system is improved through adding more sensors and actuators. Hence, the control system should be able to handle multiple sensors and actuators at once. The Linear Quadratic Gaussian controller (LQG) is one the more modern approaches to the problem. This controller is made to be optimal with respect to a quadratic cost function. This basically converts the problem into that of reducing a quadratic cost function to achieve optimal control schemes. This also takes into account the various disturbances in the system.

The LQG is based on the Linear Quadratic Regulator (LQR) and the Linear Quadratic Estimator (LQE) or the Kalman filter [2]. This has been further elaborated on in [3],[4] and [5].This allows the LQR to regulate performance, while receiving relatively noise free full state measurement from the Kalman filter. Both the LQR and LQE can be designed independently, allowing simplicity in the design. The design of the kalman filter requires the calculation of noise covariances, which is only accurate if done experimentally. Hence the controller has to be robust enough to adjust to the modeling errors in the noise.

2 Flight Equations

An aircraft typically has six degrees of freedom for motion. Three of these specify the position and the other three specify orientation. The change in orientation with respect to the passing air can be denoted by *Yaw*, *Pitch* and *Roll*. *Pitch* refers to the rotary motion of the aircraft about its lateral axis. *Yaw* refers to the rotation about the vertical axis of the aircraft, while *Roll* refers to the rotation about its longitudinal axis. Changing the angle of heading or turning the aircraft however requires it to roll first and achieve the desired bank angle and then apply a reverse roll to stabilize itself [6].

2.1 Nomenclature

Symbol	Indication
γ	Path Angle
θ	Pitch Angle
α	Angle of Attack
ϕ	Roll Angle
ψ	Yaw Angle
β	Sideslip Angle
q	Pitch Rate
u_0	Longitudinal Velocity
m	Aircraft Mass
δ_e	Elevator Deflection
δ_r	Rudder Deflection
δ_a	Aileron Deflection

2.2 Aircraft Dynamics

Aircraft dynamics can typically be described by six differential equations. There are three kinds of control surfaces on most fixed wing aircraft i.e. *Ailerons*, *Elevators* and *Rudders*. Deflection in the *Ailerons* gives rise to change in the *Roll* angle. Deflections in the elevator controls the *Pitch* of the aircraft, and the Rudders control the *Yaw*.

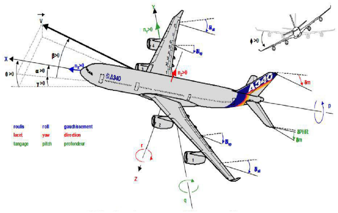

Figure 1: Aircraft dynamics nomenclature reference.

2.2.1 Velocity

The velocity can be described by the following equation:

$$\vec{v} = u\vec{i} + v\vec{j} + w\vec{k} \tag{1}$$

Where u,v and w are components of the velocity vector $\vec{v}$.

2.2.2 Forces

The forces acting on the aircraft can be described by the following equation:

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k} \tag{2}$$

Where,
x is the Axial force
y is the transverse force
z is the Normal force

2.2.3 Moments

The moments acting on the aircraft can be described by the following equation:

$$\vec{T} = l\vec{i} + m\vec{j} + n\vec{k} \tag{3}$$

Where,
l is the Axial moment
m is the transverse moment
n is the Normal moment

2.2.4 Angular Velocities

The angular velocities can be described by the following equation:

$$\vec{\omega} = p\vec{i} + q\vec{j} + r\vec{k} \tag{4}$$

Where,
p is the Roll rate
q is the Pitch rate
r is the Yaw rate

2.3 Control and sign conventions

There are typically four basic forces acting on an aircraft, i.e. Lift, Drag, Weight and Thrust. These forces can be controlled using the control surfaces on the aircraft to produce desired changes in motion. The function of the control surfaces in a broad sense is as follows:
Ailerons control Roll rate.
Elevators control the Pitch.
Rudders control the Yaw.

2.3.1 Ailerons

Ailerons are the primary control surfaces for rolling, and can be operated differentially. The downward deflection of the aileron deflects air downward, causing lift to be generated under the wing and lifting it up. Thus, the positive aileron deflection causes a corresponding negative deflection with its orientation axis
Ailerons do not actually control roll angle. They control the roll rate.

2.3.2 Elevators

Elevators are used as control surfaces for increasing or decreasing pitch. The downward deflection of the elevator causes the airflow to be deflected downward at the very end of the aircraft, causing a moment which causes the aircraft to decrease the pitch angle. Thus positive elevator deflection causes a corresponding positive deflection with its orientation axis.

2.3.3 Rudders

Rudders are the primary control surfaces for controlling the Yaw. The counterclockwise deflection causes a flow of air towards the left at the tail end of the aircraft. This causes a moment, turning it about the z axis. Here, a positive angle of deflection gives rise to a positive deflection in the Yaw angle.

2.4 Equations of Motion

The general equations of motion can be derived from the equations of mechanics. They can be generalized as forces and moments respectively:

$$m\frac{du}{dt} = \sum \vec{F_e} \tag{5}$$

$$\frac{dC}{dt} = \sum \vec{M_e} \tag{6}$$

Thus the dynamic equations can be described as longitudinal dynamics and lateral dynamics.

3 Longitudinal Dynamics

Aircraft longitudinal dynamics assume,

$$\beta = p = r = \phi = 0 \tag{7}$$

Hence, Longitudinal equations can be written as follows [7]:

$$\dot{u} = \frac{X_u}{m}u + \frac{X_w}{m}w - \frac{g\cos\theta_0}{m}\theta + \Delta X^c \tag{8}$$

$$\dot{w} = \frac{Z_u}{m - Z_{\dot{w}}}u + \frac{Z_w}{m - Z_{\dot{w}}}w - \frac{Z_q + mU_0}{m - Z_{\dot{w}}}q - \frac{mg\sin\theta_0}{m - Z_{\dot{w}}} + \Delta Z^c \tag{9}$$

$$\dot{q} = \frac{M_u + Z_u\Gamma}{Iyy}u + \frac{M_u + Z_u\Gamma}{Iyy}w - \frac{M_q + (Z_q + mU_0)\Gamma}{Iyy} - \frac{mg\sin\theta_0\Gamma}{Iyy}\theta + \Delta M^c \tag{10}$$

$$\dot{\theta} = q \tag{11}$$

Where,

$$\Delta X^c = \frac{X_{\delta e}}{m}\delta e + \frac{X_p}{m}\delta e \tag{12}$$

$$\Delta Z^c = \frac{Z_{\delta e}}{m - Z_{\dot{w}}}\delta e + \frac{X_{\delta p}}{m - Z_{\dot{w}}}\delta e \tag{13}$$

$$\Delta M^c = \frac{M_{\delta e} + Z_{\delta e}\Gamma}{Iyy}\delta e + \frac{M_{\delta p} + Z_{\delta p}\Gamma}{Iyy}\delta p \tag{14}$$

.Now, u is assumed to be approximately zero. Hence, $\dot{u} = 0$. We can now rewrite the equations in state space form:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\begin{bmatrix} \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mU_0}{m - Z_{\dot{w}}} & \frac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \frac{M_w + Z_w\Gamma}{Iyy} & \frac{M_q + (mU_0)\Gamma}{Iyy} & \frac{-mg\sin\theta_0}{Iyy} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix} \delta e \tag{15}$$

This is a non-linear state space. The author of the paper has linearized the plant about an unspecified equilibrium point to achieve the following matrix:

$$\begin{bmatrix} \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.3149 & 235.8928 & 0 \\ -0.0034 & -0.4282 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} -5.5079 \\ 0.00021 \\ 0 \end{bmatrix} \delta e \tag{16}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \tag{17}$$

The current variable of interest is the pitch angle and it is reflected in the values of the C matrix.

3.1 Stability and Controllability

The Eigen values for the longitudinal dynamic equations are as follows:

$$0.0000 + 0.0000i$$

$$-0.3715 + 0.8938i$$

$$-0.3715 - 0.8938i$$

Thus, based on the Eigen values, it can be concluded that the system is marginally stable. The controllability of the system can be deduced by the rank of the controllability matrix:

$$\begin{bmatrix} B & AB & ... & A^{n-1}B \end{bmatrix}$$

The controllability matrix for the longitudinal dynamics is as follows:

$$\begin{bmatrix} -5.5079 & 2.2298 & 3.5032 \\ 0.0021 & 0.0178 & -0.0152 \\ 0 & 0.0021 & 0.0178 \end{bmatrix}$$

Which has rank 3, i.e. full rank. Thus it can be concluded that the longitudinal dynamics of the aircraft are controllable. A similar test can be run for observability, to verify if full state feedback is possible. The observability matrix is as follows:

$$\begin{bmatrix} 0 & 0 & -0.0034 \\ 0 & 1.0000 & -0.4282 \\ 1.0000 & 0 & 0 \end{bmatrix}$$

The rank for this matrix is 3, i.e. full rank. Thus full state feedback is possible for the longitudinal dynamic states. The input to the system is the elevator deflection impulse of $0.2rad$ or $11\deg$.

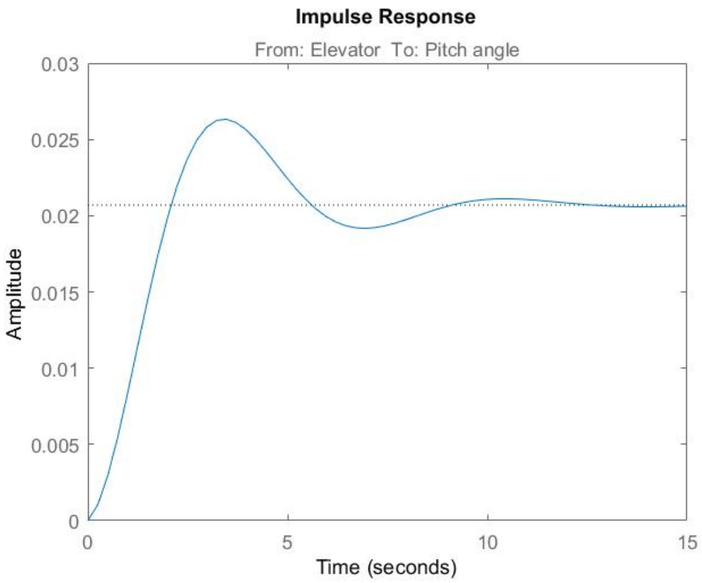

Figure 2: Open loop impulse response (Pitch Angle).

The output is in the form of the Pitch Angle θ.

4 Lateral Dynamics

Based on the equations in [], and using a similar method to the longitudinal dynamic equations, the state equations for the aircraft's lateral dynamics can be deduced. The state variables are chosen as Sideslip Angle(β), Roll rate(p), Yaw rate (r) and Roll angle(ϕ). The inputs to the system are aileron and rudder (δ_a and δ_r) deflection. These can be represented as a state space model, as follows [7]:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{Y_\beta}{U_0} & \frac{Y_p}{U_0} & \frac{Y_r}{U_0} & \frac{g\cos\theta}{U_0} \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} \frac{Y_{\delta_r}}{U_0} & \frac{Y_{\delta_a}}{U_0} \\ L_{\delta_r} & L_{\delta_a} \\ N_{\delta_r} & N_{\delta_a} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{18}$$

This is a nonlinear state space equation. The author of the paper as linearized it about a non-specific equilibrium point. The author also hasn't mentioned the values of the various constants used in the state space models used to achieve the following result:

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1.0000 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} 0.0729 & 0 \\ -4.7500 & 0.0077 \\ 0.1530 & 0.1430 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{19}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{20}$$

It can be seen that the outputs of concern are the Sideslip angle β, and the Roll angle ϕ.

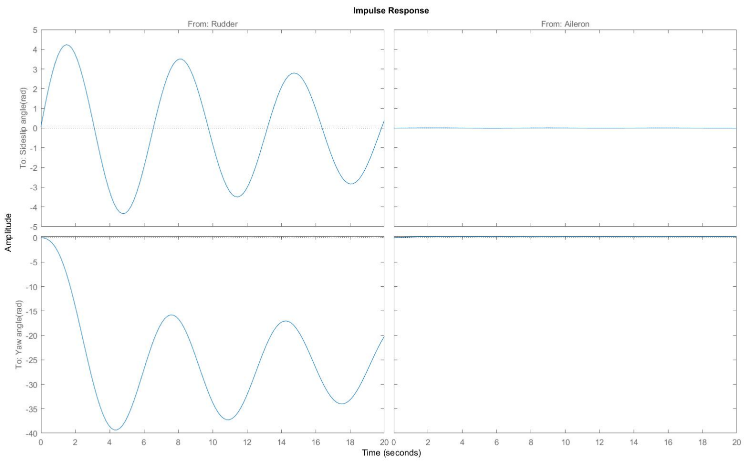

Figure 3: Open loop impulse response (Sideslip Angle and Roll Angle).

4.1 Stability and Controllability

The Eigen values for the lateral dynamic equations are as follows:

$$-0.0329 + 0.9467i - 0.0329 - 0.9467i - 0.5627 + 0.0000i - 0.0073 + 0.0000i$$

Thus, since all the real parts of the EIgen values are negative, the system is stable. The controllability of the system can be deduced by the rank of the controllability matrix:

$$\begin{bmatrix} B & AB & ... & A^{n-1}B \end{bmatrix}$$

The controllability matrix for the lateral dynamics is as follows:

$$\begin{bmatrix} 0.0729 & 0 & 4.7430 & 0.0038 & -1.0286 & 0.0061 & -3.9195 & -0.0066 \\ -4.7500 & 0.0077 & 0.5850 & -0.0054 & 2.8370 & 0.0049 & -0.5202 & 0.0026 \\ 0.1530 & 0.1430 & -2.1365 & -0.0635 & -13.2457 & 0.0159 & 10.3973 & -0.0240 \\ 0 & 0 & -0.2294 & 0.1436 & -2.0894 & -0.0639 & -13.0173 & 0.0162 \end{bmatrix}$$

The rank for this matrix is 4, i.e. the matrix is half rank. This means that not all poles can be controlled. However, the matrix is stabilizable due to its inherent stability.

Here, the impulse response for the roll angle is different from the output the author observed in the paper. The paper has an output which rises up to a value of $0.18rad$, before showing an under-damped oscillation. The simulations run during the review of the paper show that the roll angle drops below zero to a value of $-40rad$ followed by under-damped oscillations similar to those seen in the paper. The reason for this discrepancy has not been fully explored yet, but preliminary examining reveals no differences in the state space or the input to the state space from the paper.

5 LQG Controller

The Linear Quadratic Gaussian (LQG) controller is one of the most prominent optimal controllers . It can be used to control linear system with additive white Gaussian noise by reducing the state space error to a quadratic cost function which can be then minimized. The solution to this minimization is unique and is computed with relative ease. The LQG controller is one of the most popular controllers for the optimal control of nonlinear systems, after they have been linearized about an equilibrium point. It has previously been used for motion prediction [8]

LQG control allows regulation of the trade off between performance and control effort, while taking into account the noise from the system and its measurements. The LQG is simply an extended version of the Linear Quadratic Regulator (LQR). However, since the controller takes into account that all the states may not be available and the measurements may be noisy, it requires a Kalman state estimator for full state feedback.

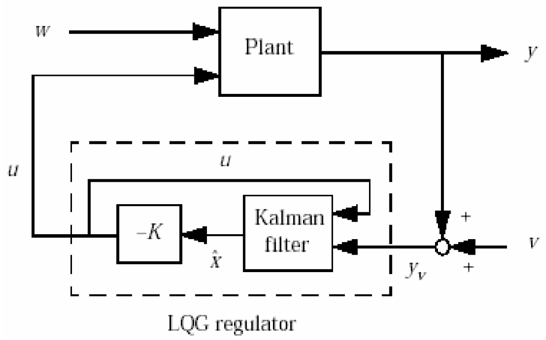

Figure 4: Block representation of the LQG controller.

State space equations for the regulator can be given as:

$$\frac{d}{dt}\hat{x} = [A - LC - (B - LD)K]\hat{x} + Ly_v \tag{21}$$

$$u = -Kx \tag{22}$$

Regulating the output y to near 0 is the goal. The plant has disturbances incorporated which are given by $y_v = y + v$. These disturbances are accounted for by the controller while generating the control signals. The state equations can be given by the following:

$$\dot{x} = Ax + Bu + Gw \tag{23}$$

$$y_v = Cx + Du + Hw + v \tag{24}$$

Here, v and w are modeled as white noise. As mentioned above, the LQG consists of both optimal state feedback and a Kalman filter. For the purposes of the LQG, the Kalman filter and the Optimal state feedback can be designed independently.

5.1 Optimal State feedback gain

The LQG reduces the state error to a quadratic function which can be minimized. This quadratic performance criterion can be given by:

$$J(u) = \int_0^\infty [x^T Qx + 2x^T Nx + u^T Ru] \tag{25}$$

The regulator is an infinite horizon regulator. Hence the limits to the integration are from 0 to ∞. The function $J(u)$ is to be minimized by the regulator and regulated to near 0. The matrices Q,N and R are user defined, and are used to specify the trade off between control effort and regulation performance. This optimal feedback gain is called the LQ optimal gain. This is basically the optimal full state feedback gain K which minimized the error function. It is multiplied with the feedback and subtracted from the reference signal.

5.2 Kalman State Estimator

The control scheme $u = -kx$ is not possible to implement without full state feedback x. The states may not always be directly measurable. Hence, we derive the states from the output of the system and the state transition matrix of the system, while taking into account the noise from the process and the measurements. This is the basic function of the Kalman filter or the Linear Quadratic Estimator (LQE). The Kalman filter generates the state estimate $\hat{x}$ which remains optimal for the output-feedback problem. The state equation is given by:

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y_v - C\hat{x} - Du) \tag{26}$$

Where L is the observer gain.
For the controller and estimator, there are two separate sets of Riccati equations to be solved. This allows their design to be independent of each other.
For the control signal u, and the measurement y_v, noise covariance data is as follows:

$$E(ww^T) = Q_n \tag{27}$$

$$E(vv^T) = R_n \tag{28}$$

$$E(wv^T) = Nn \tag{29}$$

These covariances are calculated as follows:
$E(x)$ = Expected value of x
$E(y)$ = Expected value of y

$$E(x, y) = E[x - E(x)][y - E(y)] = E[x] - E[x]E[y]$$

The Kalman gain L can be determined through the algebraic Riccati equation. Kalman filters are the optimal estimators when dealing with white Gaussian noise. The minimization of estimation error is given by:

$$\lim_{t\to\infty} E((x - \hat{x})(x - \hat{x})^T) \tag{30}$$

For the regulation of the plant to near zero, the input disturbance at low frequency with Power Spectral Density(PSD) below $10rad/sec.$ was chosen.

6 Linear Quadratic Regulator

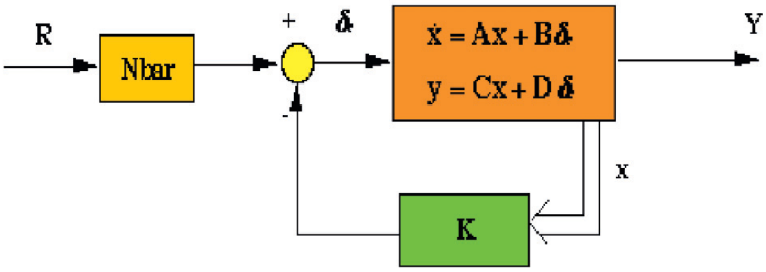

Figure 5: Block representation of the LQ Regulator.

The LQR assumes an absence of disturbance in the system. It requires the state space approach to analyze a system. This makes it easy to work with multi-output system. The LQR relies on full state feedback. Thus if the pair (A, B_k) is stabilizable, then we look for a feedback gain k which minimizes the following quadratic cost function [9]:

$$J(k, \vec{x}(0)) = \int_0^\infty \vec{x}(t)^T Q\vec{x}(t) + \vec{u_k}(t)^T R\vec{u_k}(t)dt \tag{31}$$

Here, Q and R are positive definite. The optimal solution is given by the following controller:

$$k = -R^{-1}{B_k}^T P \tag{32}$$

Here, P is a positive symmetric solution of the following stationary Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P = -Q \tag{33}$$

Now, the cost function $J = (k, \vec{x}(0))$ can be expressed as :

$$J(k, \vec{x}(0)) = \int_0^\infty \vec{x}(t)^T Q\vec{x}(t) + [x^T(t)k^T]R[kx(t)]dt \tag{34}$$

$$(u = kx) \tag{35}$$

$$J(k, \vec{x}(0)) = \int_0^\infty x^T(Q + k^T Rk)xdt \tag{36}$$

Substituting value of u in the state space equations:

$$\dot{x} = Ax + B_k u = Ax + B_k kx = (A + B_k)x \tag{37}$$

$$x(t) = e^{(A+B_k K)t}x(0) \tag{38}$$

$$J(k, \vec{x}(0)) = x(0)\int_0^\infty e^{(A+B_k K)^T t}(Q + k^T Rk)e^{(A+B_k K)t}dtx(0) \tag{39}$$

$$= x^T(0)Px(0) \tag{40}$$

Where P is the symmetric positive definite solution to:

$$(A + B_k k)^T P + P(A + B_k k) = -(Q + k^T Rk) \tag{41}$$

Using completion of squares, we can rewrite the equation as:

$$A^T P + PA = -k^T Rk - k^T B_k P - Q + PB_k R^{-1} {B_k}^T P \tag{42}$$

$$A^T P + PA - PB_k R^{-1} {B_k}^T P = -Q \tag{43}$$

This is the Riccati equation.

The lqr function in MATLAB can be used to design the optimal feedback gains for any system. This paper attempts to design the feedback using the same. This is done by inputting the state matrices (A, B), and the weight matrices Q and R. The Q matrix can be calculated as $Q = C^T * C$. The paper assumes the following values for the matrices in the longitudinal dynamic model:

$$R_{longitudinal} = 1, Q_{longitudinal} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{44}$$

The matrices for the lateral model have not been mentioned in the paper, but they can be calculated as follows:

$$R_{lateral} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Q_{lateral} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{45}$$

However, the response achieved using the values from the paper result in a different rise time for the response in each case. Different values of scalar multipliers for the R matrix were tried and the values for which the response matched the response in the paper are:

$$R_{longitudinal} = 5 * 10^-4, R_{lateral} = 0.02 \tag{46}$$

The k matrices for both the models are as follows:

$$K_{longitudinal} = \begin{bmatrix} -0.1391 & 31.5864 & 44.7213 \end{bmatrix}, K_{lateral} = \begin{bmatrix} 7.9929 & -2.0883 & -4.6226 & -6.5483 \\ -0.4177 & 0.1924 & 1.7464 & 2.6065 \end{bmatrix} \tag{47}$$

The response for these gains is give in figure 6.

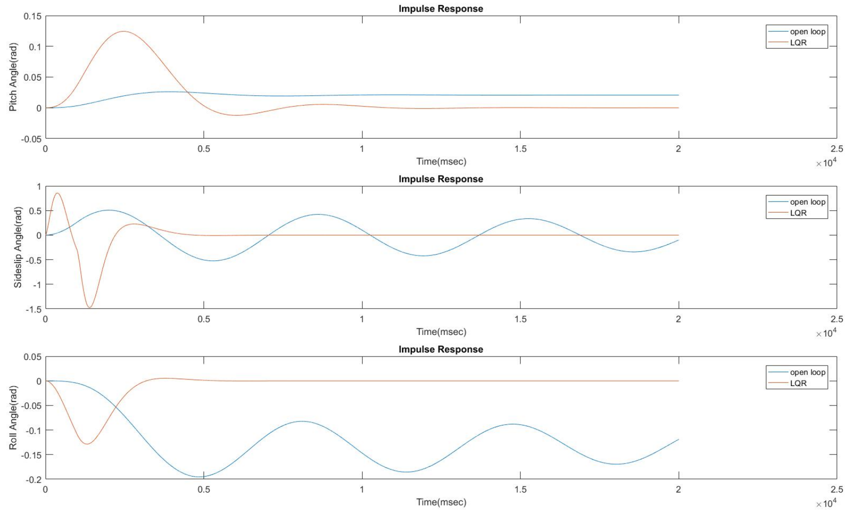

Figure 6: .(a) Comparison of Open-loop and Closed-Loop Impulse Response for the LQR (Pitch angle),(b) Comparison of Open-loop and Closed-Loop Impulse Response for the LQR (Sideslip angle,(c) Comparison of Open-loop and Closed-Loop Impulse Response for the LQR (Roll angle)

7 Kalman Filtering

For the purpose of designing the Kalman filter, we can assume the discrete plant to be represented as:

$$x(n+1) = Ax(n) + B(u(n) + w(n)) \tag{48}$$

$$y(n) = Cx(n) \tag{49}$$

$w(n)$ is the additive noise to the input. The Kalman filter should be able to estimate $y(n)$, given $u(n)$ despite $v(n)$ added to the output measurements.

$$y_v(n) = Cx(n) + v(n) \tag{50}$$

$v(n)$ is modeled as Gaussian white noise.

7.1 Discrete Kalman Filter

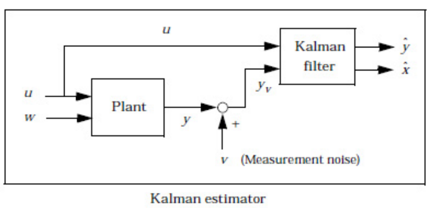

Figure 7: Block representation of the Kalman estimator.

The discrete Kalman filter has two modes i.e. Time update and Measurement update. The time update mode is responsible for the calculation of the projected value of the state based on the previous values of the state.
The measurement update is responsible for the update to the time update, so it can recalculate the projected values based on the current measurement.

The equations for the discrete steady state Kalman filter are as follows:
Measurement update:

$$\hat{x}(n/n) = \hat{x}(n/n-1) + M(y_v(n) - C\hat{x}(n/n-1)) \tag{51}$$

Time update:

$$\hat{x}(n+1/n) = A\hat{x}(n) + Bu(n) \tag{52}$$

Here,
$\hat{x}(n+1/n)$ is the estimate of $x(n)$ given past measurements up to $y_v(n-1)$.
$\hat{x}(n/n)$ is the updated measurement given the last estimate $y_v(n)$.

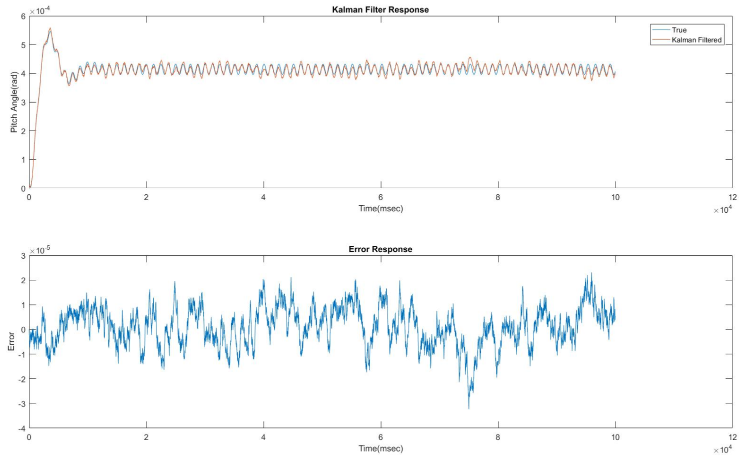

Figure 8: Kalman filter response for pitch angle θ.

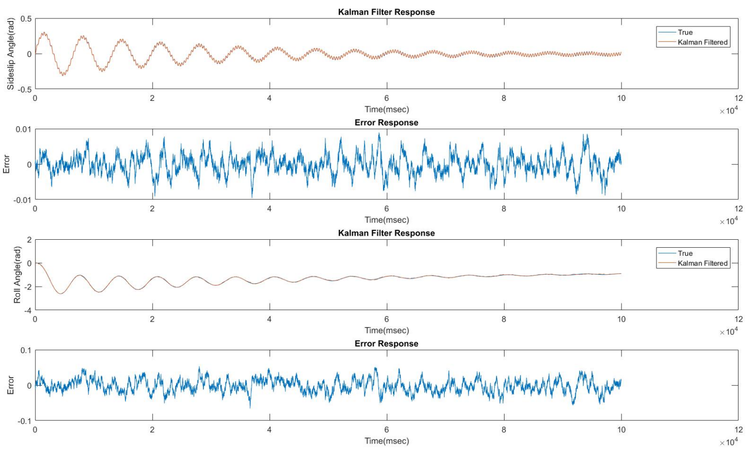

Figure 9: (a) Kalman Filter Response for the sideslip angle, β (b) Kalman Filter Response for the roll angle ϕ.

Given the current estimate $\hat{x}(n/n)$, the time update predicts the state value at the next sample $n+1$ (one-step-ahead predictor). The measurement update then adjusts this prediction based on the new measurement $y_v(n-1)$. The correction term is a function of the innovation, that is, the discrepancy between the measured and predicted values of $y(n+1)$.

$$y_v(n-1) - C\hat{x}(n/n-1) = C(x(n+1) - \hat{x}(n+1/n)) \tag{53}$$

The innovation gain M is chosen to minimize the steady-state covariance of the estimation error given the noise covariances.

$$E(w(n)w(n)^T) = Q \tag{54}$$

$$E(v(n)v(n)^T) = R \tag{55}$$

Thus the time and update equations can be bundled into one state space model, i.e. the Kalman filter:

$$\hat{x}(n+1/n) = A(I - MC)\hat{x}(n/n-1) + \begin{bmatrix} B & AM \end{bmatrix} \begin{bmatrix} u(n) \\ y_v(n) \end{bmatrix} \tag{56}$$

$$\hat{y}(n/n) = C(I_M C)\hat{x}(n/n-1) + CMy_v(n) \tag{57}$$

This generates the optimal estimate $\hat{y}(n/n)$ of $y(n)$.
State of the filter is $\hat{x}(n/n-1)$. For the design of the Kalman filter block in Simulink, the gain matrices Q, N and R had to be chosen. The matrices for the lateral and longitudinal model are as follows:

$$\hat{R}_l ongitudinal = [1], \hat{Q}_l ongitudinal = [1], \hat{N}_l ongitudinal = 0 \tag{58}$$

$$\hat{R}_l ateral = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{Q}_l ateral = [1], \hat{N}_l ateral = 0 \tag{59}$$

These values or the method to obtain them is not discussed in the paper. The simulation results for the Kalman filter output for a sine wave input for each of the models has been described in the figures 8 and 9.

8 Conclusion

The control scheme developed by the end of the paper should serve as a robust controller for the regulation of the Pitch angle, Sideslip angle and the Roll angle. Compared to a more primitive controller such as the PID, the LQR and the LQG are more optimal. This controller takes into account the various process disturbances by implementing the Kalman filter for a an accurate state measurement.
Instead of tuning gains to match performance parameters, the optimal gains can be calculated by solving the Riccati equations for the system. This is a more powerful control scheme, as the system specific features in the response are accounted for instead of manually tuning the gains.

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