Galerkin Formulation for the Exponential Covariance Operator in 2D

Notes

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1 Introduction

We consider a two-dimensional random field $u(x, y)$ defined on a rectangular domain

$$D = [a, b] \times [c, d]\,,$$

with an exponential covariance function having unit variance and a length scale equal to 1. The covariance kernel is given by

$$C\big((x, y), (\xi, \eta)\big) = \exp\left(-\sqrt{(x-\xi)^2 + (y-\eta)^2}\right).$$

Our goal is to approximate the eigenfunctions of the covariance operator using a Galerkin method with a basis composed of products of orthonormal polynomials in each variable.

2 Orthonormal Polynomial Basis

Assume that we have two families of orthonormal polynomials:

- $\{\phi_i^{(1)}(x)\}_{i=0}^{\infty}$ orthonormal on $[a, b]$ with respect to the standard L^2 inner product,
$$\int_a^b \phi_i^{(1)}(x)\phi_j^{(1)}(x)\,dx = \delta_{ij}\,,$$
- $\{\phi_j^{(2)}(y)\}_{j=0}^{\infty}$ orthonormal on $[c, d]$ with
$$\int_c^d \phi_i^{(2)}(y)\phi_j^{(2)}(y)\,dy = \delta_{ij}\,.$$

We then form a tensor-product basis for $L^2(D)$ as follows:

$$\psi_{ij}(x, y) = \phi_i^{(1)}(x)\phi_j^{(2)}(y)\,.$$

It is straightforward to verify that these basis functions are orthonormal on D:

$$\int_a^b \int_c^d \psi_{ij}(x, y)\psi_{kl}(x, y)\,dy\,dx = \left(\int_a^b \phi_i^{(1)}(x)\phi_k^{(1)}(x)\,dx\right)\left(\int_c^d \phi_j^{(2)}(y)\phi_l^{(2)}(y)\,dy\right) = \delta_{ik}\delta_{jl}\,.$$

3 The Covariance Operator and Its Eigenproblem

The covariance operator $\mathcal{C}$ is defined by

$$(\mathcal{C}f)(x,y) = \int_D C\big((x,y),(\xi,\eta)\big) f(\xi,\eta)\, d\xi d\eta\,.$$

We seek eigenpairs (λ, ϕ) satisfying

$$\int_D C\big((x,y),(\xi,\eta)\big)\phi(\xi,\eta)\, d\xi d\eta = \lambda\, \phi(x,y)\,.$$

In a Galerkin approximation, we represent the eigenfunction as

$$\phi(x,y) \approx \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij}\, \psi_{ij}(x,y)\,.$$

4 Galerkin Projection and Matrix Formulation

To obtain a finite-dimensional approximation, we project the eigenvalue problem onto the basis functions. Multiply both sides of

$$\int_D C\big((x,y),(\xi,\eta)\big)\left(\sum_{i,j} c_{ij}\, \psi_{ij}(\xi,\eta)\right) d\xi d\eta = \lambda \sum_{i,j} c_{ij}\, \psi_{ij}(x,y)$$

by $\psi_{kl}(x,y)$ and integrate over D:

$$\int_D \psi_{kl}(x,y)\left[\int_D C\big((x,y),(\xi,\eta)\big)\left(\sum_{i,j} c_{ij}\, \psi_{ij}(\xi,\eta)\right) d\xi d\eta\right] dx\, dy = \lambda\, c_{kl}\,.$$

Because the basis is orthonormal, the right-hand side simplifies to $\lambda\, c_{kl}$.

4.1 Constructing the Matrix A

Define the matrix entries by

$$A_{(kl),(ij)} = \int_D\int_D \psi_{kl}(x,y)\, C\big((x,y),(\xi,\eta)\big)\, \psi_{ij}(\xi,\eta)\, d\xi d\eta\, dx\, dy\,.$$

Thus, the projected eigenvalue problem becomes the generalized (in this case, standard) eigenvalue problem:

$$A\,\mathbf{c} = \lambda\,\mathbf{c}\,,$$

where $\mathbf{c}$ is the vector of coefficients c_{ij} (typically arranged as a column vector of length N^2).

5 Practical Steps to Build the Matrix A

1. **Choose the finite basis:** Decide on the number N of basis functions in each dimension. Then, your basis functions will be $\{\psi_{ij}(x,y) \mid i,j = 0,\ldots,N-1\}$.

2. **Compute the quadruple integrals:** For each pair of basis indices (k,l) and (i,j), compute

$$A_{(kl),(ij)} = \int_a^b \int_c^d \psi_{kl}(x,y) \left[\int_a^b \int_c^d C\big((x,y),(\xi,\eta)\big)\, \psi_{ij}(\xi,\eta)\, d\xi d\eta \right] dy\, dx\,.$$

3. **Numerical Quadrature:** In practice, the integrals are typically evaluated using numerical quadrature (such as Gauss–Legendre quadrature). This involves:
   - Selecting quadrature points $\{x_m\}$ and weights $\{w_m\}$ for the interval $[a,b]$, and similarly for $[c,d]$.
   - Approximating the integrals as weighted sums.

4. **Assembly:** Organize all the computed entries $A_{(kl),(ij)}$ into a matrix $A \in \mathbb{R}^{N^2 \times N^2}$. This matrix is symmetric and positive definite due to the properties of the covariance kernel.

6 Summary

To summarize, given a rectangular domain $D = [a,b] \times [c,d]$ and orthonormal polynomial bases in each variable:

1. We build a tensor product basis $\psi_{ij}(x,y) = \phi_i^{(1)}(x)\phi_j^{(2)}(y)$ that is orthonormal in $L^2(D)$.

2. We formulate the eigenvalue problem for the covariance operator as

$$\int_D C\big((x,y),(\xi,\eta)\big)\phi(\xi,\eta)\, d\xi d\eta = \lambda\, \phi(x,y)$$

   and approximate the eigenfunctions as finite sums over the basis.

3. By projecting onto the basis functions and exploiting orthonormality, the eigenproblem reduces to solving

$$A\,\mathbf{c} = \lambda\,\mathbf{c}\,,$$

   where the matrix A is defined by

$$A_{(kl),(ij)} = \int_D \int_D \psi_{kl}(x,y)\, C\big((x,y),(\xi,\eta)\big)\, \psi_{ij}(\xi,\eta)\, d\xi d\eta\, dx\, dy\,.$$

This formulation provides a clear path to discretize and solve the eigenproblem using standard numerical linear algebra techniques.