MULTI-ITEM ECONOMIC ORDER QUANTITY MODEL WITH AN INITIAL STOCK OF CONVERTIBLE UNITS

EDWARD A. SILVER
The University of Calgary, CANADA

ILKYEONG MOON
Pusan National University, KOREA

ABSTRACT

In this paper we consider a group of end items facing level demand patterns and opportunities for regular purchasing. In addition, there are a number of units that can be converted into any one of the end items, but at different unit costs of conversion. A constrained optimization model is developed for the problem of how best to allocate the convertible units among the end items. A solution algorithm is presented along with numerical illustrations.

INTRODUCTION

In this paper we consider the situation of a service (or purchasing) department dealing with a number of end items each satisfying the assumptions underlying the economic order quantity derivation, and particularly, having an essentially known, constant demand rate. However, in addition to initial inventories of these items, there are also a number of units that are not directly usable but which can be converted to each end item at a constant unit cost that depends upon the specific end item. Each of these end items can be purchased at unit costs that, in general, are higher than the unit conversion costs. This research was motivated by a situation, observed by one of the authors in a consulting assignment for a telecommunications organization, where customers returned used telephone units that could be converted into other usable units (through repairs, adding a different colored plastic cover, etc.). Another less obvious application is in a supply chain context where partially processed items can be converted to different end items, e.g. personal computer printers for sale in different countries (see Lee, et al. [4] and Feitzinger and Lee [3]). Part of the manufacturing is done centrally and units are shipped to various destinations where localized finishing (e.g. addition of special power source, user manual, etc.) is done. Partially processed units at a particular location could be processed further there or transshipped to other locations for completion.

The convertible situation is clearly related to the contexts of service parts and repairable items for which there is a substantial literature. References, that include broad surveys of modeling efforts in these contexts, include Brown [1], Diks, et al. [2], Mabini and Gelders [5], Nahmias [6], Sherbrooke [7], Silver, et al. [9], and Verrijdt [10].

Another closely related situation is the so-called "special opportunity to buy" where there is a one-time opportunity to acquire one or more end items at a unit cost lower than will be the case in the future. Silver, et al [9] present a heuristic solution procedure for the multi-item case subject to a **budget** restriction on the total value of the units acquired under the special opportunity (and provide a set of references on the single item situation). In the current paper the constraint is in the form of the total **number** of convertible units available and an optimal solution algorithm is provided

Silver and Moon [8] have addressed a convertible context that differs in two respects from the current paper. First, a single period horizon is considered. Secondly, each end item faces probabilistic demand with a known distribution. In other words, they have dealt with the multi-item newsvendor context.

In the first section we present the notation and assumptions, then develop a mathematical model of the decision situation. The solution algorithm is the subject of the second section. Also included is a numerical example. The final section provides some brief concluding remarks.

PRELIMINARIES

NOTATION

The notation to be used is as follows:

n = number of different end items

j = index for end items ($j = 1, 2, \ldots, n$)

c_j = unit conversion cost from the convertible item to end item j, in dollars/unit

v_j = unit purchase cost of end item j, in dollars/unit

h_j = inventory carrying cost rate of end item j, in dollars/unit/unit time

A_j = fixed ordering cost associated with a replenishment of item j, in dollars

D_j = known demand rate for end item j, in units/unit time

α = discount rate per unit time

N = available number of units of the convertible item

I_j = inventory of item j ($j \neq 1$) at $t = 0$ where $t = 0$ is defined as the moment where the first item runs out of inventory (the items are indexed so that item 1 runs out of inventory first)

R_j = number of convertible units that are converted into end item j (decision variables)

Q_j = number of units of end item j that are purchased (decision variables)

DEVELOPMENT OF THE MATHEMATICAL MODEL OF THE DECISION PROBLEM

As indicated earlier, we can think of a service department (or purchasing department), which is in charge of providing items to other departments or factories, having N units of an item that are not directly usable. Each unit can be converted to an end item j (j = 1, 2, ... , n) at a cost of c_j. The end item can also be purchased at a unit cost of v_j. We need to decide how many units should be converted for each item, as well as the timing of the conversions.

First, we derive the present cost of a single regular purchasing cycle for item j discounted to its starting point as follows:

$$A_j + Q_j v_j + \int_0^{\frac{Q_j}{D_j}} (Q_j - D_j t) h_j e^{-\alpha t} dt = A_j + Q_j \left(v_j + \frac{h_j}{\alpha} \right) - \frac{h_j D_j}{\alpha^2} \left(1 - e^{-\alpha \frac{Q_j}{D_j}} \right) \tag{1}$$

Since there are infinite number of cycles, we compute the present value of total cost for item j, without conversion and at the start of a cycle (i.e. with zero inventory on hand) as follows:

$$\begin{aligned} PV(Q_j) &= \left[A_j + Q_j \left(v_j + \frac{h_j}{\alpha} \right) - \frac{h_j D_j}{\alpha^2} \left(1 - e^{-\alpha \frac{Q_j}{D_j}} \right) \right] \left(1 + e^{-\alpha \frac{Q_j}{D_j}} + e^{-2\alpha \frac{Q_j}{D_j}} + \ldots \right) \\ &= \frac{A_j + Q_j \left(v_j + \frac{h_j}{\alpha} \right)}{1 - e^{-\alpha \frac{Q_j}{D_j}}} - \frac{h_j D_j}{\alpha^2} \end{aligned} \tag{2}$$

We can prove that $PV(Q_j)$ is convex in Q_j. If we compute,

$$\frac{dPV(Q_j)}{dQ_j} = 0$$

we obtain the following equation.

$$\left(v_j + \frac{h_j}{\alpha} \right) \left(1 - e^{-\alpha \frac{Q_j}{D_j}} \right) - \frac{\alpha A_j}{D_j} e^{-\alpha \frac{Q_j}{D_j}} - \frac{\alpha}{D_j} \left(v_j + \frac{h_j}{\alpha} \right) e^{-\alpha \frac{Q_j}{D_j}} Q_j = 0 \tag{3}$$

Equation (3) can be rewritten as

$$\left(v_j + \frac{h_j}{\alpha} \right) \left(e^{\alpha \frac{Q_j}{D_j}} - 1 \right) - \frac{\alpha A_j}{D_j} - \frac{\alpha}{D_j} \left(v_j + \frac{h_j}{\alpha} \right) Q_j = 0 \tag{4}$$

From Eq. (3), the optimal Q_j^* satisfies

$$\frac{A_j + Q_j^*\left(v_j + \frac{h_j}{\alpha}\right)}{1 - e^{-\alpha \frac{Q_j^*}{D_j}}} = \frac{v_j + \frac{h_j}{\alpha}}{\frac{\alpha}{D_j} e^{-\alpha \frac{Q_j^*}{D_j}}} \tag{5}$$

By substituting Eq. (5) into Eq. (2), we obtain the optimal present value of future costs for item j without any conversion:

$$PV(Q_j^*) = \frac{\left(v_j + \frac{h_j}{\alpha}\right) e^{\alpha \frac{Q_j^*}{D_j}}}{\frac{\alpha}{D_j}} - \frac{h_j D_j}{\alpha^2} \tag{6}$$

From Eq. (4), we get the following:

$$\frac{\left(v_j + \frac{h_j}{\alpha}\right) e^{\alpha \frac{Q_j^*}{D_j}}}{\frac{\alpha}{D_j}} = A_j + \left(V_j + \frac{h_j}{\alpha}\right) Q_j^* + \frac{v_j D_j}{\alpha} + \frac{h_j D_j}{\alpha^2} \tag{7}$$

By substituting Eq. (7) into Eq. (6), we obtain

$$PV(Q_j^*) = A_j + \left(v_j + \frac{h_j}{\alpha}\right) Q_j^* + \frac{v_j D_j}{\alpha} \tag{8}$$

Consider the following approximation:

$$e^{\alpha \frac{Q_j}{D_j}} \cong 1 + \frac{\alpha}{D_j} Q_j + \frac{\left(\alpha \frac{Q_j}{D_j}\right)^2}{2}$$

It is likely to be very accurate as $\alpha Q_j / D_j$ is almost certainly well below unity. Substituting it into Eq. (4), and simplifying, we get the following approximate solution, which is essentially an EOQ:

$$Q_j \cong \sqrt{\frac{2 A_j D_j}{\alpha v_j + h_j}} \tag{9}$$

By substituting Eq. (9) into Eq. (8), we obtain

$$W_j \equiv PV(Q_j^*) \equiv A_j + \frac{v_j D_j}{\alpha} + \sqrt{\frac{2A_j D_j\left(v_j + {h_j}/{\alpha}\right)}{\alpha}} \tag{10}$$

We now consider the use of the convertible units. Without loss of generality, we reorder items in the order of inventory shortage time. Let $I_j(j \neq 1)$ be the inventory level of item j when item 1's inventory level becomes 0, i.e. $I_1 = 0$. Then item $j(j \neq 1)$ will run out at time I_j/D_j, and convertible units can be replenished at that time. After convertible units have been consumed, the regular economic order quantities can be replenished thereafter. Let *PVTRC* ($R_1,\ldots, R_n$) be the present value of total costs including convertible decisions. Then the costs consist of three elements:

(i) *Convertible Cost*

$$c_j R_j e^{-\alpha\frac{I_j}{D_j}} \tag{11}$$

(ii) *Holding Cost for Convertible Units*

$$h_j e^{-\alpha\frac{I_j}{D_j}} \int_0^{R_j/D_j} (R_j - D_j t)e^{-\alpha t} dt = e^{-\alpha\frac{I_j}{D_j}}\left[\frac{h_j R_j}{\alpha} - \frac{h_j D_j}{\alpha^2}\left(1 - e^{-\alpha\frac{R_j}{D_j}}\right)\right] \tag{12}$$

(iii) *Relevant Cost after Consuming Convertible Units*

$$W_j e^{-\alpha\frac{I_j+R_j}{D_j}} \tag{13}$$

Note that at time $(I_j + R_j)/D_j$, regular economic order quantity replenishments will be resumed. Then, our decision problem can be expressed as follows:

$$\text{Minimize } PVTRC\,(R_1,\ldots, R_n)$$

$$= \sum_{j=1}^{n}\left[\left(c_j R_j + \frac{h_j R_j}{\alpha} - \frac{h_j D_j}{\alpha^2} + \frac{h_j D_j}{\alpha^2} e^{-\alpha\frac{R_j}{D_j}}\right)e^{-\alpha\frac{I_j}{D_j}} + W_j e^{-\alpha\frac{I_j+R_j}{D_j}}\right] \tag{14}$$

$$\text{subject to } \sum_{j=1}^{n} R_j \leq N$$

$$R_j \geq 0 \quad \forall_j$$

ALGORITHM AND NUMERICAL EXAMPLE

We can prove that the above objective function is convex as follows:

$$\frac{\partial^2 PVTRC}{\partial R_i \partial R_j} = 0 \quad \forall i, j (i \neq j)$$

Consequently, the Hessian matrix is diagonal. The diagonal elements are given by

$$\frac{\partial^2 PVTRC}{\partial R_j^2} = \left(\frac{h_j}{D_j} + \frac{\alpha^2}{D_j^2} W_j \right) e^{-\alpha \frac{l_j + R_j}{D_j}} \ldots\ldots \forall j$$

The determinant of the Hessian matrix becomes

$$\prod_{j=1}^{n} \left(\frac{h_j}{D_j} + \frac{\alpha^2}{D_j^2} W_j \right) e^{-\frac{l_j + R_j}{D_j}} > 0$$

Therefore the determinant of the Hessian matrix is positive definite, hence, the objective function is convex. Now let λ be the Lagrangian multiplier associated with the constraint. The Lagrangian function is then

$$L(R_1, \ldots, R_n, \lambda) = PVTRC(R_1, \ldots, R_n) + \lambda \left(\sum_{j=1}^{n} R_j - N \right)$$

By computing $\partial L / \partial R_j$, we obtain the following equations:

$$\frac{\partial L}{\partial R_j} = \left[c_j + \frac{h_j}{\alpha} \left(1 - e^{-\alpha \frac{R_j}{D_j}} \right) \right] e^{-\alpha \frac{l_j}{D_j}} - \frac{\alpha}{D_j} W_j e^{-\alpha \frac{l_j + R_j}{D_j}} + \lambda = 0 \ldots\ldots \forall j$$

We need to find the smallest nonnegative λ^* such that the $R_j(\lambda)$ satisfy Eq. (14). A line search algorithm can be used to find this optimal value of λ. It represents the rate of reduction of the optimal cost as N is increased.

LINE SEARCH ALGORITHM

STEP 1: First, check if the following independent solution satisfies the constraint. If it satisfies the constraint in Eq. (14), it is indeed optimal. Otherwise, go to STEP 2.

$$R_j = \frac{D_j}{\alpha}\ln\left(\frac{\frac{h_j}{\alpha}+\frac{\alpha W_j}{D_j}}{c_j+\frac{h_j}{\alpha}}\right) \tag{15}$$

STEP 2: Start with an arbitrary $\lambda > 0$.
STEP 3: Compute $R_j s$ as follows:

$$R_j = \frac{D_j}{\alpha}\ln\left[\frac{\frac{h_j}{\alpha}+\frac{\alpha W_j}{D_j}}{\lambda+\left(c_j+\frac{h_j}{\alpha}\right)e^{-\alpha\frac{I_j}{D_j}}}\right] - I_j$$

STEP 4: If $\sum_j R_j < N$, then decrease λ and go to STEP 3.
If $\sum_j R_j > N$, then increase λ and go to STEP 3.
If $\sum_j R_j = N$, we have found an optimal solution.

EXAMPLE:
We assume that there are 1000 units of the convertible items available. The detailed data for this example are given in TABLE 1. We set $\alpha = 0.25$/unit time.

TABLE:1. Data for the Example (N =1000 and λ=0.25)

Item	c_j	v_j	h_j	A_j	I_j	D_j
Item 1	\$20	\$30	\$9	\$75	100	2000
Item 2	\$30	\$50	\$15	\$120	50	500
Item 3	\$15	\$20	\$6	\$110	100	8000
Item 4	\$100	\$120	\$36	\$200	200	5000

First we compute the unconstrained convertible quantities using Eq. (15), and obtain (R_1, R_2 R_3 R_4) = (1449,467,4260,1750). The total exceeds 1,000 (i.e. N). If we apply the line search algorithm, we find the following optimal convertible quantities (R_1, R_2 R_3 R_4) = (0,213,0,787). with the optimal Lagrangian multiplier, λ^*, being 11.910 for this case. The λ^* value approximately represents the savings in costs associated with increasing N from 1,000 to 1,001.

If we increase the holding cost rate of Item 4 to h_4 = \$50, the unconstrained convertible quantities become (R_1, R_2 R_3 R_4) = (1449,467,4260,1449). The following optimal convertible quantities are then obtained (R_1, R_2 R_3 R_4) = (27,236,0,737). Here the optimal Lagrangian multiplier, λ^*, becomes 10.762.

We notice that R_4 is decreased somewhat as expected.

If we increase the conversion cost of Item 4 to $c_4 = \$110$, the unconstrained convertible quantities become (R_1, R_2 R_3 R_4) = (1449,467,4260,946). The following optimal convertible quantities are then obtained (R_1, R_2 R_3 R_4) = (379,285,0,326). Here the optimal Lagrangian multiplier, λ^*, becomes 7.917. We notice that R_4 is decreased somewhat as expected and the marginal value of an extra unit of the convertible item is reduced somewhat due to the higher unit cost of conversion to end Item 4.

CONCLUSIONS

We have considered the situation of a group of end items facing level demand patterns and the possibility of regular purchasing. In addition, there are a number of units that can be converted into any one of the end items, but at different unit costs of conversion. An algorithm has been developed for optimally allocating the convertible units to the various end items.

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BIOGRAPHICAL SKETCHES

EDWARD A. SILVER holds the Carma Chair at the University of Calgary. He received a Bachelor of Civil Engineering from McGill University, and subsequently, a Science Doctorate in Operations Research at M.I.T. Prior to the University of Calgary, his work experience included positions with Arthur D. Little Inc., and the University of Waterloo. Dr. Silver has held several visiting positions including at Stanford University, the Swiss Federal Institute of Technology, the University of Canterbury, New Zealand (Visiting Erskine Fellow) and the University of Auckland, New Zealand (Auckland Foundation Visitor). Professor Silver has consulted and provided executive training and other workshops for a wide range of organizations. These activities have covered a broad range of topics including inventory management, supply chain management, business process improvement, and the use of quantitative modeling to aid in decision-making. He has co-authored three editions of the textbook Inventory Management and Production Planning and Scheduling. He and has had over 130 articles published in professional journals. Dr. Silver is a past president of the Canadian Operational Research Society and received the 1990 Award of Merit from the Society. He was elected a Fellow of the Institute of Industrial Engineering in 1995 and as an Inaugural Fellow of the Manufacturing and Services Operations Management Society in 2000. Professor Silver is a past president of the International Society for Inventory Research.

ILKYEONG MOON is an associate professor and chair of Industrial Engineering at Pusan National University in Korea. He received his B.S. and M.S. in Industrial Engineering from Seoul National University, Korea, and Ph.D. in Operations Research from Columbia University in 1991. His research interests include production planning and control (especially, JIT applications), logistics (especially, supply chain management), modeling and analysis of manufacturing systems, and inventory control. His work has appeared in journals such as *Computers & Industrial Engineering, Computers & Operations Research, European Journal of Operational Research, IIE Transactions, International*

Journal of Production Economics, International Journal of Production Research, Journal of the Operational Research Society, Management Science, Naval Research Logistics, Omega, Operations Research, and *The Engineering Economist*. He currently serves on the editorial boards of *IIE Transactions (Scheduling & Logistics), International Journal of Industrial Engineering* and *International Journal of Operations and Quantitative Management.*