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Production-Inventory control model for a supply chain network with economic production rates under no shortages allowed

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ABSTRACT

This paper presents a new production-inventory control (PIC) model for a supply chain network (SCN) where multiple suppliers, manufacturers, and buyers are vertically integrated to provide multiple items to the market. The objective of the model is to simultaneously determine optimal replenishment quantities, replenishment cycles, and production rates in order to minimize the total network cost allowing no shortages of inventory and shutdown periods. We propose closed-form functions of the average annual inventory levels. A novel mixed-integer linear programming (MILP) formulation is developed based on these functions to solve the PIC model. In addition, an algorithm based on a decomposition approach was developed to solve a special case of the PIC model with a less computational burden. The results obtained by the algorithm are compared to the results of the existing models in literature to see if there is any improvement. A case study for designing an integrated SCN and controlling operational decisions is presented to demonstrate the application of the proposed PIC model.

1. Introduction

With the growing focus on supply chain management (SCM), many firms recognize that the inventory system across the related supply chain network (SCN) should be controlled to trim undesirable costs caused by order sizes or inventory levels that are larger or smaller than necessary. To address this, several supply chain coordination models, including vendor-managed inventory (VMI), have been assessed in previous studies (Zahran, Jaber, and Zanoni, 2017). VMI is a replenishment strategy whereby manufacturers (or vendors) minimize ordering and inventory handling costs and monitor and manage buyers' inventory levels to reduce undesirable or fluctuating inventory—called the bullwhip effect. Walmart, Kmart, and JCPenney are examples of companies that have successfully implemented the VMI strategy (Simchi-levi, D. Kaminsky, 2004).

In VMI implementation, a *production-inventory control (PIC) model* (or *coordination policy)* is devised to provide a simple but reasonable solution to both production and inventory problems (Woo, 2018). In general, a PIC model includes two rules related to both ordering and production. The first rule concerns replenishing inventory to keep up with demand. The second rule concerns the timing and quantity of production and the management of existing inventory by filling pending orders.

The pioneering research of VMI implementation addressed a joint PIC problem with a single manufacturer and a single buyer in a supply chain over a continuous planning horizon (Goyal, 1977, 1995; Hill, 1997, 1999). Goyal (1977) introduced a PIC model in which the buyer placed orders based on the economic order quantity (EOQ) model, and the manufacturer (or supplier) decided on a production cycle based on the economic production quantity (EPQ) model. This model suggested creating inventory that filled buyers' orders in every batch run. However, rather than considering the production cycle, which consists of both an active and an idle period of a facility, only the batch production volume was measured by assuming an infinite production rate. The assumption of the infinite production rate is relaxed in some subsequent models (Goyal, 1995; Hill, 1997, 1999). Goyal (1995) proposed another PIC model, which allowed for geometrically increasing the sizes of shipments (DS) based on a given factor. Hill (1997) generalized the policy of Goyal (1995) by considering the factor as a decision variable. By doing so, he showed that neither of the two PIC models outperforms the other in some instances. Hill (1999) developed a more cost-effective policy in which non-decreasing shipments are constrained.

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Later, Khouja (2003) and Lee (2005) extended the PIC problem with a three-echelon supply chain in which work-in-process (WIP) for a raw material needed to create a product is also jointly controlled. To address inventory requirements for each item—the raw materials needed, and the final products themselves—they applied an integer-ratio cycle (IRC) policy to their PIC model. Similarly, in the transitional coordination policy, the concept of EOQ and EPQ models is applied to the IRC policy. Meanwhile, the production cycles of downstream members, such as manufacturers and suppliers, are stipulated by integer-multiplied values of a buyer's replenishment or the adjacent member's production cycle. In particular, the IRC policy for the PIC problem is more cost-effective than the common cycle policy, in which production cycles of members are set to the same cycle and compared to the integer-power-of-two cycle policy—a policy in which a production cycle is set to an integer-power-of-two multiple of the basic cycle time by Khouja (2003).

Table 1 lists the previous studies that introduced the earliest forms of coordination policies and the advanced studies that scrutinize some complex configurations of SCNs or investigate special cases of PIC problems encountered in the past. Many studies followed the IRC policy to determine the most economically efficient lot size of members. Jaber and Goyal (2008) introduced a generalized PIC model in which there are multiple suppliers and buyers. Ben-Daya, As'Ad, and Seliaman (2013) extended the problem of Lee (2005) to a problem in which the production and shipment schedules of the supplier are also jointly managed. Ben-Daya et al. (2013) investigated a consignment stock with delay-in-payments and a payment scheme in the PIC problem of a single vendor and multiple buyers. Kim and Glock (2013) developed a generalized model of a serial multi-stage supply chain for the PIC problem. Shafieezadeh and Sadegheih (2014) considered another generalized problem of a three-echelon SCN, in which multiple suppliers, manufacturers, and buyers exist. They described a nonlinear structure of the PIC model and proposed a heuristic algorithm owing to the intractability of the nonlinear structure. Zhao, Wu, and Yuan (2016) proposed a PIC model wherein the pipeline costs of inventory are charged after considering lead time. Zahran and Jaber (2017) investigated a consignment scheme in the PIC problem of a three-echelon supply chain. From their findings, they presented four generalized coordination policies that applied combinations of consignment and no-consignment terms to subsequent members of the SCN. Aljazzar, Jaber, and Moussawi-Haidar (2017) considered a PIC problem that includes permissible delay in payments and price discounts in a three-echelon supply chain. Alfares and Attia (2017) extended the PIC model of Ben-Daya et al. (2013) into a more general model with quality defects and inspection errors. Kumar, Khan, and Mandal (2017) addressed a PIC problem wherein a vendor supplies multiple products to and buyers. They introduced a nonlinear structure of the PIC model and proposed a metaheuristic algorithm. Darvish and Coelho (2018) addressed a discrete version of PIC problem where the delivery time window and dynamic location decision are considered in distribution system. Their PIC problem is related to a lot sizing problem where the amount of production, inventory, and delivery should be decided for a discrete period under given unit production cost and unit inventory holding cost. Gharaei, Karimi, and Shekarabi (2019) investigated a multi-product and multi-buyer supply chain model with quality control and green policies. In their model, each replenishment ordered by the buyers contains the fixed percentage of the vendor's defective products. The inventory profiles of their model involve disposals of the defective products. Chen and Bidanda (2019) developed a PIC model for a closed loop supply chain considering component recovery and emission control. They investigated the optimal decision of integrated production-inventory of beer industry. Their PIC model referred to the IRC policy where an idle period of production is allowed. Tarhini, Karam, and Jaber (2020) investigated a PIC problem of a vendor and multiple buyers where transhipments are allowed among the buyers. They presented a structural property useful in searching the optimal solution to the PIC problem and proposed a metaheuristic algorithm that effectively searches for continuous decision variables under given combinatorial variables using the property. Mishra, Wu, and Sarkar (2020) discussed a PIC model for a closed loop supply chain with carbon emission issue was discussed to reduce the waste of an imperfect production system. Their model, named the sustainable EPQ carbon tax and cap model, is develpoed to determine optimal cycle time and green technology investment, and the period of positive inventory level considering several strategies on shortages. Agustiandi, Aritonang, and Rikardo (2021) addressed a PIC problem for a single-vendor multi-buyer supply chain under the consideration of capital and warehouse constraints. Each replensishment of buyer is limited by a fixed capacity fo

Table 1
A summary of the literature on coordination policy for PIC model.

Author	Configuration of SCN	Coordination policy:	Production rate	Inventory types		
				R	WIP	FP
Goyal (1977)	1 M−1B	(EOQ, EPQ)	Infinite			√
Goyal (1995)	1 M−1B	(DS, EPQ)	Fixed			√
Hill (1997)	1 M−1B	(EOQ, EPQ)	Fixed			√
Hill (1999)	1 M−1B	(DS, EPQ)	Fixed			√
Khouja (2003)	1S-nM-nB[a]	IRC	Fixed	√	√	√
Lee (2005)	1S-1 M−1B	IRC	Fixed	√	√	√
Jaber and Goyal (2008)	nS-1 M−nB	IRC	Fixed	√	√	√
Ben-Daya, As'Ad, and Seliaman (2013)	1S-1 M−1B	IRC	Fixed	√	√	√
Ben-Daya et al. (2013)	1 M−nB	IRC	Fixed			√
Kim and Glock (2013)	1S-1 M-...-1B	IRC	Fixed	√	√	√
Shafieezadeh (2014)	nS-nM-nB	IRC	Fixed	√	√	√
Zhao, Wu, and Yuan (2016)	1S-1 M−1B	IRC	Fixed	√	√	√
Zahran and Jaber (2017)	nS-1 M−nB	IRC	Fixed	√	√	√
Aljazzar, Jaber, and Moussawi-Haidar (2017)	1S-1 M−1B	IRC	Fixed	√	√	√
Alfares and Attia (2017)	1 M−nB	IRC	Fixed			√
Kumar, Khan, and Mandal. (2017)	1 M−nB	IRC	Fixed			√
Darvish and Coelho (2018)	nS-nM-nB	Lot sizing	Adjustable[b]	√		√
Gharaei, Karimi, and Shekarabi (2019)	1 M−nB	IRC	Fixed			√
Chen and Bidanda (2019)	nS-nM-nB	IRC	Fixed	√	√	√
Tarhini, Karam, and Jaber (2020)	1 M−nB	IRC	Fixed			√
Mishra, Wu, and Sarkar (2020)	1 M−1B	EPQ	Fixed			√
Agustiandi, Aritonang, and Rikardo (2021)	1 M−nB	IRC	Fixed			√
Aazami and Saidi-Mehrabad (2021)	nM-nB	Lot sizing	Adjustable[b]	√		√
This study	nS-nM-nB	(EOQ, EPQ)	Adjustable[b]	√	√	√

[a] A distribution system in which a member ships an item to multiple members in the next echelon.
[b] Adjustable production rate with the maximum production capacity ($< P_{\max}$).

the buyer. They attained optimal solutions for the PIC problem by using the Lagrange multipliers method. Aazami and Saidi-Mehrabad (2021) investigated a production and distribution planning problem with perishability and freshness. They proposed a seller-buyer model for perishable products under competitive factors which is based on leader and follower model. Due to the intractibility of the problem they developed a hierarchical heuristic approach using the genetic algorithm and Benders decomposition algorithm.

Although many studies have used the PIC models to determine cost savings in an SCN, existing models may still not be effective in reducing overall network costs. This is because the models assume that a production rate is fixed with a maximum production rate. Setting the production rate this way makes a manufacturing member maintain a cycle that includes both an active and an idle period. In the active period, the member saves inventory and releases inventory required for a single batch simultaneously. In the idle period, all the remaining inventory is released. However, in implementing these models, this manufacturing policy creates inefficiency and undesirable inventory because inventory is produced too early in the active period. This can lead to a decrease in circulating capital due to excessive inventory of products to manufacturers even if all relevant members are vertically integrated.

To address the issue, this paper develops a novel PIC model for controlling variable production rates as well as replenishment quantities and replenishment cycles—an approach, to the best of our knowledge, that has not yet been addressed. In addition, our model considers multiple suppliers, manufacturers, and buyers; multiple channels of item orders and delivery; and multiple items, including raw materials, intermediate products, and final products. The following questions will be addressed in the paper:

- Is an inventory level non-negative when an item is replenished (or shipped) via multiple channels and is consumed (or produced)?
- Does a closed-form function exist to calculate the average annual inventory of an item?
- Is it possible to reduce the relevant costs by applying the proposed model?

In summary, the main contributions of this paper are as follows. First, we propose a PIC model for general SCN which is to determine replenishment cycles of orders between members and production rates of multiple products. Second, we introduce an optimization model with closed-form functions of inventory considering a set of replenishments and operations. Third, we develop an algorithm based on a decomposition approach to solve a special case. Moreover, we present the application of the production-inventory control model.

The rest of this paper is organized as follows. Section 2 introduces the PIC problem and some preliminary results. Section 3 proposes an MILP as a main solution approach for solving the problem and an algorithm based on a decomposition approach to solve the PIC model for a special case. Section 4 looks at computational experiments that were conducted to determine the cost savings from the proposed model. Section 4 also offers insights into modeling difficulties. Section 5 shows the case study on an integrated SCN to demonstrate the application of the proposed PIC model. Section 6 concludes with a look at directions research may take in the future.

2. Production-inventory control problem

2.1. Problem description, assumption, and notation

We looked first at the PIC problem of how to order lot (replenishment) sizes accurately, determine replenishment cycles, and maintain efficient production rates. Fig. 1 illustrates a general SCN, where multiple suppliers, manufacturers, and buyers are vertically integrated. A supplier provides raw materials to manufacturers with a supplement rate. A manufacturer orders *materials,* including raw materials and intermediate products, from preceding supply chain members (suppliers and manufacturers), and converts these materials into *products,* including intermediate products and final products. These products are then delivered to succeeding supply chain members (manufacturers and buyers). Each manufacturer is characterized by a technology facility type. In accordance with a type, a manufacturer makes a product by processing a material with a production rate. In placing orders for an *item* (material or product) between two adjacent members, a constant replenishment quantity and replenishment cycle should be determined. The series of replenishments of an item between the two members is called a *schedule* or *replenishment schedule*. If a member receives replenishments based on a schedule, then the schedule (or replenishment) is said to be *inbound* associated with the member. At the same time, the schedule (or replenishment) is said to be *outbound* associated with the other member who sends the item to the member. The objective of the PIC problem is to minimize the total network cost (TNC), which consists of ordering costs, inventory holding costs, production costs, and annual investment costs.

The following assumptions are made in the problem under consideration:

- Demand rate of a buyer is deterministic and constant (Ben-Daya et al., 2013; Zahran and Jaber, 2017; Zhao et al., 2016).
- Production rate of a manufacturer is a constant decision variable.
- No shortages are allowed (Ben-Daya et al., 2013; Sarkar, 2013; Zahran and Jaber, 2017; Zhao et al., 2016).
- Lead time for replenishments is zero (Ben-Daya et al., 2013; Zhao et al., 2016).
- Idle time of manufacturers is not allowed.
- For the inbound schedule, equal-sized replenishments occur at the beginning of every cycle. For the outbound schedule, equal-sized replenishments occur at the end of every cycle (see Fig. 2). Section

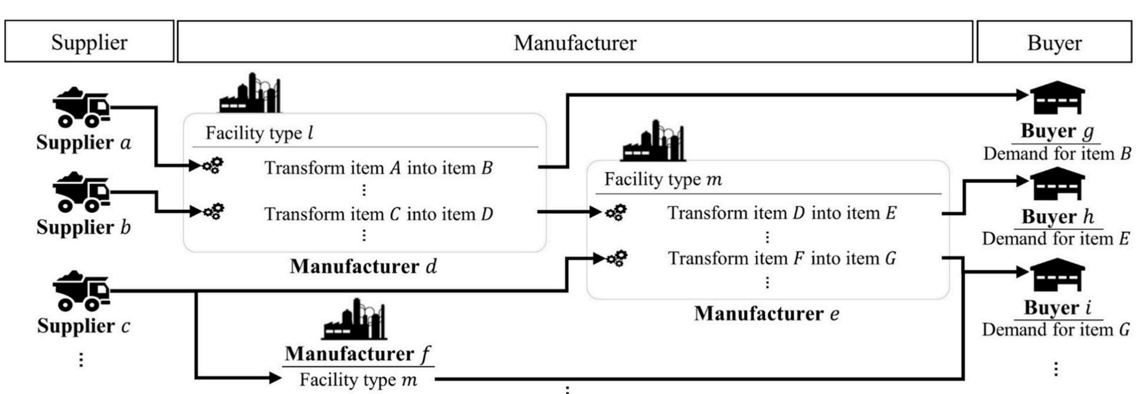


Fig. 1. General supply chain network.

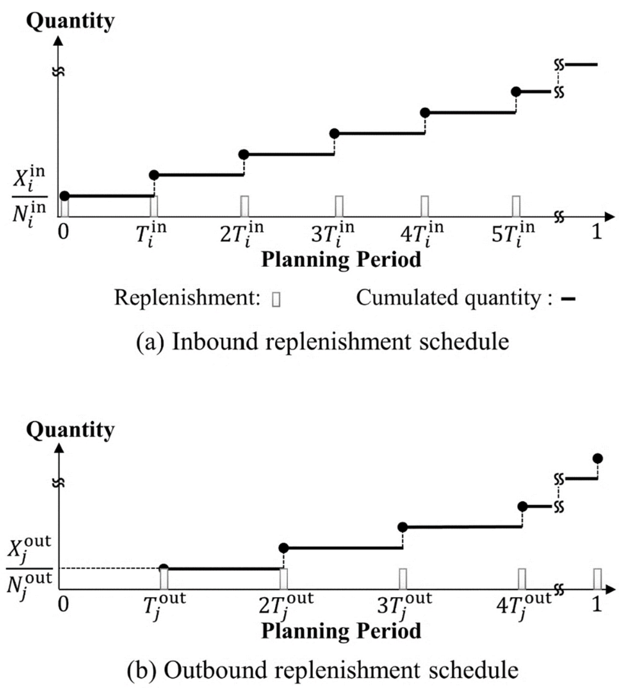


Fig. 2. Replenishment points of inbound and outbound schedules.

3 will prove that under such policy guarantees, no shortages occur in the supply chain.

- Alternatives to replenishment cycles exist, such as $\{x = n/365(\text{year}) | n \in \mathbb{Z}^+\}$.

The rigorous assumptions in the IRC policy, such as (i) production rate determined as the maximum value, and (ii) IRCs of a common cycle, are relaxed in our model (Ben-Daya et al., 2013; Sarkar, 2013; Zahran and Jaber, 2017).

The following notations are used to describe a new PIC model:

p	number of inbound schedules
q	number of outbound schedules
X_i^{in}	total quantity in inbound schedule i over a planning period (i.e., a year)
X_j^{out}	total quantity in outbound schedule j over a planning period
T_i^{in}	replenishment cycle of inbound schedule i
T_j^{out}	replenishment cycle of outbound schedule j
N_i^{in}	frequency of inbound schedule i ($N_i^{\text{in}} = 1/T_i^{\text{in}}$)
N_j^{out}	frequency of outbound schedule j ($N_j^{\text{out}} = 1/T_j^{\text{out}}$)
O_t^{in}	sum of quantities of inbound replenishments at time t
O_t^{out}	sum of quantities of outbound replenishments at time t
I_t	inventory level at time t
C_t^{in}	cumulative quantity of all inbound replenishments at time t ($C_t^{\text{in}} = \sum_{a=1}^{t} O_a^{\text{in}}$)
C_t^{out}	cumulative quantity of all outbound replenishments at time t ($C_t^{\text{out}} = \sum_{a=1}^{t} O_a^{\text{out}}$)

The following notations are used to develop a mathematical model:

2.2. Sets

G	set of members, $G = H \cup I \cup J$
H	set of suppliers
I	set of manufacturers
J	set of buyers
K	set of items
L	set of facility types
M	set of replenishment cycles

2.3. Parameters

$\alpha_{kk'l}$	conversion factor from item k to item k' at facility l
β_{ik}^S	maximum annual supplement rate of item k at supplier $i \in H$
$\beta_{lkk'}$	maximum annual production rate from item k to item k' at manufacturer $i \in J$
β_{ik}^B	annual demand rate of item k at buyer $i \in J$
γ_{il}	1 if manufacturer $i \in I$ has facility type l, otherwise 0
$\delta_{ii'k}$	ordering cost per order of item j from member i to member i'
ε_{ik}	inventory holding cost of item k per unit at member i
ζ_{ik}	operating cost of item k per unit at member i
τ_m	cycle time of replenishment cycle m

2.3.1. Decision variables

x_{ijkm}	total amount of item k that is shipped from member i to member j with cycle m
$y_{ilkk'}$	annual production rate of item k' made from item k at manufacturer i with facility type l
z_{ijkm}	1 if item k is shipped from member i to member j with cycle m, otherwise 0

2.4. Preliminary results

The following presents some preliminary results on inventory levels of both a raw material and a product that will be used in subsequent formulas. Generalizing broadly, we look at some replenishment and shipment schedules of an item associated with a supply chain member, as shown in Fig. 3 below. The member replenishes the item with p inbound schedules and furnishes them with q outbound schedules. In Fig. 3 (a), one bar graph represents a replenishment schedule. For a schedule $i = 1, \cdots, p$ (or $j = 1, \cdots, q$), there are a total quantity of inbound replenishments X_i^{in}, frequency of replenishments N_i^{in}, and replenishment cycle T_i^{in} (or X_j^{out}, N_j^{out}, and T_j^{out}). Aggregating the above schedules, inventory level for time $t \in [0, 1]$, I_t is recorded as shown in Fig. 3 (b). The curved arrow represents the following condition: if there are both non-zero inbound and outbound replenishments for time $t = a + \Delta$, i.e., $O_{a+\Delta}^{\text{in}}$ and $O_{a+\Delta}^{\text{out}}$, the inventory level $I_{a+\Delta}$ equals to $I_{a+\Delta} = I_a + O_{a+\Delta}^{\text{in}} - O_{a+\Delta}^{\text{out}}$, where Δ is an extremely small positive value. This produces the following results for inventory levels.

Lemma 1. *Inventory level of an item material at time t is equal to the difference between the cumulated quantity of all inbound replenishments and the cumulated quantity of all outbound replenishments at time t.*

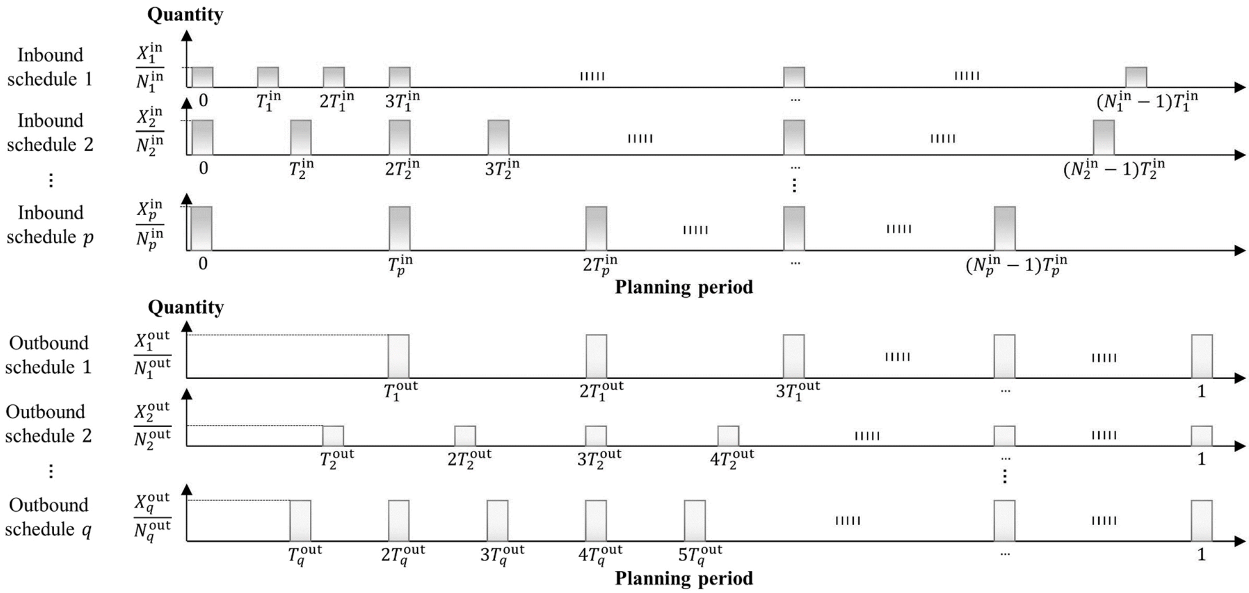
Proposition 1. *Inventory levels of an item associated with a member are always non-negative during the planning period if the following conditions are met:*

a) There are the number of inbound schedules, p, and the number of outbound schedules, q, including constant replenishment quantities and replenishment cycles.
b) The total quantity of inbound replenishments is equal to or greater than the total quantity of outbound replenishments, i.e., ($\sum_{i=1}^{p} X_i^{in} \geq \sum_{j=1}^{q} X_j^{out}$).
c) For inbound schedules, an order is placed at the beginning of every cycle; for outbound schedules, an order is placed at the end of every cycle.

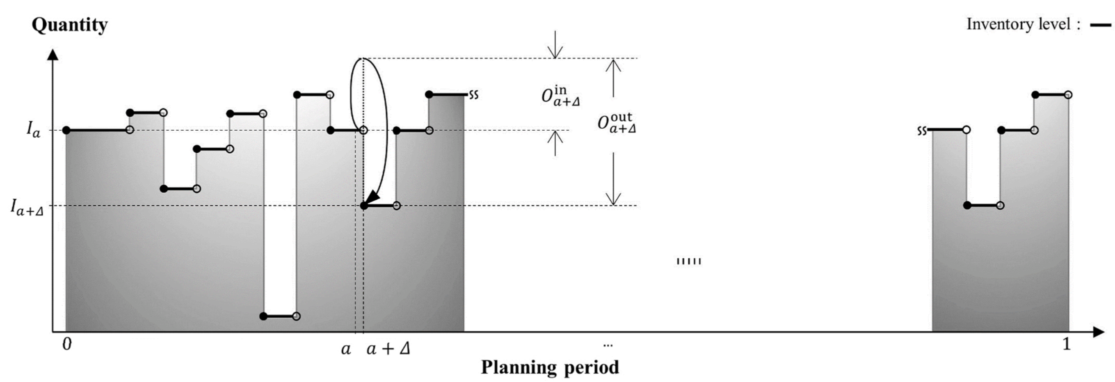
Lemma 2. *The total annual cumulated quantity of inbound replenishments associated with schedule i is*

$$X_i^{in}\left(\frac{1 + T_i^{in}}{2}\right),$$

and the total annual cumulated quantity of outbound replenishments associated with schedule j is

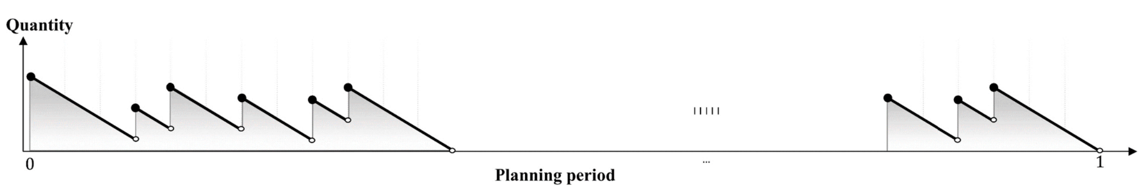


(a) Inbound and outbound replenishment schedules of an item

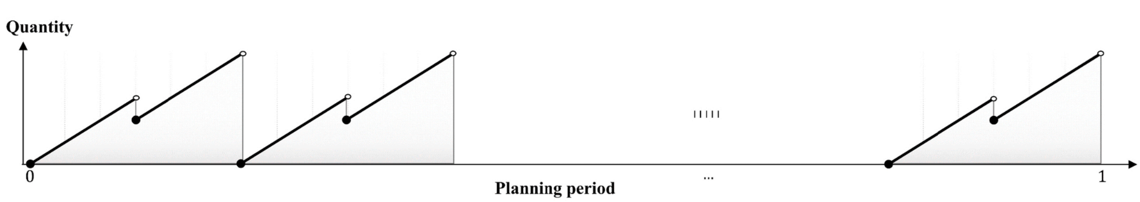


(b) Inventory profile of an item

Fig. 3. Schedules and inventory profile of an item for a supply chain member.



(a) Inventory profile of a WIP considering a consumption rate



(b) Inventory profile of a product considering a production rate

Fig. 4. General inventory profiles of a WIP and a product for a supply chain member.

$$X_j^{out}\left(\frac{1-T_j^{out}}{2}\right)$$

Proposition 2. *If conditions (a) through (c) in Proposition 1 are met, then the average annual inventory level of an item is*

$$\int_0^1 f(t)dt = \sum_{i=1}^{p} X_i^{in}\left(\frac{1+T_i^{in}}{2}\right) - \sum_{j=1}^{q} X_j^{out}\left(\frac{1-T_j^{out}}{2}\right) \quad (1)$$

This proposition means that the average annual inventory level of an item (i.e., the right-hand side of Equation (1)) can be calculated if the total quantities of all the inbound and outbound schedules and replenishment cycles of the schedules are known. The following results extend the above closed-form function on standing inventory levels to inventory levels where both the consumption and production rates are constant. To ensure that facilities are being used at a high utilization rate, the production rate is calculated by $\sum_{j=1}^{q} X_j^{\text{out}}$, meeting planned outputs exactly. Fig. 4 shows inventory records of an item at such a manufacturing member during the period. Fig. 4 (a) represents an inventory profile of an item to be processed (i.e., WIP). Fig. 4 (b) illustrates an inventory profile of a product.

Proposition 3. *If conditions (a) through (c) in Proposition 1, as well as the following condition (d), are met, then the inventory levels of a WIP and a product at a specific member are always non-negative.*

(a) The hypothetical annual average level of a produced item is given by $\omega^P/2$ where ω^P (unit/year) is the total quantity of outbound replenishments of the product the same as the production rate of the item, and the hypothetical annual average level of a consumed WIP is given by $\omega^{WIP}/2$ where ω^{WIP} (unit/year) is the total quantity of inbound replenishments of the WIP the same as the production rate of the WIP.

Proposition 4. *If conditions (a) through (d) are met, then the average annual inventory levels of a WIP and a product at a member are as follows.*

$$\int_0^1 f^{\text{WIP}}(t)dt = \sum_{i=1}^{p} X_i^{\text{in}}\left(\frac{1+T_i^{\text{in}}}{2}\right) - \frac{\omega^{\text{WIP}}}{2}, \quad (2)$$

$$\int_0^1 f^{\text{P}}(t)dt = \frac{\omega^{\text{P}}}{2} - \sum_{j=1}^{q} X_j^{\text{out}}\left(\frac{1-T_j^{\text{out}}}{2}\right) \quad (3)$$

The proofs of the above lemmas and propositions can be found in Appendix A.

3. Model formulation and solution approach

3.1. Model formulation

The main claim of this study is that a novel MILP formulation was developed by using the lemmas and propositions in Section 2. Although our PIC model is to simultaneously determine replenishment quantities, replenishment cycles, and production rates, the formulation developed for solving the PIC model has no nonlinear functions.

The objective of the PIC problem is to minimize the *TNC* under the stipulated assumption. The *TNC* consists of ordering costs, inventory holding costs, and operation costs. The detailed formulations for each cost term are presented as follows.

Ordering cost: Because the ordering cost accounts for both the administrative and transportation costs of an item, the cost per order depends on the product type and the distance between the two members. Hence, the total ordering cost is given in Equation (4).

$$ODC = \sum_{i\in G}\sum_{j\in G\setminus i}\sum_{k\in K}\sum_{m\in M}\frac{\delta_{ijk}}{\tau_m} z_{ijkm} \quad (4)$$

Inventory holding cost: The inventory holding cost accounts for a supplier's raw material holding cost; a manufacturer's WIP holding cost and product holding cost; and a buyer's inventory holding cost, as described in Equation (5). Based on Proposition 4, relevant functions of inventory holding costs are given by Equations (6) through (9). For a supplier, a total inventory holding cost of products is calculated by Equation (6). We note that the formulation is obtained by substituting $\sum_{j\in I}\sum_{m\in M} x_{ijkm}$ for ω^{WIP} in Equation (2). For a manufacturer, a total inventory holding cost of WIPs and products is obtained by Equations (7) and (8), respectively. Similarly, the formulations are obtained by substituting $\sum_{k'\in K}\sum_{l\in L} y_{ilkk'}$ and $\sum_{k'\in K}\sum_{l\in L} y_{ilk'k}$ for ω^{WIP} and ω^{P} in Equations (2) and (3), respectively. Equation (9) describes a total inventory holding cost of products for a buyer.

$$IHC = \sum_{i\in H} IHC_i^P + \sum_{i\in I}\left(IHC_i^{WIP} + IHC_i^P\right) + \sum_{i\in J} IHC_i^P \quad (5)$$

$$IHC_i^P = \sum_{k\in K}\varepsilon_{ik}\left[\sum_{j\in I}\sum_{m\in M} x_{ijkm}\left(\frac{\tau_m}{2}\right)\right], \forall i\in H \quad (6)$$

$$IHC_i^{WIP} = \sum_{k\in K}\varepsilon_{ik}\left[\sum_{\substack{j\in G\\ i\neq j}}\sum_{m\in M} x_{jikm}\left(\frac{1+\tau_m}{2}\right) - \sum_{k'\in K}\sum_{l\in L}\frac{y_{ilkk'}}{2}\right], \forall i\in I \quad (7)$$

$$IHC_i^P = \sum_{k\in K}\varepsilon_{ik}\left[\sum_{k'\in K}\sum_{l\in L}\frac{y_{ilk'k}}{2} - \sum_{\substack{j\in I\cup J\\ i\neq j}}\sum_{m\in M} x_{ijkm}\left(\frac{1-\tau_m}{2}\right)\right], \forall i\in I \quad (8)$$

$$IHC_i^P = \sum_{k\in K}\sum_{m\in M}\varepsilon_{ik}\left[\sum_{\substack{j\in I\cup J\\ i\neq j}} x_{jikm}\left(\frac{1+\tau_m}{2}\right) - \sum_{\substack{j\in I\cup J\\ i\neq j}} x_{ijkm}\left(\frac{1-\tau_m}{2}\right) - \frac{\rho_i}{2}\right], \forall i\in J \quad (9)$$

Operating cost: The process of converting a material into a product by using a specific facility incurs an operating cost. Thus, the total operation cost is given by Equation (10).

$$OPC = \sum_{i\in I}\sum_{l\in L}\sum_{k\in K}\sum_{k'\in K}\zeta_{lk'}\alpha_{lkk'} y_{ilkk'} \quad (10)$$

Finally, a novel MILP for solving the PIC model is developed as follows.

$$\text{Min.} TNC = ODC + IHC + OPC \quad (11)$$

$$\text{S.t.} \sum_{j\in I}\sum_{m\in M} x_{ijkm} \le \beta_{ik}^S \forall i\in H, \forall k\in K \quad (12)$$

$$\sum_{\substack{j\in G\\ i\neq j}}\sum_{m\in M} x_{jikm} = \sum_{k'\in K}\sum_{l\in L} y_{ilkk'} + \sum_{\substack{j\in I\cup J\\ i\neq j}}\sum_{m\in M} x_{ijkm}, \forall i\in I, \forall k\in K \quad (13)$$

$$\sum_{k'\in K}\sum_{l\in L}\alpha_{lk'k}\times y_{ilk'k} = \sum_{\substack{j\in I\cup J\\ i\neq j}}\sum_{m\in M} x_{ijkm} \forall i\in I, \forall k\in K \quad (14)$$

$$\sum_{\substack{j\in I\cup J\\ i\neq j}}\sum_{m\in M} x_{jikm} = \beta_{ik}^B + \sum_{\substack{j\in I\cup J\\ i\neq j}}\sum_{m\in M} x_{ijkm}, \forall i\in J, \forall k\in K \quad (15)$$

$$x_{ijkm} \le \Pi\times z_{ijkm}, \forall i\in G, \forall j\in G\backslash\{i\}, \forall k\in K, \forall m\in M \quad (16)$$

$$\sum_{m\in M} z_{ijkm} + \sum_{m\in M} z_{jikm} \leq 1 \forall i \in G, \forall j \in G\backslash\{i\}, \forall k \in K \tag{17}$$

$$y_{ilkk'} \leq \beta_{lkk'} \times \gamma_{il} \forall i \in I, \forall k \in K \tag{18}$$

$$x_{ijkm} \geq 0 \forall i \in G, \forall j \in G\backslash\{i\}, \forall k \in K, \forall m \in M \tag{19}$$

$$y_{ilkk'} \geq 0 \forall i \in G, \forall l \in L, \forall k, k' \in K \tag{20}$$

$$z_{ijkm} \in \{0,1\} \forall i \in G, \forall j \in G\backslash\{i\}, \forall k \in K, \forall m \in M \tag{21}$$

Objective function (11) is to minimize the *TNC* consisting of *ODC*, *IHC*, and *OPC*. Constraint (12) ensures that a supplier can provide a raw material within the related maximum supplement rate. Constraint (13) guarantees that the mass balance of an item, which can be replenished from a preceding member, can be processed to make a product, and can be shipped to a succeeding member. Constraint (14) figures the annual amount of products processed with the conversion rate of a material. The processed product is transported to succeeding members. Constraint (15) indicates that each buyer must receive as many products as have been ordered and delivered to other members. Constraint (16) represents the condition that replenishments of an item can be supplied only if a specific replenishment cycle has been determined. Constraint (17) stipulates that only one replenishment cycle, at most, can be determined for supplying replenishments of an item between two members. Also, Constraint (17) prevents infeasible cycles of an item. Constraint (18) figures the maximum production rate for a manufacturer who has a specific type of technology associated with a product. Constraints (19) through (21) define decision variables.

The proposed MILP formulation is an extended-form formulation because all discrete replenishment cycles are defined as columns of the formulation. In general, as the number of columns increases, the node tree that branch and bound algorithm searches for increases, resulting in significant computational complexity. To resolve this drawback of the formulation, in the next, we introduce a decomposition approach for a special case of the PIC problem.

3.2. Decomposition approach for a special case

In this section, a solution approach is developed for a special case of the PIC problem. The special case is as follows. Suppose all the supply chain members are going to receive or deliver a given amount of an item, i.e., there is no need to determine the appropriate flow of an item from a succeeding member to another following member. Then, annual shipments to be carried between members and the production rate of a specific item can be calculated by aggregating the relevant demand frontward. If all the annual shipments and the production rates are given, then we can derive the alternative formulation the same as the formulation in Equations (11)-(21) as follows:

$$\text{Min.} TNC = ODC + IHC + OPC$$

S.t. Constraints (16), (17), and (21)

$$\sum_{m\in M} x_{ijkm} = \overline{x}_{ijk} \forall i \in G, \forall j \in G\backslash\{i\}, \forall k \in K, \forall m \in M \tag{22}$$

$$y_{ilk'k} = \overline{y}_{ilk'k} \forall i \in G, \forall l \in L, \forall k, k' \in K \tag{23}$$

where $\overline{x}_{i'jk}$ is the given total amount of item k to be shipped from member i to member j and $\overline{y}_{ilkk'}$ is the given annual production rate of item k' made from item k at manufacturer i with facility type l.

The above alternative PIC problem can be decomposed to several subproblems that are to determine the replenishment cycle of a replenishment schedule for item k on an arc between members p and q. It is because that the replenishment cycle for an arc only affect the objective terms related to the ordering cost of all the replenishments, the inventory holding cost incurred by item k as a product from succeeding member p, and the inventory holding cost incurred by item k as a WIP from following member p. In other words, the subproblems are independent of each other. Since the replenishment cycle for the arc does not affect any objective terms associated with the other arc, the alternative PIC problem becomes a set of subproblems which of each optimizes the replenishment cycle for an arc. Subproblem SP^{H2I}_{pqk} is to determine the replenishment cycle for an acr associated with item k and a pair of suppler p and manufacturer q.

$$\begin{aligned}[SP^{H2I}_{pqk}(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})]\text{Min.}TNC^{H2I}_{pqk} = &\sum_{m\in M}\frac{\delta_{pqk}}{\tau_m} + \varepsilon_{pk}\left[\sum_{m\in M} x_{pqkm}\left(\frac{\tau_m}{2}\right)\right] \\ &+ \varepsilon_{qk}\left[\sum_{m\in M} x_{pqkm}\left(\frac{1+\tau_m}{2}\right) - \sum_{k'\in K}\sum_{l\in L}\frac{y_{qlkk'}}{2}\right] \\ &+ \sum_{l\in L}\sum_{k'\in K}\zeta_{lk'}\alpha_{lkk'}y_{qlkk'}\end{aligned} \tag{24}$$

S.t. Constraints (16)-(17) and (21)-(23)

Objective function (24) is to minimize the *TNC* associated with suppler p, manufacturer$q$, and item k. The first term represents the ordering cost incurred in the replenishments between supplier p and manufacturer$q$. The second term describes the inventory holding cost calculated for the supplier. The third term is the inventory holding cost calculated for the manufacturer. The last term represents the operating cost of the manufacturer.

Similarly, subproblems SP^{I2I}_{pqk}, and SP^{I2B}_{pqk} are for an arc associated a pair of a manufacturer and another manufacturer, and a pair of a manufacturer and a buyer, respectively.

$$\begin{aligned}[SP^{I2I}_{pqk}(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})]\text{Min.}TNC^{I2I}_{pqk} = &\sum_{m\in M}\frac{\delta_{pqk}}{\tau_m} + \varepsilon_{pk}\left[\sum_{k'\in K}\sum_{l\in L}\frac{y_{plk'k}}{2} - \sum_{m\in M} x_{pqkm}\left(\frac{1-\tau_m}{2}\right)\right] \\ &+ \varepsilon_{qk}\left[\sum_{m\in M} x_{pqkm}\left(\frac{1+\tau_m}{2}\right) - \sum_{k'\in K}\sum_{l\in L}\frac{y_{qlkk'}}{2}\right] \\ &+ \sum_{l\in L}\sum_{k'\in K}\zeta_{lk'}\alpha_{lkk'}y_{qlkk'}\end{aligned} \tag{25}$$

S.t. Constraints (16)-(17) and (21)-(23)

$$\begin{aligned}[SP^{I2J}_{pqk}(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})]\text{Min.}TNC^{I2B}_{pqk} = &\sum_{m\in M}\frac{\delta_{pqk}}{\tau_m} + \varepsilon_{pk}\left[\sum_{k'\in K}\sum_{l\in L}\frac{y_{plk'k}}{2} - \sum_{m\in M} x_{pqkm}\left(\frac{1-\tau_m}{2}\right)\right] \\ &+ \varepsilon_{qk}\left[x_{pqkm}\left(\frac{1+\tau_m}{2}\right) - \frac{\rho_q}{2}\right]\end{aligned} \tag{26}$$

S.t. Constraints (16)-(17) and (21)-(23)

Now we have decomposed the alternative formulation and got the subproblems, namely SP^{H2I}_{pqk}, SP^{I2I}_{pqk}, and SP^{I2J}_{pqk} which compose the special case of the PIC problem. However, each subproblem still is an extended-form formulation because it has to determine the replenishment cycle from an infinite discrete set. In each subproblem, if the integrality of the replenishment cycle is relaxed, the continuos-version formulations for the subproblems can be obtained as follows:

$$[SP^{H2I}_{\cdot pqk}(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})]\underset{\tau>0}{\text{Min.}}TNC^{H2I}_{\cdot pqk}(\tau) = \frac{\delta_{pqk}}{\tau} + \varepsilon_{pk}\frac{\overline{x}_{pqk}\tau}{2} + \varepsilon_{qk}\left[\overline{x}_{pqk}\left(\frac{1+\tau}{2}\right) - \sum_{k'\in K}\sum_{l\in L}\frac{y_{qlkk'}}{2}\right] \tag{27}$$

$$\begin{aligned}[SP^{I2I}_{-pqk}(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})]\underset{\tau>0}{\text{Min.}}TNC^{I2I}_{-pqk}(\tau) = &\frac{\delta_{pqk}}{\tau} + \varepsilon_{pk}\left[\sum_{k'\in K}\sum_{l\in L}\frac{y_{plk'k}}{2} - \overline{x}_{pqk}\left(\frac{1-\tau}{2}\right)\right] \\ + \varepsilon_{qk}\left[\overline{x}_{pqk}\left(\frac{1+\tau}{2}\right) - \sum_{k'\in K}\sum_{l\in L}\frac{y_{qlkk'}}{2}\right] &+ \sum_{l\in L}\sum_{k'\in K}\zeta_{lk'}\alpha_{lkk'}y_{qlkk'}\end{aligned} \tag{28}$$

$$\left[SP_{pqk}^{I2J}(\overline{x},\overline{y})\right]_{\tau>0}^{Min} \quad TNC_{pqk}^{I2B}(\tau)=\sum_{m\in M}\frac{\delta_{pqk}}{\tau}+\varepsilon_{pk}\left[\sum_{k'\in K}\sum_{l\in L}\frac{y_{plk'k}}{2}-\overline{x}_{pqk}\left(\frac{1-\tau}{2}\right)\right]+\varepsilon_{qk}\left[\overline{x}_{pqk}\left(\frac{1+\tau}{2}\right)-\frac{\rho_q}{2}\right] \tag{29}$$

where τ is the continuos replenishment cycle to be determined.

Since the subproblems are still independent, we can now easily search for the relaxed optimal replenishment cycle for each arc. In Formulation (27), the decision variable $\tau > 0$ is continuous and differentiated, and satisfies $\partial^2 TNC_{pqk}^{H2I}(\tau)/\partial\tau^2 = 2\delta_{pqk}/\tau^3 > 0$. The relaxed optimal replenishment cycle associated with suppler p, manufacturer$q$, and item k can be obtained by taking the derivative of $TNC_{pqk}^{H2I}(\tau)$ and setting it to 0 as follows.

$$\tau^* = \sqrt{2\sigma_{pqk}/(\varepsilon_{pk}+\varepsilon_{qk})\overline{x}_{pqk}} \tag{30}$$

Similarly, in the relaxed subproblems SP_{pqk}^{I2I} and SP_{pqk}^{I2J}, we can get the relaxed optimal solutions. Corresponding, $\tau^* = \sqrt{2\sigma_{pqk}/(\varepsilon_{pk}+\varepsilon_{qk})\overline{x}_{pqk}}$ is the relaxed optimal replenishment cycle associated with a specific item and two consecutive members. Since $TNC_{pqk}^{H2I}(\tau^*)$, $TNC_{pqk}^{I2I}(\tau^*)$, and $TNC_{pqk}^{I2J}(\tau^*)$ provide the lower bound of SP_{pqk}^{H2I}, SP_{pqk}^{I2I}, and SP_{pqk}^{I2J}, respectively, for each arc the feasible optimal replenishment cycle can be obtained by one of discrete replenishment cycles that calculates the objective value closest to the lower bound. Using the decomposition of the alternative problem for the special case where all the annual shipments between members and the production rate are predetermined, Algorithm 1 to provide the optimal replenishment cycles for the special case is developed as follows:

Algorithm 1

Input: total annual amount of items between members $\overline{x}$ and annual production rate of items $\overline{y}$

Output: the optimal replenishment cycles

```
For all p ∈ G, q ∈ G/{p}, k ∈ K,
  If x̄_pqk > 0,
    Initialize a relaxed subproblem associated with arc(p, q) and itemk
    Calculate the relaxed optimal replenishment cycleτ*
    If the relaxed objective value of ⌊τ*⌋ is less than the relaxed objective value of ⌈τ*⌉
      Let the optimal replenishment cycle be ⌊τ*⌋ for arc(p, q) and itemk
    Otherwise
      Let the optimal replenishment cycle be ⌈τ*⌉ for arc(p, q) and itemk
END
```

4. Computational experiments

This section investigates the solutions stipulated by the proposed PIC model. To attain the optimal solution of the model, we solved the developed MILP formulation using the commercial optimization solver, CPLEX 12.8.0 on 3.1 GHz CPU. The proposed algorithm for solving a special case, Algorithm 1 is also used if it is possible to calculate both the annual shipments and production rate. For comparison, we referred to the algebraic-approach-based solution algorithms, which were proposed to solve the existing PIC models introduced by Ben-Daya, As'Ad, and Seliaman (2013) and Zahran and Jaber (2017).

4.1. Three-level supply chain system with multiple buyers

In the existing literature on the PIC model, Ben-Daya, As'Ad, and Seliaman (2013) examined the PIC model that determines the common basic cycle of buyers and the integer multiples of the basic cycle for a supplier and a manufacturer. In this PIC model, the supplier and manufacturer follow the EPQ model; thus, their facilities operate with the given production rates. This implies that a setup cost relative to a production cycle exists. In the following, we compare the solutions of this PIC model with those of our model, taking into consideration production rates, as well as replenishment quantities and replenishment cycles.

We consider an example problem of a three-level supply chain including one supplier, one manufacturer, and seven buyers is addressed. The values of the parameters are described in Table 2. The data are the same as the data in the study by Ben-Daya, As'Ad, and Seliaman (2013). In this example, because they considered inventory of the supplier's WIP, we added a dummy supplier, S0, in our PIC model and classified S1 as a manufacturer. Therefore, the costs associated with S0 are ignored.

Since in the example problem, all the annual shipments between members and the production rate can be predetermined, the example problem corresponds to the special case that Algorithm 1 can be applied. Hence, we use Algorithm 1 to obtain the optimal solution based on the proposed PIC model.

The results of the optimal cost configurations obtained by Algorithm 1 are shown in Table 3. The results from Ben-Daya, As'Ad, and Seliaman (2013) also can be seen in this table. As a result of the optimal cost configurations, the average TNC, as determined by Algorithm 1, is 17.9 % more cost-effective than that obtained by the existing algorithm. Among the members, the supplier and manufacturer both have significantly lower total costs than they had in results obtained from the former algorithm. However, not every cost of every member of the supply chain was lowered. The proposed model incurs a setup cost for a facility during the planning period because that facility operated without any shutdown. Table 4 describes a set of decisions that are determined by each of the two models.

To provide more insights, we conducted further comparison experiments for different values of the ordering costs and the holding costs. Table 5 shows the results obtained by Algorithm 1 under varying ordering costs for the example problem described in Table 2. All the TNCs obtained by Algorithm 1 were lower than those obtained from the model proposed by Ben-Daya, As'Ad, and Seliaman (2013). A managerial insight is that the replenishment cycles of the buyers, as determined by Algorithm 1, did not fluctuate much on varying ordering costs. This is because the inventory holding cost per unit for the buyers is more than other costs. Table 6 shows the results obtained by Algorithm 1 under varying inventory holding costs for the example problem. In this experiment, too, all TNCs calculated by Algorithm 1 are less than those of the existing algorithm. An interesting result is that the optimal replenishment cycles of the buyers are reduced when the inventory holding cost of the manufacturer's finished product, K3, is as high as the cost charged to buyers. The main reason for this is that, despite an increase in the total ordering costs of buyers, the manufacturer's faster replenishment cycle lowered the total inventory holding cost of the finished product.

4.2. Three-level supply chain system with multiple suppliers and multiple buyers

This section looks at a three-level supply chain problem with multiple suppliers and multiple buyers, as designed by Zahran and Jaber (2017). They addressed some PIC models that determined the basic cycle of a manufacturer and the number of shipments for each supplier and buyer during that manufacturer's cycle. In their study, they primarily looked at an extended PIC model connected with a consignment inventory. The major focus of their paper, however, was not to study consignment inventory but to see if there was any improvement in production efficiency when the production rate was a decision variable. Hence, we referred to the PIC model of Zahran and Jaber (2017), which allowed for no consignment inventory, and we compared the solutions of this PIC model with those of our model.

An example problem of a three-level supply chain including two suppliers, one manufacturer, and three buyers in Zahran and Jaber (2017) is given in Table 7. In the problem, the final product, K3, is composed of raw materials, K1 and K2, and two units of each material are consumed in making the product.

We could not refer to the computational results in Zahran and Jaber

Table 2
Data for the example problem of a three-level supply chain with a single supplier, a single manufacturer, and seven buyers.

Level	1	2	3						
Member index	S1	M1	B1	B2	B3	B4	B5	B6	B7
Setup cost ($ / cycle)	800	200							
Type of work-in-process	K1	K2	–	–	–	–	–	–	–
Ordering cost ($ / order)	600	300							
Work-in-process holding cost ($)	0.08	0.8							
Type of finished product	K2	K3	–	–	–	–	–	–	–
Finished product holding cost ($)	0.8	2							
Annual production rate (M units/year)	399	140							
Type(s) of demand product	–	–	K3	K3	K3	K3	K3	K3	K3
Demand product holding cost ($)			5	5	5	5	5	5	5
Ordering cost ($ / order)			50	50	50	50	50	50	50
Annual demand rate (M units/year)			10	20	40	12	24	9	18

Table 3
Comparison of cost configurations between the two models for the example problem in Table 2.

Model	Algorithm 1					Ben-Daya, As'Ad, and Seliaman (2013)				
Cost ($)	Holding cost		Ordering cost	Setup cost	Total cost	Holding cost		Ordering cost	Setup cost	Total cost
	WIP	Product				WIP	Product			
S1	1,792.8	2,784.5	1,780.5	800.0	7,157.8	344.0	4,901.4	3,093.3	4,124.3	12,463.0
M1	2,769.3	3,600.4	5,763.2	200.0	12,333.0	3,267.7	7,787.9	1,546.6	1,031.1	13,633.3
B1	–	958.9	1,303.6	–	2,262.5	–	692.8	1,804.4	–	2,497.2
B2	–	1,369.9	1,825.0	–	3,194.9	–	1,385.5	1,804.4	–	3,189.9
B3	–	1,917.8	2,607.1	–	4,524.9	–	2,771.0	1,804.4	–	4,575.4
B4	–	1,068.5	1,403.8	–	2,472.3	–	831.3	1,804.4	–	2,635.7
B5	–	1,479.5	2,027.8	–	3,507.3	–	1,662.6	1,804.4	–	3,467.0
B6	–	924.7	1,216.7	–	2,141.4	–	623.5	1,804.4	–	2,427.9
B7	–	1,232.9	1,825.0	–	3,057.9	–	1,247.0	1,804.4	–	3,051.4
					40,651.8					47,940.7

Table 4
Comparison of operational decisions between the two models for the example problem in Table 2.

Model	Algorithm 1					Ben-Daya, As'Ad, and Seliaman (2013)					
Solution	Replenishment			Production		Replenishment				Production	
	Cycle[1]		Lot Size	Cycle	Lot Size	Cycle[1]		No. of WIP replenishments for a cycle	Lot Size	Cycle	Lot Size
	(year)	(days)	(unit/cycle)	(days)	(unit/cycle)	(year)	(days)		(unit/cycle)	(days)	(unit/cycle)
S1	0.33699	123	44,819.2	365	133,000	0.19397	70.8	1	25,798.0	70.8	25,798.0
M1	0.05206	19	6,923.3	365	133,000	0.19397	70.8	2	8,599.3	70.8	25,798.0
B1	0.03836	14	383.6	–	–	0.02771	10.1	–	277.1	–	–
B2	0.02740	10	548.0	–	–	0.02771	10.1	–	554.2	–	–
B3	0.01918	7	767.2	–	–	0.02771	10.1	–	1,108.4	–	–
B4	0.03562	13	427.4	–	–	0.02771	10.1	–	332.5	–	–
B5	0.02466	9	591.8	–	–	0.02771	10.1	–	665.0	–	–
B6	0.04110	15	369.9	–	–	0.02771	10.1	–	249.4	–	–
B7	0.02740	10	493.2	–	–	0.02771	10.1	–	498.8	–	–

[1] In this study, truncated replenishment cycles are considered as $T\subset\{x = n/365(\text{year})|n\in\mathbb{Z}\}$ while annual basis and untruncated cycles are assumed in the literature; Hence, the related values are transcribed in both daily and annual basis unit.

Table 5
Comparison of the two models under varying ordering costs for the example problem in Table 2.

Model		Algorithm 1										Ben-Daya, As'Ad, and Seliaman (2013)
Ordering cost ($)		Replenishment cycle (days)									TNC ($)	TNC ($)
$\sigma_{S1,K1}$	$\sigma_{M1,K2}$	S1	M1	B1	B2	B3	B4	B5	B6	B7		
100	50	50	8	14	10	7	13	9	15	10	30,840	41,307
200	100	71	11	14	10	7	13	9	15	10	33,358	43,003
300	200	86	16	14	10	7	13	9	15	10	36,528	45,186
1,000	500	157	25	14	10	7	13	9	14	10	43,979	51,762
2,000	800	224	32	14	10	7	13	9	15	10	49,759	58,099
3,000	1,000	274	35	14	10	7	13	9	15	10	53,411	62,750

Table 6
Comparison of the two models under varying inventory holding costs for the example problem in Table 2.

Model			Algorithm 1										Ben-Daya, As'Ad, and Seliaman (2013)
Holding cost ($)			Replenishment cycle (days)									TNC	TNC
$\delta_{S1,K1}$	$\delta_{S1,K2}$	$\delta_{M1,K2}$	S1	M1	B1	B2	B3	B4	B5	B6	B7	($)	($)
0.2	0.8	2.0	77	19	14	10	7	13	9	15	10	41,725	52,355
0.4	0.8	4.0	55	19	12	9	6	11	8	13	9	47,391	57,732
0.8	0.8	5.0	39	19	12	8	6	11	7	12	9	52,243	61,492
0.4	2.0	4.0	56	12	12	9	6	11	8	13	9	53,970	70,528
0.8	2.0	5.0	39	15	12	8	6	11	7	12	9	55,889	73,664

(2017) as target values for comparisons because their PIC model maximizes the total network profit. Therefore, we set all profit coefficient in their model to zero and solved the example problem using Zahran and Jaber's algebraic-approach-based solution algorithm. Since this example problem also belonging to a special case, Algorithm 1 was used to search for the optimal solution based on the proposed PIC model.

Table 8 illustrates the results of optimal cost configurations Algorithm 1 found for the problem. The results from Zahran and Jaber (2017) also can be seen in this table. The average TNC determined by Algorithm 1 is 17.6 % lower than that obtained from the existing model. Among the members, the total cost of all buyers is lower than the cost obtained from the former algorithm; however, the total costs of the suppliers and the manufacturer increase significantly.

To provide more insights, further comparison experiments were conducted for combinations of different ordering costs and inventory holding costs. Using the set of ordering costs and inventory holding costs in Table 7 as a baseline, we designed 400 combinations of example problems by multiplying each input for ordering costs and inventory holding costs. Each input is from the range $[0.1, 2.0]$, and the interval is set at 0.1. Using Algorithm 1 and the solution algorithm of Zahran and Jaber (2017), all TNCs were calculated. Table 9 compares the objectives obtained by the two methods for some combinations of the multipliers. This table shows that all the TNCs obtained by Algorithm 1 are less than those obtained by the algorithm. Moreover, the gap value between the two objectives for an instance became larger with the growth of multipliers. This can be seen in Fig. 5. This implies that the solutions stipulated by the proposed PIC model outperform those stipulated by the existing PIC model in the comparison experiments.

Table 7
Data for the example problem of a three-level supply chain with two suppliers, a single manufacturer and three buyers.

Level	1		2		3		
Member index	S1	S2	M1		B1	B2	B3
Type(s) of WIP	-	-	K1	K2			
Ordering cost ($ / order)			15.5	20.5			
Work-in-process holding cost ($)			0.14	0.26			
Required numbers of WIPs to make the final product			2	2			
Type of finished product	K1	K2	K3		–	–	–
Finished product holding cost ($)	0.14	0.26	0.59				
Annual production rate (M units/year)	750	700	600				
Type(s) of demand product	–	–	–		K3	K3	K3
Demand product holding cost ($)					0.80	0.77	0.83
Ordering cost ($ / order)					30.5	51	70.7
Annual demand rate (M units/year)					100	75	50

5. Case study

Here we introduce a further extension of the proposed PIC model. In the previous section, the PIC model, which determines both replenishment quantities and cycles without any nonlinear terms simultaneously, provides a more effective solution than the existing model (or algorithm). The PIC model can be integrated with not only operational decisions but also strategic decisions such as the configuration of a SCN in the MILP formulation in Equations. (11)-(22). To demonstrate the capability of the proposed PIC model in practice, a hypothetical case study is considered. The case study concerns a SCN that supplies the final product to the market. Referring to a set of reasonable scales of parameters in the literature (Ahmadi Javid and Azad, 2010; Sadjady and Davoudpour, 2012), such an application example is designed: there are five given suppliers, three potential locations for manufacturing facilities, five potential locations for distribution centers, and thirty given buyers. Fig. 6 describes the locations of the entire members of the application example, in which a diamond symbol represents a member. Red symbols are existing suppliers "S1"-"S5", blue symbols are potential locations of manufacturers "M1"-"M3", gray symbols are potential locations of distribution centres "D1"-"D5", and then yellow symbols are buyers "B1"-"B30". In the example, the buyers demand the final product, "K1", which is made of two types of raw material, "K2" and "K3" at a manufacturing facility. For making one unit of "K1", three "K2" and one "K3" are required. There are two alternative facility types, "P1" and

Table 8
Comparison of the two models for the example problem in Table 7.

Model	Algorithm 1						Zahran and Jaber (2017)					
Solution	Replenishment cycle		Cost ($)				Cycle		Cost ($)			
			Holding cost ($)		Ordering cost	Total cost			Holding cost ($)		Ordering cost	Total cost
	(year)	(day)	WIP	Product			(year)	(day)	WIP	Product		
S1	0.00274	1		345.21		345.21	0.02826	10.3		8.54		8.54
S2	0.02192	8		560.96		560.96	0.02826	10.3		22.19		22.19
M1	0.01918	7	906.16	1,214.84	1,776.12	3,897.12	0.02826	10.3	170.71	5,503.96	1,274.08	6,948.75
B1	0.01370	5		1,917.81	2,226.50	4,144.31	0.02826	10.3		3,955.79	1,079.43	5,035.22
B2	0.01918	7		1,992.12	2,659.29	4,651.41	0.02826	10.3		2,935.05	1,804.95	4,740.00
B3	0.03014	11		2,132.19	2,345.95	4,478.15	0.02826	10.3		1,999.09	2,502.16	4,501.24
						18,077.15	0.02826	10.3				21,255.95

Table 9
Comparison of the two models under varying inventory holding costs and ordering cost for the example problem in Table 7.

Multipliers		TNC ($)		Multipliers		TNC ($)	
Holding cost	Ordering cost	Algorithm 1	Zahran and Jaber (2017)	Holding cost	Ordering cost	Algorithm 1	Zahran and Jaber (2017)
1.0	0.2	8,105	9,506	0.2	1.0	8,080	9,506
1.0	0.4	11,443	13,443	0.4	1.0	11,428	13,443
1.0	0.6	14,006	16,465	0.6	1.0	14,006	16,465
1.0	0.8	16,189	19,012	0.8	1.0	16,178	19,012
1.0	1.0	18,077	21,256	1.0	1.0	18,077	21,256
1.0	1.2	19,812	23,285	1.2	1.0	19,830	23,285
1.0	1.4	21,386	25,150	1.4	1.0	21,385	25,150
1.0	1.6	22,862	26,887	1.6	1.0	22,870	26,887
1.0	1.8	24,248	28,518	1.8	1.0	24,287	28,518
1.0	2.0	25,557	30,060	2.0	1.0	25,634	30,060

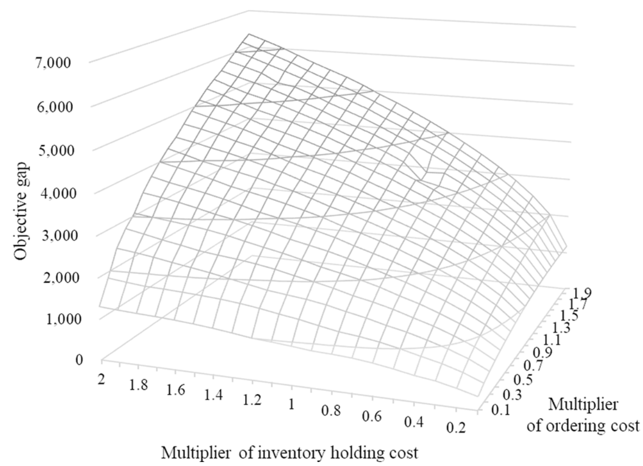


Fig. 5. Objective gap between Algorithm 1 and the model referring to Zahran and Jaber (2017).

"P2", that have different production capacities and investment costs. Distribution center can be established as a hub for conveying the product to buyers. However, it is not mandatory to build a distribution center in SCN. There are three types of distribution center, "W1", "W2", and "W3", that have different capacities and investment costs. In more detail, the parameters of the application problem are described in Appendix B. Currently, manufacturing facility and distribution center have not yet been established in SCN. Replenishment schedule, which consists of replenishment quantity and cycle (see Section 2), has not yet been provided either. In this case study, we will deal with the design of network configuration and determination of a replenishment schedule simultaneously using an extended MILP formulation.

The current MILP formulation cannot provide a solution for the application problem because it does not include information on the quantity of a material required to make a product—the bill of materials. It is common in the supply chain that a product, such as "K1" in the example, is made up of more than two types of materials. Moreover, for implementing the proposed PIC model with determining the configuration of an SCN, some decision variables and equations on investments of facilities are required on the MILP formulation in Equations (11)-(21). In regards, some notations on subsets, parameter, and decision variable for additional constraints are presented as the following:

6. Subsets

I^P	set of potential locations for manufacture, $I^P \subset I$
I^D	set of potential locations for distribution center, $I^D \subset I$
L^P	set of types of manufacturing facility, $L^D \subset L$
L^D	set of types of distribution center, $L^D \subset L$

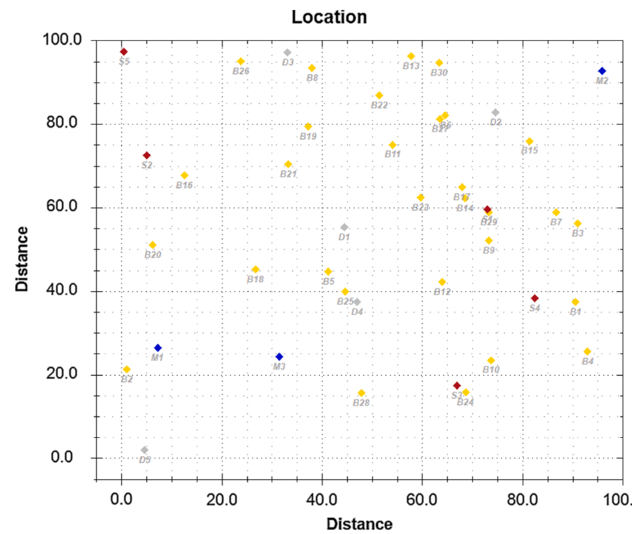


Fig. 6. Locations of members for the application problem.

6.1. Parameter

$\eta_{lkk'}$	required quantity of material k to make one unit of material k' at facility $l \in L^P$
$\theta_{lkk'}$	proportion of material k among all materials in making one unit of material k' at facility $l \in L^P$.
ϱ_l	annualized investment cost of manufacturing facility $l \in L^P$
σ_l	annualized investment cost of distribution center $l \in L^D$
ς_l	annual capacity of types of distribution center $l \in L^D$
∂	transportation cost per distance
Λ_{ij}	distance between node i and node j

6.1.1. Decision variable

w_{il}	1 if manufacturing facility $l \in L^P$ (or distribution center $l \in L^D$) is established in location $i \in I^P$ (or $\in I^D$), otherwise 0.

The application problem concerns strategic decisions along with operational decisions. When designing the network configuration, it is important to decide where to place an appropriate facility. Both the investment cost and the transportation cost incurred in the distance to other related members have to be considered. Hence, the facility investment cost term and the transportation cost term, which would be added to *TNC*, are presented as follows.

Facility investment cost: since the other cost terms are defined with annual cost, the investment cost of a facility should be annualized as well. Assuming that the interest rate is 0.04 and the lifetime of a facility is 20 years, we consider annualized facility investment cost. The total annualized investment cost is calculated with Equation (31).

$$FIC = \sum_{\forall i \in I^P} \sum_{l \in L^P} \varrho_l w_{il} + \sum_{\forall i \in I^D} \sum_{l \in L^D} \sigma_l w_{il} \tag{31}$$

Transportation cost: there is a transportation cost including fuel cost and maintenance cost during the planning periods. Equation (32) calculates the total transportation cost.

$$TPC = \sum_{i \in G} \sum_{j \in G \setminus i} \sum_{k \in K} \sum_{m \in M} \partial \left(\frac{\Lambda_{ij}}{\tau_m} z_{ijkm} \right) \tag{32}$$

The followings introduce some constraints to handle the bill of materials and the strategic decisions.

$$y_{ilk'k} \geq \eta_{lk'k} \sum_{\substack{j \in I \cup J \\ i \neq j}} \sum_{m \in M} x_{ijkm} \forall i \in I, \forall k, k' \in K, \forall l \in L \tag{33}$$

$$\sum_{k' \in K} \sum_{l \in L} \theta_{lk'k} \times y_{ilk'k} = \sum_{\substack{j \in I \cup J \\ i \neq j}} \sum_{m \in M} x_{ijkm} \forall i \in I, \forall k \in K \tag{34}$$

$$\sum_{l \in L^P} w_{il} \leq 1, \forall i \in I^P \tag{35}$$

$$\sum_{l \in L^D} w_{il} \leq 0, \forall i \in I^P \tag{36}$$

$$\sum_{l \in L^D} w_{il} \leq 1, \forall i \in I^D \tag{37}$$

$$\sum_{l \in L^P} w_{il} \leq 0, \forall i \in I^D \tag{38}$$

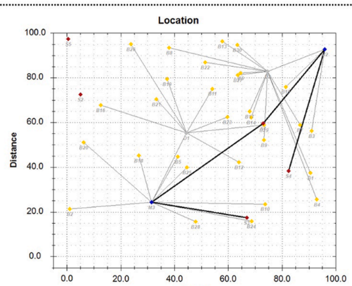
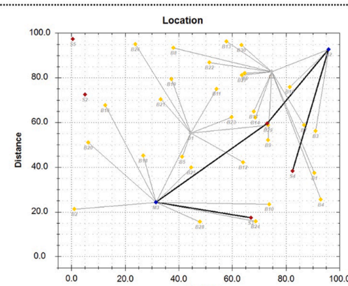
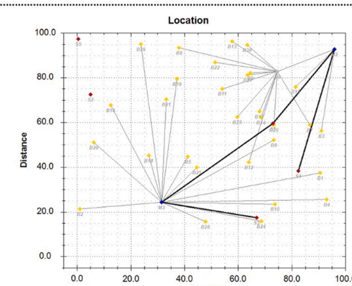
$$\sum_{j \in G} \sum_{k \in K} \sum_{m \in M} x_{jikm} \leq \sum_{l \in L^D} \varsigma_l \times w_{il}, \forall i \in I^D \tag{39}$$

Constraints (33) and (34) are for replacing Constraint (13) to deal with the bill of materials in the MILP formulation. Constraint (33) forces a specific manufacturer to consume the entire quantity of required components (i.e., WIPs) to supply a quantity of the corresponding product. Constraint (34) shows that the quantity of the products must be equal to the sum of the products of the consumed quantity of a WIP and a corresponding proportion of the WIP. Constraints (35)-(39) are for handling strategic decisions in the formulation. Constraints (35) and (36) show that a manufacturing facility can be established at a location for manufacturer, not for distribution center. Similarly, Constraints (37) and (38) represent that a distribution center can be assigned to a location for distribution center. Constraint (39) stipulates that a distribution center should handle the inventory of the product within its annual capacity. Finally, an extended MILP formulation for solving the application problem is arranged as follows.

$$\text{Min.} TNC = ODC + IHC + FIC + TPC + OPC$$

S.t. Constraints (2)-(13), (15)-(21), and (31)-(39)

To investigate how changes of parameters affect on the configuration of SCN, Sensitivity analysis was also carried out. Setting the application example as a base scenario, we generated four scenarios by changing a set of parameters on inventory handling cost, ordering cost, transportation cost, and facility investment cost, respectively. For each set of parameters, a multiplier is multiplied. Each multiplier is one of the range

Multiplier for *IHC*						0.2, 0.6					1.0, 1.4, 1.8				
Multiplier for *ODC*	0.2					0.6					1.0, 1.4, 1.8				
Network Configuration															
Number of Facilities	P1	P2	W1	W2	W3	P1	P2	W1	W2	W3	P1	P2	W1	W2	W3
	2	-	1	1	-	2	-	1	1	-	2	-	-	1	-

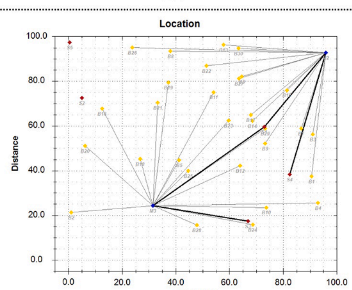
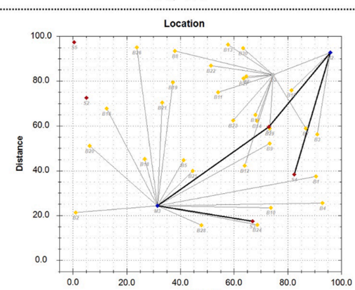
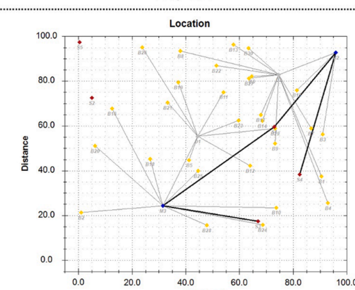
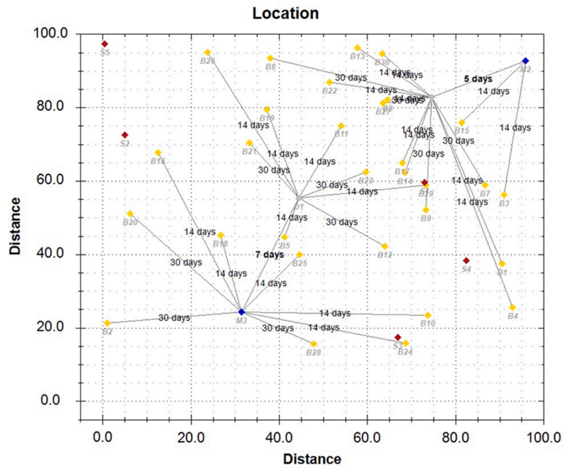
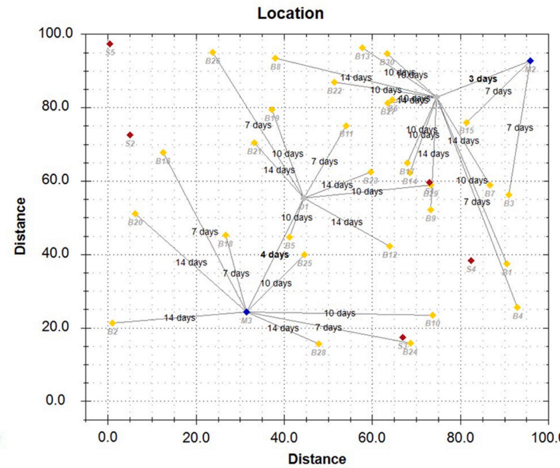
Multiplier for *TPC*	0.2, 0.6					1.0					1.4, 1.8				
Multiplier for *FIC*	1.8					1.4, 1.0					0.6, 0.2				
Network Configuration															
Number of Facilities	P1	P2	W1	W2	W3	P1	P2	W1	W2	W3	P1	P2	W1	W2	W3
	2	-	-	-	-	2	-	-	1	-	2	-	1	1	-

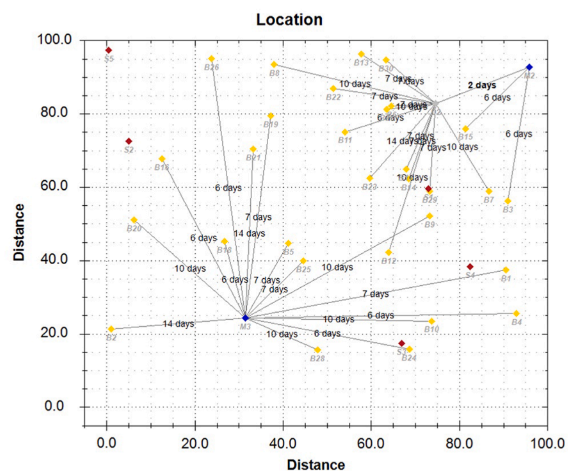
Fig. 7. Network configurations corresponding to the 17 scenarios of application problem.



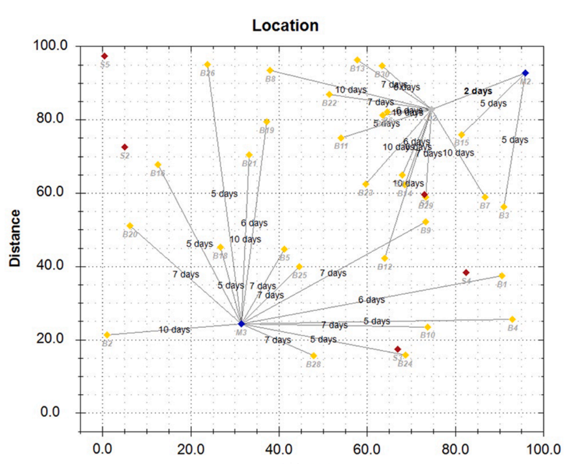
(a) Replenishment cycles with $0.2 \times IHC$



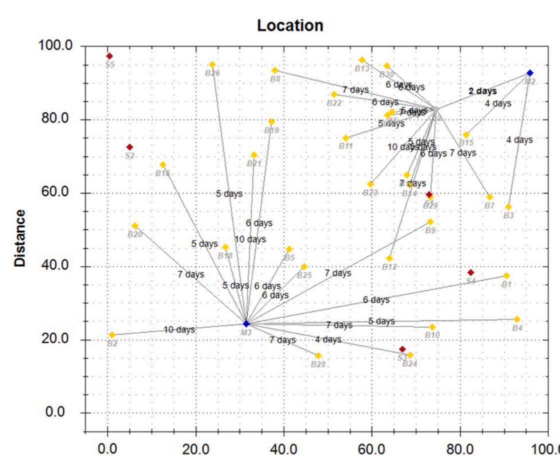
(b) Replenishment cycles with $0.6 \times IHC$



(c) Replenishment cycles with $1.0 \times IHC$

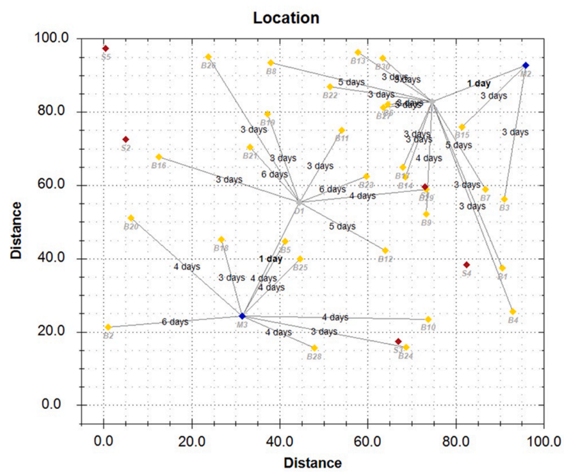


(c) Replenishment cycles with $1.4 \times IHC$

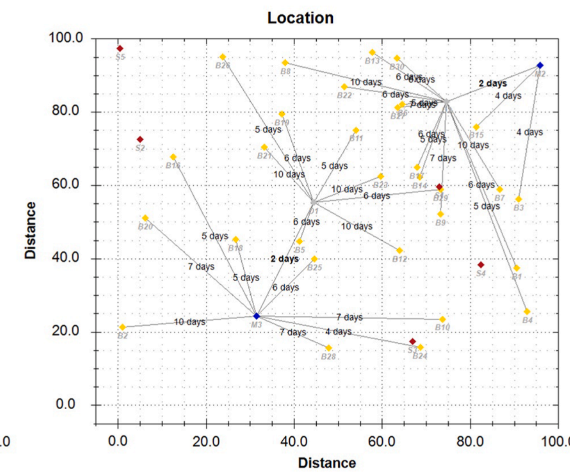


(d) Replenishment cycles with $1.8 \times IHC$

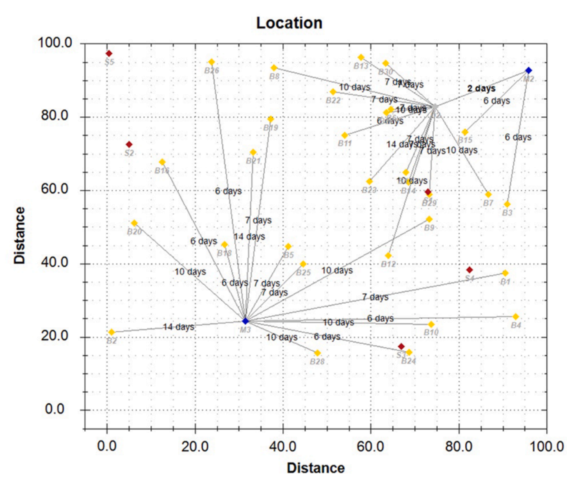
Fig. 8. Replenishment cycles under varying multipliers related to.*IHC*



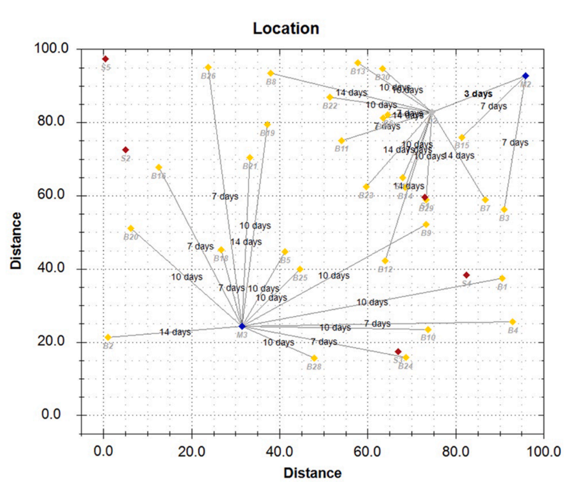
(a) Replenishment cycles with 0.2 × ODC



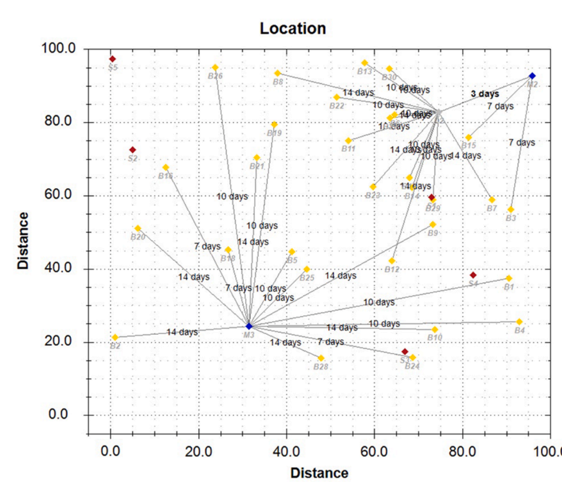
(b) Replenishment cycles with 0.6 × ODC



(c) Replenishment cycles with 1.0 × ODC

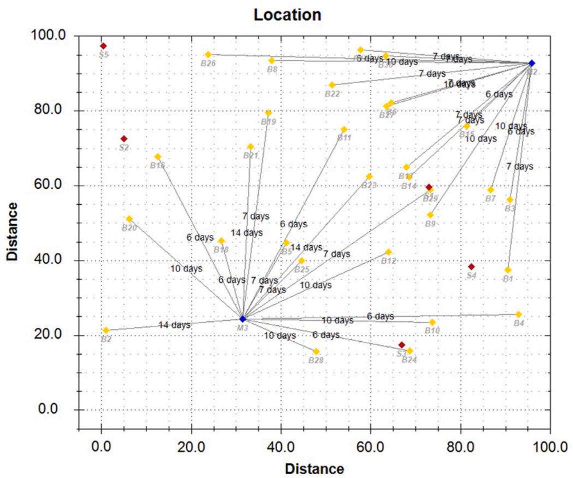


(c) Replenishment cycles with 1.4 × ODC

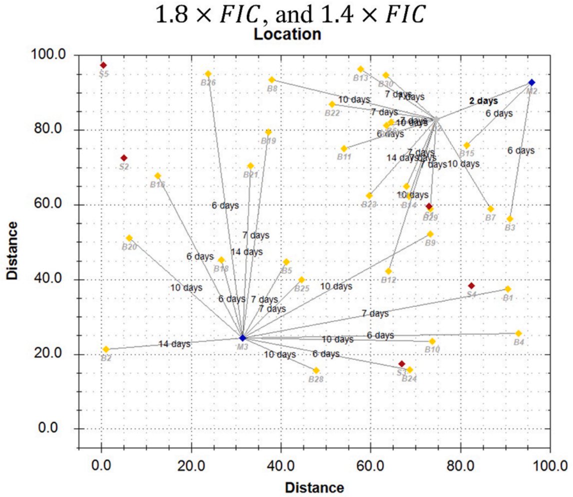


(d) Replenishment cycles with 1.8 × ODC

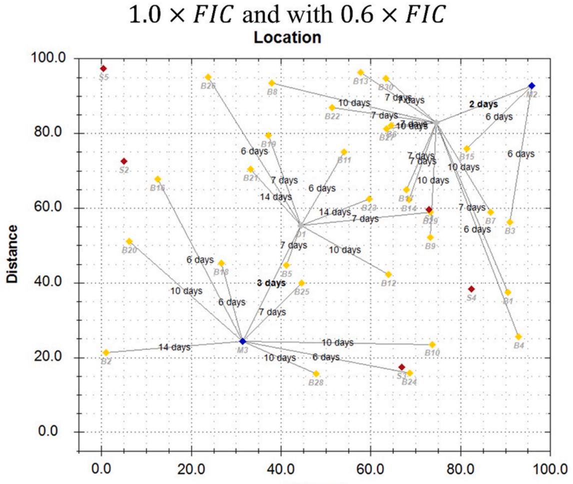
Fig. 9. Replenishment cycles under varying multipliers related to.ODC



(a) Replenishment cycles with $0.2 \times TPC$, $0.6 \times TPC$, $1.8 \times FIC$, and $1.4 \times FIC$



(b) Replenishment cycles with $1.0 \times TPC$, $1.0 \times FIC$ and with $0.6 \times FIC$



(c) Replenishment cycles with $1.4 \times TPC$, $1.8 \times TPC$, and with $0.2 \times FIC$

Fig. 10. Replenishment cycles under varying multipliers related to *TPC* and.*FIC*

$[0.2, 1.8]$, and its interval is set by 0.4. Subsequently, a total of 17 scenarios were arranged including the base scenario.

All scenarios were solved by ILOG CPLEX 12.8 solver engine within an hour. Fig. 7 shows that the solutions to the strategic decisions for all the scenarios. A multiplier value is described in the first and second rows. Each series of multipliers were laid out in a row. A graphic network configuration is described in the third row. In each graphic configuration shows the operating suppliers, the locations of the established facilities, and the supply chains. Bold solid lines represent supplies between the supplier and manufacturers. Thin solid lines represent supplies between the manufacturers, distribution centers, and buyers. The number of facilities is represented in the fourth row. The results show that the configuration becomes centralized as multipliers related to *IHC*, *ODC*, and *FIC* increase. For *IHC* and *ODC*, this is because when their multipliers are low, establishing a distribution center helps to minimize the transportation cost, but when the multipliers are high, the inventory holding cost and ordering cost became more expensive than the transportation cost saved by locating the distribution center. For *FIC*, it is uneconomical to establish a distribution center if the multiplier increases. While the configuration becomes decentralized as multipliers related to *TPC* increase. Although the production capacity of the large-sized manufacturing facility “P2” is large enough to handle all demand of the final product “K3”, however, “P2” is not suitable to establish it in this sensitivity analysis.

Figs. 8 through 10 show that the solutions of the operational decisions for all the scenarios. The replenishment cycles are described in the graphic configuration. Fig. 8 lays out the change of replenishment cycles under varying the multiplier for *IHC*. As the multiplier increases, the replenishment cycles become shorter. This is trivial because relatively low inventory levels of a member are demanded when the inventory holding costs are expensive. In Fig. 8 (c), the network configuration becomes centralized by disregarding the establishment of distribution center “D1”. However, distribution center “D2” remains even if the multiplier increases to 1.6.

Fig. 9 shows the change of replenishment cycles under varying the multiplier for *ODC*. While the above replenishment cycles become shorter as the multiplier related to *IHC* increases, the replenishment cycles in this figure become longer as the multiplier related to *ODC* increases.

Fig. 10 illustrates replenishment cycles the change of replenishment cycles under varying the multiplier for *TPC* and *FIC*. Whereas in Figs. 8 and 9, the replenishment cycles vary according to the multipliers, the replenishment cycles are not sensitive to the multipliers in Fig. 10. The only change of configuration is observed in the figure. This indicates that varying multipliers for *TPC* and *FIC* do not affect *IHC* and *ODC*.

7. Conclusions

This study presented a novel coordination policy for controlling a production-inventory system of a general SCN where multiple suppliers, manufacturers, and buyers are vertically integrated to provide multiple items to the market. Unlike the IRC policy applied to most PIC models in existing literature, the proposed PIC model relaxed the assumption that a production rate is fixed with the maximum value. The objective of the PIC model is to simultaneously determine optimal order sizes, replenishment cycles, and production rates while allowing no shortages of inventory. We proposed two closed-form functions of the average annual inventory of WIPs and products. As one of our contributions, we developed a novel MILP formulation using the closed-form functions in order to find an optimal solution to the PIC model. In addition, an algorithm based on a decomposition approach was developed to solve a special case of the PIC problem with less computational burden. We showed that the solutions for the new coordination policy, i.e., the PIC model, dominate the solutions provided by Ben-Daya, As’Ad, and Seliaman (2013) and Zahran and Jaber (2017). A case study on optimal design and operations of an integrated SCN was presented to demonstrate the application. These results have significant managerial implications. First, the existing coordination policies that assume constant production rates of manufacturers are not able to provide cost-effective solutions. Thus, managers who make decisions according to the constant production rate model limit their facility's manufacturing efficiency and cost-savings. This is because the practice of allowing for a constant production rate with a maximum speed gives rise to undesirable inventory and makes manufacturers activate facilities unnecessarily. This, in turn, results in manufacturers incurring large setup and holding costs. Second, the new coordination policy also addresses the limitations imposed by the IRC policy in determining dependent replenishment cycles between supply chain members. Hence, for coordination, our model shows that it is no longer necessary to make replenishment cycles dependent on other supply chain members. This guarantees that manufacturers only need to consider replenishing raw materials of WIPs for their production operations, independent of other supply chain members.

This work might be extended in several interesting directions. As shown in other studies, retailers' orders do not correspond to the best economic quantity—that is, the individual optimizer of the cost structure. This means that there is no incentive to coordinate with other members except in vertically integrated situations. To deal with these coordination issues, further research might look at possibilities of revenue-sharing among supply chain members. Another extension is considering location-allocations of manufacturing facilities and distribution centers into the problem. In practical situations, it is important how to configure an SCN matching steady demands such as energy with low costs. Therefore, an extended PIC model, which may consider various facilities types occurring different costs or producing demanded items with different yields, is expected to provide more cost-effective decisions.

CRediT authorship contribution statement

Young-Bin Woo: Conceptualization, Methodology, Software, Formal analysis, Visualization. **Ilkyeong Moon:** Writing - review & editing, Validation, Supervision. **Byung Soo Kim:** Conceptualization, Writing - original draft, Writing - review & editing, Validation, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

***Proof* of Lemma 1**. *In general, inventory level at time* $t \in [0, 1]$ *is*

$$I_t = \begin{cases} O_0^{\text{in}} - O_0^{\text{out}}, t = 0 \\ \lim\limits_{x \to t-} I_x + O_t^{\text{in}} - O_t^{\text{out}}, otherwise \end{cases}$$

Let $S = \{x = nT_i^{\text{in}} | n = 1, 2, \cdots, N_i^{\text{in}}, \forall i\} \cup \{x = nT_j^{\text{out}} | n = 1, 2, \cdots, N_j^{\text{out}}, \forall j\}$. Since for $t \in S^c$ O_t^{in} (or O_t^{out}) must be zero, it is definite that both non-negative O_t^{in} and O_t^{out} are calculated with $\lim\limits_{x \to t-} I_x$ only for $t \in S$.

Then, if $t_1, t_2 \in S$ where $t_1 < t_2$ and $\{x | t_1 < x < t_2, x \in S\} = \varnothing$, inventory level at the following time t_2 is

$$I_{t_2} = I_{t_1} + O_{t_2}^{\text{in}} - O_{t_2}^{\text{out}}.$$

Because $I_0 = O_0^{\text{in}} - O_0^{\text{out}}$ and both O_t^{in} and O_t^{out} are density function for time t where $O_t^{\text{in}}, O_t^{\text{out}} = 0, \forall t \in S^c$, an inventory level at time t is

$$I_t = \sum_{\substack{x \in S \\ x \le t}} O_x^{\text{in}} - \sum_{\substack{x \in S \\ x \le t}} O_x^{\text{out}}. = \int_0^t O_x^{\text{in}} dx - \int_0^t O_x^{\text{out}} dx.$$

Since both cumulated quantities quantity of all inbound replenishments and all outbound replenishments at time t are $C_t^{\text{in}} = \int_0^t O_x^{\text{in}} dx$ and $C_t^{\text{out}} = \int_0^t O_x^{\text{out}} dx$, respectively,

we have

$$I_t = C_t^{\text{in}} - C_t^{\text{out}}.$$

Hence, this lemma is established. □

***Proof* of Proposition 1**. *Let functions, f, g, and h be an inventory level for time t and two cumulated quantities for inbound schedule i (or outbound schedule j) and time t as*

$$f(t) = I_t,$$

$$g(i, t) = \frac{X_i^{\text{in}}}{N_i^{\text{in}}} \left\lfloor 1 + \frac{t}{T_i^{\text{in}}} \right\rfloor,$$

and

$$h(j,t) = \frac{X_j^{\text{out}}}{N_j^{\text{out}}}\left\lfloor \frac{t}{T_j^{\text{out}}} \right\rfloor.$$

We can obtain the following equation from Lemma 1 as

$$f(t) = \sum_{i=1}^{p} \frac{X_i^{\text{in}}}{N_i^{\text{in}}}\left\lfloor 1 + \frac{t}{T_i^{\text{in}}} \right\rfloor - \sum_{j=1}^{q} \frac{X_j^{\text{out}}}{N_j^{\text{out}}}\left\lfloor \frac{t}{T_j^{\text{out}}} \right\rfloor.$$

Then, by refining the right side of above equation we have

$$f(t) = \sum_{i=1}^{p}\left[\frac{X_i^{\text{in}}}{N_i^{\text{in}}} - X_i^{\text{in}}\left\{t \bmod T_i^{\text{in}}\right\} + X_i^{\text{in}} t\right] - \sum_{j=1}^{q}\left[-X_j^{\text{out}}\left\{t \bmod T_j^{\text{out}}\right\} + X_j^{\text{out}} t\right]$$

$$= \sum_{i=1}^{p} \frac{X_i^{\text{in}}}{N_i^{\text{in}}} - \sum_{i=1}^{p} X_i^{\text{in}}\left\{t \bmod T_i^{\text{in}}\right\} + \sum_{j=1}^{q} X_j^{\text{out}}\left\{t \bmod T_j^{\text{out}}\right\} + \left(\sum_{i=1}^{p} X_i^{\text{in}} - \sum_{j=1}^{q} X_j^{\text{out}}\right)t.$$

Since

$$0 \le \sum_{i=1}^{p} X_i^{\text{in}}\left\{t \bmod T_i^{\text{in}}\right\} < \sum_{i=1}^{p} X_i^{\text{in}} T_i^{\text{in}}$$

and

$$0 \le \sum_{j=1}^{q} X_j^{\text{out}}\left\{t \bmod T_j^{\text{out}}\right\} \le \sum_{j=1}^{q} X_j^{\text{out}} T_j^{\text{out}},$$

we have

$$f(t) \ge \sum_{i=1}^{p} \frac{X_i^{\text{in}}}{N_i^{\text{in}}} - \sum_{i=1}^{p} X_i^{\text{in}} T_i^{\text{in}} + \left(\sum_{i=1}^{p} X_i^{\text{in}} - \sum_{j=1}^{q} X_j^{\text{out}}\right)t.$$

Then, by refining the right side of above equation we have

$$f(t) \ge \left(\sum_{i=1}^{p} X_i^{\text{in}} - \sum_{j=1}^{q} X_j^{\text{out}}\right)t.$$

As condition (b) stipulates that $\sum_{i=1}^{p} X_i^{\text{in}} \ge \sum_{j=1}^{q} X_j^{\text{out}}$, inventory levels of a material be always non-negative for all time t. □

***Proof* of Lemma 2.** *Since functions g and h are step functions, we have*

$$\int_0^1 g(i,t)dt = \int_0^{T_i^{\text{in}}} \frac{X_i^{\text{in}}}{N_i^{\text{in}}}dt + \int_{T_i^{\text{in}}}^{2T_i^{\text{in}}} \frac{2X_i^{\text{in}}}{N_i^{\text{in}}}dt + \cdots + \int_{\left(N_i^{\text{in}}-1\right)T_i^{\text{in}}}^{1} X_i^{\text{in}} dt$$

and

$$\int_0^1 h(i,t)dt = \int_0^{T_j^{\text{out}}} \frac{X_j^{\text{out}}}{N_j^{\text{out}}}dt + \int_{T_j^{\text{out}}}^{2T_j^{\text{out}}} \frac{2X_j^{\text{out}}}{N_j^{\text{out}}}dt + \int_{2T_j^{\text{out}}}^{3T_j^{\text{out}}} \frac{3X_j^{\text{out}}}{N_j^{\text{out}}}dt + \cdots + \int_{\left(N_j^{\text{out}}-1\right)T_i^{\text{in}}}^{1} \frac{\left(N_j^{\text{out}}-1\right)X_j^{\text{out}}}{N_j^{\text{out}}}dt.$$

Now by using arithmetic sequence formula, we have

$$\int_0^1 g(i,t)dt = \frac{X_i^{\text{in}}}{N_i^{\text{in}}}T_i^{\text{in}} + \frac{2X_i^{\text{in}}}{N_i^{\text{in}}}T_i^{\text{in}} + \frac{3X_i^{\text{in}}}{N_i^{\text{in}}}T_i^{\text{in}} + \cdots + X_i^{\text{in}}T_i^{\text{in}} = \frac{X_i^{\text{in}}}{N_i^{\text{in}}}T_i^{\text{in}} \times \frac{N_i^{\text{in}}\left(N_i^{\text{in}}+1\right)}{2} = X_i^{\text{in}}\left(\frac{1+T_i^{\text{in}}}{2}\right)$$

and

$$\int_0^1 h(i,t)dt = \frac{X_j^{\text{out}}}{N_j^{\text{out}}}T_j^{\text{out}} + \frac{2X_j^{\text{out}}}{N_j^{\text{out}}}T_j^{\text{out}} + \frac{3X_j^{\text{out}}}{N_j^{\text{out}}}T_j^{\text{out}} + \cdots + \left(N_j^{\text{out}}-1\right)\frac{X_j^{\text{out}}}{N_j^{\text{out}}}T_j^{\text{out}} = X_j^{\text{out}}\left(\frac{1-T_j^{\text{out}}}{2}\right),$$

respectively. □

***Proof* of Proposition 2.** *By using Lemma* 1, *total inventory level is given by*

$$\int_0^1 f(t)dt = \sum_{i=1}^{p}\int_0^1 g(i,t)dt - \sum_{j=1}^{q}\int_0^1 h(j,t)dt$$

Subsequently by using Lemma 2, we easily obtain the total inventory levels such as

$$\sum_{i=1}^{p} X_i^{\text{in}}\left(\frac{1+T_i^{\text{in}}}{2}\right) - \sum_{j=1}^{q} X_j^{\text{out}}\left(\frac{1-T_j^{\text{out}}}{2}\right)$$

Hence, this proposition is established. □

***Proof* of Proposition 3.** *For a type of WIPs, let functions, f^{WIP}, g^{WIP}, and h^{WIP} for continuous time $t \in [0,1]$ and inbound schedule i be an inventory level of* *the WIP, total cumulated quantity of inbound replenishments associated with schedule i, and cumulated consumption quantity for time t as*

$$f^{\mathrm{WIP}}(t)=\sum_{i=1}^{p}g^{\mathrm{WIP}}(i,t)-h^{\mathrm{WIP}}(t),$$

$$g^{\mathrm{WIP}}(i,t)=\frac{X_i^{\mathrm{in}}}{N_i^{\mathrm{in}}}\left\lfloor 1+\frac{t}{T_i^{\mathrm{in}}}\right\rfloor,$$

and

$$h^{\mathrm{WIP}}(t)=\omega^{WIP}t=\sum_{i=1}^{p}X_i^{\mathrm{in}}t.$$

Furthermore, we have

$$f^{\mathrm{WIP}}(t)=\sum_{i=1}^{p}\frac{X_i^{\mathrm{in}}}{N_i^{\mathrm{in}}}\left\lfloor 1+\frac{t}{T_i^{\mathrm{in}}}\right\rfloor-\sum_{i=1}^{p}X_i^{\mathrm{in}}t.$$

Then, by refining the right side of above equation we have

$$\sum_{i=1}^{p}\left[\frac{X_i^{\mathrm{in}}}{N_i^{\mathrm{in}}}-X_i^{\mathrm{in}}\left\{t\mathrm{mod}T_i^{\mathrm{in}}\right\}+X_i^{\mathrm{in}}t\right]-\sum_{i=1}^{p}X_i^{\mathrm{in}}t$$

$$=\sum_{i=1}^{p}\frac{X_i^{\mathrm{in}}}{N_i^{\mathrm{in}}}-\sum_{i=1}^{p}X_i^{\mathrm{in}}\left\{t\mathrm{mod}T_i^{\mathrm{in}}\right\}$$

Since

$$0\leq\sum_{i=1}^{p}X_i^{\mathrm{in}}\left\{t\mathrm{mod}T_i^{\mathrm{in}}\right\}\leq\sum_{i=1}^{p}X_i^{\mathrm{in}}T_i^{\mathrm{in}},$$

we have

$$f^{\mathrm{WIP}}(t)\geq\sum_{i=1}^{p}\frac{X_i^{\mathrm{in}}}{N_i^{\mathrm{in}}}-\sum_{i=1}^{p}X_i^{\mathrm{in}}T_i^{\mathrm{in}}=0.$$

For a type of processed products let functions, f^{P}, g^{P}, and h^{P} be an inventory level of the product, cumulated production quantity, and total cumulated quantity of outbound replenishments associated with schedule j for time t as

$$f^{\mathrm{P}}(t)=g^{\mathrm{P}}(t)-\sum_{j=1}^{q}h^{\mathrm{P}}(j,t),$$

$$g^{\mathrm{P}}(t)=\omega^{\mathrm{P}}t=\sum_{j=1}^{q}X_j^{\mathrm{out}}t,$$

and

$$h^{\mathrm{P}}(j,t)=\frac{X_j^{\mathrm{out}}}{N_j^{\mathrm{out}}}\left\lfloor\frac{t}{T_j^{\mathrm{out}}}\right\rfloor.$$

Thus, we have

$$f^{\mathrm{P}}(t)=\sum_{j=1}^{q}X_j^{\mathrm{out}}t-\sum_{j=1}^{q}\frac{X_j^{\mathrm{out}}}{N_j^{\mathrm{out}}}\left\lfloor\frac{\omega t}{T_j^{\mathrm{out}}}\right\rfloor.$$

On the same principle, we have

$$f^{\mathrm{P}}(t)=\sum_{j=1}^{q}X_j^{\mathrm{out}}t-\sum_{j=1}^{q}\left[-X_j^{\mathrm{out}}\left\{t\mathrm{mod}T_j^{\mathrm{out}}\right\}+X_j^{\mathrm{out}}t\right]$$

$$=\sum_{j=1}^{q}X_j^{\mathrm{out}}\left\{t\mathrm{mod}T_j^{\mathrm{out}}\right\}\geq 0.$$

Hence, this proposition is established. □

***Proof* of Proposition 4.** *It is simply obtained using Lemmas* 1 *and* 2. □

Appendix B

A set of reasonable scales of parameters is generated by referring to the literature (Ahmadi Javid and Azad, 2010; Sadjady and Davoudpour, 2012). Annual demand rate of each buyer uniformly obtained by [1,000, 10,000]; Capacities of manufacturing facilities are calculated by $D\times 0.75$ and $D\times 1.50$ where D is the total annual demand rate of all buyers associated with “K3”. Annual establishment costs of manufacturing facilities are obtained by

$87{,}500 \times 0.75$, and $87{,}500 \times 1.50$. Capacities of distribution centers are calculated by $D \times 0.25$, $D \times 0.375$, and $D \times 0.5$. Annual establishment costs of manufacturing facilities are obtained by $7{,}500 \times 0.65$, $7{,}500 \times 0.9$, and $7{,}500 \times 1.1$. All ordering costs, all inventory handling costs of "K1" and "K2", and all inventory handling costs of "K3" are fixed 15, 2, and 5, respectively. For "K1", suppliers "S1", "S2", and "S5" provide the material with annual maximum capacities, $D \times 1.5$, $D \times 1.0$, and $D \times 0.15$, respectively. For "K2", suppliers "S2", "S3", "S4", and "S5" provide the material with annual maximum capacities, $D \times 0.35$, $D \times 0.5$, $D \times 0.8$, and $D \times 1.0$, respectively. For "K3", all types of manufacturing facilities can produce single "K3" with single "K1" and single "K2".

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