INT. J. PROD. RES., 1994, VOL. 32, NO. 12, 2795–2801

Multiproduct economic lot size models with investment costs for setup reduction and quality improvement: review and extensions

I. MOON

There is a rapidly growing literature on modelling the effects of investment strategies to control *givens* such as setup time, setup cost, quality level. Recently, Hwang *et al.* (1993) studied the multiproduct economic lot size models in which setup reduction and quality improvement can be achieved with one-time initial investment. The aim here is to review and extend their work. First, we point out that their algorithms can produce irrelevant solutions, and clarify the cause of the problem. Second, we provide a complete formulation and analyse for the general investment functions using the Lagrangian method. The algorithm to find an optimal solution is provided, and an example problem is solved to illustrate the algorithm.

1. Introduction

As pointed out by Silver (1991) in his review, if the quantitative models are to be more useful as aids for managerial decision making, they must represent more realistic problem formulations, particularly permitting some of the usual *givens* to be treated as decision variables. *Givens* can be defined as the parameters which have been treated as fixed or given, for example, setup times, setup costs, production rate, etc. Silver (1991) listed a wide variety of possible improvements to undertake (equivalently, usual *givens* to change) in the operations of manufacturing, such as setup reduction, higher quality level, controllable production rates, etc. There is a rapidly growing literature on modelling the effects of changing the *givens* in the manufacturing decision situations. Most of the models include the cost of the change, usually amortized as part of total relevant cost per unit time.

Recently, researchers have studied investment strategies to control *givens* such as setup time, setup cost, quality level. This is consistent with the just-in-time (JIT) manufacturing philosophy. Porteus (1985) developed a framework to reduce setup costs in the Economic Order Quantity (EOQ) model. He developed an extension of the EOQ model in which setup cost is viewed as a decision variable, rather than as a parameter, and the cost of selecting different values of the setup is included explicitly in the formulation. Porteus (1986) studied the relationship between lot sizing and quality control. He concluded that quality control can be improved by reducing setups. Gallego and Moon (1992) developed a model that considers the economic effect of externalizing internal setup operations in the economic lot scheduling problem (ELSP). Beek and Putten (1987) showed how Operations Research (OR) models can contribute to quantifying the integral effects of investment decisions with respect to production systems. They gave several examples illustrating opportunities reducing setup times, increasing production capacity, cutting supply leadtimes, etc.

Revision received December 1993.
Department of Industrial Engineering, Pusan National University, Pusan 609-735, Korea.

0020-7543/94 $10·00 © 1994 Taylor & Francis Ltd.

Spence and Porteus (1987) applied the setup time reduction concept to the multi-item capacitated EOQ model. They mentioned that '... the solution presented here is probably not directly implementable, because of the difficulties, ignored by our model, of scheduling lot sizes for many products on one facility'. Moon (1991) attacked this problem by applying the setup time reduction concept to the ELSP, and developed heuristics to minimize total relevant costs including amortized one time investment cost. Recently, Hwang *et al.* (1993) generalized Spence and Porteus' (1987) work by adopting the common cycle approach of the ELSP. They also included the defective rate into decision variables to investigate the effects of investment in quality improvement.

The aim here is to review and extend Hwang *et al.*'s (1993) work. First, we point out by a numerical example that their algorithms can produce irrelevant solutions, and identify the cause of the problem. Second, we provide a complete formulation and prove the optimality of the Karush–Kuhn–Tucker (KKT) conditions for the general investment functions. The algorithm to find an optimal solution is addressed, and an example problem is solved to illustrate the algorithm.

2. Review of the the procedure in Hwang *et al.* (1993)

We first review the procedure developed by Hwang *et al.* (1993). From now on, we call the procedure an HKK procedure. The assumptions are same as HKK procedures which are similar to those for the common cycle approach of the ELSP. We use the same notations to avoid any possible confusion.

Data:

Index for the products:	$j=1,\ldots,N$
Demand rates:	$D_j \quad j=1,\ldots,N$
Production rates:	$P_j \quad j=1,\ldots,N$
Setup times:	$S_j \quad j=1,\ldots,N$
Setup costs:	$K_j \quad j=1,\ldots,N$
Unit inventory holding costs:	$h_j \quad j=1,\ldots,N$
Unit production costs:	$c_j \quad j=1,\ldots,N$
Initial defective rate for all products:	R_0
Total time available for productions and setups:	u_0
Sum of the initial setup costs for all products:	K_0
Sum of the initial setup times for all products:	S_0
Capital cost rate per unit time:	i

$C_p=\Sigma_{j=1}^{N} c_jD_j$

$H=\Sigma_{j=1}^{N} h_jD_j(1-D_j/P_j)/2$

$H_j=h_jD_j(1-D_j/P_j)/2.$

Decision variables:

Common cycle time:	T
Ratio of the reduced setup cost to the initial one:	k
Ratio of the reduced setup time to the initial one:	s
Ratio of the improved defective rate to the initial one:	r

Define $\kappa \equiv u_0 - \Sigma_{j=1}^{N} D_j/P_j$. Note that κ is the long run proportion of time available for setups. We first consider the problem SQ formulated by Hwang *et al.* (1993).

(SQ)

$$\min C(k, s, r, T) = \frac{K_0 k}{T} + HT + C_p R_0 r + iak^{-b}s^{-c}r^{-d}$$

$$\text{subject to } \frac{S_0 s}{\kappa T} \leqslant 1.$$

Consider the following three products' example.

Demand rate: $D_1 = 10000$, $D_2 = 6000$, and $D_3 = 8000$ units/year.

Production rate: $P_1 = 40000$, $P_2 = 20000$, and $P_3 = 30000$ units/year.

Production cost: $c_1 = 20$, $c_2 = 30$, and $c_3 = \$50$/unit.

Holding cost: $h_1 = 6$, $h_2 = 9$, and $h_3 = \$10$/unit, year.

Setup cost: $K_1 = 1000$, $K_2 = 2000$, and $K_3 = \$3000$.

Setup time: $S_1 = 0{\cdot}002$, $S_2 = 0{\cdot}003$, and $S_3 = 0{\cdot}005$ year.

Defective rate: $R_0 = 0{\cdot}1$.

Current time available for production processing: $u_0 = 1{\cdot}0$.

Fractional opportunity cost of capital: $i = 0{\cdot}15$.

Parameters for investment cost function: $a = \$5000$, $b = c = d = 1$.

By substituting above parameters into HKK solution, we obtain $(k^*, s^*, r^*, T^*) = (0{\cdot}266, 3{\cdot}895, 0{\cdot}096, 0{\cdot}212)$ with total cost \$37 568·2 (excluding production cost C_p). Clearly, $s^* = 3{\cdot}895$ is irrelevant since it means that increasing the setup time about 4 times the current level is optimal. The cause of producing this kind of irrelevant solution using HKK procedure is very simple: the following three constraints which impose that we cannot gain money by deteriorating setup levels or quality level must be included into SQ to produce meaningful solutions.

$$k \leqslant 1, \quad s \leqslant 1, \quad r \leqslant 1.$$

By adding the above three constraints to SQ, the degree of difficulty of problem SQ becomes three which precludes the possibility of the closed form solutions using the geometric programming (GP) technique.

3. Formulation and algorithm: new approach

Here we consider the case in which the setup costs are proportional to the setup times. Then the setup cost K_j can be denoted as $C_K S_j$, where C_k is the cost per unit time for setups. Here, we consider the general investment function $g(s,r)$ which is assumed to be twice differentiable and strictly convex. Note that the investment function $g(s,r)=as^{-c}r^{-d}-e$, which is considered by Hwang *et al.* (1993), is twice differentiable and strictly convex. We denote $\partial g(s,r)/\partial s$ by $g_s(s,r)$ and $\partial^2 g(s,r)/\partial s\partial r$ by $g_{r,j}(s,r)$, respectively. The complete formulation of SQP is as follows:

(SQP)

$$\min\ C(s,r,T)=\frac{C_k S_0 s}{T}+HT+C_p R_0 r+ig(s,r)$$

$$\text{subject to } \frac{S_0 s}{T}\leqslant\kappa$$

$$s\leqslant 1$$

$$r\leqslant 1.$$

Note that the degree of difficulty of SQP becomes three which prevents us from using GP technique. We solve SQP by Lagrangian analysis.

Let λ, v_s and $v_r\geqslant 0$ be the Lagrange multipliers corresponding to $s_0 s/T\leqslant\kappa, s\leqslant 1$, and $r\leqslant 1$, respectively. Then the KKT conditions for program SQP are:

$$ig_s(s,r)+\frac{(\lambda+C_k)S_0}{T}=-v_s \tag{1}$$

$$v_s\geqslant 0 \text{ complementary slackness (c.s.) with } s\leqslant 1 \tag{2}$$

$$ig_r(s,r)+C_p R_0=-\mu_r \tag{3}$$

$$v_r\geqslant 0 \text{ c.s. with } r\leqslant 1 \tag{4}$$

$$H-\frac{(\lambda+C_k)S_0 s}{T^2}=0 \tag{5}$$

$$\frac{S_0 s}{T}\leqslant\kappa \text{ c.s. with } \lambda\geqslant=0. \tag{6}$$

Even though the objective function is not convex in (s,r,T), the following proposition shows that a point satisfying conditions (1–6) is a strict local minimum.

Proposition 1. The point (s^*, r^*, T^*) satisfying conditions (1–6) is a strict local minimum of SQP.

Proof. See Appendix.

If the specific form of the $g(s,r)$ is given, we can find a KKT point satisfying conditions (1–6) by a variety of techniques designed to solve systems of non-linear equations in several variables (Ortega and Rheinboldt 1970). We present an algorithm that finds a KKT point satisfying conditions (1–6) for a power functional form investment function such that $g(s,r)=as^{-c}r^{-d}-e$. Note that it requires only a one-dimensional line search.

Algorithm SQP

Step 1. (Check if $\lambda = 0$ gives an optimal solution.)

Solve the following equation for T

$$HT^2 = C_k S_0 s$$

using a line search on T where

$$s = \min\left\{\left[\frac{icaT}{C_k S_0}\right]^{d+1/c+d+1}\left[\frac{iad}{C_p R_0}\right]^{-d/c+d+1}, \quad 1\right\}.$$

If $S_0 s/T \leqslant \kappa$, stop. (s, T) is optimal. Else, go to step 2.

Step 2. Start from an arbitrary $\lambda > 0$.

Step 3. Solve the following equation for T

$$HT^2 = (C_k + \lambda) S_0 s$$

using a line search on T

where

$$s = \min\left\{\left[\frac{icaT}{(\lambda + C_k) S_0}\right]^{d+1/c+d+1}\left[\frac{iad}{C_p R_0}\right]^{-d/c+d+1}, \quad 1\right\}.$$

Step 4. If $S_0 s/T < \kappa$, reduce λ. Go to step 3.

If $S_0 s/T > \kappa$, increase λ. Go to step 3.

If $S_0 s/T = \kappa$, stop. (s, T) is optimal.

An optimal r is as follows:

$$r = \min\left\{\left[\frac{iads^{-c}}{C_p R_0}\right]^{1/d+1}, \quad 1\right\}.$$

The following proposition shows that the point satisfying the above algorithm is a global minimum of SQP.

Proposition 2. The point (s^*, r^*, T^*) satisfying conditions (1–6) is a global minimum of SQP.

Proof. Clearly the point obtained using Algorithm SQP satisfies conditions (1–6). HT^2 is an unbounded strictly increasing function of T starting at zero for $T = 0$. Consequently, there is a unique solution to $HT^2 = (C_k + \lambda) S_0 s$. By proposition 1, the point is a local and hence a global minimum of SQP. □

Example 1. We solve the following problem using algorithm SQP.

Demand rate: $D_1 = 10000$, $D_2 = 6000$, and $D_3 = 8000$ units/year.

Production rate: $P_1 = 24\,000$, $P_2 = 20\,000$, and $P_3 = 30\,000$ units/year.

Production cost: $c_1 = 20$, $c_2 = 30$, and $c_3 = \$50$/unit.

Holding cost: $h_1=6$, $h_2=9$, and $h_3=\$10$/unit, year.

Setup time: $S_1=0{\cdot}001$, $S_2=0{\cdot}003$, and $S_3=0{\cdot}005$ year.

Cost per unit time (year) for setups: $C_k=\$500\,000$.

Defective rate: $R_0=0{\cdot}1$.

Current time available for production processing: $u_0=1{\cdot}0$.

Fractional opportunity cost of capital: $i=0{\cdot}15$.

Parameters for investment cost function: $a=\$5000$, $b=c=d=1$.

After applying algorithm SQP, we get $(s^*, r^*, T^*)=(0{\cdot}3596, 0{\cdot}1635, 0{\cdot}1942)$ with total cost \$46 606·6. Note that the result is same as that of HKK procedure. The optimal solution without investment is $T^*=0{\cdot}5400$ with total cost \$121 829·3. Refer to Table 1 for the cost components.

4. Conclusion

For the facilities that operate at or near capacity, Japanese manufacturing companies have exerted themselves to reduce setup times. In contrast, US factories have traditionally reacted to capacity problems by increasing production rates, often at a very large cost. This contrast has been appropriately identified by Maxwell (1989). 'The Japanese recognized, from the point of view of manufacturing control, that engineering the setup often has a higher payoff than engineering the per unit run time'. Intensive attention has been paid to this observation in order to provide meaningful modellings to decision makers. We have reviewed one of the models, and have extended it to consider general investment functions. A heuristic algorithm to squeeze out a better solution by adopting the time-varying lot-size approach of the ELSP can be developed.

Investing in reducing setup times/setup costs are widely proven to be very worth while for the facilities that operate near or at their capacity limits. Interestingly, the opposite was observed in Silver (1990), Moon *et al.* (1991), and Gallego (1993) where production rate reduction was shown to be more profitable in under utilized facilities.

Acknowledgements

The author gratefully acknowledges the comments made by two anonymous referees on an earlier version of the paper. This work has been supported by the Korea Science and Engineering Foundation (KOSEF) under grant 941–1000–020–1.

(\$/year)	SQP	SQP (investment)
Investment cost	0	12 755·5
Holding cost	35 496·0	12 762·3
Setup cost	8333·3	8333·3
Defective cost	78 000·0	12 755·5
Total average cost	121 829·3	46 606·6

Table 1. Cost components for the example.

Appendix. Proof of proposition 1

We will show that the Hessian matrix of Lagrangian function $L(s,r,T,\lambda,v_s,v_r) = (C_kS_0s)/T+HT+C_pR_0r+ig(s,r)+\lambda(S_0s/T-\kappa)+v_s(s-1)+v_r(r-1)$ at (s^*,r^*,T^*) is positive definite on the subspace $M=\{\mathbf{y}:\mathbf{y}=(y_s,y_r,y_T)\in R^3$ s.t. $(S_0/T^*,0,-S_0s^*/T^{*2})\,\mathbf{y}=0\}$. Then it follows from Second-Order Sufficient Conditions (Luenberger 1984) that the point is a strict local minimum.

$$\nabla^2 L(s^*,r^*,T^*,\lambda,v_s,v_r)=\begin{pmatrix} ig_{s,s}(s^*,r^*) & ig_{s,r}(s^*,r^*) & \dfrac{-(C_k+\lambda)S_0}{T^{*2}} \\ ig_{r,s}(s^*,r^*) & ig_{r,r}(s^*,r^*) & 0 \\ \dfrac{-(C_k+\lambda)S_0}{T^{*2}} & 0 & \dfrac{2(C_k+\lambda)S_0s^*}{T^{*3}} \end{pmatrix}$$

Then any $\mathbf{y}\in M$ and $\mathbf{y}\neq 0$,

$$\begin{aligned}\mathbf{y}^T\nabla^2 L\mathbf{y} &= [ig_{s,r}(s^*,r^*)y_sy_r+ig_{s,s}(s^*,r^*)(y_s)^2+ig_{r,s}(s^*,r^*)y_ry_s+ig_{r,r}(s^*,r^*)(y_r)^2] \\ &\quad +\frac{2(C_k+\lambda)S_0s^*}{T^{*3}}(y_T)^2-\frac{2(C_k+\lambda)S_0}{T^{*2}}y_sy_T \\ &= i(y_s,y_r)\nabla^2 g(s^*,r^*)(y_s,y_r)^T>0,\end{aligned}$$

since for any $\mathbf{y}\in M$, $y_s=s^*/T^*\,y_T$, the cross terms are eliminated. The inequality then follows from the strict convexity of $g(s,r)$. Consequently, (s^*,r^*,T^*) is a strict local minimum. □

References

BEEK, P., and PUTTEN, C., 1987, OR contributions to flexibility improvement in production-/inventory systems. *European Journal of Operations Research*, **31**, 52–60.

GALLEGO, G., 1993, Reduced production rates in the economic lot scheduling problem. *International Journal of Production Research*, **31**, 1035–1046.

GALLEGO, G., and MOON, I., 1992, The effect of externalizing setups in the economic lot scheduling problem. *Operations Research*, **40**, 614–619.

HWANG, H., KIM, D. and KIM, Y., 1993, Multiproduct economic lot size models with investment costs for setup reduction and quality improvement. *International Journal of Production Research*, **31**, 691–703.

LUENBERGER, D., 1984, *Linear and Nonlinear Programming* (New York: Addison-Wesley).

MAXWELL, W., 1989, A research agenda for models to plan and schedule manufacturing systems. Working paper, School of OR and IE, Cornell University.

MOON, I., 1991, Some issues in the economic lot scheduling problem. Ph.D thesis, Columbia University.

MOON, I., GALLEGO, G., and SIMCHI-LEVI, D., 1991, Controllable production rates in a family production context. *International Journal of Production Research*, **29**, 2459–2470.

ORTEGA, J., and RHEINBOLDT, C., 1970, *Iterative Solution of Nonlinear Equations in Several Variables* (New York: Academic).

PORTEUS, E., 1985, Investing in reduced setups in the EOQ model. *Management Science*, **31**, 998–1010.

PORTEUS, E., 1986, Optimal lot sizing, process quality improvement and setup cost reduction. *Operations Research*, **34**, 137–144.

SILVER, E., 1990, Deliberately slowing down output in a family production context. *International Journal of Production Research*, **28**, 17–27.

SILVER, E., 1991, Modelling in support of continuous improvements towards achieving world class operations. Working Paper. University of Calgary.

SPENCE, A., and PORTEUS, E., 1987, Setup reduction and increased effective capacity. *Management Science*, **33**, 1291–1301.