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# The storage capacity expansion and space leasing for container depots

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**Abstract** Positioning empty containers is one of the most effective ways to solve the container imbalance problem and is affected by the capacity of depots. A shipping company will be more competitive if the depot capacity is large. This study provides a decision tool for planning the expansion of depot capacity. Mathematical models are utilized to minimize the total relevant costs that include the capacity expansion cost, storage space leasing cost, inventory holding cost, container leasing cost, and positioning cost. The problem is formulated as a mixed integer program. Then, we develop a heuristic algorithm that is based on Lagrangian relaxation. Computational experiments are conducted to evaluate the performance of the proposed algorithm.

**Keywords** Empty container positioning · Capacity expansion · Lagrangian relaxation

## 1 Introduction

With advances in the world economy and consequent increase in global trade, the transportation of goods has grown. Containers are effective and inexpensive transportation commodities. For inland transportation, several options can be used to transport goods, but for ocean transportation, only containers and container vessels can be utilized. A container that is fully loaded with goods from the supplier is transported through the oceans to the destination port then delivered to the customer and unloaded. Subsequently, the container is stored at the destination port, until it is required for another consignment. When there is an imbalance in the number of import and export containers, some ports have a surplus of empty containers, while others

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have a deficit. When ports have a deficit of empty containers, shipping companies must lease or purchase empty containers. When ports have a surplus of empty containers, the empty containers which are not in use are stored in depots. Strategically, positioning empty containers is therefore one of the most effective ways to solve container imbalance problem. Unfortunately, one of the reasons why we cannot position as many empty containers as possible is that the depot capacity is limited. Moreover, the demand is usually fluctuating. Empty containers should be kept in the depot during low demand periods and used when the demand becomes high. A shipping company will be more competitive if the depot capacity is large. Depot capacity expansion plays a significant role in the activities of shipping companies.

Capacity expansion is not a new issue. Many studies have considered this problem, for example, Berman and Ganz (1994), Luss (1982), and Herrera et al. (2009). However, there have been no studies on the capacity expansion of container depots. In this research, we provide a decision tool to plan the expansion of depot capacity. If the capacity is expanded too early, it will be wasteful and we need to pay for the unused capacity. The budget for expansion could be invested in other projects to yield more profit. On the other hand, it is very hard and costly for the company to satisfy the demand for empty containers if the depot capacity is expanded too late. Another situation is that the company needs to lease empty containers, which usually entails higher cost. Therefore, we need a plan to determine when we have to expand the capacity and how large it should be. An alternative way to enlarge the depot capacity is to lease storage space from other depots. Adequate storage space can be leased in adequate periods of time depending on the demand and be returned later.

As mentioned above, depot capacity expansion affects the number of empty containers to be held in the company's own depot or repositioned from other ports. Hence, in this research, we consider not only capacity expansion planning, but also plans for positioning and leasing. The mathematical model is to minimize the total relevant costs that consist of the capacity expansion cost, storage space leasing cost, inventory holding cost, container leasing cost, and positioning cost. There is a limitation on the capacity expansion and leased storage space. The decision variables involve the amount of expanded capacity, the amount of leased storage space, the number of leased containers, and the number of positioned containers. We formulate the problem as a mixed integer program. Then, we develop a heuristic algorithm based on Lagrangian relaxation. Computational experiments are conducted to evaluate the performance of the proposed algorithm. The paper is structured as follows. Section 2 will present the problem description that involves a review of the literature, problem definition, and mathematical model. The solution algorithm will be discussed in Sect. 3, and numerical experiments will be presented in Sect. 4. Finally, Sect. 5 presents some conclusions.

## 2 Problem descriptions

### 2.1 Literature review

Recently, many studies related to empty container positioning have been conducted. Crainic et al. (1989) considered a multi-commodity capacitated location problem with balancing requirements (MCLB). They proposed models for MCLB with inter-depot balancing requirements. The decision variables in MCLB include a set of binary variables, which refer to the opening or closing of depots, and a set of continuous variables that represent the flows of empty containers between supply customers, depots, and demand customers. The objective function is to minimize the total cost that involves the cost of opening the depot and transportation costs. The demand for empty containers must be satisfied. Many studies have attempted to solve the MCLB problem. Crainic and Delorme (1993) developed dual-ascent procedures for the proposed model. Crainic et al. (1993) solved the problem by using a Tabu search procedure. Gendron and Crainic (1997) presented a parallel branch and bound approach, which is based on the dual-ascent procedure previously proposed by Crainic and Delorme (1993). Gendron et al. (2003) also solved the problem using a Tabu search procedure, but used the slope scaling method to identify the starting solution. Li et al. (2004) studied the management of empty containers in a port with stochastic demand. Their analysis was based on a multistage inventory problem and Markov decision processes with discrete time. They focused on the optimization of the pair-critical policy, (U, D). In this policy, if the number of empty containers at a port is less than U, empty containers are imported up to the amount U; if it is more than D, empty containers are exported until D amount of containers are left behind. Li et al. (2007) have since extended the problem for multi-ports applications. Shen and Khoong (1995) proposed a decision support system (DSS) for empty container distribution planning. The DSS is based on network optimization models. In the network, they considered the leasing-in, off-leasing, positioning-in, and positioning-out at a port. The problem was broken down into three levels, namely, terminal (port) planning, intra-regional planning, and inter-regional planning. They considered a single type of container. Francesco et al. (2009) studied the empty container repositioning problem with several uncertainty parameters. The distribution of an uncertainty parameter can be represented by a set of a limited number of values. A scenario is formed by a combination of uncertainty parameters. The experiments show that the multi-scenario model is more adaptable to the requirements of a shipping company rather than the deterministic model. Dong and Song (2009) conducted a research on container fleet sizing and empty container repositioning. They considered the multi-port, multi-vessel, and multi-routing systems under uncertain demand. They proposed an algorithm that combines simulation and GA to solve the problem. Imai et al. (2009) figured out the design of container liner shipping networks including empty container repositioning.

### 2.2 Problem definition

In Moon et al. (2010), they developed the empty container positioning problem considering the leasing and purchasing. Also in this research, sensitivity analysis was conducted; it is thus evident that the total cost will decrease if the maximum level of inventory increases. This means that when demand is continuously increasing and fluctuating, the maximum level of inventory (or the capacity of inventory) will help to reduce the total cost. However, it is costly to increase the capacity of inventory. Therefore, the issue of expanding the capacity of inventory should be investigated in empty container positioning.

The purpose of this paper is to investigate the expansion of the capacity of inventory at a port or depot (from now on, in this paper, we use the term "depot capacity" to refer to the capacity of inventory at a port). This research will help the decision maker to decide when the depot capacity should be expanded as well as how much the expanded capacity should be. In this research, we also consider the positioning and leasing of empty containers. The problem is categorized as lying between tactical and strategic planning. There are several ports/depots and each of them has its own demand for empty containers in each period. When a port has a shortage of empty containers, leasing will be carried out to satisfy demand. On the other hand, if the port has a surplus of empty containers, these containers will be either repositioned to other depots that have a shortage of empty containers or stored in inventory in reserve for the future demand. There are two alternatives for expanding the depot capacity: expanding the current depot or leasing storage space. The amount of expanded capacity is limited but the number of leased containers is not. The following section will develop a mathematical model of the problem.

### 2.3 Mathematical model

The mathematical model is built by considering multiple-ports, multiple-commodities, and multiple-periods. The assumptions of the mathematical model are as follows.

- The demand must be satisfied; no backlog is allowed.
- When there is a shortage of empty containers, containers will be leased.
- We consider a long-term leasing of containers.
- There is no limitation on the number of leased containers.
- Full containers will be unloaded and available after they have arrived at the destination port in one period.
- The vessel capacities and vessel routes are not considered because this model concentrates on the flow of empty containers irrespective of the transport modes.
- The expansion cost will be affected by the time value of money and is pre-determined.

The model is described in detail as follows.

**Indices**

| | |
|---|---|
| $t$ | periods, $t = 1, 2, 3,\ldots, T$ |
| $i, j$ | ports, $i, j = 1, 2,\ldots, P$ |
| $v$ | type of container, $v = 1, 2, \ldots, V$ |

**Parameters**

| | |
|---|---|
| $D_{ijvt}$ | demand of type $v$ container (full containers) in period $t$, which should be transported form port $i$ to port $j$ |

| | |
|---|---|
| $K_i$ | current depot capacity at port $i$ (TEU unit) |
| $c_{iv}^{H}$ | unit holding cost of type $v$ containers at port $i$ |
| $c_{iv}^{L}$ | unit leasing cost of type $v$ containers at port $i$ |
| $c_{ijv}^{T}$ | unit transportation cost of type $v$ containers from port $i$ to port $j$ |
| $K_i^E$ | maximum depot capacity that can be expanded at port $i$ |
| $K_i^L$ | maximum storage space that can be leased at port $i$ |
| $T_{ij}$ | transportation time from port $i$ to port $j$ |
| $f_{it}^{E}$ | fixed cost for expanding the depot capacity at port $i$ in period $t$ |
| $v_{it}^{E}$ | variable cost for expanding the depot capacity at port $i$ in period $t$ |
| $v_{it}^{L}$ | variable cost for leasing storage space at port $i$ in period $t$ |
| $TEU_v$ | conversion rate from container type $v$ to TEUs |

### Decision variables

| | |
|---|---|
| $I_{ivt}$ | inventory level of empty containers, for type $v$ at port $i$ in period $t$ |
| $y_{ivt}$ | number of type $v$ containers that are leased at port $i$ in period $t$ |
| $x_{ijvt}$ | number of type $v$ empty containers, transported from port $i$ to port $j$ in period $t$ |
| $s_{it}^{E}$ | $\begin{cases} 1, & \text{if the depot capacity at port } i \text{ is expanded in period } t \\ 0, & \text{otherwise} \end{cases}$ |
| $w_{it}^{E}$ | amount of expansion of the depot capacity at port $i$ in period $t$ |
| $w_{it}^{L}$ | amount of storage space that will be leased at port $i$ in period $t$ |

### *The mathematical model*

$$\text{Min}\, Z = \sum_{i,t} f_{it}^{E} s_{it}^{E} + \sum_{i,t} v_{it}^{E} w_{it}^{E} + \sum_{i,t} v_{it}^{L} w_{it}^{L} + \sum_{i,j\neq i,v,t} c_{ijv}^{T} x_{ijvt} + \sum_{i,v,t} c_{iv}^{L} y_{ivt} + \sum_{i,v,t} c_{iv}^{H} I_{ivt}$$

Subject to

$$I_{ivt} = I_{iv(t-1)} + \sum_{j\neq i, t > T_{ji}} x_{jiv\left(t-T_{ji}\right)} + \sum_{j\neq i, t > T_{ji}+1} D_{jiv\left(t-T_{ji}-1\right)} - \sum_{j\neq i} \left(x_{ijvt} + D_{ijvt}\right) + y_{ivt} \quad \forall\, i, v, t \tag{1}$$

$$\sum_{v} TEU_v I_{ivt} \leq K_i + \sum_{k\leq t} w_{ik}^{E} + w_{it}^{L} \quad \forall\, i, t \tag{2}$$

$$w_{it}^{E} \leq K_i^{E} s_{it}^{E} \quad \forall\, i, t \tag{3}$$

$$\sum_{t} w_{it}^{E} \leq K_i^{E} \quad \forall\, i \tag{4}$$

$$w_{it}^{L} \leq K_{i}^{L} \quad \forall i, t \tag{5}$$

$$I_{ivt}, y_{ivt}, x_{ijvt}, w_{it}^{E}, w_{it}^{L} \geq 0 \quad \text{and} \quad s_{it}^{E} = \{0, 1\} \quad \forall i, j, v, t. \tag{6}$$

The objective function is to minimize the total relevant cost. The total cost involves the cost of expanding depot capacity, the cost of leasing storage space, the cost of positioning empty containers, leasing cost, and holding cost. Constraint (1) is inventory balance equations. In the inventory at port $i$ in period $t$ for each type of container:

- The number of empty containers going into inventory includes the number of empty containers in the preceding period (period $t-1$), the number of empty containers that are received in that period, the number of full containers that are received in the preceding period, and the number of containers leased in that period.
- The number of empty containers going out of inventory includes the number of empty and full containers that are transported to other ports.

Constraint (2) ensures that the inventory level cannot exceed the total capacity that consists of the current depot capacity, the total expanded capacity, and the amount of leased storage-space. Constraints (3) and (4) limit the amount of expanded capacity. Finally, Constraint (5) prevents the amount of leased storage-space from exceeding the limit. As in many studies on empty container movement, especially those of Crainic et al. (1989, 1993) and Crainic and Delorme (1993), the variables that relate to the flow of empty containers are considered continuous variables in this study. The values of these variables are large and can be rounded to the nearest integer.

## 3 The solution algorithm

The problem is a Mixed Integer Programming (MIP) problem. This means that when the size of the problem increases, optimization software cannot solve it in reasonable time. Several meta-heuristic algorithms can be explored to solve this kind of problem such as Genetic Algorithms (GAs), Ant Colony Optimization (ACO), Tabu Search (TS), etc. There are also some methods for breaking down the problem into sub-problems that are smaller in size and easier to solve. Lagrangian relaxation is one of those methods. By using Lagrangian multipliers and relaxing some constraints, the Lagrangian relaxation (LR) of the original problem can be dealt with as smaller problems. The optimal value of the objective of the LR problem is a lower bound on that of the original problem. The Lagrangian multipliers will be updated at each iteration, and the optimal objective value for the LR problem will become closer to the optimal objective value of the original problem.

Many studies have applied GAs for solving MIP problems. However, it is very hard to evaluate the performance of GAs for large-sized problems since there are no optimal solutions for comparison. This is established in Moon et al. (2010) and the authors had to use the Linear Relaxation method to find a lower bound on the optimal objective value. LR can yield a lower bound; furthermore, it is shown that the lower bound obtained by LR can be better than that obtained by Linear Relaxation. In this research, we propose to use LR to solve the original problem. The reason is that not only can LR provide a better lower bound for evaluating the performance of the algorithm but also the following can be recognized from the mathematical model.

- The objective function can be divided into three parts for three activities.
  - The cost of expanding
  - The cost of leasing storage space
  - The cost that relates to the operation of the shipping company.
- The constraints can also be divided into three parts.
  - Constraints (3) and (4) relate to capacity expansion.
  - Constraint (5) is for leasing storage space.
  - Constraint (1) relates to the operation of the shipping company.
- The three activities are linked by Constraint (2).

The proposed problem can be broken down if Constraint (2) is relaxed. Therefore, it is reasonable to apply LR in solving this problem. The following section will briefly introduce LR.

## 3.1 Lagrangian relaxation

Suppose that we have a problem as follows.

$$(\mathrm{P}) \qquad \mathrm{Min} \qquad Z = cx$$

$$\text{Subject to}$$

$$Ax \geq b \tag{1}$$

$$\begin{aligned} Dx &\geq e \\ x &\geq 0. \end{aligned} \tag{2}$$

The Lagrangian multiplier $\lambda$ corresponds to Constraint (1). The LR problem is formed as:

$$(\mathrm{LR}(\lambda)) \qquad \mathrm{Min} \qquad Z_{\mathrm{LR}} = cx + \lambda(b - Ax) = (c - \lambda A)x + \lambda b$$

$$\text{Subject to}$$

$$\begin{aligned} Dx &\geq e \\ x &\geq 0. \end{aligned} \tag{2}$$

Furthermore, there is a relation between (LR($\lambda$)) and (P) as follows: $Z = \max_{\lambda \geq 0} Z_{\mathrm{LR}}$ (Fisher 1985). The LR problem yields a lower bound for (P) and it should be easily solved. Then, we try to identify a method to search for the value of $\lambda$ that maximizes the objective value of the LR problem. Sub-gradient optimization is a popular approach. The procedure to solve the original problem by using Lagrangian relaxation is described below.

Step 0. Initialization

- Set LB, the lower bound for (P), equal to $-\infty$.
- Set UB, the upper bound for (P), equal to $+\infty$.
- Set $\pi$, the parameter that controls the step size, equal to 2.
- Set $\lambda$, the Lagrangian multiplier, equal to 0.

Step 1. Calculate the lower bound.

- Solve the LR problem with the given value of $\lambda$ and obtain the optimal value $Z_{\mathrm{LB}}$, and a solution x'.
- If LB $< Z_{\mathrm{LR}}$, then set LB equal to $Z_{\mathrm{LR}}$.

Step 2. Calculate the upper bound.

- Find a feasible solution of (P). Usually, the feasible solution is found based on the solution to the LR problem. $Z_{\mathrm{UB}}$ is the optimal value corresponding to the feasible solution.
- If UB $> Z_{\mathrm{UB}}$, then set UB equal to $Z_{\mathrm{UB}}$ and store the feasible solution as the incumbent solution to (P).

Step 3. Update the Lagrangian multiplier.

- Calculate the subgradient $G$ by the formula: $G = b - Ax'$.
- Calculate the step size: $S = \frac{\pi(Z_{UB} - Z_{LB})}{G^2}$.
- Update the Lagrangian multiplier: $\lambda_i = \max(0,\ \lambda_{i-1} + SG)$, where $i$ is the iteration counter.

Step 4. Check the stopping criteria.

- The solution to the LR problem, $x'$, satisfies the relaxed constraint: $G = 0$.
- The parameter $\pi$ is very small, usually $\pi < 0.005$.
- The gap between UB and LB is very small. Usually, we calculate (UB − LB)/UB. If this value is less than a desired value, then we stop.

If any stopping criterion is satisfied, then we stop; otherwise, we go back to Step 1. The best solution is the one that is stored in Step 2.

Fisher (1985) introduced a practical guide for the use of LR with many examples and illustration. Subsequently, many researchers applied LR in their studies. Among those Guignard (2003) reviewed in-depth the use of LR to solve optimization problems. She also analyzed several interesting questions in her paper. Chen (2007) applied LR to a production planning problem with uncertain demand.

## 3.2 The solution algorithm

As mentioned above, we applied Lagrangian relaxation in solving this problem. Constraint (2), viz., the inventory capacity constraint, was selected for relaxation. The reason is that this is the linking constraint and if we relax it, we can break down the original problem into three sub-problems. We define $\lambda_{it}$ as the Lagrangian multiplier corresponding to constraint (2) for port $i$ and period $t$. Then, the LR problem will be as follows.

- Objective function

$$\begin{aligned}\operatorname{Min} Z_{\mathrm{LR}} = & \sum_{i,t} f_{it}^E s_{it}^E + \sum_{i,t} v_{it}^E w_{it}^E + \sum_{i,t} v_{it}^L w_{it}^L + \sum_{i,j\neq i,v,t} c_{ijv}^T x_{ijvt} + \sum_{i,v,t} c_{iv}^L y_{ivt} + \sum_{i,v,t} c_{iv}^H I_{ivt} \\ & + \sum_{i,t} \lambda_{it}\left(\sum_v TEU_v I_{ivt} - K_i - \sum_{k\leq t} w_{ik}^E - w_{it}^L\right)\end{aligned}$$

or

$$\begin{aligned}\operatorname{Min} Z_{\mathrm{LR}} = & \sum_{i,t} f_{it}^E s_{it}^E + \sum_{i,t}\left(v_{it}^E - \sum_{k\geq t}\lambda_{ik}\right) w_{it}^E + \sum_{i,t}\left(v_{it}^L - \lambda_{it}\right) w_{it}^L \\ & + \sum_{i,j\neq i,v,t} c_{ijv}^T x_{ijvt} + \sum_{i,v,t} c_{iv}^L y_{ivt} + \sum_{i,v,t}\left(c_{iv}^H + \lambda_{it} TEU_v\right) I_{ivt} - \sum_{i,t}\lambda_{it} K_i.\end{aligned}$$

We denote $v_{it}^{\prime E} = v_{it}^E - \sum_{k\geq t}\lambda_{ik}$, $v_{it}^{\prime L} = v_{it}^L - \lambda_{it}$, and $c_{ivt}^{\prime H} = c_{iv}^H + \lambda_{it} TEU_v$. Then, the objective function can be rewritten as follows.

$$\begin{aligned}\operatorname{Min} Z_{\mathrm{LR}} = & \sum_{i,t} f_{it}^E s_{it}^E + \sum_{i,t} v_{it}^{\prime E} w_{it}^E + \sum_{i,t} v_{it}^{\prime L} w_{it}^L + \sum_{i,j\neq i,v,t} c_{ijv}^T x_{ijvt} + \sum_{i,v,t} c_{iv}^L y_{ivt} \\ & + \sum_{i,v,t} c_{ivt}^{\prime H} I_{ivt} - \sum_{i,t}\lambda_{it} K_i\end{aligned}$$

- Constraints

$$I_{ivt} = I_{iv(t-1)} + \sum_{j\neq i, t > T_{ji}} x_{jiv(t-T_{ji})} + \sum_{j\neq i, t > T_{ji}+1} D_{jiv(t-T_{ji}-1)} - \sum_{j\neq i}\left(x_{ijvt} + D_{ijvt}\right) + y_{ivt} \quad \forall i, v, t \tag{1}$$

$$w_{it}^E \leq K_i^E s_{it}^E \quad \forall i, t \tag{3}$$

$$\sum_t w_{it}^E \leq K_i^E \quad \forall i \tag{4}$$

$$w_{it}^L \leq K_i^L \quad \forall i, t \tag{5}$$

$$I_{ivt}, y_{ivt}, x_{ijvt}, w_{it}^E, w_{it}^L \geq 0 \quad \text{and} \quad s_{it}^E = \{0, 1\} \quad \forall i, j, v, t. \tag{6}$$

The LR problem can be decomposed into three sub-problems.

- The empty container positioning problem
- The expansion problem
- The leasing storage space problem

### 3.2.1 The empty container positioning problem

The mathematical model

$$\text{Min } Z_{\text{LR1}} = \sum_{i,j\neq i,v,t} c^T_{ijv} x_{ijvt} + \sum_{i,v,t} c^L_{iv} y_{ivt} + \sum_{i,v,t} c'^H_{ivt} I_{ivt}.$$

Subject to

$$I_{ivt} = I_{iv(t-1)} + \sum_{j\neq i, t > T_{ji}} x_{jiv\left(t-T_{ji}\right)} + \sum_{j\neq i, t > T_{ji}+1} D_{jiv\left(t-T_{ji}-1\right)} - \sum_{j\neq i} \left(x_{ijvt} + D_{ijvt}\right) + y_{ivt} \quad \forall i, v, t$$

$$\sum_v TEU_v I_{ivt} \leq K_i + K^E_i + K^L_i \quad \forall i, t.$$

The second constraint can be derived from constraints (2), (4), and (5) of the original problem. This constraint is added to prevent the circumstance that the solution of $Z_{\text{LR1}}$ violates constraint (2) of the original problem. This problem is easily solved by optimization software.

### 3.2.2 The expansion problem

The mathematical model

$$\text{Min } Z_{\text{LR2}} = \sum_{i,t} f^E_{it} s^E_{it} + \sum_{i,t} v'^E_{it} w^E_{it}.$$

Subject to

$$w^E_{it} \leq K^E_i s^E_{it} \quad \forall i, t$$

$$\sum_t w^E_{it} \leq K^E_i \quad \forall i.$$

This problem can be broken down into $i$ problems.

$$\text{Min } Z^i_{\text{LR2}} = \sum_t f^E_{it} s^E_{it} + \sum_t v'^E_{it} w^E_{it}$$

Subject to

$$w^E_{it} \leq K^E_i s^E_{it} \quad \forall t$$

$$\sum_t w^E_{it} \leq K^E_i.$$

Before the procedure for solving these sub-problems is shown, the following lemmas are introduced. From the objective function, the cost to expand the capacity in period $t$ will be as follows:

- zero when $s^E_{it} = 0$
- $f^E_{it} + v'^E_{it} w^E_{it}$ when $s^E_{it} = 1$.

The objective function of the sub-problem is to minimize the total expansion cost. Therefore, it can be easily seen that $s^E_{it} = 0$ if $v'^E_{it} \geq 0$. Moreover, if $w_t = -\frac{f^E_{it}}{v'^E_{it}} \geq K^E_i$

then $s_{it}^E = 0$ since $f_{it}^E + v_{it}'^E w_{it}^E \geq 0 \quad \forall w_{it}^E,\ 0 < w_{it}^E \leq K_i^E$. Hence, in the following lemmas, we only consider the case of $v_{it}'^E < 0$ and $0 < w_t < K_i^E$.

There are several terminologies to be defined before the lemmas are presented. Firstly, the terminology "dominate" is discussed as follows. The following lemmas are used for solving the abovementioned sub-problems. The decision variables are used for determining when the capacity of a depot needs to be extended and what the extended capacity should be. Considering two periods $k$ and $m$, the terminology "$k$ is dominated by $m$" means that the capacity of a certain depot should be extended in period $m$ rather than in period $k$. Secondly, we want to mention "the break-even point of a period" in this study. When $w_{it}^E = w_t$, the expansion cost at period $t$ $f_{it}^E + v_{it}'^E w_{it}^E$ will be zero. Therefore, from now on, it will be called the break-even point (BEP) of period $t$. Finally, the value of $v_{it}'^E$ can be considered as the incremental cost of expansion in period $t$. Hence, $v_{it}'^E$ can be called "the incremental cost of period $t$".

**Lemma 1** *A period is dominated by another period if it has higher BEP and incremental cost. Let k and m be the two periods that can be candidates for expanding capacity. If $w_k < w_m$ and $v_{ik}'^E < v_{im}'^E$, then either m is dominated by k or $s_{im}^E = 0$.*

*Proof* Let $w_{ik}^E + w_{im}^E = K \leq K_i^E$. We will prove that

$$f_{ik}^E s_{ik}^E + v_{ik}'^E w_{ik}^E + f_{im}^E s_{im}^E + v_{im}'^E w_{im}^E \geq f_{ik}^E s_{ik}^E + v_{ik}'^E K.$$

We have either $w_{ik}^E + w_{im}^E = K$ or $w_{ik}^E = K - w_{im}^E$. Substituting $K - w_{im}^E$ for $w_{ik}^E$ in the LHS of the above inequality, we have

$$\begin{aligned} & f_{ik}^E s_{ik}^E + v_{ik}'^E \left(K - w_{im}^E\right) + f_{im}^E s_{im}^E + v_{im}'^E w_{im}^E \\ & \quad = f_{ik}^E s_{ik}^E + v_{ik}'^E K + f_{im}^E s_{im}^E + \left(v_{im}'^E - v_{ik}'^E\right) w_{im}^E \\ & \quad \geq f_{ik}^E s_{ik}^E + v_{ik}'^E K \left(\text{since } f_{im}^E \geq 0 \text{ and } v_{im}'^E > v_{ik}'^E\right) \end{aligned}$$

"=" occurs when $w_{im}^E = 0$, $s_{im}^E = 0$, $w_{ik}^E = K$, and $s_{ik}^E = 1$.

Because the objective function is to be minimized, in the optimal solution, the above inequality will become an equality. This completes the proof of Lemma 1.

**Lemma 2** *Considering the expansion cost with the maximum expanded amount, even though its incremental cost is lower, a period is dominated by another period if its expansion cost is higher. Let k and m be two periods that can be candidates for expanding capacity. If $v_{ik}'^E < v_{im}'^E$ and $f_{ik}^E + v_{ik}'^E K_i^E > f_{im}^E + v_{im}'^E K_i^E$, then either k is dominated by m or $s_{ik}^E = 0$.*

*Proof* Let $w_{ik}^E + w_{im}^E = K \leq K_i^E$. We will prove that

$$f_{ik}^E s_{ik}^E + v_{ik}'^E w_{ik}^E + f_{im}^E s_{im}^E + v_{im}'^E w_{im}^E \geq f_{im}^E s_{im}^E + v_{im}'^E K.$$

Since $f_{ik}^E + v_{ik}'^E K_i^E > f_{im}^E + v_{im}'^E K_i^E$, $\left(f_{ik}^E - f_{im}^E\right) > \left(v_{im}'^E - v_{ik}'^E\right) K_i^E$.

When $w_{ik}^E = w_{im}^E = w \leq K_i^E$ we have

$$\begin{aligned} \left(f_{ik}^E + v_{ik}'^E w\right) - \left(f_{im}^E + v_{im}'^E w\right) &= \left(f_{ik}^E - f_{im}^E\right) - \left(v_{im}'^E - v_{ik}'^E\right) w \\ &> \left(v_{im}'^E - v_{ik}'^E\right) K_i^E - \left(v_{im}'^E - v_{ik}'^E\right) w \geq 0. \end{aligned}$$

Therefore, $\left(f_{ik}^{E} + v_{ik}^{\prime E} w\right) > \left(f_{im}^{E} + v_{im}^{\prime E} w\right) \quad \forall w, w \leq K_{i}^{E}$.

From the first constraint of the sub-problem, it can be seen that $s_{ik}^{E} = 1$ if $w_{ik}^{E} > 0$ and $s_{ik}^{E} = 0$ if $w_{ik}^{E} = 0$. Hence, we can claim the following

$$\left(f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E} w\right) \geq \left(f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} w\right) \quad \forall w, w \leq K_{i}^{E}.$$

We have $w_{ik}^{E} + w_{im}^{E} = K$ or $w_{im}^{E} = K - w_{ik}^{E}$. Substituting $K - w_{ik}^{E}$ for $w_{im}^{E}$ in the LHS of the above inequality, we have

$$\begin{aligned} A &= f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E} w_{ik}^{E} + f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E}\left(K - w_{ik}^{E}\right) \\ &= f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E} w_{ik}^{E} + f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E}\left(K - w_{ik}^{E}\right). \end{aligned}$$

If $w_{ik}^{E} = K$, then $w_{im}^{E} = 0$, $s_{im}^{E} = 0$, and $A = f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E} K > f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} K$.
If $w_{ik}^{E} = 0$, then $w_{im}^{E} = K$, $s_{ik}^{E} = 0$, and $A = f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} K$.
If $0 < w_{ik}^{E} < K$, then $s_{ik}^{E} = s_{im}^{E} = 1$, and $A > f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} w_{ik}^{E} + f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E}\left(K - w_{ik}^{E}\right) > f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} K$.

Therefore, $f_{ik}^{E} s_{ik}^{E} + v_{im}^{\prime E} w_{ik}^{E} + f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} w_{im}^{E} \geq f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} K$.

Furthermore, equality occurs when $w_{ik}^{E} = 0$, $s_{ik}^{E} = 0$, $w_{im}^{E} = K$, and $s_{im}^{E} = 1$.

Because the objective function is to be minimized, in the optimal solution, the above inequality will become an equality. This completes the proof of Lemma 2.

**Lemma 3** *Among the non-dominated candidates, the candidate that has the lowest incremental cost will be chosen for expansion with the maximum expanded amount. Let C be the set of non-dominated candidates.*

*If k is the period whereby* $v_{ik}^{\prime E} = \min\left\{v_{it}^{\prime E} | t \in C\right\}$,
*then* $s_{ik}^{E} = 1$, $w_{ik}^{E} = K_{i}^{E}$ *and* $s_{im}^{E} = 0$, $w_{im}^{E} = 0$, *where* $m \neq k, m \in C$.

*Proof* Recursion will be used to prove this lemma.

When $|C| = 2$, the sub-problems can be shown as follows.

$$\operatorname{Min} Z_{\mathrm{LR2}}^{i} = f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E} w_{ik}^{E} + f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} w_{im}^{E}$$

Subject to

$$w_{ik}^{E} \leq K_{i}^{E} s_{ik}^{E} \quad \text{and} \quad w_{im}^{E} \leq K_{i}^{E} s_{im}^{E}$$
$$w_{ik}^{E} + w_{im}^{E} \leq K_{i}^{E}$$

$k$ and $m$ are two members of $C$ and their values are pre-determined.

We assume that $v_{ik}^{\prime E} < v_{im}^{\prime E}$. Let $w_k$ and $w_m$ ($0 \leq w_k$, $w_m \leq K_i^E$) be solutions to this sub-problem. It can be easily seen that $w_k + w_m = K_i^E$ or $w_m = K_i^E - w_k$.

Substituting $w_{ik}^{E} = w_k$ and $w_{im}^{E} = w_m = K_i^E - w_k$ to the objective function, we have:

$$\begin{aligned} Z_{\mathrm{LR2}}^{i} &= f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E}\left(K_{i}^{E} - w_{m}^{E}\right) + f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} w_{m}^{E} \\ &= f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E} K_{i}^{E} - v_{ik}^{\prime E} w_{m}^{E} + f_{im}^{E} s_{im}^{E} + v_{im}^{\prime E} w_{m}^{E} \\ &\geq f_{ik}^{E} s_{ik}^{E} + v_{ik}^{\prime E} K_{i}^{E} \end{aligned}$$

Thus, $\operatorname{Min} Z^i_{\mathrm{LR2}} = f^E_{ik} s^E_{ik} + v'^E_{ik} K^E_i$ and the solution will be $w^E_{ik} = K^E_i$, $s^E_{ik} = 1$, $w^E_{im} = 0$, and $s^E_{im} = 0$. Therefore, Lemma 3 can be applied to the case, $|C| = 2$.

Suppose that Lemma 3 can be applied to the case, $|C| = n$. We prove that Lemma 3 can be also applied to the case, $|C| = n + 1$.

The sub-problem when $|C| = n + 1$.

$$\operatorname{Min} Z^i_{n+1} = \sum_{t=1}^{n+1} \left( f^E_{it} s^E_{it} + v'^E_{it} w^E_{it} \right)$$

Subject to
$$w^E_{it} \le K^E_i s^E_{it} \quad \forall t = 1, 2\ldots, n+1$$
$$\sum_{t=1}^{n+1} w^E_{it} \le K^E_i.$$

Assume that $v'^E_{ik} = \min\left\{ v'^E_{it} | t \in C \right\}$ and $w_k$ $(0 \le w_k \le K^E_i)$ is the solution to $w^E_{ik}$. When $w^E_{ik} = w_k$, the sub-problem can be rewritten as follows.

$$\operatorname{Min} Z^i_{n+1} = \operatorname{Min} Z(w_k) = \sum_{t=1, t\neq k}^{n+1} \left( f^E_{it} s^E_{it} + v'^E_{it} w^E_{it} \right) + \left( f^E_{ik} s^E_{ik} + v'^E_{ik} w_k \right)$$

Subject to
$$w^E_{it} \le K^E_i s^E_{it} \quad \forall t \neq k, t = 1, 2\ldots, n+1$$
$$\sum_{t=1, t\neq k}^{n+1} w^E_{it} \le K^E_i - w_k.$$

Suppose $v'^E_{il} = \min\left\{ v'^E_{it} | t \in C \backslash \{k\} \right\}$. It can be seen that the sub-problem, $Z(w_k)$, is similar to the sub-problem in the case, $|C| = n$. Hence, the solution to the sub-problem, $Z(w_k)$, will be $s^E_{il} = 1$, $w^E_{il} = K^E_i - w_k$ and $s^E_{im} = 0$, $w^E_{im} = 0$, where $m \neq l, m \in C \backslash \{k\}$. The sub-problem will be equivalent to the following problem.

$$\operatorname{Min} Z^i_{n+1} = \left( f^E_{il} s^E_{il} + v'^E_{il} w^E_{il} \right) + \left( f^E_{ik} s^E_{ik} + v'^E_{ik} w_k \right)$$

Subject to
$$w^E_{ik} \le K^E_i s^E_{ik} \text{ and } w^E_{il} \le K^E_i s^E_{il}$$
$$w^E_{il} = K^E_i - w_k$$

It is noted that $v'^E_{ik} < v'^E_{il}$. We can see that the problem $Z^i_{n+1}$ is similar to the problem in the case, $|C| = 2$. Therefore, the optimal solution will be $s^E_{ik} = 1$, $w^E_i = K^E_i$ and $s^E_{im} = 0$, $w^E_{im} = 0$, where $m \neq k, m \in C$. This means that Lemma 3 can be also applied to the case, $|C| = n + 1$. This completes the proof of Lemma 3.

From the above lemmas, the procedure to solve the sub-problems is described as follows.

Step 0. Initialization

For each period $t$, we calculate $w_t = -\frac{f_{it}^E}{v_{it}'^E}$.

Let $C$ be the set of periods that can be a candidate for expanding capacity.

Set $C = \{t | 0 < w_t \le K_i^E\}$.

Step 1. Consider each pair of periods, $k$ and $m$.

If $w_k < w_m$ and $v_{ik}'^E < v_{im}'^E$, then $C = C \setminus \{m\}$

If $\frac{f_{ik}^E - f_{im}^E}{v_{ik}'^E - v_{im}'^E} > K_i^E$ and $v_{ik}'^E < v_{im}'^E$, then $C = C \setminus \{k\}$

Step 2. Among the candidate periods in the set $C$, we select the period $k$ where

$$v_{ik}'^E = \min\left\{v_{it}'^E | t \in C\right\}.$$

Step 3. Set $s_{ik}^E = 1$, $w_{ik}^E = K_i^E$ and $s_{im}^E = 0$, $w_{im}^E = 0$, $m \neq k$.

### 3.2.3 *The leasing storage space problem*

The mathematical model

$$\text{Min}\, Z_{\text{LR3}} = \sum_{i,t} v_{it}'^L w_{it}^L.$$

Subject to

$$w_{it}^L \le K_i^L \quad \forall\, i, t.$$

This problem can be broken down into $i*t$ problems and solved as follows.

If $v_{it}'^L < 0$, then $w_{it}^L = K_i^L$, otherwise $w_{it}^L = 0$.

The optimal value of LR is a lower bound for the MIP problem. To find an upper bound, a feasible solution should be identified. In this study, we propose to use the solution of $I_{ivt}$, $x_{ijvt}$, and $y_{ivt}$ from the empty container positioning problem that was mentioned above. Then, these solutions are substituted into the original problem to make a new problem. The new problem is called the upper bound problem and is solved by a heuristic method. The upper bound is described as follows.

$$\text{Min}\, Z_{UB} = \sum_{i,t} f_{it}^E s_{it}^E + \sum_{i,t} v_{it}^E w_{it}^E + \sum_{i,t} v_{it}^L w_{it}^L + A$$

$$\text{where } A = \sum_{i,j \neq i,v,t} c_{ijv}^T x_{ijvt} + \sum_{i,v,t} c_{iv}^L y_{ivt} + \sum_{i,v,t} c_{iv}^H I_{ivt}$$

Subject to

$$\sum_{v} TEU_v I_{ivt} \le K_i + \sum_{k \le t} w_{ik}^E + w_{it}^L \quad \forall\, i, t \tag{2}$$

$$w_{it}^E \le K_i^E s_{it}^E \quad \forall\, i, t \tag{3}$$

$$\sum_{t} w_{it}^E \le K_i^E \quad \forall\, i \tag{4}$$

$$w_{it}^L \le K_i^L \quad \forall\, i, t \tag{5}$$

$$w_{it}^E, w_{it}^L \ge 0 \text{ and } s_{it}^E = \{0,\ 1\} \quad \forall\, i, j, v, t. \tag{6'}$$

The upper bound problem can be decomposed into $i$ sub-problems. Each sub-problem is solved by the proposed heuristic algorithm. We consider each period from the beginning to the end of a planning horizon. The expanding depot capacity will be considered when there is a shortage of storage space in that period. Either expanding depot capacity or leasing storage space will be chosen by comparing the expansion cost to the leasing storage space cost. If an expanding depot is selected, the expanded amount will be determined with the lowest cost. Then, the next period will be considered.

Each sub-problem is solved by the procedure shown below.

Step 0. Initialization

- Set $t$, the period, as $t = 1$.
- Set $E$, the total capacity that is expanded from previous periods, as $E = 0$.
- Set $R$, the remaining capacity that is available for expansion, as $R = K_i^E - E$.
- Calculate the lack of inventory in each period:

$$G_k = \mathrm{Max}\left(\sum_v TEU_v I_{ivk} - K_i - E, 0\right) k \geq t.$$

Step 1. If $G_k = 0\ \forall k \geq t,$ then stop; otherwise, go to Step 2.

Step 2. If $G_t = 0,$ then go to Step 7; otherwise, do the following.

- Calculate $G'_k = \min(G_k, R), k \geq t.$
- Calculate $H_k = f_{it}^E + v_{it}^E G'_k + \sum_{u \geq t} \max\left(G_u - G'_k, 0\right) v_{iu}^L.$
- Calculate $L = \sum_{u \geq t} G'_u v_{iu}^L.$

Step 3. If $L \leq H_k\ \forall\ k \geq t$, then

- $w_{it}^L = \min(G_t,\ K_i^L)$
- If $G_t > K_i^L$, then $w_{it}^E = G_t - w_{it}^L$, $s_{it}^E = 1$; otherwise, $s_{it}^E = w_{it}^E = 0$
- Go to Step 6.

Step 4. Define $k*$ so that $H_{k*} = \min(H_k)$, $k \geq t$ ($k*$ is the closest period to $t$).

- Step 4.1.

Define the set $C = \left\{G'_u \middle| G'_u < G_{k*}, t \leq u < k*\right\}$ (if $H_i = H_j$ and $i < j$, then select $i$)

- Step 4.2. Select $H_u \in C$ where $u$ is the biggest value.

If $v_{it}^E\left(G'_{k*} - G'_u\right) > f_{ik*}^E + v_{ik*}^E\left(G'_{k*} - G'_u\right) + \sum_{t \leq n < k*} \max\left[\min\left(G_{k*}, G'_n\right) - G_u, 0\right]$ $v_{in}^L$, then $k* = u$; go back to Step 4.1.

- Step 4.3.

$C = C \backslash \{G_u{}'\}$

If $C = \varnothing$, then go to Step 5; otherwise, go back to Step 4.2.

Step 5. Set $w_{it}^E = G'_{k*}, s_{it}^E = 1.$

Step 6. Calculate

- $E = E + w_{it}^{E}$
- $R = K_i^E - E$
- $G_k = Max\left(\sum_v TEU_v I_{ivk} - K_i - E, 0\right) k \geq t$
- $w_{it}^{L} = G_t - w_{it}^{E}$

Step 7. Set $t = t + 1$. If $t \leq T$, then go back to Step 1; otherwise, stop. The procedure as shown in Fig. 1 is based on Lagrangian relaxation.

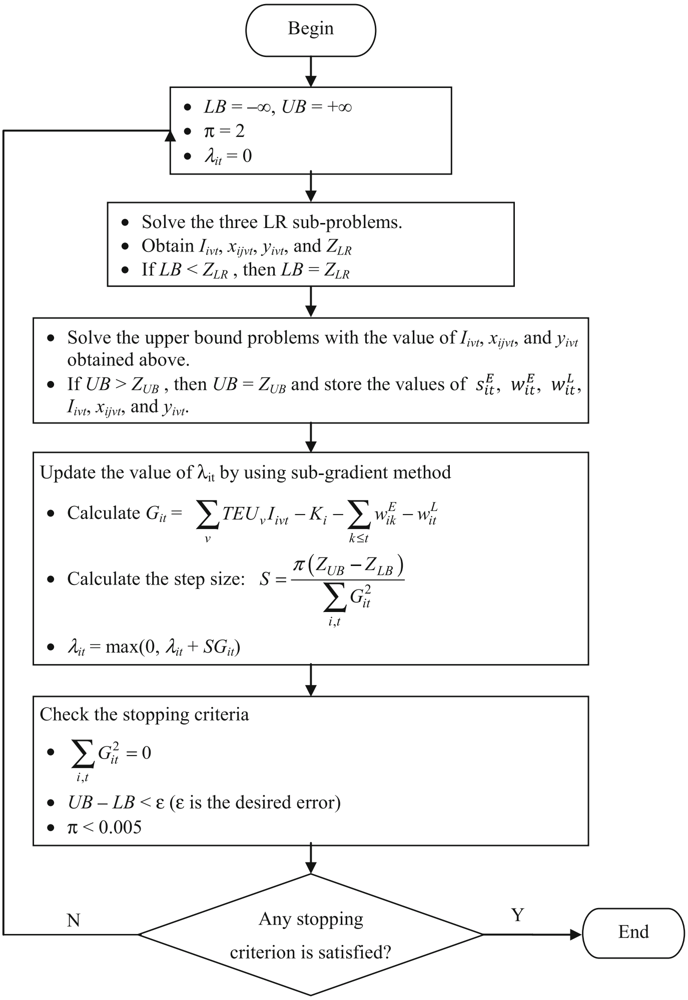


**Fig. 1** Procedure to solve the problem based on Lagrangian relaxation

**Table 1** Result of the pilot experiment

| Problem size | MIP | | LR-based algorithm | | Lower bound |
|---|---|---|---|---|---|
| | Objective value | Comp. time (s) | Objective value | Comp. time (s) | |
| P = 3,C = 4,Pr = 8 | 6,496,070 | 0.3906 | 6,499,637 | 0.1094 | 6,474,361 |
| P = 3,C = 4,Pr = 24 | 12,614,224 | 2.5781 | 12,643,477 | 0.2188 | 12,589,621 |
| P = 5,C = 4,Pr = 30 | 23,652,843 | 5.2031 | 23,795,544 | 0.3125 | 23,636,988 |

## 4 Computational experiments

Initial experiments are conducted to show the performance of the proposed algorithm. Three problems with a range of sizes are generated for the test. All the experiments are conducted on a computer with an Intel Core 2 Quad 2.4 GHz and 3.24 GB of RAM. LINGO is used to solve the empty container positioning problem. The problem size and the result of the pilot experiment are shown in Table 1. *P*, *C*, and *Pr* are denoted as the number of ports (depots), the number of container types, and the number of periods, respectively. When we compare the objective values shown in Table 1, we found out that the gap between the objective value of the proposed incumbent solution and the optimal value does not exceed 0.6%. The gap between the lower bound and the optimal value is less than 0.7%. Moreover, the computational time of the proposed algorithm is faster than that of the MIP. This persuasive evidence shows that the proposed algorithm can solve the problem much faster with an acceptable error (in our opinion, an error of less than 5% is acceptable).

In order to evaluate the performance of the LR-based algorithm, we also randomly generate 30 problems that have different relevant costs, maximum expanded capacities, demands, and other parameters. The problem size varies from small to large and is shown in Tables 2 and 3. LB and UB stand for the lower and upper bounds, respectively. In Table 3, the problem instances 1 through 10 are of small size, 11 through 20 are of medium size, and 21 through 30 are of large size. For each problem, the proposed algorithm is run 10 times. The comparison between the objective values obtained from the LR-based algorithm and the optimal objective values as well as the computation times are presented in Table 4.

In Table 4, the cells containing a "–" symbol indicate that the corresponding problems cannot be solved within a day and no results can be obtained. For each problem instance, the computational time for the LR-based algorithm is the average

**Table 2** Range of the problem size for 30 randomly generated problems

| Problem size | Number of ports | | Number of container types | | Number of periods | |
|---|---|---|---|---|---|---|
| | LB | UB | LB | UB | LB | UB |
| Small | 5 | 10 | 4 | 6 | 30 | 50 |
| Medium | 11 | 20 | 6 | 8 | 50 | 75 |
| Large | 21 | 30 | 8 | 10 | 75 | 100 |

**Table 3** Problem sizes of 30 randomly generated problems

| Problem instance | Problem size | | | Number of variables | | | Number of constraints |
|---|---|---|---|---|---|---|---|
| | Number of ports | Number of container types | Number of periods | Integer variables | Continuous variables | Total variables | |
| 1 | 6 | 4 | 40 | 240 | 7,728 | 7,968 | 1,038 |
| 2 | 6 | 5 | 41 | 246 | 9,900 | 10,146 | 1,326 |
| 3 | 8 | 6 | 31 | 248 | 14,976 | 15,224 | 1,640 |
| 4 | 5 | 6 | 31 | 155 | 6,570 | 6,725 | 1,025 |
| 5 | 9 | 5 | 47 | 423 | 23,355 | 23,778 | 2,259 |
| 6 | 6 | 6 | 32 | 192 | 9,288 | 9,480 | 1,266 |
| 7 | 8 | 4 | 32 | 256 | 10,304 | 10,560 | 1,128 |
| 8 | 10 | 6 | 32 | 320 | 23,160 | 23,480 | 2,110 |
| 9 | 6 | 6 | 40 | 240 | 11,592 | 11,832 | 1,554 |
| 10 | 7 | 6 | 44 | 308 | 16,716 | 17,024 | 1,981 |
| 11 | 14 | 7 | 66 | 924 | 103,684 | 104,608 | 6,776 |
| 12 | 13 | 6 | 68 | 884 | 79,716 | 80,600 | 5,551 |
| 13 | 17 | 8 | 70 | 1,190 | 181,152 | 182,342 | 9,945 |
| 14 | 15 | 8 | 72 | 1,080 | 147,120 | 148,200 | 9,015 |
| 15 | 20 | 8 | 61 | 1,220 | 215,040 | 216,260 | 10,260 |
| 16 | 12 | 8 | 73 | 876 | 98,304 | 99,180 | 7,308 |
| 17 | 14 | 6 | 66 | 924 | 88,872 | 89,796 | 5,810 |
| 18 | 11 | 8 | 63 | 693 | 72,248 | 72,941 | 5,819 |
| 19 | 14 | 8 | 70 | 980 | 125,664 | 126,644 | 8,190 |
| 20 | 20 | 6 | 71 | 1,420 | 187,680 | 189,100 | 8,900 |
| 21 | 29 | 9 | 81 | 2,349 | 655,893 | 658,242 | 21,953 |
| 22 | 24 | 9 | 82 | 1,968 | 460,944 | 462,912 | 18,384 |
| 23 | 30 | 10 | 87 | 2,610 | 835,800 | 838,410 | 27,030 |
| 24 | 27 | 10 | 99 | 2,673 | 775,710 | 778,383 | 27,567 |
| 25 | 23 | 8 | 81 | 1,863 | 372,968 | 374,831 | 15,479 |
| 26 | 21 | 8 | 86 | 1,806 | 332,640 | 334,446 | 14,973 |
| 27 | 26 | 10 | 96 | 2,496 | 699,400 | 701,896 | 25,766 |
| 28 | 27 | 9 | 82 | 2,214 | 578,340 | 580,554 | 20,682 |
| 29 | 26 | 10 | 90 | 2,340 | 655,720 | 658,060 | 24,206 |
| 30 | 26 | 8 | 98 | 2,548 | 571,168 | 573,716 | 21,034 |

computational time over ten runs. Similarly, the percentage penalty for the LR-based algorithm is calculated using the average objective value of ten runs that is shown in Table 4. The percentage penalty is computed using the following formula:

$$\frac{\text{average objective value of LR}-\text{based} - \text{objective value of MIP}}{\text{objective value of MIP}} \times 100.$$

In problems where we cannot obtain the optimal solution, the percentage penalty is calculated by using the lower bound. It can be seen that the LR-based algorithm can solve large-sized problems very quickly. Moreover, the percentage difference

**Table 4** Comparison between MIP and the LR-based algorithm for 30 problems

| Problem instance | MIP | | LR-based algorithm | | | % Penalty |
|---|---|---|---|---|---|---|
| | Objective value | Comp. time (s) | Objective value | Lower bound | Comp. time (s) | |
| 1 | 43,194,933 | 0.88 | 43,218,808 | 43,154,601 | 0.45 | 0.06 |
| 2 | 52,856,104 | 1.05 | 53,063,513 | 52,856,104 | 0.53 | 0.39 |
| 3 | 96,082,710 | 1.41 | 97,380,452 | 96,082,710 | 0.76 | 1.35 |
| 4 | 40,845,607 | 0.66 | 41,033,128 | 40,845,607 | 0.38 | 0.46 |
| 5 | 144,904,752 | 2.52 | 145,457,779 | 144,904,752 | 1.14 | 0.38 |
| 6 | 37,223,996 | 0.89 | 38,513,462 | 37,223,996 | 0.50 | 3.46 |
| 7 | 70,857,347 | 1.02 | 71,628,511 | 70,857,347 | 0.52 | 1.09 |
| 8 | 169,171,301 | 2.31 | 169,520,179 | 169,171,301 | 1.20 | 0.21 |
| 9 | 48,420,821 | 1.13 | 49,036,631 | 48,420,821 | 0.61 | 1.27 |
| 10 | 87,530,512 | 1.69 | 87,538,216 | 87,530,512 | 0.84 | 0.01 |
| 11 | 442,382,553 | 20.28 | 443,018,610 | 442,382,553 | 6.10 | 0.14 |
| 12 | 281,808,360 | 13.47 | 283,083,705 | 281,808,360 | 4.45 | 0.45 |
| 13 | 756,934,208 | 52.11 | 757,421,433 | 756,934,208 | 11.62 | 0.06 |
| 14 | 508,866,914 | 36.53 | 516,318,625 | 508,866,914 | 9.16 | 1.46 |
| 15 | 1,011,116,279 | 63.75 | 1,018,409,781 | 1,011,116,279 | 15.00 | 0.72 |
| 16 | 404,814,572 | 22.05 | 405,503,520 | 404,814,572 | 5.86 | 0.17 |
| 17 | 361,452,650 | 16.23 | 362,556,652 | 361,452,650 | 5.08 | 0.31 |
| 18 | 315,560,152 | 13.25 | 315,726,649 | 315,560,152 | 4.31 | 0.05 |
| 19 | 454,018,478 | 30.89 | 454,018,478 | 454,018,478 | 8.41 | 0.00 |
| 20 | 765,818,852 | 56.42 | 766,254,414 | 765,818,852 | 13.33 | 0.06 |
| 21 | 2,371,634,377 | 522.80 | 2,372,819,917 | 2,371,634,377 | 67.23 | 0.05 |
| 22 | 1,685,537,098 | 423.77 | 1,689,039,500 | 1,685,537,098 | 42.35 | 0.21 |
| 23 | – | – | 3,016,387,571 | 3,015,983,049 | 97.33 | 0.01 |
| 24 | – | – | 2,478,193,528 | 2,475,390,529 | 87.03 | 0.11 |
| 25 | 1,486,644,563 | 296.28 | 1,488,192,517 | 1,486,644,563 | 28.18 | 0.10 |
| 26 | 1,217,690,558 | 245.22 | 1,218,345,115 | 1,217,690,558 | 25.56 | 0.05 |
| 27 | – | – | 2,314,683,724 | 2,314,178,688 | 76.98 | 0.02 |
| 28 | 1,972,930,283 | 493.63 | 2,002,020,808 | 1,972,930,283 | 58.35 | 1.47 |
| 29 | 2,203,155,300 | 586.45 | 2,211,802,376 | 2,203,155,300 | 67.99 | 0.39 |
| 30 | 1,753,921,241 | 674.00 | 1,755,812,518 | 1,753,921,241 | 57.62 | 0.11 |

between the objective value obtained from the LR-based algorithm and the optimal objective value does not exceed 4%. These features indicate the good performance of the proposed algorithm.

## 5 Conclusion

We have studied the capacity expansion problem for container terminals or depots. A mathematical model is built to minimize the total relevant cost for multiple-ports, multiple-commodities, and multiple-periods. A solution algorithm based on Lagrangian relaxation is proposed. Some pilot experiments are conducted; they also indicate that the proposed algorithm contributes effectively to solving the problem. This study can be used to support decision making between short-term and long-term planning horizons. The experiments are conducted for up to 30 ports, 10 container types, and 100 periods. The experiments demonstrated that the proposed algorithm can solve the problem very quickly (in approximately 1 min) with an error of less than 5%. These are persuasive indications of the effectiveness of the proposed algorithm. Extending the current models by accommodating vessel capacities and vessel routes might be an interesting research problem.

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