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Viewpoint

Comment on Bose S, Goswami A and Chaudhuri KS (1995). An EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting

In an article published in *JORS*, Bose *et al*[1] considered the inventory replenishment problem for a deteriorating item with linear (positive) trend in demand, finite shortage cost and equal replenishment intervals over a fixed planning horizon under inflation and time discounting. They provided the procedure of finding the optimal solution of the problem and also studied the sensitivity of the decision variables to changes in the parameter values of the model. Unfortunately, their model contains mathematical errors in the formulation of the holding cost and the purchase cost which lead to incorrect total cost function over the fixed planning horizon. In this Viewpoint, we point out the errors and present the appropriate theory for the problem. Here we use the following assumptions and notation, most of which are similar to Bose *et al*:[1]

(a) H is taken to be the fixed time horizon.
(b) n orders are placed during the time horizon H and the replenishment rate is infinite, ie replenishment is instantaneous.
(c) The demand $f(t)$ at time t is a continuous function of time.
(d) A constant fraction $\theta (0 < \theta < 1)$ of on-hand inventory deteriorates per unit of time.
(e) r is the discount rate representing the time value of money.
(f) At time $t = 0$, c_{11} and c_{12} are internal and external holding costs per unit item per unit time; c_{21} and c_{22} are internal and external shortage costs per unit item per unit time.
(g) At time $t = 0$, A is the fixed internal ordering cost per order and p is the external purchase cost.
(h) Internal and external inflation rates are denoted by i_1 and i_2 respectively.
(i) C_o, C_h, C_s and C_p are the present worth of total replenishment cost, total inventory holding cost, total shortage cost and total purchase cost during the fixed time horizon H, respectively.
(j) K is the fraction of the replenishment interval for which there is no shortage and K is same for each replenishment cycle.
(k) $t_j = (j-1)H/n$ is the time of the jth replenishment, $j = 1, 2, \ldots, n$.
(l) $s_j = t_j + KH/n = (K + j - 1)H/n$ is the time at which the inventory level in the jth replenishment cycle drops to zero, $j = 1, 2, \ldots, n-1$ and $s_n = H$.
(m) $I_1(t)$ is the inventory level at any time t in $[t_j, s_j]$, $j = 1, 2, \ldots, n$.
(n) $I_2(t)$ is the shortage level at any time t in $[s_j, t_{j+1}]$, $j = 1, 2, \ldots, n-1$.
(o) Shortages are allowed and are fully backlogged.
(p) Shortages are not allowed in the last replenishment cycle.

A typical variation of the inventory level with time for increasing demand pattern are shown in Figure 1.

During the period of positive inventory, the differential equation describing the instantaneous state of $I_1(t)$ is given by

$$\frac{dI_1(t)}{dt} = -f(t) - \theta I_1(t) \quad t_j \leqslant t \leqslant s_j, \; j = 1, 2, \quad , n \quad (1)$$

with the boundary condition $I_1(s_j) = 0$.

During the period of shortage, the differential equation governing the system is given by

$$\frac{dI_2(t)}{dt} = f(t) \quad s_j \leqslant t \leqslant t_{j+1}, \; j = 1, 2, \quad , n - \quad (2)$$

with the initial condition $I_2(s_j) = 0$.

The solutions of the differential equations (1) and (2) are given respectively by

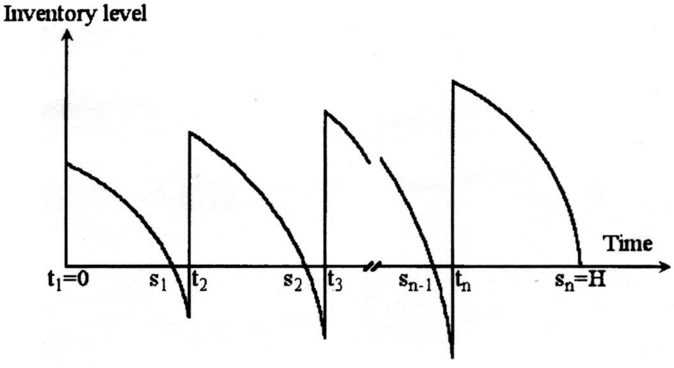

Figure Inventory level variation with time.

$$I_1(t) = \int^{s_j} \qquad ,2, \quad ,n \quad (3)$$

$$I_2(t) = \int_{s_j}^{t} f(u)\mathrm{d}u \quad s_j \leqslant t \leqslant t_{j+1}, \; j = 1, 2, . \quad , n-1 \quad (4)$$

The total cost during the fixed time horizon consists of replenishment cost, inventory holding cost, shortage cost and purchase cost.

Bose *et al*[1] considered the present worth of the holding cost over the period $[(j-1)T, jT]$, $(j = 1, 2, \ldots, n-1)$ as

$$H_j = I_{j1} + I_{j2}$$

$$I_{jm} \quad c_{1m} \int_{(j-1)T}^{(K+j-1)T} [t-(j-1)T](a+bt)\mathrm{e}^{-R_m t}\mathrm{e}^{\theta t}\,\mathrm{d}t$$

$$= c_{1m} \int_{t_j}^{s_j} \left[\int_{t}^{s_j} \mathrm{e}^{(\theta - R_m)u} f(u)\mathrm{d}u \right]$$

$$t_j = (j-1)H/n \quad s_j = (K+j-1)H/n \quad T = H/n$$

$$f(u) = a + bu \quad R_m = r - i_m \; (m = 1, 2)$$

If there is no inflation and time discounting, ie when $R_m = 0$, we find from above that the inventory level at any time t in $[t_j, s_j]$ as

$$I_1(t) = \int_{t}^{s_j} \mathrm{e}^{\theta u} f(u)\mathrm{d}u$$

This implies that the instantaneous state of $I_1(t)$ is governed by the differential equation

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} \quad -\mathrm{e}^{\theta t} f(t)$$

which is clearly wrong. Hence the expression giving the holding cost in Bose *et al*'s[1] model is erroneous.

Again, Bose *et al*[1] considered the present worth of the purchase cost for purchasing at time $(j-1)T$ for the period $[(j-1)T, (K+j-1)T]$ as

$$(P_{j-1})_1 = p\mathrm{e}^{-R_2(j-1)T} \int_{(j-1)T}^{(K+j-1)T} (a+bt)\mathrm{e}^{\theta t}\,\mathrm{d}t$$

which can be written as

$$(P_{j-1})_1 \quad p\mathrm{e}^{-R_2 t_j} \int_{t_j}^{s_j}$$

This gives

$$I_1(t_j) \quad \int^{s_j} \mathrm{e}^{\theta t} f(t)\mathrm{d}t$$

implying

$$I_1(t) \quad \int^{s_j} \mathrm{e}^{\theta u} f(u)\mathrm{d}u$$

which is wrong. Hence both the expressions for the holding cost and the purchase cost in Bose *et al*'s[1] model are incorrect.

The correct expression for the present worth of the total inventory holding cost during H will be

$$C_h = \sum_{j=1}^{n} \sum_{m=1}^{2} c_{1m} \int^{s_j} I_1(t)\mathrm{e}^{-(r-i_m)t}\,\mathrm{d}t$$

$$\sum_{j=1}^{n} \sum_{m=1}^{2} c_{1m} \int_{t_j}^{s_j} \left[\int_{t}^{s_j} \mathrm{e}^{\theta(u-t)} f(u)\mathrm{d}u \right] \mathrm{e}^{-R_m t}\,\mathrm{d}t$$

$$\sum_{j=1}^{n} \sum_{m=1}^{2} \frac{c_{1m}}{\theta + R_m} \int_{t_j}^{s_j} (\mathrm{e}^{\theta(t-t_j) - R_m t_j} - \mathrm{e}^{-R_m t}) f(t)\mathrm{d}t \quad (5)$$

where

$$t_j = (j-1)H/n \quad s_j = (K+j-1)H/n \quad R_m = r - i_m \; (m = 1, 2)$$

The correct expression for the present worth of the total purchase cost during H will be

$$C_p = p\left[\sum_{j=1}^{n} I_1(t_j)\mathrm{e}^{-R_2 t_j} + \sum_{j=1}^{n-1} I_2(t_{j+1})\mathrm{e}^{-R_2 t_{j+1}} \right]$$

$$= p\left[\sum_{j=1}^{n} \mathrm{e}^{-R_2 t_j} \int_{t_j}^{s_j} \mathrm{e}^{\theta(t-t_j)} f(t)\mathrm{d}t + \sum_{j=1}^{n-1} \mathrm{e}^{-R_2 t_{j+1}} \int_{s_j}^{t_{j+1}} f(t)\mathrm{d}t \right] \quad (6)$$

where

$$R_2 = r - i_2$$

However the expressions for C_o, the present worth of the total replenishment cost during H and C_s, the present worth of the total shortage cost during H given in Bose *et al*'s[1] model are correct which are respectively

$$C_o = \frac{A(1 - e^{-R_1 H})}{1 - e^{-R_1 H/n}} \quad \text{where } R_1 = r - i_1 \tag{7}$$

$$C_s = \sum_{j=1}^{n-1} \sum_{m=1}^{2} \frac{c_{2m}}{R_m} \int_{s_j}^{t_{j+1}} (e^{-R_m t} - e^{-R_m t_{j+1}}) f(t) dt \tag{8}$$

Hence, the present worth of the total variable cost of the system during the entire time period H is given by

$$\begin{aligned} TC(n, K) &= C_o + C_h + C_s + C_p \\ &= \frac{A(1 - e^{-R_1 H})}{1 - e^{-R_1 H/n}} \\ &\quad + \sum_{j=1}^{n} \sum_{m=1}^{2} \frac{c_{1m}}{\theta + R_m} \int_{t_j}^{s_j} (e^{\theta(t-t_j) - R_m t_j} - e^{-R_m t}) f(t) dt \\ &\quad + \sum_{j=1}^{n-1} \sum_{m=1}^{2} \frac{c_{2m}}{R_m} \int_{s_j}^{t_{j+1}} (e^{-R_m t} - e^{-R_m t_{j+1}}) f(t) dt \\ &\quad + p \left[\sum_{j=1}^{n} e^{-R_2 t_j} \int_{t_j}^{s_j} e^{\theta(t-t_j)} f(t) dt \right. \\ &\qquad \left. + \sum_{j=1}^{n-1} e^{-R_2 t_{j+1}} \int_{s_j}^{t_{j+1}} f(t) dt \right] \end{aligned} \tag{9}$$

where

$$t_j = (j-1)H/n \quad s_j = (K + j - 1)H/n \quad R_m = r - i_m \ (m = 1, 2)$$

Our objective is to determine the optimal values of n and K that minimize the total variable cost $TC(n,K)$.

Since $TC(n,K)$ is a function of two variables n and K where K $(0 < K < 1)$ is a continuous variable and n is a discrete variable, therefore, for a given value of n, the necessary condition for $TC(n, K)$ to be minimum is

$$\frac{\partial TC(n, K)}{\partial K} = 0$$

which gives

$$\begin{aligned} G(n, K) \equiv & \sum_{j=1}^{n-1} \sum_{m=1}^{2} \frac{c_{1m}}{\theta + R_m} \left[e^{\frac{H}{n} K\theta - R_m (j-1)\frac{H}{n}} - e^{-R_m \frac{H}{n}(K+j-1)} \right] \\ & - \sum_{j=1}^{n-1} \sum_{m=1}^{2} \frac{c_{2m}}{R_m} \left[e^{-R_m \frac{H}{n}(K+j-1)} - e^{-R_m \frac{jH}{n}} \right] \\ & + p \sum_{j=1}^{n-1} \left[e^{-R_2 (j-1)\frac{H}{n} + \frac{H}{n} K\theta} - e^{-R_2 \frac{jH}{n}} \right] = 0 \end{aligned} \tag{10}$$

Note that $G(n,1)$ is positive. Hence for a given value of n, $G(n, 0)$ should be negative in order to guarantee the existence of a solution to Equation (10). The condition for $G(n, 0)$ to be negative gives

$$\left(p - \frac{c_{22}}{R_2}\right)\left(1 - e^{-R_2 \frac{n-1}{n} H}\right) < \frac{c_{21}}{R_1}\left(1 - e^{-R_1 \frac{n-1}{n} H}\right) \quad n \geqslant 2 \tag{11}$$

If the condition (11) is satisfied then there exists a solution $K^* \in (0, 1)$ which can be obtained from Equation (10) by applying the line search technique.

In particular, when $p < \frac{c_{22}}{R_2}$, condition (11) is always satisfied and the possibility of satisfying $p < \frac{c_{22}}{R_2}$ increases when R_2 tends to zero. Further we observe that the optimal value of K does not depend on the demand function parameters, in contrast to the value obtained by the solution procedure of Bose *et al.*[1]

The sufficient condition for the minimization of $TC(n, K)$ is also satisfied as we find that

$$\begin{aligned} \left. \frac{\partial^2 TC(n, K)}{\partial K^2} \right|_{K=K^*} = \frac{H^2}{n^2} f(s_j) \Bigg\{ & \sum_{j=1}^{n-1} \sum_{m=1}^{2} \frac{c_{1m}}{\theta + R_m} \\ & \times \left[\theta e^{\frac{H}{n} K^* \theta - R_m (j-1)\frac{H}{n}} + R_m e^{-R_m \frac{H}{n}(K^* + j - 1)}\right] \\ & + \sum_{j=1}^{n-1} \sum_{m=1}^{2} c_{2m} e^{-R_m \frac{H}{n}(K^* + j - 1)} \\ & + p\theta \sum_{j=1}^{n-1} e^{-R_2 (j-1)\frac{H}{n} + \frac{H}{n} K^* \theta} \Bigg\} > 0 \end{aligned}$$

where

$$s_j = \frac{H}{n}(K^* + j - 1)$$

Here we consider the same numerical example of Bose *et al*[1] in order to make a clear-cut comparison. The parameter values which are considered in their paper are as follows:

$$\begin{aligned} & f(t) = a + bt, \quad a = 200, \quad b = 50, \quad p = 5, \\ & c_{11} = 0.2, \quad c_{12} = 0.4, \quad c_{21} = 0.8, \quad c_{22} = 0.6, \\ & A = 80, \quad r = 0.2, \quad i_1 = 0.08, \quad i_2 = 0.14, \\ & H = 10, \quad \theta = 0.01 \end{aligned}$$

Using these parameter values, we get the optimal solution of the corrected model as $n^* = 13$, $K^* = 0.497\,381$ and $TC(n^*, K^*) = 17\,219.14$. Table 1 shows a comparison of

Table 1 Comparison of the optimal results

Model	n^*	K^*	TC^*	*Percentage of savings*
Bose *et al*[1]		0.288 911	17 648.09	
Corrected model		0.497 381	17 219.14	2.43%

the optimal results between Bose *et al*'s[1] model and the corrected model.

If we put the optimum values of n and K of Bose *et al*'s[1] model into the total cost function of the corrected model, we get $TC(n^*, K^*) = 17\,297.95$, which is 0.46% higher than the original cost.

Reference

1 Bose S, Goswami A and Chaudhuri KS (1995). An EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting. *J Opl Res Soc* **46**: 771–782.

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