CHAPTER

9

Semi-analytical Array Models

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9.1 INTRODUCTION

Semi-analytical techniques were utilized in the first approaches to the problem of determining wave energy converter (WEC) array interactions. Since then, several periods of intense research activity have resulted in a range of such methods, suited to different situations and with different computational characteristics. This chapter attempts to compile into a single narrative the key developments in the area to date. Of course, it will not be possible to include all of the interesting and useful results that have been derived along the way, less still to describe an end point to this active area of research. However, after reading this chapter, the reader should have an appreciation of the main techniques available in this area and know where to look to find out more.

The motivation for the kind of approach described here stems from the fact that it is clearly attractive to be able to write down explicitly the solution to any given mathematical problem. This applies in particular to the problem of determining the hydrodynamic interaction effects in an array of WECs, although as may be seen by examining the numerical approaches described in this book, most sets of assumptions do not permit such a representation. However, with the starting point of linear wave theory, it will be shown that there are certain 'semi-analytical' techniques that are analytical in that the solution can be written down explicitly in terms of mathematical formulae, albeit requiring further approximation or the truncation of an infinite series to allow computation in practice. Nevertheless, such is the convergence of these series that often only a few terms are needed, meaning that semi-analytical methods can allow efficient calculation of array interactions.

At the time of the first developments in semi-analytical techniques as applied to wave energy (Budal, 1977; Evans, 1979), linear wave theory was widely used in WEC analysis. However, boundary element methods (BEMs) had not yet been developed into the form commonly used today and would not in any case have been practical for array analysis given the computational resources available. Although commercial BEMs may now be used to calculate array interactions on a standard desktop or laptop computer, the computational burden quickly becomes too great as the number of devices increases. Semi-analytical techniques, which are capable of much greater efficiency, are therefore still an important part of the landscape of array modelling tools. They also have the advantages of allowing relatively straightforward translation into computer code, providing understanding and insight into the problem for the user and allowing remarkable

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flexibility for modification, extension and coupling with other numerical techniques.

This chapter begins in Section 9.2 with a review of some general concepts relevant to many of the methods presented here. This includes the key notion of 'partial waves', which allow a decomposition of the fluid flow in manner similar to a Fourier series. Following this, four semi-analytical techniques are presented:

- the point absorber method
- the plane wave method
- the multiple scattering method
- the direct matrix method

Each approach is accompanied by a description of the context in which it has been developed and used, its formulation and a commentary on its main features.

In Section 9.3, the point absorber method is described, which considers array interactions to involve only the waves radiated by each device. This approximation is best suited to widely spaced arrays, where waves scattered by each device are small relative to the radiated waves. Further simplifying assumptions allow the maximum total power absorbed to be easily derived for any incident wave frequency.

Section 9.4 contains a description of the plane wave method, whereby scattered and radiated diverging waves emanating from each device are approximated by plane waves upon reaching others in the array. This also assumes wide spacing in the array layout.

The multiple scattering technique is detailed in Section 9.5, which attempts to form an 'exact' solution (with respect to linear wave theory) by considering a set of sequential interaction events at each device, the solution being reached via iteration.

Finally, Section 9.6 describes the direct matrix method, which solves the problem in largely the same way as the multiple scattering method, except that boundary conditions representing device interactions are applied to the entire wave field incident to each device simultaneously. This leads to a single matrix equation for the unknown hydrodynamic solution which can be directly inverted.

Following the presentation of the main methods, their capabilities and limitations are summarized and compared in Section 9.7, including a description of relevant verification and validation activities that have taken place. Finally, a summary concludes the chapter in Section 9.8.

9.2 GENERAL FORMULATION

A number of assumptions, concepts and derivations are common to more than one of the formulations described in this chapter. These are described in this section, along with (as far as is practical) a common notational system and set of conventions. Further details of the theoretical development may be found in Child (2011). It should be assumed that the conventions contained within this section apply to all of the subsequent derivations, except where explicitly stated otherwise.

9.2.1 Mathematical Model

Before any interaction theory may be applied, a mathematical model of the wave energy devices and their environment is required. The array is considered to be an arrangement of N devices deployed at sea, with predefined mean locations relative to one another. No other objects that could modify the wave field, no currents and no sources or sinks of energy apart from the devices themselves are assumed to exist in the vicinity of the array. The water is assumed to be of constant depth. Finite depth will be assumed for the purposes of the derivations unless specified at the beginning of the relevant section.

A global Cartesian coordinate system is defined, with the z-axis pointing vertically upwards. For derivations involving finite depth,

$z = 0$ is considered to be at the seabed. Furthermore, the wave heading angle β is taken to be the direction in which the waves are propagating, measured anti-clockwise from the positive x-axis. See Fig. 9.1 for a diagram illustrating the main variable definitions for the problem.

In addition to the global coordinate system, local cylindrical polar coordinate systems relative to each device j are defined (r_j, θ_j, z) with the origin at a predefined reference point on the device, sharing the z-coordinate with the global coordinate system. The global Cartesian coordinates of the origin of device j are given by (x_j, y_j). For convenience, L_{ij} is defined to be the distance between origins of devices i and j, α_{ij} the angle at device i between the positive x-axis and line joining the origin of i to that of j in the given angular measurement convention.

A notional cylinder surrounding device j, centred at its origin and tangent to it at its largest radius a_j, is also defined (if the device is axisymmetric, then the centre of the coordinate system is on its axis of symmetry with a_j its greatest radius). The domain $0 \leq r_j \leq a_j$ will be referred to as the *interior* region and that for which $r_j \geq a_j$ as the *exterior* region to the device.

Although much of the existing literature concentrates on application to single body oscillators with one degree of freedom, there is no reason why the theory cannot be extended to devices consisting of multiple bodies. This would involve diffraction characteristics being derived for the collection of bodies that constitute a single device and radiation characteristics being derived for all modes of motion permitted by it. Therefore, the term *device* is used here to denote the body or collection of bodies involved in the absorption of wave energy at a particular array location.

The standard assumptions of linear wave theory are used in all of the theories contained in this chapter. That is to say, the fluid is assumed incompressible and inviscid; the flow is irrotational; forces due to surface tension are small; and the resulting waves are of small amplitude (see Chapter 2). Moreover, a frequency-domain

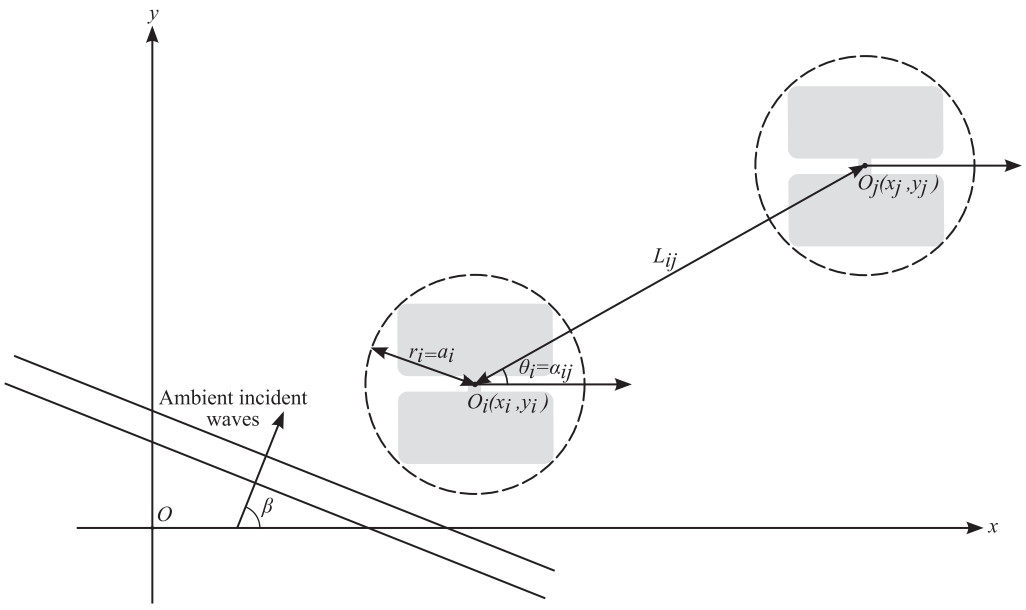

FIG. 9.1 Array definition sketch.

representation is assumed for the purposes of deriving the hydrodynamic properties of an array. Thus, there exists a time-dependent velocity potential Φ describing the flow, which may be written in the form:

$$\Phi(x,y,z,t) = \mathrm{Re}\{\phi(x,y,z)\mathrm{e}^{-\mathrm{i}\omega t}\} \tag{9.1}$$

and all other quantities dependent on time are expected to be time-harmonic, with the same angular frequency, ω (wave number k_0).

Although the hydrodynamics are derived here in the frequency domain, the equations of motion may be solved in the frequency, spectral or time domain. If they are to be solved in the frequency or spectral domain, the methods described in Chapters 2 and 4, respectively, may be followed either separately to or alongside solution of the hydrodynamics. However, in the case of a time-domain approach, the hydrodynamic problem for the excitation forces and added mass and damping matrices (incorporating array interaction effects), must be first solved. Only then may the time-domain solution of the equations of motion proceed as described in Chapter 3.

9.2.2 Partial Wave Representation of Velocity Potentials

This section provides a framework for the representation of the velocity potential that is used in the subsequent semi-analytical array models. In general the wave field can be constructed using the following key components:

- an ambient incident wave potential
- a scattered wave potential relative to each device
- a radiation wave potential relative to each device

Each of these components can be represented using partial waves, a process which is detailed in the following subsections. Prior to this, it is convenient to review the governing equations for the velocity potential.

9.2.2.1 *Governing Equations*

As a result of the incompressibility condition, the time-independent velocity potential ϕ, satisfies Laplace's equation within the fluid, subject to Neumann boundary conditions on the seabed and to the free surface condition at $z = h$:

$$\nabla^2\phi = 0 \text{ throughout the fluid} \tag{9.2}$$

$$\frac{\partial\phi}{\partial z} = 0 \text{ on } z = 0 \tag{9.3}$$

$$\frac{\partial\phi}{\partial z} = \frac{\omega^2}{g}\phi \text{ on } z = h,\ \ r_j \geq a_j,\ \ j = 1,\ldots,N \tag{9.4}$$

Furthermore, scattered and radiated potentials are subject to the Sommerfeld radiation condition (Martin, 2006, (1.25)), which states that all such waves must propagate away from the originating bodies towards infinity:

$$\lim_{k_0 r_j \to \infty} \sqrt{r_j}\left(\frac{\partial\phi}{\partial r_j} - \mathrm{i}k_0\phi\right) = 0,\ \ j = 1,\ldots,N \tag{9.5}$$

The boundary conditions on the wetted surfaces of the device are not considered here since they form part of the isolated device problem which may be treated separately from the array interaction problem.

9.2.2.2 *Ambient Incident Wave Potential*

A plane progressive wave of amplitude η can be represented by the free surface elevation

$$\zeta(x^{\mathrm{A}}, y^{\mathrm{A}}, t) = \mathrm{Re}\left\{\eta\mathrm{e}^{\mathrm{i}\left(k_0 x^{\mathrm{A}} - \omega t + \psi\right)}\right\} \tag{9.6}$$

where x^{A} and y^{A} are coordinates parallel and perpendicular to the wave direction respectively and ψ is some phase shift. If the corresponding velocity potential is assumed to comprise the product of separate functions of x^{A}, y^{A} and z, Laplace's equation (9.2) yields independent partial differential equations for each function. Upon solution of these, the dynamic boundary condition $\partial\Phi/\partial t = -g\zeta$ on $z = h$ may be applied to define the scaling of the reconstructed velocity potential. More generally, if the ambient

incident wave makes an angle β with the positive x-axis, the associated flow may be represented by the following potential, expressed in polar coordinates relative to device j:

$$\phi_j^{\mathrm{A}} = \frac{g\eta}{\omega}\frac{\cosh k_0 z}{\cosh k_0 h} I_j \mathrm{e}^{\mathrm{i}k_0 r_j \cos(\theta_j - \beta)} \tag{9.7}$$

where

$$I_j = \mathrm{e}^{\mathrm{i}k_0(x_j \cos\beta + y_j \sin\beta)} \tag{9.8}$$

is a factor representing the change in phase between the global origin and origin of device j and ω is related to the wave number by the dispersion relation ($\omega^2 = k_0 g \tanh k_0 h$ for finite depth water). Using an identity (Abramowitz and Stegun, 1964, (9.1.41)), this can be written as

$$\phi_j^{\mathrm{A}} = \frac{g\eta}{\omega}\frac{\cosh k_0 z}{\cosh k_0 h} I_j \sum_{n=-\infty}^{\infty} J_n(k_0 r_j)\mathrm{e}^{\mathrm{i}n\left(\frac{\pi}{2}+\theta_j-\beta\right)} \tag{9.9}$$

where J_n is the Bessel function of the first kind (order n). For convenience, this expression is separated into the scalar product of a vector of coefficients, $\mathbf{a}_j$, and a vector containing a set of spatial functions, $\boldsymbol{\Psi}_j^{\mathrm{I}}$ or 'partial waves' incident to device j (Kagemoto and Yue, 1986). In fact, all possible waves incident to j may be represented in this way, so evanescent modes with wave numbers k_m ($m = 1, 2, \ldots$) given by positive real roots of $\omega^2 = -k_m g \tan k_m h$ for finite depth water are also included here for completeness. Let $\boldsymbol{\Psi}_j^{\mathrm{I}}(r_j, \theta_j, z)$ be such a vector of partial waves where the element corresponding to z-mode m and θ-mode n is given by

$$\left(\boldsymbol{\Psi}_j^{\mathrm{I}}\right)_m^n = \begin{cases} \dfrac{\cosh k_0 z\, J_n(k_0 r_j)}{\cosh k_0 h\, J_n(k_0 a_j)}\mathrm{e}^{\mathrm{i}n\theta_j}, & m = 0 \\ \cos k_m z \dfrac{I_n(k_m r_j)}{I_n(k_m a_j)}\mathrm{e}^{\mathrm{i}n\theta_j}, & m \geq 1 \end{cases} \tag{9.10}$$

for all integers $0 \leq m < \infty$, $-\infty < n < \infty$ and where I_n represents the modified Bessel function of the first kind (order n). Note that some degree of parity between the typical magnitudes of these entries has been achieved by normalizing the spatial functions contained within them. This assists in the accurate computation of results but does not affect the nature of the solution and is omitted by some authors. The shapes of the first few partial wave modes are illustrated in Fig. 9.2.

Defining the elements of the vector $\mathbf{a}_j$ to be

$$\left(\mathbf{a}_j\right)_m^n = \begin{cases} I_j J_n(k_0 \mathbf{a}_j)\mathrm{e}^{\mathrm{i}n\left(\frac{\pi}{2}-\beta\right)}, & m = 0 \\ 0, & m \geq 1 \end{cases} \tag{9.11}$$

Eq. (9.9) then reduces to

$$\phi_j^{\mathrm{A}} = \frac{g\eta}{\omega}\mathbf{a}_j^{\top}\boldsymbol{\Psi}_j^{\mathrm{I}} \tag{9.12}$$

where the superscript $\top$ is the transpose operator.

9.2.2.3 *Scattered Potential*

The scattered potential considered here is the flow generated in addition to the incident wave field when a single device is held fixed under wave incidence. Note that this is distinct from the total scattered wave field in an array, which results from all scattering events at all devices within the array. In the region exterior to device i, using the method of separation of variables again on a representation of the velocity potential in cylindrical polar coordinates leads to the general solution

$$\phi_i^{\mathrm{S}} = \frac{g\eta}{\omega}\left[\frac{\cosh k_0 z}{\cosh k_0 h}\sum_{n=-\infty}^{\infty}\left(\mathbf{A}_i^{\mathrm{S}}\right)_0^n \frac{H_n(k_0 r_i)}{H_n(k_0 a_i)}\mathrm{e}^{\mathrm{i}n\theta_i} + \sum_{m=1}^{\infty}\cos k_m z \sum_{n=-\infty}^{\infty}\left(\mathbf{A}_i^{\mathrm{S}}\right)_m^n \frac{K_n(k_m r_i)}{K_n(k_m a_i)}\mathrm{e}^{\mathrm{i}n\theta_i}\right] \tag{9.13}$$

where H_n is the Hankel function of the first kind and K_n is the modified Bessel function of the second kind (each of order n), both of which satisfy Eq. (9.5). Note that Eq. (9.13) is an entirely general representation of the scattered wave field, where the response to wave incidence of any nature may be accounted for by altering only the unknown coefficients $(\mathbf{A}_i^{\mathrm{S}})_m^n$. As with the ambient incident potential, the preceding expression may be written concisely as the product of a vector $\mathbf{A}_i^{\mathrm{S}}$ containing coefficients $\left(\mathbf{A}_i^{\mathrm{S}}\right)_m^n \in \mathbb{C}$ and a vector of

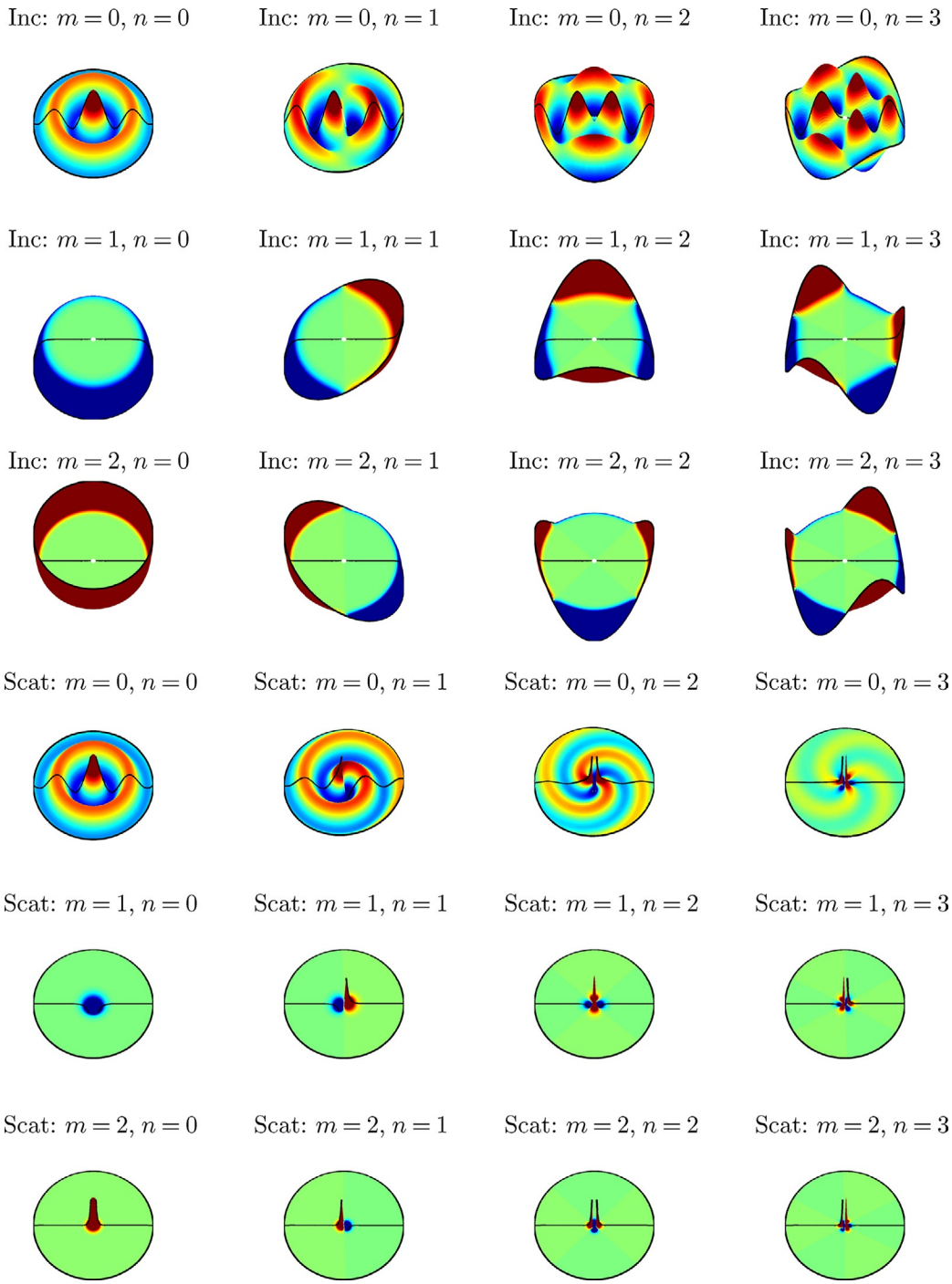

FIG. 9.2 Free surface elevation in the locality of the device, corresponding to selected partial wave modes for a given wave frequency. Incident ('Inc') and scattered ('Scat') partial waves of vertical mode m and angular mode n. Vertical coordinates and corresponding colours have been re-scaled for clarity.

all possible partial waves scattered from device i, $\boldsymbol{\Psi}_i^{\mathrm{S}}(r_i,\theta_i,z)$, with elements

$$\left(\boldsymbol{\Psi}_i^{\mathrm{S}}\right)_m^n = \begin{cases} \dfrac{\cosh k_0 z\, H_n(k_0 r_i)}{\cosh k_0 h\, H_n(k_0 a_i)} \mathrm{e}^{\mathrm{i}n\theta_i}, & m=0 \\ \cos k_m z \dfrac{K_n(k_m r_i)}{K_n(k_m a_i)} \mathrm{e}^{\mathrm{i}n\theta_i}, & m \geq 1 \end{cases} \tag{9.14}$$

These partial waves are also illustrated in Fig. 9.2. Hence Eq. (9.13) becomes

$$\phi_i^{\mathrm{S}} = \frac{g\eta}{\omega}\left(\mathbf{A}_i^{\mathrm{S}}\right)^{\top} \boldsymbol{\Psi}_i^{\mathrm{S}}, \quad r_i \geq a_i \tag{9.15}$$

9.2.2.4 *Radiation Potential*

The radiation potential is that generated by the motion of the devices in the absence of the ambient incident wave field. The potentials $R^{p_i}(r_i,\theta_i,z)$, resulting from body oscillations in mode of motion p_i associated with device i of a predetermined amplitude, are known as the *radiation characteristics* (Kagemoto and Yue, 1993). In the exterior region, the radiation potential for any oscillation may be written in terms of the scattered partial wave basis:

$$\phi_{p_i}^{\mathrm{R}} = \frac{g\eta}{\omega} R^{p_i}(r_i,\theta_i,z) \tag{9.16}$$

$$= \frac{g\eta}{\omega} \widehat{X}^{p_i} \left(\mathbf{R}^{p_i}\right)^{\top} \boldsymbol{\Psi}_i^{S}, \quad r_i \geq a_i \tag{9.17}$$

where $\widehat{X}_i^{p_i}$ is a nondimensional complex amplitude related to that of the displacement in mode p_i of device i by

$$X^{p_i} = \eta \widehat{X}^{p_i} \tag{9.18}$$

The expression of the radiated potential for each mode of motion in the form of Eq. (9.17) results from isolated device calculations.

9.2.3 Partial Wave Operators

In addition to the partial waves themselves, it is convenient to define two partial wave operators to express processes that are represented in semi-analytical array methods. Specifically, a coordinate transformation operator, which transforms coordinates between different systems defined within the array, and a diffraction transfer operator, which describes the transfer of incident to scattered waves at each device, are defined. These operators may most conveniently be expressed as matrices that operate on the vectors of partial waves (Eqs 9.10, 9.14), as shown in the following.

9.2.3.1 *Coordinate Transformation Operator*

Outgoing wave fields from a particular device in the array are usefully written in terms of a polar coordinate system relative to that device, allowing a natural characterization of scattering and radiation processes. However, when these outgoing waves are subsequently considered as incoming waves relative to a particular device, the complication of the associated differing coordinate systems should be dealt with in order to expedite further analysis.

The mathematical expression of this is to write partial waves representing the outgoing wave field from i ($\boldsymbol{\Psi}_i^{\mathrm{S}}$) in terms of those belonging to the set of incoming partial waves at j ($\boldsymbol{\Psi}_j^{\mathrm{I}}$). This can be achieved using a set of relations between Bessel functions called Graf's addition formulae (Abramowitz and Stegun, 1964, (9.1.79)):

$$H_n(k_0 r_i)\mathrm{e}^{\mathrm{i}n\theta_i} = \sum_{l=-\infty}^{\infty} H_{n+l}(k_0 L_{ij}) J_l(k_0 r_j)\, \mathrm{e}^{\mathrm{i}\left[\alpha_{ij}(l+n)+l\left(\pi-\theta_j\right)\right]} \tag{9.19}$$

$$K_n(k_m r_i)\,\mathrm{e}^{\mathrm{i}n\theta_i} = \sum_{l=-\infty}^{\infty} K_{n+l}(k_m L_{ij}) I_l(k_m r_j)\, \mathrm{e}^{\mathrm{i}\left[\alpha_{ij}(l+n)+l\left(\pi-\theta_j\right)\right]} \tag{9.20}$$

which hold for any integer n, and any nonnegative integer m, when $r_j \leq L_{ij}$.

Note that the condition $r_j \leq L_{ij}$ means that we cannot express incident waves to j in local polar coordinates, at a distance from the origin of j any further than the distance to the origin of the nearest neighbour of j. If the nearest origin were to be enclosed by the escribed cylinder to device j, centred at its origin, then there would be parts of the device outside the region of validity of the incident wave representation. This is problematic for the multiple scattering and direct matrix methods where the incident potential to the entire device is needed in order to derive the scattered wave field along with forces on the bodies themselves. However, the restriction is a mild one with respect to array layout, considering the likely distances that would be needed for installation and operations and maintenance procedures, not to mention the avoidance of collision between devices. This coordinate transformation is not used in the point absorber method and it is not directly relevant to the plane wave method since that involves a wide-spacing approximation which would forbid such arrangements in any case.

These equations lead to a coordinate transformation matrix T_{ij} between the two representations, for every i,j pair except $i=j$. Since the depth variation is unchanged by this transformation, for fixed z-mode m, the element that multiplies incident partial waves at j of θ-mode l in the representation of scattered partial waves from i of θ-mode n is

$$\left(\mathrm{T}_{ij}\right)_{mm}^{nl} = \begin{cases} \dfrac{J_l(k_0 a_j)}{H_n(k_0 a_i)} H_{n-l}(k_0 L_{ij})\, \mathrm{e}^{\mathrm{i}\alpha_{ij}(n-l)}, & m=0 \\ \dfrac{I_l(k_m a_j)}{K_n(k_m a_i)} K_{n-l}(k_m L_{ij}) \mathrm{e}^{\mathrm{i}\alpha_{ij}(n-l)}(-1)^l, & m \geq 1 \end{cases} \tag{9.21}$$

Thus for all i,j $(i \neq j)$ scattered partial waves may be rewritten as incident partial waves in the following manner:

$$\boldsymbol{\Psi}_i^{\mathrm{S}} = \mathrm{T}_{ij} \boldsymbol{\Psi}_j^{\mathrm{I}} \tag{9.22}$$

9.2.3.2 *Diffraction Transfer Operator*

To concisely characterize the diffraction properties of the devices, consider the following general representations of the wave fields incident to and scattered by a particular device j (see Eqs 9.12, 9.15):

$$\phi_j^I = \frac{g\eta}{\omega} \left(\mathbf{A}_j^{\mathrm{I}}\right)^{\top} \boldsymbol{\Psi}_j^{\mathrm{I}} \tag{9.23}$$

$$\phi_j^{\mathrm{S}} = \frac{g\eta}{\omega} \left(\mathbf{A}_j^{\mathrm{S}}\right)^{\top} \boldsymbol{\Psi}_j^{\mathrm{S}} \tag{9.24}$$

Kagemoto and Yue (1986) stated that there exists a matrix $\mathbf{S}_j$ (the diffraction transfer matrix) relating the coefficients of $\boldsymbol{\Psi}_j^{\mathrm{I}}$ in the preceding incident potential and those of $\boldsymbol{\Psi}_j^{\mathrm{S}}$ such that:

$$\mathbf{A}_j^{\mathrm{S}} = \mathbf{S}_j \mathbf{A}_j^{\mathrm{I}} \tag{9.25}$$

This matrix encapsulates the behaviour of the potential scattered by an isolated device under known incident waves. These characteristics may be calculated in advance by means of other numerical methods or from experiment for an isolated device and then utilized in one of the interaction techniques contained within this chapter.

Note that this transformation results in a representation of the scattered wave field valid where only the seabed and free surface boundary conditions apply in the vertical direction. It cannot therefore be used to derive the wave field scattered from one device (or indeed body), which will then be considered an incident wave upon another body overlapping with it in the water column. Therefore, we require that the vertical projections of the devices represented by diffraction transfer matrices do not overlap with one another. This is a mild condition on the array layout for the same reasons as discussed with respect to the coordinate transformation matrix.

With the definition of the partial wave representation of velocity potentials, together with the partial wave operators, it is now more convenient to consider the foundations of the semi-analytical methods that have been developed for the analysis of WEC arrays.

9.3 POINT ABSORBER METHOD

9.3.1 Background

Research into arrays of WECs was initiated by Budal (1977) who studied a collection of point absorber type devices. In this work, an assertion was made that became known as the *point absorber approximation*: that the diameter of the devices is small in comparison to the wavelength and device seperation. Hence the wave field radiated by the motion of each device may be calculated without reference to the diffraction effect of other elements in the array. In the ensuing analysis, the power output from the array was partially optimized, assuming that the amplitude of oscillation is the same for each device.

Subsequently, except in special cases, Evans (1979, 1980) showed that the equal amplitude assumption of Budal (1977) is not optimal. The amended analysis used point absorber theory to provide a condition for optimal power absorption on the body velocity amplitudes, a natural extension of the equivalent single device condition (see Chapter 2). This was also arrived at independently by Falnes (1980).

Although the point absorber approximation is capable of determining optimal absorption characteristics without reference to the exact geometry of the devices, Thomas and Evans (1981) noted that no such result is available for their displacements. These quantities provide important information regarding the applicability of linear wave theory and may be crucial in informing the design process. Hence for a specific device geometry, the authors numerically maximized power capture from the array, limiting the motion amplitudes to a fixed multiple of the incident wave amplitude. It was found that if this factor was set to a value of three then performance was not detrimentally diminished. However, if it was reduced to a value of two, there was found to be a significant degradation in the power output.

Point absorber theory has since been used as the basis for an optimization of array layout with respect to the array interaction factor (the ratio of the power absorbed by the array to that absorbed by the same number of devices in isolation, q) in regular waves (Fitzgerald, 2006; Fitzgerald and Thomas, 2007). Folley and Whittaker (2009) have also used this approximation to analyse one of the arrays presented by Fitzgerald and Thomas (2007) and, separately, to optimize the position of devices in irregular waves.

9.3.2 Formulation

Point absorber theory, strictly speaking, encompasses all derivations that utilize the following key assumption:

1. The device is small enough relative to the wavelength and interdevice spacing such that the effect on each device of scattered waves from other devices within the array may be neglected with respect to radiated waves.

If this assumption is satisfied, then the far field radiated wave from each device may be computed without reference to the other devices in the array (see Eq. (9.36)). This is the key step in the derivation to follow. However, the development presented here largely follows that of Evans (1980) who, along with many others that have employed the technique, uses a number of additional assumptions in order to simplify the presentation of the derivation and make further progress. These assumptions are:

2. The water depth is infinite.

3. All devices in the array are identical.
4. The devices consist of a single oscillating body.
5. The devices absorb energy and oscillate principally in a single mode of motion (heave).
6. The devices are vertically axisymmetric.
7. The external (nonhydrodynamic) forces on the bodies are linear functions of incident wave height, allowing an entirely frequency-domain approach.
8. Power take-off (PTO) forces are applied that lead to optimal power absorption for each wave frequency.

For consistency with existing literature, the notational conventions used in this section are:

1. A global Cartesian coordinate system is defined, with the x and y coordinates lying in a horizontal plane and the z-axis pointing vertically upwards from the mean water level.
2. The wave heading β is defined to be the direction from which plane waves are coming, measured anti-clockwise from the positive x-axis.
3. All time-varying quantities have their time dependence governed by the complex factor $e^{i\omega t}$ (cf. Eq. (9.1)).

As long as all hydrodynamic forces are time-harmonic with the same angular frequency as the ambient incident wave ω, the total force experienced by the bodies belonging to each device will therefore be too (Assumption 7). Hence the resulting body velocities will respond in a similar manner:

$$\mathbf{u}(t) = \mathrm{Re}\left\{\mathbf{U}(\omega, \theta)e^{i\omega t}\right\} \tag{9.26}$$

where the column vector $\mathbf{u}$ contains the body velocities in the defined modes of motion and $\mathbf{U}$ is a column vector of complex coefficients.

From the assumptions of linear wave theory (see Section 9.2) and the time-dependence of the body motions (9.26), it can be confirmed that all hydrodynamic forces are time-harmonic with the same frequency as the incident wave. Time-dependent excitation, radiation and total hydrodynamic forces are therefore given by the following expressions respectively:

$$\mathbf{f}^{\mathrm{E}}(t) = \mathrm{Re}\left\{\mathbf{F}(\omega, \beta)e^{i\omega t}\right\} \tag{9.27}$$

$$\mathbf{f}^{\mathrm{R}}(t) = -[\mathbf{M}(\omega)\dot{\mathbf{u}}(t) + \mathbf{B}(\omega)\mathbf{u}(t)] \tag{9.28}$$

$$\mathbf{f}^{\mathrm{TOT}}(t) = \mathbf{f}^{\mathrm{E}}(t) + \mathbf{f}^{\mathrm{R}}(t) \tag{9.29}$$

In the preceding, symbols in bold serif font are column vectors and matrices, with rows and columns both corresponding to each mode of motion. Symbols in bold sans-serif are matrices indexed by the same modes of motions in both the rows and columns. Here $\mathbf{F}$ is a complex quantity with amplitude and phase, whereas the remaining variables are real. The variables $\mathbf{M}$ and $\mathbf{B}$ are the added mass and damping matrices, respectively.

The mean rate at which the hydrodynamic force does work on the bodies over one wave cycle (the average power) is therefore:

$$\overline{P} = \frac{1}{T}\int_0^T \left(\mathbf{f}^{\mathrm{TOT}}(t)\right)^{\top} \mathbf{u}(t)\mathrm{d}t \tag{9.30}$$

where $\top$ is the matrix transpose operator which leads to summation of the product of velocity and total hydrodynamic force over all modes of motion.

Using the definitions (9.26)–(9.29), and the fact that the matrix $\mathbf{B}$ is real and symmetric (Section 5.2.1; Falnes, 2002), some manipulation gives the following expression for $\overline{P}$:

$$\overline{P} = \frac{1}{2}\mathrm{Re}\left\{\mathbf{F}^{\dagger}\mathbf{U}\right\} - \frac{1}{2}\mathbf{U}^{\dagger}\mathbf{B}\mathbf{U} \tag{9.31}$$

where $\dagger$ is the complex conjugate transpose operator. This is the natural extension of a similar result for absorption in a single mode of motion (see Chapter 2). Moreover, it can also be shown by rewriting the preceding expression that the maximum power capture is given by:

$$\overline{P}_{\max} = \frac{1}{8}\mathbf{F}^{\dagger}\mathbf{B}^{-1}\mathbf{F} = \frac{1}{2}\mathbf{U}^{\dagger}\mathbf{B}\mathbf{U} \tag{9.32}$$

which occurs when the following optimality condition on the motions is satisfied:

$$\mathbf{U} = \frac{1}{2}\mathbf{B}^{-1}\mathbf{F} \tag{9.33}$$

Note that this step, which uses Assumption 8, is not fundamental to the point absorber method; however, it allows considerable further progress to be made with little extra effort. In particular, the external forces required to achieve the prescribed motions are not required for the following analysis.

The problem now reduces to determining **F** and **B**, which can be approximated using reciprocity relations. In particular, application of Green's Theorem and the method of stationary phase leads to an expression for the coefficients of the added damping matrix B_{qp} in terms of the excitation force coefficients F_q (Srokosz, 1979):

$$B_{qp}(\omega) = \frac{1}{8\lambda J(\omega)} \int_0^{2\pi} F_q(\omega,\theta) F_p^*(\omega,\theta)\, d\theta \tag{9.34}$$

where * is the complex conjugate operator and $J(\omega)$ is the power transmitted by the incident wave per unit width across the wave crest.

Conversely, the excitation force coefficients may be expressed in terms of the solution to the radiation problem. Specifically, the far-field amplitudes in each mode of motion for radiated waves propagating in the direction from which we are seeking wave incidence in the excitation problem, $b_q(\omega,\beta)$, are used to form the following expression:

$$F_q(\omega,\beta) = \frac{\rho g \eta}{k(\omega)} \sqrt{2\pi} e^{-i\pi/4} b_q(\omega,\beta) e^{ik(\omega) L_{0j} \cos(\beta - \alpha_{0j})} \tag{9.35}$$

where L_{0j} is distance between the origin and the location of device j associated with mode q and α_{0j} is the angle that is made between the positive x-axis and the line joining these two points. Eqs (9.34), (9.35) are instances of what are known as the Haskind relations.

The point absorber approximation (Assumption 1), along with Eq. (9.35) implies that the far-field radiated wave amplitudes are unchanged by the presence of other bodies in the array. That is to say that the bodies are small enough relative to the wavelength of the incoming wave train and far enough apart not to significantly disrupt the radiated waves in the array. This can be written:

$$b_q(\omega,\beta) = b_q^{\text{isol}}(\omega,\beta) \tag{9.36}$$

where $b_q^{\text{isol}}(\omega,\beta)$ is the far-field radiated wave amplitude of an isolated body oscillating in mode q. Thus, the hydrodynamic problem reduces to finding the far-field radiation characteristics of a single device, which is attractive since there such solutions are known for simple geometries.

Although not strictly necessary for point absorber analysis, further progress may be made by forming additional assumptions, which are common in the literature. Namely, all absorbers are identical, axisymmetric and oscillate only in heave (Assumptions 3, 5 and 6). Under such circumstances, $b_q(\omega,\beta) = b(\omega)$ (a constant with respect to wave heading) for all devices. If these further assumptions are made, the dependence on b disappears entirely from the expression for mean absorbed power (Eq. 9.32), which simplifies to:

$$\overline{P}_{\max} = J \frac{\lambda}{2\pi} \mathbf{I}^{\dagger} \mathbf{C}^{-1} \mathbf{I} \tag{9.37}$$

where the vector **I** and matrix **C** contain the following elements:

$$I_j = e^{ikL_{0j}\cos(\beta - \alpha_{0j})} \tag{9.38}$$

$$C_{ji} = J_0(kL_{ji}) \tag{9.39}$$

J_0 is the zeroth-order Bessel function of the first kind and L_{ji} is the distance between devices i and j.

Note that the vector **I** in Eq. (9.38) contains phase factors relating to each device location (that is to say, the difference in phase for the incident wave between the global origin and the origin of device j). Note too, with reference to Eq. (9.10), that the matrix **C** contains the relative magnitudes of waves at each of the device locations that would be needed in order to result in a unit amplitude progressive incoming circular

wave at each of the remaining device locations. Hence Eq. (9.37) represents a calculation of maximum power, just as in Eq. (9.32), except that only elementary features of the excitation and radiation solutions are used.

The point absorber approximation therefore allows a theoretical limit on the power absorption from an array of simplified devices without having to compute any hydrodynamic parameters whatsoever. In particular, the precise geometry of the device is not required. However, the optimal motions in Eq. (9.33) (which can be calculated if the far-field radiated wave field is known) may lead to unrealistically large body excursions. This may violate the small amplitude assumption for body motions used in linear wave theory, invalidating the hydrodynamic approach. As a potential remedy, it is possible to maximize power in Eq. (9.32) numerically subject to chosen constraints on displacement amplitudes (Fitzgerald, 2006; Thomas and Evans, 1981).

In order to ensure that the point absorber assumption is satisfied, bounds may be placed on certain characteristics of the array. The small size of devices compared to the wavelength may be expressed as $k_0 a \ll 1$, where a is the greatest radius of an axisymmetric device. However, Mavrakos and McIver (1997) found the point absorber method to accurately predict the interaction factor up to $k_0 a = 0.8$. Another restriction relating to the scattered wave field concerns its decrease in magnitude further away from the originating device. For the scattered waves to remain small at other devices in the array, the distance between devices therefore cannot become too small. Fitzgerald (2006) reported a lower limit on the ratio of device radius to spacing of $7.5 < \frac{L_{ij}}{a}$ to be required.

Note that the optimum power absorption expression of Eq. (9.37) is valid for a single monochromatic wave component. Different wave frequencies and directions will inevitably lead to different solutions for the optimal motions and power absorption characteristics. Just as for the single device case, polychromatic wave incidence would require future knowledge of the sea surface elevation in order for optimal absorption of all wave components to take place (Price, 2009). Coupled with the fact that many practical PTO systems do not even attempt to absorb optimally, the upper bound is likely to significantly overestimate realistically attainable power absorption.

9.4 PLANE WAVE METHOD

9.4.1 Background

The analogy between ocean and electromagnetic waves has proved to be particularly fruitful over the years. The diffraction of an electromagnetic wave field by several cylinders was the focus of research that culminated in the invention of a direct matrix method by Zaviska (1913). This was subsequently resurrected by Spring and Monkmeyer (1974) to deal with diffraction of water waves by bottom-mounted, surface-piercing circular cylinders. The technique uses the body boundary conditions to simultaneously determine the unknown scattered wave amplitudes.

Simon (1982) adopted a direct matrix approach to analyse wave energy devices but approximated the diverging wave from one device as a plane wave upon reaching other devices in the array, ignoring evanescent waves. The resulting plane wave method requires that the spacing between elements (nondimensionalized using the incident wave number) is large. This restriction was judged to be a worthwhile sacrifice as a result of the improved efficiency of the method. The results suggested a significant modification in the forces felt by devices when scattering is included in the calculations. Therefore, the plane wave method represents a marked improvement on point absorber theory in the analysis of wave energy devices.

McIver and Evans (1984) later added a 'first correction' term to the expression of the plane wave and derived significantly improved

accuracy of results for bottom-mounted, surface-piercing cylinders with little extra effort. On application to floating axisymmetric bodies, McIver (1984) found the method to be accurate in comparison to a numerical solution, even when the wide spacing assumption is violated.

9.4.2 Formulation

The plane wave method uses the following key assumption:

1. The spacing between the devices is large compared to the wavelength.

If this assumption is satisfied then the diverging wave from one device can be modelled as a plane wave at another device. This assumption also means that the effect of the evanescent waves generated by each device on one another can generally be neglected.

The formulation presented in this section is based upon the derivation given by McIver and Evans (1984). A number of further assumptions are common in the literature:

2. The water depth is finite.
3. All devices in the array are identical.
4. The devices consist of a single oscillating body.
5. The devices absorb energy and oscillate principally in a single mode of motion (heave).
6. The devices are vertically axisymmetric.

For the purposes of the derivation, the following notational conventions are used within this section:

1. A global Cartesian coordinate system is defined, with the x and y coordinates lying in a horizontal plane and the z-axis pointing vertically upwards from the seabed (as in Section 9.2).
2. The wave heading β is defined to be the direction that plane waves are travelling towards, measured anti-clockwise from the positive x-axis (as in Section 9.2).
3. All time-varying quantities have their time dependence governed by the complex factor $e^{-i\omega t}$ (as in Eq. (9.1) of Section 9.2).

Since incident, scattered and radiated wave potentials (Eqs 9.12, 9.15, 9.17, respectively), can be expressed in terms of the summation of separate functions of vertical and horizontal coordinates, so too can the total velocity potential in the vicinity of the array. Taking the progressive wave part of this representation only (that is to say, ignoring evanescent waves), gives rise to a single function ϕ_{xy} of the horizontal coordinates to solve for:

$$\phi(x,y,z) = \frac{g\eta}{\omega}\sigma_z(z)\phi_{xy}(x,y) \tag{9.40}$$

where $\sigma_z(z) = \cosh k_0 z / \cosh kh$ for finite depth water.

Consider the horizontal variation in the potential corresponding to a plane wave travelling in the direction χ, centred at a predefined origin on device j, written in a local polar coordinate system (r_j, θ_j) (see Eq. (9.7)):

$$\phi_{xy,j} = e^{ik_0 r_j \cos(\theta_j - \chi)} \tag{9.41}$$

The scattered wave field resulting from this incident potential at device j can be represented for any arbitrarily shaped device as follows (see Eq. (9.13)):

$$\phi^S_{xy,j} = S(r_j, \theta_j, \chi) = \sum_{n=-\infty}^{\infty} S_n(\chi)\, H_n(k_0 r_j) e^{in\theta_j} \tag{9.42}$$

where H_n is the Hankel function of the first kind of order n. Note that this only includes the progressive part of the potential, since local waves are neglected here. Note too that the normalizing factors in the radial coordinate have been omitted here for simplicity of notation. It is assumed that the function S (or equivalently the complex coefficients S_n) may be readily computed for any angle of incidence, χ. For the purposes of this derivation, each device is taken to be identical, such that S is equal for all members

of the array. This is not a necessary restriction, however, and a more general set of equations may easily be generated by simply distinguishing notationally between scattering and radiation properties of the different device types.

In order to reexpress the scattered wave field from one device j as an incident wave field at another device i, the plane wave approximation is used. This involves first applying Graf's addition formula (Eq. 9.19) at j to Eq. (9.42). Changing the order of summation in the resulting expression then gives:

$$\phi^{S}_{xy,j} = \sum_{l=-\infty}^{\infty} J_l(k_0 r_i)\mathrm{e}^{\mathrm{i}l\left(\alpha_{ji}+\pi-\theta_i\right)} \sum_{n=-\infty}^{\infty} S_n(\chi)\, H_{n+l}\left(k_0 L_{ji}\right)\mathrm{e}^{\mathrm{i}n\alpha_{ji}} \tag{9.43}$$

For a large separation distance relative to the wavelength $k_0 L_{ji} \gg 1$, (Assumption 1), the Hankel function can be expressed as (Eq. WA221(5), p. 962, Gradshteyn and Ryztik, 1965):

$$H_{n+l}\left(k_0 L_{ji}\right) = (-\mathrm{i})^l H_n\left(k_0 L_{ji}\right) \left\{1+\mathrm{i}\left(k_0 L_{ji}\right)^{-1}\left(nl+\frac{1}{2}l^2\right)\right\} + O\left(k_0 L_{ji}\right)^{-\frac{5}{2}} \tag{9.44}$$

So for large $k_0 L_{ji}$ ($k_0 r_i = O(1)$), substitution of Eq. (9.44) into Eq. (9.43), taking just the first term in the curly braces, yields:

$$\phi^{S}_{xy,j} = \sum_{l=-\infty}^{\infty} J_l(k_0 r_i)\mathrm{e}^{\mathrm{i}l\left(\frac{\pi}{2}+\alpha_{ji}-\theta_i\right)} \sum_{n=-\infty}^{\infty} S_n(\chi) H_n\left(k_0 L_{ji}\right)\mathrm{e}^{\mathrm{i}n\alpha_{ji}} \tag{9.45}$$

On using the relation $J_{-l} = (-1)^l J_l$ (Abramowitz and Stegun, 1964, Eq. (9.1.5)), replacing l with $-l$ and using the definition of S from Eq. (9.42), the expression for the scattered potential simplifies to:

$$\phi^{S}_{xy,j} = S\left(L_{ji}, \alpha_{ji}, \chi\right) \sum_{l=-\infty}^{\infty} J_l(k_0 r_i)\mathrm{e}^{\mathrm{i}l\left(\frac{\pi}{2}+\theta_i-\alpha_{ji}\right)} \tag{9.46}$$

As was seen in Eq. (9.9), this expansion represents a plane wave, referenced to (that is to say, with zero phase at) i, travelling in the direction from j to i, with complex amplitude equal to the horizontal potential scattered by j and evaluated at i:

$$\phi^{S}_{xy,j} = S\left(L_{ji}, \alpha_{ji}, \chi\right)\mathrm{e}^{\mathrm{i}k_0 r_i \cos\left(\theta_i-\alpha_{ji}\right)} \tag{9.47}$$

This is known as the plane wave approximation, due to the replacement of the diverging scattered wave field with a plane wave upon reaching other devices in the array. Here, the 'first correction' of McIver and Evans (1984) corresponding to the last term in the curly braces of Eq. (9.44) has been omitted to simplify the derivation.

The array scattering problem is now addressed, where the bodies belonging to all devices are held fixed in the presence of ambient incident waves. From Eq. (9.41), the ambient incident potential at device j in local polar coordinates takes the form:

$$\phi^{I,A}_{xy,j} = I_j \mathrm{e}^{\mathrm{i}k_0 r_j \cos\left(\theta_j-\beta\right)} \tag{9.48}$$

where the phase factor, I_j given by Eq. (9.8), has been used to represent the change in phase experienced by the wave travelling in the direction β between the origin and the position of device j.

Applying the plane wave approximation to all scattered wave fields in the array, the incident potential at device j due to the scattering by other devices can be written in terms of as yet unknown complex amplitudes A_{jl} representing plane waves travelling directly between device origins l and j:

$$\phi^{I,S}_{xy,j} = \sum_{\substack{l=1 \\ l\neq j}}^{N} A_{jl}\mathrm{e}^{\mathrm{i}k_0 r_j \cos\left(\theta_j-\alpha_{lj}\right)} \tag{9.49}$$

Each of the components of the potential incident to device j (Eqs 9.48, 9.49) is then subject to the scattering process. The associated potential may be calculated by simply multiplying the plane wave amplitudes by the canonical scattered wave solution given by S (Eq. 9.42) for the appropriate

angle of incidence. Hence the scattered potential at device j is given by:

$$\phi^{S}_{xy,j} = I_j S(r_j, \theta_j, \beta) + \sum_{\substack{l=1 \\ l \neq j}}^{N} A_{jl} S(r_j, \theta_j, \alpha_{lj}) \quad (9.50)$$

In order to derive the plane wave amplitude A_{ij} of the total wave field scattered by j and arriving at another device i, we apply the plane wave approximation once again. This means (with reference to Eq. (9.47)) simply evaluating each of the functions S at i:

$$A_{ij} = I_j S(L_{ji}, \alpha_{ji}, \beta) + \sum_{\substack{l=1 \\ l \neq j}}^{N} A_{jl} S(L_{ji}, \alpha_{ji}, \alpha_{lj}) \quad (9.51)$$

This may be written in matrix form as follows:

$$\mathbf{A} = \mathbf{s} + \mathbf{SA} \quad (9.52)$$

where $\mathbf{S}$ is an $N(N-1) \times N(N-1)$ matrix containing the entries $S(L_{ji}, \alpha_{ji}, \alpha_{lj})$, $\mathbf{s}$ is an $N(N-1)$ column vector containing the elements $I_j S(L_{ji}, \alpha_{ji}, \beta)$ and $\mathbf{A}$ is an $N(N-1)$ column vector containing the unknown quantities A_{ij}, all of which are appropriately indexed with respect to devices i, j and l. The unknown plane wave coefficients may then be solved for by simple matrix inversion, after rearrangement of Eq. (9.52).

Note that in practice, the infinite summation of Eq. (9.42) cannot be fully computed, so the series must be truncated. A sensitivity study should be undertaken as part of any implementation to ensure that the truncation does not lead to significant loss of accuracy. A similar comment applies to the multiple scattering and direct matrix methods (see Sections 9.5 and 9.6).

Hydrodynamic forces experienced by the devices may be calculated by considering the influence of the derived collection of incident plane waves on each body. A number of methods are available for this purpose, including the use of the Haskind relation (Eq. 9.35) if the far-field radiation solution is known and using a general-purpose BEM solver. Axisymmetry and the restriction to heave motion allows all plane wave amplitudes incident to a particular device to be added together before being considered as a single plane wave (Simon, 1982), but this step is not intrinsic to the plane wave method.

The array radiation problem may be solved analogously to the array scattering problem by considering forced oscillation in each mode of motion separately in the absence of incident waves. The total wave field (both radiated and scattered as a result of the initial disturbance) transferred between one device and another is then interpreted as a plane wave, as in the preceding derivation. This gives rise to a similar equation to Eq. (9.51) (see Eq. 17, McIver, 1984), where the first term (representing the ambient incident wave field) is replaced by a contribution from the radiated wave field (of the form of Eq. (9.17)) to the summation in the second term.

Note that the diffraction properties are encapsulated in a matrix $\mathbf{S}$ which takes as input the angle of incidence of a plane wave and outputs plane wave amplitudes for each of the pairs of discrete directions between devices. Hence it is the analogue of the diffraction transfer matrix used in the multiple scattering and direct matrix methods (Eq. 9.51), which operates on input and output angular distribution modes. The matrix $\mathbf{S}$ used here however also encapsulates the coordinate transformation between devices, represented by the matrix $\mathbf{T}$ given in Eq. (9.21).

9.5 MULTIPLE SCATTERING METHOD

9.5.1 Background

Twersky (1952) discovered a way of solving the multibody diffraction problem in the field of acoustics by considering the interactions to be a series of consecutive scattering events. Intuitively, contact with each body reduces the amplitude of the resulting wave and so a

solution is reached by the convergence of an iterated sequence. This multiple scattering technique was applied to water waves by Ohkusu (1972, 1974) in the study of offshore mobile platforms with multiple supporting bodies.

Work on the multiple scattering method continued with Mavrakos and Koumoutsakos (1987) and Mavrakos (1991), in whose articles the scattering and radiation problems were solved respectively. In these investigations, the interaction theory was combined with an analytic solution for axisymmetric bodies in isolation. An application of this theory to wave energy devices was made by Mavrakos and Kalofonos (1997) with several device and array geometries being assessed. Later, several absorbing floats placed in front of a reflecting breakwater were analyzed using the same method (Mavrakos et al., 2004).

9.5.2 Formulation

The following assumptions are required for formulation of the multiple scattering method:

1. The escribed cylinder to each device, centred at its origin, does not enclose the origin of any other device.
2. The vertical projections of the devices do not overlap with one another.
3. The magnitude of the successive scattering potentials reduces sufficiently to ensure convergence to the total potential.

The formulation presented in this section is based upon Mavrakos and Koumoutsakos (1987). The following additional assumptions are commonly made in order to simplify the development:

4. The water depth is finite.
5. All devices in the array are identical.
6. The devices consist of a single oscillating body.
7. The devices absorb energy and oscillate principally in a single mode of motion (heave).
8. The devices are vertically axisymmetric.
9. The external (nonhydrodynamic) forces on the bodies are linear functions of incident wave height, allowing an entirely frequency-domain approach.

For the purposes of the derivation, the following notational conventions are used within this section:

1. A global Cartesian coordinate system is defined, with the x and y coordinates lying in a horizontal plane and the z-axis pointing vertically upwards from the seabed (as in Section 9.2).
2. The wave heading β is defined to be the direction that plane waves are travelling towards, measured anti-clockwise from the positive x-axis (as in Section 9.2).
3. All time-varying quantities have their time dependence governed by the complex factor $e^{-i\omega t}$ (as in Eq. (9.1) of Section 9.2).

The following derivation considers the array scattering problem, although the incorporation of radiation effects will be discussed at the end of the section.

Consider one of the devices in the array j to be initially excited by only the undisturbed ambient incident wave field ${}^{(1)}\phi_j^{\mathrm{I}}$ (which will be referred to as the *first order excitation*). Consider now the scattered wave field ${}^{(1)}\phi_j^{\mathrm{S}}$ that would result if the device were in isolation (the *first order scattering*). Both of these wave fields may be expressed in local coordinates, centred at j, and their sum satisfies the imposed body surface boundary conditions at j. A first approximation to the total velocity potential across the array may then be formed as the sum of the scattered potentials (as described previously) from all the devices in the array and the original incident wave potential:

$$ {}^{(1)}\phi_j = {}^{(1)}\phi_j^I + \sum_{i=1}^{N} {}^{(1)}\phi_i^S \tag{9.53}$$

All of the first order scattered waves from devices other than j may now be interpreted as

contributions to the incident wave field (*second order excitation*) for another diffraction problem.

$$^{(2)}\phi_j^{\mathrm{I}} = \sum_{\substack{i=1 \\ i\neq j}}^{N} {}^{(1)}\phi_i^{S} \tag{9.54}$$

This results in a second order scattered potential $^{(2)}\phi_j^{\mathrm{S}}$ such that the sum of the excitation and scattering potentials at second order again satisfies the body surface boundary conditions at j.

Continuing with the sequence at successive orders of interaction l leads to the following definitions of excitation, scattered and diffracted (incident and scattered) potential local to j, respectively:

$$^{(l)}\phi_j^{\mathrm{I}} = \sum_{\substack{i=1 \\ i\neq j}}^{N} {}^{(l-1)}\phi_i^{\mathrm{S}}, \quad \text{for } l \geq 2 \tag{9.55}$$

$$^{(l)}\phi_j^{\mathrm{S}} \tag{9.56}$$

$$^{(l)}\phi_j = {}^{(l)}\phi_j^{\mathrm{I}} + {}^{(l)}\phi_j^{S} \tag{9.57}$$

Letting the order l tend to infinity yields expressions for the total incident wave field to j and the total scattered wave field from j (assuming convergent series, Assumption 3):

$$\phi_j^{\mathrm{I}} = \sum_{l=1}^{\infty} {}^{(l)}\phi_j^{\mathrm{I}} \tag{9.58}$$

$$\phi_j^{\mathrm{S}} = \sum_{l=1}^{\infty} {}^{(l)}\phi_j^{\mathrm{S}} \tag{9.59}$$

Now the sum of the preceding two expressions, that is to say the total diffracted potential in the vicinity of device j, may readily be shown to equal:

$$\phi_j = \phi_j^{\mathrm{I}} + \phi_j^{\mathrm{S}} = \sum_{l=1}^{\infty} {}^{(l)}\phi_j \tag{9.60}$$

Therefore the total diffracted potential in the vicinity of j is the sum of the diffracted potentials at all orders l. Each of these individual diffracted potentials has been shown to satisfy the body surface boundary conditions at j, hence so does their total. A solution to the array scattering problem has therefore been found.

It remains to determine the scattered potential at each order of interaction, given the incident potential. At each interaction order l, the incident and scattered potentials relative to j may be expressed in the general form described in Section 9.2.2 as summations of the product of partial wave functions with complex coefficients. In vector notation these potentials are, respectively:

$$^{(l)}\phi_j^{\mathrm{I}} = \frac{g\eta}{\omega} \left({}^{(l)}\mathbf{A}_j^{\mathrm{I}} \right)^{\top} \boldsymbol{\Psi}_j^{\mathrm{I}} \tag{9.61}$$

$$^{(l)}\phi_j^{\mathrm{S}} = \frac{g\eta}{\omega} \left({}^{(l)}\mathbf{A}_j^{\mathrm{S}} \right)^{\top} \boldsymbol{\Psi}_j^{\mathrm{S}} \tag{9.62}$$

From Section 9.2.3.2, there is a relationship between the coefficients belonging to these vectors for a given device, which can be expressed in the form of the diffraction transfer matrix $\mathbf{S}$. Note that there are several alternative methods for finding this matrix, as described in relation to the plane wave method. Here for simplicity, all devices are taken to have the same diffraction properties (although this is not a necessary condition). Then the following matrix equation relates incident and scattered wave coefficients at each device and interaction order:

$$^{(l)}\mathbf{A}_j^{S} = \mathbf{S}\,{}^{(l)}\mathbf{A}_j^{\mathrm{I}} \tag{9.63}$$

Finally, in order to apply the diffraction transfer matrix, it is necessary to be able to express incident waves to the device in question j in its local coordinate system. A plane incident wave (the first order excitation) may be expressed using the coefficients defined in Eq. (9.11) as follows:

$$^{(l)}\mathbf{A}_j^{\mathrm{I}} = \mathbf{a}_j, \quad \text{for } l = 1 \tag{9.64}$$

Furthermore, the scattered waves at other devices i may be expressed in the local

coordinate system of device j using the coordinate transformation matrix T_{ij} defined in Eq. (9.21). Therefore from Eq. (9.55), the potential incident to j is:

$$^{(l)}\mathbf{A}_j^{\mathrm{I}} = \sum_{\substack{i=1 \\ i\neq j}}^{N} \mathrm{T}_{ij}^{\mathsf{T}}\, {}^{(l-1)}\mathbf{A}_i^{S}, \quad \text{for } l \geq 2 \tag{9.65}$$

Eqs (9.63), (9.64), (9.65) together allow the solution to progress from one interaction order to the next. The complex coefficients of the partial wave representations may thus be determined at all orders. The associated velocity potentials may then be summed to give the total diffraction potential.

The array radiation problem (as described in Mavrakos, 1991) may be treated in a similar way to the array scattering problem. A series of separate problems is solved, whereby forced oscillation in each mode of motion is considered in the absence of ambient incident waves. A solution of the form of Eq. (9.17) for the radiated potential is taken to be the 'zeroth-order radiation'. Using the coordinate transformation as in Eq. (9.65), this is reinterpreted as the first order excitation at other devices within the array in place of the ambient incident wave of Eq. (9.64). The solution then proceeds with iteration at increasing interaction orders exactly as the preceding, with the diffracted potential at each order satisfying the stationary body surface boundary conditions at all devices other than that which is undergoing forced oscillation. Moreover, the diffracted potential at each order leaves the inhomogeneous body boundary condition, relating to the forced mode of motion and which has already been satisfied by the zeroth order radiation potential, unchanged. The total of the diffracted potential at each order and the radiation potential therefore satisfies the prescribed boundary conditions on all of the bodies in the array.

Forces on each of the bodies belonging to devices, in the form of standard hydrodynamic coefficients, may be calculated via integration of the velocity potential over wetted surfaces. Mavrakos and Koumoutsakos (1987) facilitated this by expressing the potential interior to the escribed cylinder to the device in its own coordinate system by matching solutions on the boundary of that region.

9.6 DIRECT MATRIX METHOD

9.6.1 Background

Kagemoto and Yue (1986) combined the physical concepts associated with the multiple scattering approach and a direct formulation to form what is often referred to as the direct matrix method. This technique is in principle exact within the context of linear wave theory, subject to the truncation of an infinite summation. The scattering characteristics of each individual body under incident progressive and evanescent waves are encapsulated in the diffraction transfer matrix, which allows the unknown wave amplitudes to be solved for simultaneously.

Following its formulation, the theory of Kagemoto and Yue (1986) has been widely applied, especially in the field of very large floating structures (VLFSs). These were the subject of a study by the same authors (Kagemoto and Yue, 1993) who incorporated the radiation problem from the motion of the bodies. Modifications have also been made in order to efficiently apply the method to a much greater number ($\sim$5000) of columns supporting a VLFS (Kashiwagi, 2000). Another extension of the direct matrix method is in the case of infinite water depth (Peter and Meylan, 2004), where the diffraction transfer matrix becomes a linear integral operator.

Yilmaz (1998) incorporated the single body solution of Garrett (1971) into the array interaction procedure of Kagemoto and Yue (1986), in order to analyse tension-leg platforms (TLPs). They also added the effects of radiation in the case that all bodies move in synchrony and included evanescent waves. A similar

combination of techniques was first applied in the wave energy field by Child and Venugopal (2007). In that work (later expanded upon in Child and Venugopal, 2010), the independent movement of converters is accounted for and the effects of energy extraction included.

Recently, McNatt et al. (2013) provided a methodology to express outputs from BEM solvers for an isolated device in the format needed in the direct matrix method. This opens up the interaction technique to arbitrary device geometries, allowing a more simplified application to the general case than was previously possible.

9.6.2 Formulation

The following assumptions are required in order to derive the direct matrix method:

1. The escribed cylinder to each device, centred at its origin, does not enclose the origin of any other device.
2. The vertical projections of the devices do not overlap with one another.

The derivation presented in this section largely follows Kagemoto and Yue (1986, 1993). The following assumptions are also common in the literature:

3. The water depth is finite.
4. All devices in the array are identical.
5. The devices consist of a single oscillating body.
6. The devices absorb energy and oscillate principally in a single mode of motion (heave).
7. The devices are vertically axisymmetric.
8. The external (nonhydrodynamic) forces on the bodies are linear functions of incident wave height, allowing an entirely frequency-domain approach.

For the purposes of the derivation the following notational conventions are used within this section:

1. A global Cartesian coordinate system is defined, with the x and y coordinates lying in a horizontal plane and the z-axis pointing vertically upwards from the seabed (as in Section 9.2).
2. The wave heading β is defined to be the direction that plane waves are travelling towards, measured anti-clockwise from the positive x-axis (as in Section 9.2).
3. All time-varying quantities have their time dependence governed by the complex factor $e^{-i\omega t}$ (as in Eq. (9.1) of Section 9.2).

The total potential incident to any particular device j in the array may be expressed as the summation of the ambient incident wave field (Eq. 9.12), the waves scattered from other devices (Eq. 9.15) and the waves radiated by their motion (Eq. 9.17), using the coordinate transformation matrix of Eq. (9.22):

$$\phi_j^{\mathrm{I}} = \frac{g\eta}{\omega}\left[\mathbf{a}_j^{\top} + \sum_{\substack{i=1\\ i\neq j}}^{N}\left(\mathbf{A}_i^{\mathrm{S}} + \sum_{p_i=1}^{P_i}\widehat{X}^{p_i}\mathbf{R}^{p_i}\right)^{\top}\mathrm{T}_{ij}\right]\boldsymbol{\Psi}_j^{\mathrm{I}} \quad (9.66)$$

where P_i is the number of modes of motion for the ith device.

At each device, the coefficients of $\boldsymbol{\Psi}_j^{\mathrm{I}}$ in the preceding incident potential and those of $\boldsymbol{\Psi}_j^{\mathrm{S}}$ in the appropriate scattered potential of Eq. (9.15) are related by the diffraction transfer matrix, $\mathbf{S}_j$, (see Eq. (9.25)) as follows:

$$\mathbf{A}_j^{\mathrm{S}} = \mathbf{S}_j\left[\mathbf{a}_j + \sum_{\substack{i=1\\ i\neq j}}^{N}\mathrm{T}_{ij}^{\top}\left(\mathbf{A}_i^{\mathrm{S}} + \sum_{p_i=1}^{P_i}\widehat{X}^{p_i}\mathbf{R}^{p_i}\right)\right],$$

$$j = 1,\ldots,N \quad (9.67)$$

These are the scattering equations, which represent a system of N vector equations in the

$P = \sum_{i=1}^{N} P_i$ scalar unknowns $\widehat{X}^{p_i}$ and the N vector unknowns $\mathbf{A}_i^{\mathrm{S}}$.

There are two main options to form a complete system of equations that can be solved:

1. Form a set of equations of motion, that is to say a set of P scalar equations in the P scalar unknowns $\widehat{X}^{p_i}$ and the N vector unknowns $\mathbf{A}_i^{\mathrm{S}}$. This would lead to two sets of equations which may be solved simultaneously in order to completely determine the motion and wave amplitudes in the vicinity of the array. The equations of motion require hydrodynamic forces on the bodies, which in turn require an expression of the velocity potential in the interior region to the enclosing cylinder. This may be achieved using special solutions for the diffraction transfer matrix and radiation characteristics in the interior domain (see Child and Venugopal (2010) for further details). Note that if this approach is taken, all external forces must be linear (Assumption 8).
2. Prescribe the motion amplitudes $\widehat{X}^{p_i}$ and solve for just the partial wave amplitudes $\mathbf{A}_i^{\mathrm{S}}$ for a number of special cases in order to determine the hydrodynamic interactions alone. The usual way of performing this is to:
   - **(a)** First solve the array scattering problem by setting $\widehat{X}^{p_i} = 0$ for all p_i in Eq. (9.67).
   - **(b)** Next, for each mode of motion in the array, solve a separate radiation problem such that in Eq. (9.67), $\mathbf{a}_j = 0$ (that is to say, in the absence of incident waves) and $\widehat{X}^{p_i} = 1$ for a single value of p_i, taking the value zero otherwise.
   - **(c)** Use the exact solution for the potential in each of the preceding cases to derive forces on relevant bodies and hence calculate excitation coefficients and added mass and damping matrices for use in dedicated multibody dynamics codes.

Whichever path is chosen to form a complete system of equations, all of the unknown complex amplitudes may be grouped together in a single vector. The scattering equations in Eq. (9.67) along with any equations of motion may then be grouped into a single matrix of factors and an inhomogeneous right-hand side. The solution to the system is then most easily found by direct matrix inversion.

Note that the diffraction transfer matrix $\mathbf{S}_j$ in its most general form explicitly describes the transformation of progressive and evanescent incident waves in all angular modes into progressive and evanescent scattered waves in another set of angular modes. It is hence a relatively straightforward matter to modify the theory underpinning the direct matrix (and multiple scattering) method to neglect some of these components, if not considered significant, by removing the appropriate entries from the diffraction transfer matrix as well as the remaining vectors and matrices. For example, it is possible to keep only progressive modes, which corresponds to a 'Kochin function' representation of outgoing waves (Babarit et al., 2013).

Note that the direct matrix method essentially considers exactly the same processes as the multiple scattering technique (see Section 9.6). The main difference is that Eqs (9.63), (9.64), (9.65) of the multiple scattering technique represent an iterative solution to Eq. (9.67) of the direct matrix method. This is less accurate in principle, relying on the convergence of a series of interaction orders instead of computing the solution directly. However, it may be more computationally efficient in certain circumstances, where convergence with respect to interaction order is fast and where matrix inversion is impractical.

9.7 CAPABILITIES AND LIMITATIONS

9.7.1 Comparison Between Semi-analytical Methods

Although the limitations of each of the semi-analytical methods have been discussed in the preceding sections of this chapter, it is

TABLE 9.1 Comparison of Limitations and Capabilities of All Semi-analytical Techniques

Aspect	Point Absorber	Plane Wave	Multiple Scattering	Direct Matrix
Layout constraints	Wide spacing (device is small compared to wavelength and spacing) such that scattered waves can be neglected	Wide spacing (spacing is large compared to wavelength) such that circular waves can be approximated as plane waves at other devices	Escribed vertical cylinder to each device origin cannot contain another origin. Vertical projections of device geometries cannot overlap	Escribed vertical cylinder to each device origin cannot contain another origin. Vertical projections of device geometries cannot overlap
Array radiation	Included	Included	Included	Included
Array scattering	Neglected	Included (as an approximation)	Included	Included
Evanescent waves	Not used	Neglected	Included	Included
Isolated device hydrodynamic solution required	Far-field radiated wave amplitudes. Optimal power may be found without reference to exact geometry	Scattered wave field under plane progressive wave incidence. Radiated wave field for unit amplitude motion in each mode	Scattered wave field under progressive and (optionally neglected) evanescent wave incidence. Radiated wave field for unit amplitude motion in each mode	Scattered wave field under progressive and (optionally neglected) evanescent wave incidence. Radiated wave field for unit amplitude motion in each mode
Series needed to converge	None	Angular mode expansion	Angular and vertical mode expansions, interaction order	Angular and vertical mode expansions
Matrix size to compute complete solution[a]	Inversion of matrix of size N	$O(P)$ inversions of matrices of size $N(N-1)$	$O(PNQ)$ matrix-vector multiplications using square matrices of size M_0N_0	$O(P)$ inversions of matrices of size NM_0N_0

[a] N *is the number of devices in the array,* P *is the total number of modes of motion in the array,* Q *is number of interaction orders,* M_0, N_0 *the number of vertical and angular modes considered, respectively.*

instructive to consider them alongside one another (as shown in Table 9.1).

Overall, it is clear that the point absorber method cannot be as accurate as the other methods, due to the omission of scattered wave effects in the array. However, for the simplest case, no device geometry or PTO characteristics are necessary in order to efficiently derive an upper bound on power absorption and thus it may be useful for initial optimization and parametric studies.

The plane wave method approximates the solution to the array scattering problem rather than computing it exactly as for the multiple scattering and direct matrix methods. Although results from these methods may be indistinguishable for relevant portions of parameter space (Mavrakos and McIver, 1997), the technique has largely fallen out of favour, with authors preferring methods that can produce the exact linear wave theory solution or are more computationally attractive.

The multiple scattering technique, on the other hand, does produce the exact solution with respect to linear wave theory but relies on the convergence of the solution over interaction

orders. Furthermore, Linton and McIver (2001) state that the multiple scattering method 'rapidly becomes unmanageable as the number of scatterers increases'.

The direct matrix method again produces the exact linear wave theory solution but exchanges the convergence of an iterative series for the direct solution of a larger matrix equation. This trade-off has been considered worthwhile for a number of recent authors in the field (Child and Venugopal, 2010; McNatt et al., 2015; Ruiz et al., 2015).

9.7.2 Comparison With Other Methods

Considering now the collection of semi-analytical methods described in this chapter, in comparison to other array interaction procedures, a number of advantages are apparent. First, all of the methods described are capable of representing the effects of phase differences between wave components, including those belonging to incident and interacted wave fields. This is a feature shared by BEMs (see Chapter 8) but not by modified spectral wave models (see Chapter 11). In fact two of the methods in this chapter (the multiple scattering and direct matrix techniques) are capable of producing the 'exact' hydrodynamic solution with respect to linear wave theory.

Due to the explicit way in which the specific physical processes associated with array interaction are represented, relatively few variables are needed to completely determine the solutions. Therefore there is an efficiency advantage when solving for the solution with semi-analytical techniques over BEMs, for example. Hence these methods are more suited to optimization studies as well as the analysis of greater numbers of devices than BEMs.

Although the derivation of the mathematical expressions contained in this chapter may seem complicated, the numerical implementation of the methods is generally straightforward. Thus, semi-analytical techniques provide a widely accessible gateway to the research and analysis of array interactions for the community of wave energy numerical modellers.

The explicit formulation of the semi-analytical techniques also has a final, significant advantage over other methods. That is to say that the close link between numerically calculated quantities and intuitive physical quantities (for example, the amplitude of the radially symmetric part of the travelling wave scattered by each device) allows a deeper level of understanding of array processes than other 'black box' solvers. This bestows great flexibility in the approach taken, allowing features to be suppressed if their effects are not considered significant, as well as naturally extended via, for example, coupling with other solvers.

The limitations on the collection of semi-analytical techniques include the following key considerations. Firstly, the techniques concentrate on finding the hydrodynamic array interaction solution in the frequency domain under the assumptions of linear wave theory. Hence if not included in the solution procedure, body dynamics (and power estimates) must be solved for separately using the derived hydrodynamic properties and, typically a frequency-, spectral- or time-domain model (see Chapters 2, 4 and 3 respectively). The restriction to linear wave theory and the absence of depth variation, currents, sources and sinks of energy such as wind forcing and other nonlinear processes such as wave breaking mean that the derived interacted wave field may not be entirely representative of realistic conditions if these phenomena are significant. Furthermore, the associated assumption that the forces on each device may be approximated by those in its mean position may lead to inaccuracies for floating devices that deviate significantly from their installed locations in the course of operation. Finally, some restrictions on the array layout that can be analyzed exist for all methods, although these limitations are mild for at least multiple scattering and direct matrix methods.

With respect to the accuracy of the derived solution, some of the methods (that is to say the point absorber and plane wave techniques) involve approximations which could lead to deviations from the true solution attainable under linear wave theory (see Section 9.7.3). Slow convergence in particular cases of the infinite summations used in all techniques apart from the point absorber method may lead to a loss of accuracy or alternatively an increased computational time for the procedure. All of the semi-analytical techniques, apart from the multiple scattering method, rely on the inversion of a matrix which is at least the size of the number of devices in the array. Even though the techniques have favourable efficiency characteristics in comparison to BEMs, the solution time will still inevitably rise rapidly as the number of devices increases. Furthermore, the number of devices in the array may affect the rate of convergence with respect to scattering order in the multiple scattering technique, which in turn will affect solution time.

Many of the semi-analytical techniques were demonstrated in the original works with known analytical solutions for simplified WECs. One of the difficulties in applying these techniques to arbitrary device geometries therefore is creating characterizations of the isolated device with respect to scattering and radiation processes. However, the output of fluid pressure at user-defined field points in recent versions of popular BEM codes has allowed arbitrary geometries to be described in a manner compatible with semi-analytical techniques (McNatt et al., 2013).

9.7.3 Verification and Validation

A comparison of the multiple scattering, plane wave and point absorber methods has been carried out by Mavrakos and McIver (1997) in the context of wave power. They measured the accuracy of the latter two techniques against that of the former 'exact' method for arrays of point absorber type devices. Fig. 9.3 shows the results of their comparison for an array of five devices, equally spaced by a separation distance of five device radii a in a line perpendicular to the oncoming waves, in terms of the array interaction factor q (measured against nondimensional progressive wave number ka). In general, the plane wave method was found

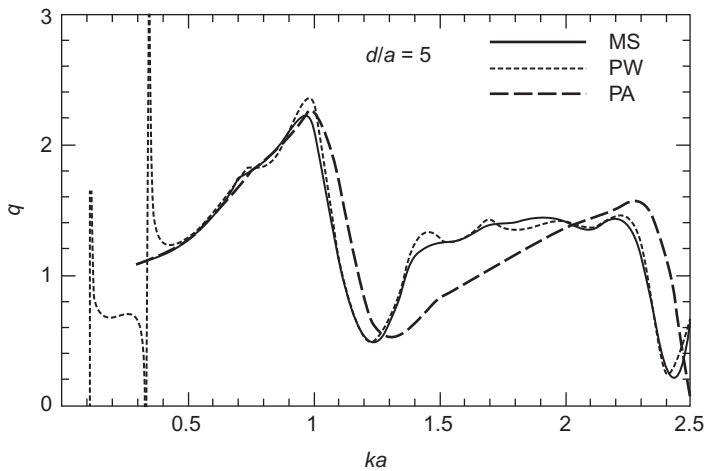

FIG. 9.3 Comparison of results from multiple scattering (MS), plane wave (PW) and point absorber (PA) methods: q-factor *versus* ka for an array of five devices. Taken from Mavrakos and McIver (1997).

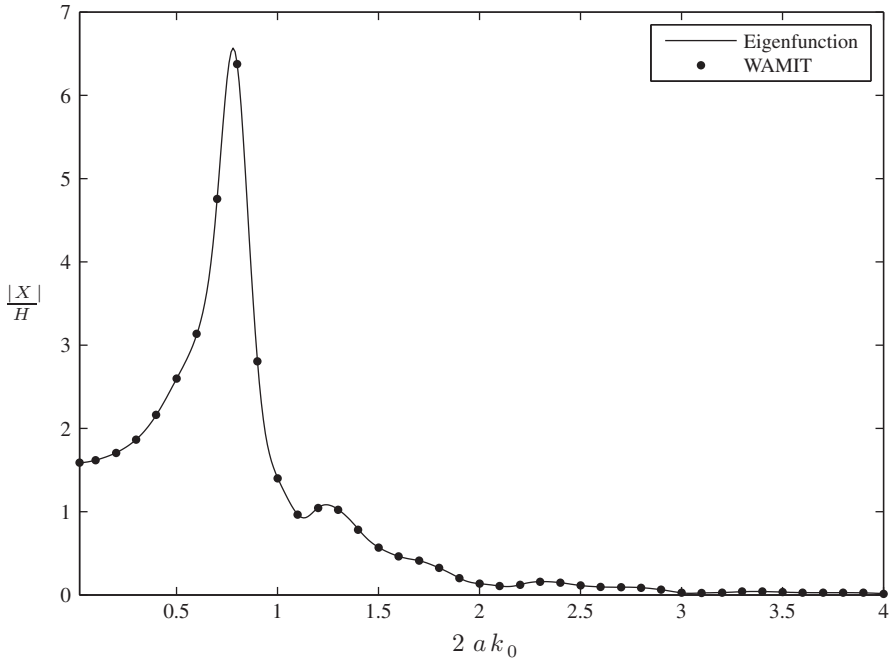

FIG. 9.4 Comparison of results from direct matrix (eigenfunction) method and BEM (WAMIT) simulation. Variation of response amplitude operator in heave with non-dimensional ambient incident wave number for an array of two devices. Taken from Child (2011).

to break down in long waves but perform well in the high frequency range, whilst for the point absorber approximation the converse was seen to be true. The direct matrix method should theoretically yield identical results to the multiple scattering method, although the accuracy of the latter is also subject to the summation over scattering order having sufficiently converged.

Child (2011) performed a verification exercise on the direct matrix method, whereby an array of five point absorber type devices were again arranged in a line perpendicular to the oncoming wave direction, this time separated by a centre-to-centre distance of 12 device radii. The response amplitude operator (RAO) of the central device in heave was monitored as nondimensional ambient incident wave number $2ak_0$ was varied. Calculations were performed using the direct matrix method as well as commercial BEM code WAMIT (2000), as shown in Fig. 9.4. It can be seen that the results match very closely, confirming that both codes have successfully found the same linear wave theory solution.

Further direct verification and validation of the semi-analytical tools described in this chapter are perhaps not as crucial as additional investigations comparing predictions derived from linear wave theory (via BEM, semi-analytical technique, or otherwise) to experimental results (at partial as well as full scale) and more detailed simulation techniques (such as computational fluid dynamics, CFD).

9.8 SUMMARY

- Four semi-analytical methods for computing hydrodynamic interactions in arrays of WECs have been presented:

  - the point absorber method
  - the plane wave method
  - the multiple scattering method
  - the direct matrix method
- All of these semi-analytical methods are based on linear wave theory.
- The methods presented have the following key characteristics:
  - The point absorber method is likely to be least accurate but also least computationally demanding.
  - The plane wave method is likely to be reasonably accurate provided that the separation distances in the array are sufficiently large.
  - The multiple scattering and direct matrix methods can provide 'exact' solutions to the array interaction problem.
- Semi-analytical methods are generally more computationally efficient in calculating array interactions than methods based on BEM models, whilst potentially attaining the same degree of accuracy.
- Semi-analytical methods can provide insight into the mechanisms of array interaction due to their explicit formulation, which can be linked to specific physical processes.
- The direct matrix method is perhaps the currently most popular semi-analytical method, although all techniques are best suited to different contexts.

References

Abramowitz, M., Stegun, I.A. (Eds.), 1964. Handbook of Mathematical Functions. Government Printing Office, Washington, DC.

Babarit, A., et al., 2013. On the modelling of WECs in wave models using far field coefficients. In: Proceedings of the 10th European Wave and Tidal Energy Conference, Aalborg, Denmark.

Budal, K., 1977. Theory for absorption of wave power by a system of interacting bodies. J. Ship Res. 21 (4), 248–253.

Child, B.F.M., 2011. On the configuration of arrays of floating wave energy converters. Ph.D. Thesis, The University of Edinburgh, Edinburgh.

Child, B.F.M., Venugopal, V., 2007. Interaction of waves with an array of floating wave energy devices. In: Proceedings of the 7th European Wave and Tidal Energy Conference, Porto, Portugal.

Child, B.F.M., Venugopal, V., 2010. Optimal configurations of wave energy device arrays. Ocean Eng. 37 (16), 1402–1417.

Evans, D.V., 1979. Some theoretical aspects of three dimensional wave-energy absorbers. In: Proceedings of the First Symposium on Wave Energy Utilization, Chalmers University of Technology, Gothenburg, Sweden.

Evans, D.V., 1980. Some analytic results for two and three dimensional wave-energy absorbers. In: Count, B.M. (Ed.), Power From Sea Waves. Academic Press Inc., London, pp. 213–249

Falnes, J., 1980. Radiation impedance matrix and optimum power absorption for interacting oscillators in surface waves. Appl. Ocean Res. 2 (2), 75–80.

Falnes, J., 2002. Ocean Waves and Oscillating Systems: Linear Interactions Including Wave-Energy Extraction. Cambridge University Press, Cambridge.

Fitzgerald, C., 2006. Optimal configurations of arrays of wave-power devices. Master's Thesis, National University of Ireland, Cork.

Fitzgerald, C., Thomas, G., 2007. A preliminary study on the optimal formation of an array of wave power devices. In: Proceedings of the 7th European Wave and Tidal Energy Conference, Porto, Portugal.

Folley, M., Whittaker, T.J.T., 2009. The effect of sub-optimal control and the spectral wave climate on the performance of wave energy converter arrays. Appl. Ocean Res. 31 (4), 260–266.

Garrett, C.J.R., 1971. Wave forces on a circular dock. J. Fluid Mech. 46 (1), 129–139.

Gradshteyn, I.S., Ryztik, I.M., 1965. Table of Integrals, Series and Products. Academic Press, London.

Kagemoto, H., Yue, D.K.P., 1986. Interactions among multiple three-dimensional bodies in water waves: an exact algebraic method. J. Fluid Mech. 166, 189–209.

Kagemoto, H., Yue, D.K.P., 1993. Hydrodynamic interaction analyses of very large floating structures. Mar. Struct. 6, 295–322.

Kashiwagi, M., 2000. Hydrodynamic interactions among a great number of columns supporting a very large flexible structure. J. Fluids Struct. 14, 1013–1034.

Linton, C.M., McIver, P., 2001. Handbook of Mathematical Techniques for Wave/Structure Interactions. Chapman & Hall/CRC, London.

Martin, P.A., 2006. Multiple Scattering: Interaction of Time-Harmonic Waves With *N* Obstacles. Cambridge University Press, Cambridge.

Mavrakos, S.A., 1991. Hydrodynamic coefficients for groups of interacting vertical axisymmetric bodies. Ocean Eng. 18, 485–515.

Mavrakos, S.A., Kalofonos, A., 1997. Power absorption by arrays of interacting vertical axisymmetric wave-energy devices. J. Offshore Mech. Arct. Eng. 119 (4), 244–250.

Mavrakos, S.A., Katsaounis, G.M., Nielsen, K., Lemonis, G., 2004. Numerical performance investigation of an array of heaving wave power converters in front of a vertical breakwater. In: Proceedings of the Fourteenth International Offshore and Polar Engineering, pp. 238–245.

Mavrakos, S.A., Koumoutsakos, P., 1987. Hydrodynamic interaction among vertical axisymmetric bodies restrained in waves. Appl. Ocean Res. 9 (3), 128–140.

Mavrakos, S.A., McIver, P., 1997. Comparison of methods for computing hydrodynamic characteristics of arrays of wave power devices. Appl. Ocean Res. 19, 283–291.

McIver, P., 1984. Wave forces on arrays of floating bodies. J. Eng. Math. 18, 273–285.

McIver, P., Evans, D.V., 1984. Approximation of wave forces on cylinder arrays. Appl. Ocean Res. 6 (2), 101–107.

McNatt, J.C., 2014. mwave: A water wave and wave energy converter computation package [Online]. Available at: https://github.com/camalamadingdong/mwave (accessed 11.07.15).

McNatt, J.C., Venugopal, V., Forehand, D., 2013. The cylindrical wave field of wave energy converters. In: Proceedings of the 10th European Wave and Tidal Energy Conference, Porto, Portugal.

McNatt, J.C., Venugopal, V., Forehand, D., 2015. A novel method for deriving the diffraction transfer matrix and its application to multi-body interactions in water waves. Ocean Eng. 94, 173–185.

Ohkusu, M., 1972. Wave action on groups of vertical circular cylinders. J. Soc. Naval Arch. Jpn.. 131.

Ohkusu, M., 1974. Hydrodynamic forces on multiple cylinders in waves. In: Proceedings of the International Symposium on Dynamics of Marine Vehicles and Structures in Waves, pp. 107–112.

Peter, M.A., Meylan, M.H., 2004. Infinite-depth interaction theory for arbitrary floating bodies applied to wave forcing of ice floes. J. Fluid Mech. 500, 145–167.

Price, A.A.E., 2009. New perspectives on wave energy converter control. Ph.D. Thesis, The University of Edinburgh, Edinburgh.

Ruiz, P.M., Ferri, F., Kofoed, J.P., 2015. Sensitivity analysis of WEC array layout. In: Proceedings of the 11th European Wave and Tidal Energy Conference, Nantes, France.

Simon, M.J., 1982. Multiple scattering in arrays of axisymmetric wave-energy devices. Part 1. A matrix method using a plane-wave approximation. J. Fluid Mech. 120, 1–25.

Spring, B.H., Monkmeyer, P.L., 1974. Interaction of plane waves with vertical cylinders. In: Proceedings of 14th International Conference on Coastal Engineering, pp. 1828–1845.

Srokosz, M.A., 1979. Ph.D. Thesis, University of Bristol, Bristol.

Srokosz, M.A., Evans, D.V., 1979. A theory for wave-power absorbtion by two independently oscillating bodies. J. Fluid Mech. 90 (2), 337–362.

Thomas, G.P., Evans, D.V., 1981. Arrays of three-dimensional wave-energy absorbers. J. Fluid Mech. 108, 67–88.

Twersky, V., 1952. Multiple scattering and radiation by an arbitrary configuration of parallel cylinders. J. Acoust. Soc. Am. 24, 42–46.

WAMIT Inc., 2000. WAMIT User Manual. Cambridge, MA (Online). Available at: http://www.wamit.com/

Yilmaz, O., 1998. Hydrodynamic interactions of waves with group of truncated vertical cylinders. J. Waterw. Port Coast. Ocean Eng. 124 (5), 272–279.

Zaviska, F., 1913. Uber die Beugung elektromagnetischer Wellen an parallelen, unendlich langen Kreiszylindern. Annalen der Physik, Folge.