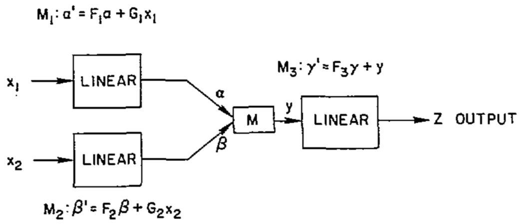

Fig. 3. A bilinear system with the "missing" memoryless bilinear map.

product $X_1 \otimes X_2$ of two spaces X_1 and X_2 can be formally represented as a vector space whose typical element can be written as

$$\alpha \otimes \beta = \sum_{i,j} \alpha_i\beta_j e_i \otimes e_j'$$

where $\alpha = \sum_i \alpha_i e_i$ and $\beta = \sum_j \beta_j e_j'$ are representations of the vectors α and β with respect to bases e_i and e_j' in X and X', respectively. Thus $\alpha \otimes \beta' = \gamma \otimes \delta$ iff $\alpha = k\gamma$ and $\beta = k^{-1}\delta$ for some nonzero scalar k, unless α or β is zero.

What we have done is to define a *bilinear* map

$$\otimes : X_1 \times X_2 \to\to X_1 \otimes X_2 : (\alpha,\beta) \mapsto \alpha \otimes \beta.$$

In fact any bilinear map $HX_1 \times X_2 \to Y$ can be factored through $X_1 \otimes X_2$, i.e., there exists a linear map $\hat{H}: X_1 \otimes X_2 \to Y$ such that $H = \hat{H} \cdot \otimes$. In particular, linear maps $F_1 : X_1 \to X_1$ and $F_2 : X_2 \to X_2$ induce a well-defined linear map

$$F_1 \otimes F_2 : X_1 \otimes X_2 \to X_1 \otimes X_2 : \alpha \otimes \beta \mapsto F_1\alpha \otimes F_2\beta.$$

To reduce the system of Fig. 3 is now straightforward. We leave M_1 and M_2 as they are, with state spaces X_1 and X_2, respectively. We replace M_3 with state space X by M_3' with state space $X \times (X_1 \otimes X_2)$. We then define the connecting maps

$$J_1 : X_1 \times U_2 \to X \times (X_1 \otimes X_2) : (\alpha, x_2) \mapsto (0, F_1\alpha \otimes G_2x_2)$$

$$J_2 : U_1 \times X_2 \to X \times (X_1 \otimes X_2) : (x_1, \beta) \mapsto (0, G_1x_1 \otimes F_2\beta)$$

$$G_2 : U_1 \times U_2 \to X \times (X_1 \otimes X_2) : (x_1, x_2) \mapsto (0, G_1x_1 \otimes G_2x_2)$$

while the linear state-transition map of M_3' is given by

$$X \times (X_1 \otimes X_2) \to X \times (X_1 \otimes X_2):$$

$$(\gamma, \alpha \otimes \beta) \mapsto (F_3\alpha + M\alpha \otimes \beta, F_1\alpha \otimes F_2\beta).$$

Then if M_1, M_2, M_3' are in states α,β and $(\alpha,\beta,\alpha \otimes \beta)$, respectively, at time t and receive input

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

then at time $t + 1$ their states will be $F_1\alpha + G_1x_1$, $F_2\beta + G_2x_2$, and

$$(F_3\alpha + \hat{M}\alpha \otimes \beta, F_1\alpha \otimes F_2\beta) + (0, F_1\alpha \otimes G_2x_2) + (0, G_1x_1 \otimes F_2\beta)$$

$$+ (0, G_1x_1 \otimes G_2x_2) = (F_3\alpha + M(\alpha,\beta), (F_1\alpha + G_1x) \otimes (F_2\beta + G_2x_2))$$

respectively, i.e., the state is updated as required. The previous construction suggests that although a map $X_1 \times X_2 \to X$ may not be necessary, it is certainly economical, and should thus be included in practical decompositions.

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An Algorithm for Inverting Linear Dynamic Systems

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Abstract—An algorithm is presented for inverting transformations associated with linear dynamic systems. The induced inverse system, when it exists, operates on outputs and generates the corresponding inputs of the system to which it is inverse. The applications for inverse systems are found in such diverse areas as control, coding, filtering, and sensitivity analysis.

I. Introduction

While [1]–[5] and [10] are all related to the present study, the algorithm of [3] is the most similar. The difference between the present algorithm and that of [3] will become clear as the treatment unfolds. In brief, the dimensions of the input, output, and state spaces are reduced systematically at each step of the present algorithm. The present algorithm also inverts the system as it proceeds and is felt to offer some advantage in conceptual simplicity.

In this study we consider a linear dynamic system of the form

$$\dot{x} = Ax + Bu$$

$$y = Cx \tag{1}$$

where $x = (x_1,\cdots,x_n)$, $u = (u_1,\cdots,u_m)$, and $y = (y_1,\cdots,y_r)$ while A, B, and C are constant matrices of compatible dimensions. While our attention here is restricted to stationary differential systems the algorithm may be extended to encompass discrete time and/or nonstationary systems. The necessary extensions are analogous in part to the developments of [3] and [6]. For the sake of brevity these are left to the interested reader.

In the following c_i, $i = 1,\cdots,r$, will denote the ith row of the matrix C while

$$\alpha_i = \min \{ j : c_iA^{j-1}B \neq 0, \quad j = 1,2,\cdots,n \}.$$

The matrix $\tilde{A}$ has as its ith row the tuplet $c_iA^{\alpha_i-1}$, $i = 1,\cdots,m$. The symbol D denotes the diagonal differential operator given by $Dy = (y_1^{\alpha_1},\cdots,y_r^{\alpha_r})$. These quantities are described in more detail in [6] and [7]. With these definitions in hand the system of (1) may be equivalently modeled in the form

$$\dot{x} = Ax + Bu$$

$$Dy = Kx + Eu \tag{2}$$

$$\Phi y = Qx$$

where the operator Φ and the matrix Q are defined below and $E = \tilde{A}B$, $K = \tilde{A}A$.

It is shown in the following that (2) may be brought to the form

$$\dot{p} = A'p + B'\mu$$

$$\eta = C'p \tag{3}$$

which is in the form of (1). In (3), however, the tuplet p has "n − rank [Q]" components, μ has "m − rank [E]" components, and η has "r − rank [E]" components, while A',B',C' are of compatible dimensions. In the transition from (1) to (3) additional information is obtained so that the process is reversible. In this sense the mapping from (1) to (3) is said to be a *reducing cycle* (on the system equations).

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II. A Reducing Cycle

In arriving at (2), recall that the typical component of y satisfies

$$
\begin{aligned}
y_i^{(0)} &= c_iA^0x \\
\vdots \quad & \quad \vdots \\
y_i^{(\alpha_i-1)} &= c_iA^{\alpha_i-1}x \\
y_i^{(\alpha_i)} &= c_iA^{\alpha_i}x + c_iA^{\alpha_i-1}Bu.
\end{aligned}
$$

The last equations, as $i = 1,\cdots,r$, are embodied in the vector matrix equation $Dy = Kx + Eu$. As to the first equations, define $y_{[\alpha_i]} = (y_i^{(0)},\cdots,y_i^{(\alpha_i-1)})$ and form the matrix Q_i by using c_i, $c_iA,\cdots,c_iA^{\alpha_i-1}$ as rows, in which case

$$y_{[\alpha_i]} = Q_ix, \quad i = 1,\cdots,r \tag{4}$$

summarizes this information. Here the matrices Q_i are a priori known and the vectors $y_{[\alpha_i]}$ are completely determined by y.

Consider now the rows of the matrices $\{Q_i\}$ taken together. Select a maximal independent set in the following manner. Form the sets $X_0 = \{c_1,\cdots,c_r\}$, $X_1 = \{c_1A,\cdots,c_rA\}$, $\cdots$, $X_j = \{c_1A^j,\cdots,c_rA^j\},\cdots,$ where c_iA^j is deleted whenever $j \geq \alpha_i$. Take all[1] the rows of X_0, add rows from X_1 in any order deleting any rows that are linearly dependent on X_0 and earlier additions from X_1. When X_1 is exhausted repeat the process with the set X_2 and continue until n rows have been included or the sets X_j have been exhausted.

The matrix Q is defined by using the resultant linearly independent set as rows. Then $\Phi y = Qx$ represents the obvious subset of (4). Inherent in the selection process is the guarantee that Φy contains the minimum number of total derivatives of components of y for any such maximal linearly independent subset of (4).

We consider now the transition from (2) to (3). First if Q is nonsingular then $x = Q^{-1}\Phi y$. This determines $\dot{x}\text{-}Ax$ and hence[1] u. In this case the function u of (1) can be determined directly from y and derivatives of its components. Similarly if E is 1:1 then (2) can be solved for u as a function of Dy. In the following we assume that neither Q nor E is 1:1.

Let T be any nonsingular matrix such that $QT = [I \mid 0]$. That such a matrix exists follows from the linear independence of the rows of Q. Similarly nonsingular matrices S,V exist such that $P = SEV$ is diagonal with an identity submatrix in the upper left corner and 0 elsewhere. Making the change of variables $r = Tx$, $u = Vv$, (2) then takes the equivalent form

$$
\begin{aligned}
\dot{r} &= Fr + Gv \\
SDy &= Jr + Pv \\
\Phi y &= [I \mid 0]r
\end{aligned}
\tag{5}
$$

where $F = T^{-1}AT, G = T^{-1}B, J = SKT$. With the obvious partitions this equation set may be written

$$
\begin{aligned}
\dot{r}_1 &= F_{11}r_1 + F_{12}r_2 + G_{11}v_1 + G_{12}v_2 \\
\dot{r}_2 &= F_{21}r_1 + F_{22}r_2 + G_{21}v_1 + G_{22}v_2 \\
(SDy)_1 &= J_{11}r_1 + J_{12}r_2 + v_1 \\
(SDy)_2 &= J_{21}r_1 + J_{22}r_2 \\
\Phi y &= r_1.
\end{aligned}
$$

By direct manipulation these equations reduce to

$$
\begin{aligned}
r_1 &= \Phi y \\
v_1 &= (SDy)_1 - J_{11}\Phi y - J_{12}r_2 \\
\dot{r}_2 &= (F_{22} - G_{21}J_{12})r_2 + G_{22}v_2 + (F_{21} - G_{21}J_{11})\Phi y + G_{21}(SDy)_1 \\
& (SDy)_2 - J_{21}\Phi y = J_{22}r_2.
\end{aligned}
\tag{6}
$$

[1] We assume B, C have maximum rank.

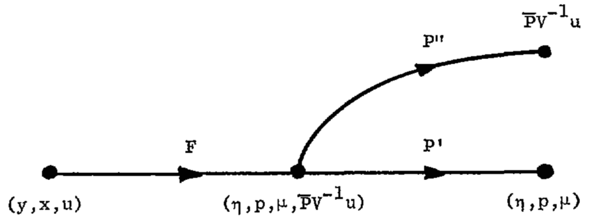

Fig. 1. Reducing cycle.

To complete the development choose $A' = F_{22} - G_{21}J_{12}$, $B' = G_{22}$, and $C' = J_{22}$. Let ρ denote the solution to the equation

$$\dot{\rho} = A'\rho + (F_{21} - G_{21}J_{11})\Phi y + G_{21}(SDy)_1, \quad \rho(0) = 0$$

and let

$$
\begin{aligned}
\eta &= (SDy)_2 - J_{21}\Phi y - C'\rho \\
p &= r_2 - \rho, \quad \mu = v_2.
\end{aligned}
$$

Then (η,p,μ) satisfies (3) as can be readily verified.

The process is reversible in the following sense. If (η,ρ,μ) satisfies (3) then $v_2 = \mu$, $r_2 = p + \rho$ together with (6) establishes a tuplet (r,v) satisfying (5), and using $p = T^{-1}r$, $u = Vv$ a solution tuplet (y,x,u) for (1).

To summarize this development let $\bar{P},\tilde{P}$ denote the orthogonal projections $v \rightarrow (v_1,0)$ and $r \rightarrow (r_1,0)$, respectively, then

$$
\begin{aligned}
A' &= (I - \tilde{P})T^{-1}[A - B\bar{P}SK]T(I - \tilde{P}) \\
B' &= (I - \tilde{P})T^{-1}B(I - \tilde{P}) \\
C' &= (I - \bar{P})SKT(I - \tilde{P}).
\end{aligned}
\tag{7}
$$

Here a slight abuse of notation is used in that $(I - \tilde{P})T^{-1}B(I - \tilde{P})$, for instance, is really an $n \times m$ matrix whose only nonzero submatrix is B'. In the same vein suppose that

$$\xi(y) = (I - \tilde{P})T^{-1}\{[A - B\bar{P}SK]T\bar{P}\Phi y + B\bar{P}SDy\}$$

while $\rho(y),\Delta(y)$ are defined by

$$
\begin{aligned}
\dot{\rho} &= A'\rho + \xi, \quad \rho(0) = 0 \\
\Delta &= \bar{P}[SD - SKT\Phi]y.
\end{aligned}
$$

Then (η,p,μ) satisfies (3) if and only if (y,x,u) satisfies (1) where

$$
\begin{aligned}
u &= [\Delta(y) - \bar{P}SKT(1 - \tilde{P})r,\mu) \\
x &= T^{-1}(\Phi y,p + \rho)
\end{aligned}
\tag{8}
$$

or conversely

$$
\begin{aligned}
\mu &= (I - \bar{P})V^{-1}u \\
P &= (I - \tilde{P})Tx + \rho.
\end{aligned}
\tag{9}
$$

We then have the following theorem.

Theorem 1

The tuplet (y,x,u) satisfies (1) if and only if the tuplet (η,p,μ) satisfies (2) where the invertable map $(y,x,u) \leftrightarrow (\eta,p,\mu,\bar{P}V^{-1}u)$ is given in (8) and (9). Moreover in the cycle $(y,x,u) \rightarrow (\eta,p,\mu)$ the partial solution

$$
\begin{aligned}
\bar{P}V^{-1}u &= \Delta(y) - \bar{P}SKT(I - \tilde{P})r \\
\tilde{P}T^{-1}x &= \Phi y
\end{aligned}
$$

to (1) is obtained.

In view of Theorem 1 the cyclic process may be depicted in the following way. Consider the sets $\{(y,x,u)\}$ and $\{(\eta,p,\mu)\}$ of all solution tuplets to (1) and (3), respectively. We have then an invertable mapping $\{(y,x,u)\} \leftrightarrow \{(\eta,p,\mu,\bar{P}V^{-1}u)\}$ which we denote by $\mathfrak{F}$ and compute by (7), (8), and (9). It is not difficult to show also that $\mathfrak{F}$ is linear, provided the domain of the original system is linear. The cycle $\mathfrak{F}$ may be depicted as in Fig. 1.

III. Inversion of Linear Systems

Consider now the problem of inverting the map of the system of (1). That is, given y find u such that (1) holds. Our system has no direct transmission (that is, $y = Cx + Eu$, $E \neq 0$). However, if direct transmission is present it may be removed by methods suggested in the previous reduction of (2) to (3). Thus, our consideration of (1) is without loss of generality.

In view of Theorem 1 an inverse mapping, if one exists, may be constructed as follows. The system is subjected to the cycle $\mathfrak{F}$ with only two possible results. The first is $\bar{P} = I$, in which case E is invertable, and the inverse is at hand:

$$u = E^{-1}\{Dy - Kx\}$$
$$\dot{x} = (A - BE^{-1}K)x + BE^{-1}Dy.$$

The second is $\bar{P} \neq I$, in which case the function $\bar{P}V^{-1}u$ is obtained and a residue system of reduced order is specified. In this case the cycle $\mathfrak{F}$ may be applied to the residue system and the process repeated.

Now let $E(j)$ denote the direct transmission matrix of (2) obtained in the jth application of $\mathfrak{F}$. Then concerning the previous iterative process $\mathfrak{F},\mathfrak{F}^2,\cdots$ we have the following lemma.

Lemma 1

The system is invertable if and only if an inverse is explicitly specified by the iterates $\mathfrak{F},\mathfrak{F}^2,\cdots,\mathfrak{F}^k$ where $\sum_{j=1}^{k} \text{rank}\,[E(j)] = m$.

In proving this lemma it is convenient also to delineate the inverse system. In this regard recall that, in terms of Fig. 1, $u = V(\bar{P}V^{-1}u + \mu)$. Similarly at the jth step of the iteration process

$$\mu(j) = V(j)[\bar{P}(j)V(j)^{-1}\mu(j) + u(j+1)] \tag{10}$$

where, for instance, the notation $V(j)$ is used to denote the matrix V as it occurs between $\mathfrak{F}^j$ and $\mathfrak{F}^{j+1}$. Thus if $E(j)$ is invertable at any step then $\mu(j+1)$ is obtained from

$$\mu(j+1) = E(j)^{-1}\{D(j)\eta(j) - K(j)q(j)\}$$
$$\dot{q}(j) = [A(j) - B(j)E^{-1}(j)K(j)]q(j) + B(j)E^{-1}(j)D(j)\eta(j). \tag{11}$$

Equation (11) then specifies $u(j)$ and in turn $\mu(j-1),\cdots,\mu(0) = u$.

We note also that if $\mu(j)$ has l components then $\mu(j+1)$ has $l - \text{rank}\,[E(j)]$ components and by induction $\mu(j)$ has $m - \sum_{l=1}^{j-1} \text{rank}\,[E(l)]$ components. Thus the components in the sequence $\{\mu(j)\}$ are monotonically reduced, provided $E(j) \neq 0$. The sequence obviously terminates at any iterate k for which $E(k)$ has full rank. It is also clear that $m = \sum_{l=1}^{k} \text{rank}\,[E(l)]$.

It remains only to consider the case where $E(j) = 0$ occurs. For this we return to the context of (1) and (2). This implies, for instance, that

$$y_i^{(0)} = c_i x,\cdots,y_i^{(n)} = c_i A^n x \tag{12}$$
$$c_i B = \cdots = c_i A^{n-1} B = 0.$$

Using the Cayley–Hamilton theorem it follows easily from (12) that y_i satisfies a homogeneous nth-order differential equation and hence is entirely independent of x and u. In this case the map $u \rightarrow y$ is not one-to-one and the lemma is proved.

Concerning the matrix Q we note that in view of X_0 it follows that rank $[C] \leq$ rank $[Q]$. In addition it is easily seen that E is singular $\Rightarrow Q$ is singular. Indeed if E is singular then $0 \neq a \in R^m$ exists such that $\tilde{A}Ba = 0$ and hence a $0 \neq d \in$ Range $[B]$ exists such that $\tilde{A}d = 0$. This implies that all vectors $\{c_i A^{\alpha_i - 1}, i = 1,\cdots,r\}$ are orthogonal to d. Since all the vectors $\{c_i A^j : j < \alpha_i - 1, i = 1,\cdots,r\}$ are orthogonal to range $[B]$, the singularity of Q follows. It is then easily seen that rank $[Q] \leq n$-nullity $[E]$, a condition which may be used to terminate the selection procedure for Q.

IV. Conclusion

In this study the problem of inverting the transformation inherent in a linear dynamic system is considered. When an inverse exists, the algorithm of Section III provides a realization of the inverse system. If the system in question is not invertable then the algorithm detects this condition.

While stationary differential equations were chosen as the vehicle for the analysis, analogous methods can be easily developed for time-varying and/or discrete time systems. Although such developments are straightforward (see [2], [6]) they are time and space consuming and as such are left to the reader.

Omitted from the present study is a definitive comparison of the algorithms of [3] and [5] with the present algorithm. While some similarities and differences are apparent such a comparison involves difficulties which cannot be adequately dealt with within the confines of the present paper. Moreover, it is felt that a broader comparison, over a suitable class of inversion algorithms, would be a more meaningful objective.

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Canonical Forms for Multiple-Input Time-Variable Systems

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***Abstract*—In this paper necessary and sufficient conditions are given for the existence of a transformation that will transform a time-variable linear system into a multiple-input canonical form. A procedure for generating such a transformation is also described.**

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