# Comments on "Mixed $L^\infty/H_\infty$ Suboptimal Controllers for SISO Continuous Time Systems"

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***Abstract*—This paper gives two remarks on the example in Section IV of the above-mentioned paper.**

***Index Terms*—Control systems, $H_\infty$ control, optimal control, suboptimal control.**

## I. Remark 1

The example illustrates a method to do tradeoffs between two design goals: $H_\infty$ and what the authors of the above paper call $L^\infty$ control. To make a fair comparison with $H_\infty$ design possible, we believe the optimal $H_\infty$ norm should have been presented. As the authors themselves point out the value given in Table I $\|T_{\zeta\omega}\|_{\mathcal{H}_\infty} = 4.31$ is suboptimal and hence just an arbitrary number that might give misleading information. The optimal $H_\infty$ norm is 3 and a controller of the form

$$C_\epsilon(s) = -\frac{3}{2}\frac{s-\epsilon}{s}\frac{1}{1-\epsilon s}$$

(which is proper and includes integral action) gives a stable closed-loop system with an $H_\infty$-norm with $\|T_{\zeta\omega}\|_{\mathcal{H}_\infty} \to 3$ when $\epsilon \to 0+$. With $\epsilon = 0.01$ the controller $C_\epsilon(s)$ gives $\|T_{\zeta\omega}\|_{\mathcal{H}_\infty} = 3.19$ and $\|e\|_{L^\infty} = 3.812$, which is better than both values for $H_\infty$ design presented in Table I. The controller obtained with the mixed $L^\infty/H_\infty$ technique gives an increase in $\|T_{\zeta\omega}\|_{\mathcal{H}_\infty}$ of $(4.77 - 3.19)/3.19 = 50\%$ compared to the controller $C_\epsilon(s)$ with $\epsilon = 0.01$. This is considerably worse than the 10% mentioned in the text. It is also claimed that the controller obtained from the mixed $L^\infty/H_\infty$ design gives a 27% reduction of the peak tracking error compared to the suboptimal $H_\infty$ controller. This is a misprint, and it should be $0.27/1.27 = 21\%$.

## II. Remark 2

The formula for the $L^\infty$-controller contains some misprints. The following controller will give the values suggested in Table I:

$$K_{L^\infty} = -\frac{\sqrt{2}s(s-2)\left(1+\sqrt{2}-(2+\sqrt{2})e^{-0.3466s}\right)}{(s-1)\left(1+\sqrt{2}\right)(1-2e^{-0.3466s})}.$$

Note that the singularities in $s = 1$ and $s = 2$ are removable in the sense of analytical functions. Such a reduction is required for the controller to stabilize the plant. The issue of implementation should be investigated further.

## III. Remark 3

Note that the $L_\infty$ controller cancels the integrator pole at $s = 0$. Hence zero steady-state error is not achieved.

Also the settling time of the controller $C_\epsilon(s)$ tends to infinity when $\epsilon \to 0$.

We would like to point out that we have no serious criticism against the theory presented in the paper. The authors present a possible design tool for finding compromises between $H_\infty$ and $L^\infty$ optimal controllers.

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# System Zeros Analysis via the Moore–Penrose Pseudoinverse and SVD of the First Nonzero Markov Parameter

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***Abstract*—A new characterization of system zeros of an arbitrary linear system described by a state-space model $S(A, B, C, D)$ is presented. The transmission zeros are characterized as invariant zeros of an appropriate strictly proper system with a smaller number of inputs and outputs than the original system. The approach is based on singular value decomposition (SVD) of the first nonzero Markov parameter. This result together with characterization of invariant and decoupling zeros, based on the Moore–Penrose inverse of the first nonzero Markov parameter and the Kalman canonical decomposition theorem, provided in the first part of the paper yield a complete characterization of system zeros of an arbitrary multi-input/multi-output system.**

***Index Terms*—Decoupling zeros, invariant zeros, linear systems, state-space methods, transmission zeros.**

## I. Introduction

The determination of zeros, because of their importance in control theory has received considerable attention in recent years ([1], [4], [7]–[10], [12]–[14], [16], [17], [20], [21], [23], and [24]). The zeros are defined in many ways (for a survey of these definitions see [10], [15], and [23]) so that the term "zero" has become ambiguous. Various approaches and methods of calculation of zeros from the state-space model parameters are given in the literature; e.g., [1], [9], [10], [12], [13], and [23]. However, none of the above-mentioned references relates invariant zeros to controllability and observability. Reference [14] constitutes an exception where in terms of a special coordinate basis (scb) decoupling zeros of strictly proper systems are related to these notions. The scb requires, however, a special high-level software. Invariant zeros of square systems $S(A, B, C)$ of uniform rank are analyzed in [4]. It is shown that these zeros may be determined as some eigenvalues of the matrix $A - B(CA^kB)^{-1}CA^{k+1}(*)$, where $CA^kB$ is the first nonzero Markov parameter.

In this paper the approach of [4] is extended on nonsquare and proper systems. Our aim is also to give a natural and simple tool for relating zeros to the notions of controllability and observability. It should indicate how different kinds of decoupling zeros are connected with individual parts of a system (controllable and unobservable, uncontrollable and unobservable, uncontrollable and observable) via appropriate state vectors (state-zero directions). On the other side, it should deliver a more general expression for real-valued output-zeroing inputs which in a compact form could convey information

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about invariant zeros and their action in a system. The question of interpreting system zeros (transmission + decoupling) of a system $S(A, B, C, D)$ (Section II) is based on the Kalman decomposition theorem [3], [11]. It is shown that in any nondegenerate system $S(A, B, C, D)$ with the first nonzero Markov parameter of full rank all system zeros are completely characterized as eigenvalues associated with the system real matrix of the order of the state matrix. That matrix may be expressed in upper-triangular block form consistent with the Kalman form of the state matrix. The submatrices of the diagonal describe successively output decoupling, transmission + zeros at infinity, input–output decoupling, and input decoupling zeros. The proposed approach is based on the Moore–Penrose inverse of the first nonzero Markov parameter. Invariant zeros are treated as the triples: complex number, state-zero direction, and input-zero direction. Strictly proper and proper systems are discussed separately. For system $S(A, B, C)$ as matrix-characterizing zeros we take matrix $K_k A$, where $K_k := I - B(CA^k B)^+ CA^k$ is idempotent and $(CA^k B)^+$ stands for the Moore–Penrose inverse of the first nonzero Markov parameter (note that $K_k A$ constitutes generalization of $(*)$). For proper systems $A - BD^+ C$ is taken as matrix-describing zeros. Since for any system, including the degenerate case, these matrices disclose all decoupling zeros, the Kalman form of $K_k A$ and $A - BD^+ C$ becomes a natural tool for relating zeros to controllability and observability. It also suggests a simple procedure for the computation of zeros of nonminimal systems. Unfortunately, when the first nonzero Markov parameter of a nondegenerate system is not of full rank, some of the transmission zeros may not appear in the spectrum of $K_k A$ or $A - BD^+ C$. In order to overcome this disadvantage a method for determining transmission zeros of minimal systems is presented (Section III). The method is based on an SVD of the first nonzero Markov parameter [24]. For a system $S(A, B, C, D)$ with $m$ inputs, $r$ outputs, and the first nonzero Markov parameter of $\operatorname{rank} p$ we form at first an auxiliary strictly proper system with $m - p$ inputs and $r - p$ outputs and then discuss relations between sets of invariant zeros of these systems. When the original system is minimal, the set of its zeros coincides with a subset of invariant zeros of the auxiliary system (nonminimal in general).

Throughout the paper $A^T$ $(A^*)$ denotes the transpose (conjugate transpose) of $A$, $I$ denotes an identity matrix of appropriate dimension, while $I_m$ stands for an identity matrix of order $m$, and $\sigma(A)$, $\operatorname{Ker} A$, and $\operatorname{Im} A$ denote the spectrum, the kernel, and the range space of a matrix $A$. By $R, C$ we denote fields of real and complex numbers. Recall [5] that for a given $r \times m$ real matrix $M$ of $\operatorname{rank} p$ a factorization $M = M_1 M_2$ with $r \times p$-dimensional $M_1$ and $p \times m$-dimensional $M_2$ is called skeleton factorization of $M$; $M^+$ is then uniquely determined as $M_2^+ M_1^+$, with $M_1^+ = (M_1^T M_1)^{-1} M_1^T$ and $M_2^+ = M_2^T (M_2 M_2^T)^{-1}$.

A system $S(A, B, C, D)$ of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du \tag{1}$$

is considered, where $\dim x = n$, $\dim u = m$, $\dim y = r$, $A, B, C, D$ are real matrices and the first nonzero Markov parameter is of $\operatorname{rank} 0 < p \leq \min\{m, r\}$. We adopt the following definitions of zeros of (1).

A number $\lambda \in C$ is an invariant zero if there exist $0 \neq x_o \in C^n$ and $g \in C^m$ such that the triple $\lambda$, $x_o$, $g$ satisfies

$$P(\lambda)\begin{bmatrix} x_o \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \text{where } P(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix}. \tag{1a}$$

The vectors $x_o$, $g$ are called the state-zero and input-zero directions. The transmission zeros coincide with invariant zeros of the minimal subsystem of (1). A system is degenerate if it has an infinite number of invariant zeros. A number $\lambda \in C$ is an output decoupling (o.d.) zero if there exists a vector $x_o \neq 0$ such that $Ax_o = \lambda x_o$ and $Cx_o = 0$. A number $\lambda \in C$ is an input decoupling (i.d.) zero if there exists a vector $z_o \neq 0$ such that $\bar{\lambda} z_o^* = z_o^* A$ and $z_o^* B = 0$. A number $\lambda \in C$ is an input–output decoupling (i.o.d.) zero if $\lambda$ is simultaneously an i.d. and an o.d. zero.

## II. Invariant and Decoupling Zeros

### A. Strictly Proper Systems $(D = 0)$

*Definition 1:* For (1) we define a projective matrix $K_k := I - B(CA^k B)^+ CA^k$ and a class of real-valued inputs $u(t, x_o) := -(CA^k B)^+ CA^{k+1} e^{tK_k A} x_o$, $x_o \in R^n$, where $CA^k B$ stands for the first nonzero Markov parameter.

*Lemma 1:* For a given $u(t, x_o)$ the solution of the state equation of (1) corresponding to an initial condition $x(0)$ equals $x(t) = e^{tA}(x(0) - x_o) + e^{tK_k A} x_o$.

*Proof:* The obvious proof by differentiation is omitted here. □

From Lemma 1 it follows that if $\lambda \in \sigma(K_k A)$ and an associated eigenvector $x_o$ of $K_k A$ is in $\operatorname{Ker} C$, then $u(t, x_o)$ constitutes an output-zeroing input for (1) (i.e., $y(t) = 0$ for $t \geq 0$ if $x(0) = x_o$ or $y(t) \to_{t\to\infty} 0$ at any $x(0)$ if $\sigma(A) \subset C^-$). As it is shown below, such a $\lambda$ is an invariant zero of (1) with $x_o$ and $g = u(0, x_o)$ as the corresponding state-zero and input-zero direction. If $\lambda = \alpha + j\omega$, then by linearity of $u(t, x_o)$ in the second variable, the real-valued output-zeroing inputs associated with $\lambda$ are generated by $\operatorname{Re} x_o$ and $\operatorname{Im} x_o$ and take the form $u(t, \operatorname{Re} x_o) = e^{\alpha t}(\operatorname{Re} g \cos \omega t - \operatorname{Im} g \sin \omega t)$, $u(t, \operatorname{Im} x_o) = e^{\alpha t}(\operatorname{Re} g \sin \omega t + \operatorname{Im} g \cos \omega t)$, where $\operatorname{Re} g = -(CA^k B)^+ CA^{k+1} \operatorname{Re} x_o$, $\operatorname{Im} g = -(CA^k B)^+ CA^{k+1} \operatorname{Im} x_o$.

Finally, $K_k A$ has a number of zero eigenvalues with the property that the subspace generated by associated eigenvectors and pseudoeigenvectors has the trivial intersection with $\operatorname{Ker} C$. These eigenvalues will represent zeros at infinity of (1).

Motivated by the above, we give a more detailed discussion of the structure of $K_k A$. Let $CA^k B = H_1 H_2$ denote a skeleton factorization of $CA^k B$. With (1) we associate an auxiliary square ($p$-input, $p$-output) system $S(A, B_D, C_D)$, with $B_D = BH_2^T$, $C_D = H_1^T C$, obtained from (1) by introducing precompensator $H_2^T$ and postcompensator $H_1^T$. The first nonzero Markov parameter $C_D A^k B_D$ of $S(A, B_D, C_D)$ is nonsingular and gives $\operatorname{rank} B_D = \operatorname{rank} C_D = p$ and $p \leq n$. The matrix $K_{k,D} := I - B_D(C_D A^k B_D)^{-1} C_D A^k$ satisfies $K_{k,D} = K_k$. $P_D(s)$ stands for the system matrix of $S(A, B_D, C_D)$.

*Lemma 2:*

$$\begin{aligned} &\det(sI - K_k A) \\ &\quad = \det\bigl(H_1^T H_1\bigr)^{-1} \det\bigl(H_2 H_2^T\bigr)^{-1} s^{p(k+1)} \det P_D(s). \end{aligned} \tag{2}$$

*Proof:* The proof follows from the definition of $K_k$, well-known identities for determinants [2, Appendix B], and from properties of matrices $H_1, H_1^+, H_2, H_2^+$ (see [25] for details). □

*Remark 1:* A number $\lambda \in C$ is an invariant zero of $S(A, B_D, C_D)$ if and only if (iff) $\det P_D(\lambda) = 0$. This follows from the definition of these zeros and the fact that $B_D$ is monic. Since $K_{k,D} A = K_k A$, this means, by virtue of (2), that each invariant zero of $S(A, B_D, C_D)$ has to be in $\sigma(K_{k,D} A)$. The number of such eigenvalues (counting multiplicity) equals $n - p(k+1)$. The remaining $p(k+1)$ zero eigenvalues of $K_{k,D} A$ represent zeros at infinity of $S(A, B_D, C_D)$. It is clear that these arguments hold also for any square system $S(A, B, C)$ of uniform rank (i.e., with nonsingular first nonzero Markov parameter) [4], [13], [17].

For a more detailed treatment of invariant zeros of (1) as eigenvalues of $K_k A$, we need some algebraic properties of $K_k$.

*Lemma 3:* The matrix $K_k$ (Definition 1) has the following properties:

$$\begin{aligned} &K_k^2 = K_k, && \Sigma_k := \{x: K_k x = x\} = \mathrm{Ker}\big(H_1^T CA^k\big) \\ &\dim \Sigma_k = n - p, && \Omega_k := \{x: K_k x = 0\} = \mathrm{Im}\big(BH_2^T\big) \\ &\dim \Omega_k = p, && C^n(R^n) = \Sigma_k \oplus \Omega_k. \end{aligned}$$

Moreover

$$K_k BH_2^T = 0, \qquad H_1^T CA^k K_k = 0$$

and

$$H_1^T C(K_k A)^l = \begin{cases} H_1^T CA^l, & \text{for } 0 \le l \le k \\ 0, & \text{for } l \ge k+1. \end{cases}$$

*Proof:* We proceed with the standard proof for $K_{k,D}$, and then we use relation $K_{k,D} = K_k$. Since $K_k$ is determined uniquely, its properties do not depend upon a particular choice of $H_1, H_2$ in the skeleton factorization of $CA^k B$. □

The $p(k+1)$ zero eigenvalues of $K_k A$ [see (2)], which we treat as representing zeros at infinity of (1), are responsible for transmitting through the system polynomial vector functions of $t$ of degree not larger than $k$. In order to illustrate this role of zeros at infinity assume $\det P_D(0) \neq 0$, $\det A \neq 0$ and consider the subspace $X_\infty = \{x: (K_k A)^{p(k+1)} x = 0\} \subset R^n (\dim X_\infty = p(k+1))$. Define subspaces $X_{\infty,1}, X_{\infty,2}, \cdots, X_{\infty,k+1}$ as

$$\begin{aligned} X_{\infty,1} &= \mathrm{Span}\{A^{-1}b_{D,1}, \cdots, A^{-1}b_{D,p}\} \\ X_{\infty,2} &= \mathrm{Span}\{A^{-1}b_{D,1}, \cdots, A^{-1}b_{D,p}, \\ &\qquad A^{-2}b_{D,1}, \cdots, A^{-2}b_{D,p}\}, \cdots, \\ X_{\infty,k+1} &= \mathrm{Span}\{A^{-1}b_{D,1}, \cdots, A^{-1}b_{D,p}, \cdots, \\ &\qquad A^{-(k+1)}b_{D,1}, \cdots, A^{-(k+1)}b_{D,p}\} \end{aligned}$$

where $b_{D,i} = B_D e_i = BH_2^T e_i$, $i = 1, \cdots, p$, stands for the $i$th column of $B_D$. For these subspaces we have $X_{\infty,1} \subset X_{\infty,2} \subset \cdots \subset X_{\infty,k+1} \subset X_\infty$. The proof of inclusion $X_{\infty,k+1} \subset X_\infty$ follows directly via equalities $K_k(A^{-l}B_D) = A^{-l}B_D$, $l = 1, 2, \cdots, k$, and $K_k B_D = 0$ (comp., Definition 1 and Lemma 3).

*Corollary 1:* Let $\det P_D(0) \neq 0$, $\det A \neq 0$ in (1). Then any input $u(t, x_o)$ generated by $x_o \in X_{\infty,1}$ is a constant vector in $R^m$ and the system response corresponding to the initial condition $x(0) = x_o$ is a constant vector in $R^r$. Furthermore, if $x_o \in X_{\infty,l}$ for some $l = 1, 2, \cdots, k+1$, then the input $u(t, x_o)$ is a polynomial vector function of $t$ of degree $\le l-1$ in $R^m$, while the system response corresponding to this input and to the initial condition $x(0) = x_o$ is a polynomial vector function of $t$ of degree $\le l-1$ in $R^r$; the degree of the output polynomial does not exceed the degree of the input polynomial.

*Proof:* Let $x_o = A^{-l}b_{D,i}$ for some $i = 1, 2, \cdots, p$ and $1 \le l \le k+1$. Then the following equalities hold:

$$e^{tK_k A}x_o = A^{-l}b_{D,i} + \frac{t}{1!}A^{-(l-1)}b_{D,i} + \cdots + \frac{t^{l-1}}{(l-1)!}A^{-1}b_{D,i}$$

$$u(t, x_o) = -(CA^k B)^+ CA^{k+1} e^{tK_k A} x_o = -\frac{t^{l-1}}{(l-1)!}H_2^T e_i$$

$$y(t) = Ce^{tA}(x(0) - x_o) + \left[CA^{-l}B + \frac{t}{1!}CA^{-(l-1)}B + \cdots + \frac{t^{l-1}}{(l-1)!}CA^{-1}B\right]H_2^T e_i.$$

The remaining part of the proof follows from linearity of $u(t, x_o)$ in the second variable. □

In order to obtain algebraic conditions characterizing invariant zeros of (1), we show at first that each state-zero direction has to belong to the subspace $\Sigma_k$ (see Lemma 3).

*Lemma 4:* If a triple $\lambda$, $x_o \neq 0$, $g$ satisfies (1a), then $x_o \in \bigcap_{l=0}^k \mathrm{Ker}\, CA^l$ (i.e., $x_o \in \Sigma_k$) and $CA^k Bg = -CA^{k+1}x_o$.

*Proof:* The proof follows by successive multiplication of $\lambda x_o - Ax_o = Bg$ (1a) from the left by $C, \cdots, CA^k$. □

A sufficient condition for invariant zeros of (1) is formulated as follows.

*Proposition 1:* If $\lambda \in \sigma(K_k A)$ and there exists an eigenvector $x_o$ of $K_k A$ associated with $\lambda$ such that $x_o \in \mathrm{Ker}\, C$, then $\lambda$ is an invariant zero of (1). Moreover, $x_o$ and $g = u(0, x_o) = -(CA^k B)^+ CA^{k+1} x_o$ are state-zero and input-zero directions associated with $\lambda$.

*Proof:* Using the definition of $K_k$ we write $\lambda x_o - K_k Ax_o = 0$ as $\lambda x_o - Ax_o = Bg$, where $g = -(CA^k B)^+ CA^{k+1} x_o$. In this way, the triple $\lambda$, $x_o$, $g$ satisfies (1a). □

As a necessary condition for invariant zeros of (1) we use the following.

*Proposition 2:* If $\lambda$, $x_o \neq 0$, $g$ satisfies (1a), then $\lambda x_o - K_k Ax_o = Bg_1$, $K_k Ax_o - Ax_o = Bg_2$, $Cx_o = 0$, where $g = g_1 + g_2$, $g_1 \in \mathrm{Ker}\, CA^k B$, $g_2 \in \mathrm{Im}\,(CA^k B)^T$, and $g_1, g_2$ are uniquely determined by $g$; moreover, $Bg_1 \in \Sigma_k$, $Bg_2 \in \Omega_k$, and $g_2 = -(CA^k B)^+ CA^{k+1} x_o$.

*Proof:* For $g$ we take the decomposition $g = g_1 + g_2$, with $g_1, g_2$ defined as $g_1 = (I_m - (CA^k B)^+ CA^k B)g$, $g_2 = (CA^k B)^+ CA^k Bg$. Then $Bg_1 = K_k Bg$, $Bg_2 = (I - K_k)Bg$ which implies $K_k Bg_1 = Bg_1$ and $K_k Bg_2 = 0$. Now $\lambda x_o - Ax_o = Bg$ can be written as $(\lambda I - K_k A)x_o + (K_k - I)Ax_o = Bg_1 + Bg_2$ with vectors $\lambda x_o - K_k Ax_o$, $Bg_1$ in $\Sigma_k$ and $(K_k - I)Ax_o$, $Bg_2$ in $\Omega_k$. From the uniqueness of this decomposition (see Lemma 3) we obtain the first two equalities of the proposition. The desired expression for $g_2$ follows from equalities $g_2 = (CA^k B)^+ CA^k Bg$ and $CA^k Bg = -CA^{k+1} x_o$. □

The subsequent result shows that all decoupling zeros of (1) are eigenvalues of $K_k A$ as well as the fact that o.d. and i.o.d. zeros are invariant zeros and the corresponding to them output-zeroing inputs $u(t, x_o)$ equal zero identically.

*Proposition 3:* If a number $\lambda$ is an o.d. or an i.d. or an i.o.d. zero of (1), then $\lambda \in \sigma(K_k A)$. Moreover, each o.d. and each i.o.d. zero of (1) is an invariant zero and the corresponding output-zeroing input is $u(t, x_o) \equiv 0$.

*Proof:* If $\lambda$ is an o.d. zero, then the definition yields $x_o \in \bigcap_{l=0}^k \mathrm{Ker}\, CA^l \subset \Sigma_k$, i.e., $K_k x_o = x_o$ (see Lemma 3). From $\lambda x_o = Ax_o$ one obtains $\lambda x_o = K_k Ax_o$, i.e., $\lambda \in \sigma(K_k A)$. Now, $\lambda$ is an invariant zero via Proposition 1. Furthermore, $g = u(0, x_o) = -(CA^k B)^+ CA^{k+1} x_o = 0$ since $CA^{k+1} x_o = CA^k(Ax_o) = \lambda CA^k x_o = 0$. This yields $u(t, x_o) \equiv 0$.

Now, let $\lambda$ be an i.d. zero of (1) and let $S(A_a, B_a, C_a)$ be the adjoint system (with $A_a = -A^T$, $B_a = C^T$, $C_a = B^T$). The first nonzero Markov parameter of the adjoint system is $(-1)^k (CA^k B)^T$. For the adjoint system we form the projection $K_{a,k}$ (see Definition 1) and (using the properties $(M^+)^T = (M^T)^+$, $(\alpha M)^+ = \alpha^{-1} M^+$ for a real $M$ and a nonzero scalar $\alpha$) we obtain $K_{a,k} = I - B_a(C_a A_a^k B_a)^+ C_a A_a^k = E_k^T$, where $E_k$ is defined as $E_k = I - A^k B(CA^k B)^+ C$. Next, we observe that $K_{a,k} A_a = -(AE_k)^T$. Now, if $\lambda$ is an i.d. zero of (1), then $(-\lambda)$ is an o.d. zero of $S(A_a, B_a, C_a)$ and $-\lambda \in \sigma(K_{a,k} A_a) = \sigma(-(AE_k)^T) = \sigma(-AE_k)$. Since $\det(sI - K_k A) = \det(sI - AE_k)$, we have $\lambda \in \sigma(AE_k) = \sigma(K_k A)$. The remaining part of the proof is obvious. □

Restricting our attention to systems with the first nonzero Markov parameter of full rank (and assuming nondegeneracy when necessary), we observe that each invariant zero has to be an eigenvalue of $K_k A$, which in turn (via Proposition 3) implies that each system zero of (1) is in $\sigma(K_k A)$.

*Corollary 2:*

1) If $m \le r$ and $\operatorname{rank} CA^kB = m$, then each system zero of (1) is in $\sigma(K_kA)$.
2) If $m > r$, $\operatorname{rank} CA^kB = r$, and system (1) is nondegenerate, then each of its system zeros is in $\sigma(K_kA)$.

In both cases, $\lambda$ is an invariant zero of (1) iff $\lambda \in \sigma(K_kA)$ and an associated eigenvector of $K_kA$ is in $\operatorname{Ker} C$.

*Proof:* In view of Proposition 3 we only need to consider invariant zeros.

1) Since $\operatorname{Ker} CA^kB = \{0\}$, Proposition 2 implies $g_1 = 0$, i.e., each invariant zero is in $\sigma(K_kA)$ (this also means that the system is nondegenerate). The last assertion of the corollary follows from Propositions 1 and 2.
2) Let the skeleton decomposition of $CA^kB$ be $H_1 = I_r$ and $H_2 = CA^kB$. We show that if a triple $\lambda$, $x_o \ne 0$, $g$ satisfies (1a), then $\lambda \in \sigma(K_kA)$ and $x_o$ is associated with $\lambda$ eigenvector of $K_kA$ satisfying $Cx_o = 0$. By virtue of Proposition 2, $\lambda x_o - K_kAx_o = Bg_1$, $Bg_1 \in \operatorname{Ker} CA^k$; moreover, $Cx_o = 0$. Assume $Bg_1 \ne 0$ and consider any $\lambda_1 \notin \sigma(K_kA)$. Let $x_1 := (\lambda_1 I - K_kA)^{-1}Bg_1$ and let $g_2^1 := -(CA^kB)^+ \, CA^{k+1}x_1$. The definition of $K_k$ enables us to write $K_kAx_1 - Ax_1 = Bg_2^1$; moreover, $K_kBg_2^1 = 0$. Since $H_1 = I_r$, Lemma 3 yields $C(sI - K_kA)^{-1}B = CA^kB/s^{k+1}$ and $Cx_1 = \lambda_1^{-(k+1)}CA^kBg_1 = 0$. In this way, $\lambda_1 x_1 - K_kAx_1 = Bg_1$, $K_kAx_1 - Ax_1 = Bg_2^1$, $Cx_1 = 0$. This means that $\lambda_1$ is an invariant zero of $S(A, B, C)$, for the triple $\lambda_1$, $x_1 \ne 0$, $g_1 + g_2^1$ satisfies (1a). Since $\lambda_1$ is arbitrary, the system is degenerate. This contradiction means that $Bg_1 = 0$. The last assertion follows as in 1) from Propositions 1 and 2. □

For further discussion of the structure of zeros we use the Kalman form of (1). Since Markov parameters are invariant under nonsingular transformations of the state space, if a change of basis $x' = Hx$ transforms $S(A, B, C)$ into $S(A', B', C')$, then $K_k' = HK_kH^{-1}$ and $K_k'A' = H(K_kA)H^{-1}$, where $K_k' := I - B'[C'(A')^kB']^+C'(A')^k$. On the other hand, invariant zeros and decoupling zeros of individual kinds (o.d., i.o.d., i.d.) are also invariant under $H$. Thus analyzing system zeros and zeros at infinity of (1), we can consider equivalently instead of (1) its Kalman canonical form with

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$
$$C = [0 \quad C_2 \quad 0 \quad C_4]$$

where subscripts 1–4 stand for controllable and unobservable, controllable and observable, uncontrollable and unobservable, and uncontrollable and observable parts of (1) and $n = n_1 + n_2 + n_3 + n_4$. By $x_i \in R^{n_i}$, $i = 1, 2, 3, 4$, we denote appropriate components of the state vector, i.e., $x = (x_1^T, x_2^T, x_3^T, x_4^T)^T$. Using (3) one obtains

$$K_k = \begin{bmatrix} I_{n_1} & x & 0 & x \\ 0 & K_k^{22} & 0 & x \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{bmatrix}$$

and

$$K_kA = \begin{bmatrix} A_{11} & x & A_{13} & x \\ 0 & K_k^{22}A_{22} & 0 & x \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \tag{4}$$

where $K_k^{22} = I_{n_2} - B_2(C_2A_{22}^kB_2)^+C_2A_{22}^k$ is a projection of $\operatorname{rank}_{n_2-p}$. The matrix $K_k^{22}A_{22}$ has $p(k+1)$ zero eigenvalues representing zeros at infinity of (1) and of $G(s)$. This follows from (2)–(4). In fact, cancelling common factors on both sides of (2) we obtain $\det(sI_{n_2} - K_k^{22}A_{22}) = \det(H_1^TH_1)^{-1} \det(H_2H_2^T)^{-1}s^{p(k+1)} \det P_D^{22}(s)$, where $P_D^{22}(s)$ stands for system matrix of the squared down minimal subsystem of (1), i.e., of $S(A_{22}, B_2H_2^T, H_1^TC_2)$.

*Corollary 3:* $\sigma(A_{11})$, $\sigma(A_{33})$, and $\sigma(A_{44})$ consist of o.d., i.o.d., and i.d. zeros of (1), respectively.

*Proof:* The assertions concerning $\sigma(A_{11})$ and $\sigma(A_{44})$ follow from (3) and definitions of o.d. and i.d. zeros. Similarly, we prove that $\sigma(A_{33})$ consists of i.d. zeros. In order to show that each element of $\sigma(A_{33})$ is an o.d. zero, we consider two cases. If $\lambda \in \sigma(A_{33})$ and $\lambda \in \sigma(A_{11})$, then $\lambda$ is an o.d. zero. In the second case, when $\lambda \in \sigma(A_{33})$ and $\lambda \notin \sigma(A_{11})$, this $\lambda$ and the vector $x_o = ((x_1^o)^T, 0_2^T, (x_3^o)^T, 0_4^T)^T$, with $\lambda x_3^o = A_{33}x_3^o$ and $x_1^o = (\lambda I_{n_1} - A_{11})^{-1}A_{13}x_3^o$, satisfy the definition of o.d. zeros. □

For minimal subsystem $S(A_{22}, B_2, C_2)$ of (3), if $\lambda \in \sigma(K_k^{22}A_{22})$ and there exists an associated eigenvector $x_2^o$ such that $C_2x_2^o = 0$, then (via Proposition 1) $\lambda$ is an invariant zero of this subsystem and a zero of $G(s)$ of (1). As a corresponding input-zero direction we take $g^{22} = -(C_2A_{22}^kB_2)^+C_2A_{22}^{k+1}x_2^o$. Now, using Lemma 1, (3), and (4) one can check that $x_o = (0_1^T, (x_2^o)^T, 0_3^T, 0_4^T)^T$ generates input $u(t, x_o) = g^{22}e^{\lambda t}$ (since the considered $x_o$ does not need to be an eigenvector of $K_kA$ (4) associated with $\lambda$, the corresponding solution of the state equation of (3) may have a nonzero component in controllable and unobservable part). Next, we show that $B_2g^{22} \ne 0$ (i.e., $u(t, x_o) \ne 0$). At first, note that $\lambda x_2^o \ne A_{22}x_2^o$; otherwise, $S(A_{22}, B_2, C_2)$ would be unobservable. Now, using the definition of $K_k^{22}$ we write $\lambda x_2^o - K_k^{22}A_{22}x_2^o = 0$ as $\lambda x_2^o - A_{22}x_2^o = B_2g^{22} \ne 0$. Let us note that $B_2g^{22} \ne 0$ even if the considered $\lambda$ is also a pole of $G(s)$ [since, for the same reason as above, $\lambda x_2^o \ne A_{22}x_2^o$ is even if $\lambda \in \sigma(A_{22})$]. This discussion holds for systems satisfying assumptions of Corollary 2 because all their transmission zeros are characterized as those eigenvalues of $K_k^{22}A_{22}$ whose eigenvectors belong to $\operatorname{Ker} C_2$.

*Corollary 4:* At the assumptions of Corollary 2 $\lambda \in C$ is a transmission zero of (1) iff $\lambda \in \sigma(K_k^{22}A_{22})$, and an associated eigenvector of $K_k^{22}A_{22}$ is in $\operatorname{Ker} C_2$.

*Remark 2:* Each transmission zero (even if it is also a pole) of any system (1) satisfying assumptions of Corollary 2 has the transmission blocking property for certain nonzero real-valued inputs. Since in the above discussion only argument of observability is involved, this property extends on all observable strictly proper systems.

The structure of $K_k^{22}A_{22}$ can be simplified. Writing $K_k^{22}$ as

$$R_{22}K_k^{22}R_{22}^{-1} = \begin{bmatrix} I_{n_2-p} & 0 \\ 0 & 0_p \end{bmatrix}$$

and $R_{22}A_{22}R_{22}^{-1}$ as

$$\begin{bmatrix} \overrightarrow{A}_{22}^{11} & \overrightarrow{A}_{22}^{12} \\ \overrightarrow{A}_{22}^{21} & \overrightarrow{A}_{22}^{22} \end{bmatrix}$$

as well as using the transformation $\operatorname{diag}(I_{n_1}, R_{22}, I_{n_3}, I_{n_4})$ to (3), we obtain matrices $K_kA$ (5) and $A$ (6)

$$\begin{bmatrix} A_{11} & x & x & x \\ 0 & \begin{matrix} \overrightarrow{A}_{22}^{11} & \overrightarrow{A}_{22}^{12} \\ 0 & 0_p \end{matrix} & 0 & x \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} A_{11} & x & A_{13} & A_{14} \\ 0 & \begin{matrix} \overrightarrow{A}_{22}^{11} & \overrightarrow{A}_{22}^{12} \\ \overrightarrow{A}_{22}^{21} & \overrightarrow{A}_{22}^{22} \end{matrix} & 0 & x \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}. \tag{6}$$

The matrix $\overline{\bar{A}}_{22}^{11}$ of (5) has $pk$ zero eigenvalues representing zeros at infinity. Among the remaining $(n_2-p)-pk$ eigenvalues of $\bar{A}_{22}^{11}$ may appear some (not necessarily all) of the transmission zeros of (1). By the assumptions of Corollary 2 all transmission zeros of (1) have to appear among $(n_2-p)-pk$ eigenvalues of $\bar{A}_{22}^{11}$.

### *B. Proper Systems* $(D \neq 0)$

This section contains a concise treatment of the proper case. Since the approach is similar to that of Section II-A, all proofs are omitted here (they may be found in [22]).

*Definition 2:* For $S(A,B,C,D)$ we define a class of real-valued inputs $v(t,x_o) := -D^+Ce^{t(A-BD^+C)}x_o,\ x_o \in R^n$.

*Lemma 5:* For a given $v(t,x_o)$, the solution of the state equation of (1) corresponding to an initial condition $x(0)$ has the form $x(t) = e^{tA}(x(0)-x_o)+e^{t(A-BD^+C)}x_o$, while $y(t) = Ce^{tA}(x(0)-x_o) + (I_r - DD^+)Ce^{t(A-BD^+C)}x_o$.

Let $D_1D_2$ denote a skeleton factorization of $D$. With (1) we associate the $p$-input/$p$-output system $S(A,B_D,C_D,D_D)$, where $B_D = BD_2^T$, $C_D = D_1^TC$, $D_D = D_1^TDD_2^T$. The symbol $P_D(s)$ stands for the system matrix of the associated system.

*Lemma 6:*

$$\begin{aligned}&\det(sI-(A-BD^+C))\\ &\quad= \det\big(D_1^TD_1\big)^{-1}\det\big(D_2D_2^T\big)^{-1}\det P_D(s).\end{aligned}\tag{7}$$

*Proposition 4:* $\lambda \in C$ is an invariant zero of (1) if $\lambda \in \sigma(A-BD^+C)$, and there exists an eigenvector $x_o$ satisfying $x_o \in \operatorname{Ker}(I_r - DD^+)C$. Then $x_o$ and $g = v(0,x_o) = -D^+Cx_o$ constitute state-zero and input-zero direction for $\lambda$.

*Proposition 5:* If a triple $\lambda,\ x_o \neq 0,\ g$ satisfies (1a), then $\lambda x_o - (A-BD^+C)x_o = Bg_1,\ x_o \in \operatorname{Ker}(I_r - DD^+)C$ and $g = g_1 + g_2$, where $g_1 \in \operatorname{Ker} D,\ g_2 = -D^+Cx_o \in \operatorname{Im} D^T$.

*Proposition 6:* If $\lambda$ is an o.d. or an i.o.d. or an i.d. zero of (1), then $\lambda \in \sigma(A-BD^+C)$. If $\lambda$ is an o.d. or an i.o.d. zero of (1), then $\lambda$ is an invariant zero and the corresponding input $v(t,x_o) \equiv 0$.

Under additional assumptions all systems zeros of (1) have to appear as some eigenvalues of $A - BD^+C$.

*Corollary 5:*

1) If $m \leq r$ and $\operatorname{rank} D = m$, then each system zero of (1) is in $\sigma(A-BD^+C)$. Moreover, $\lambda$ is an invariant zero of (1) iff $\lambda \in \sigma(A-BD^+C)$ and an associated eigenvector is in $\operatorname{Ker}(I_r - DD^+)C$.
2) If $m > r$, $\operatorname{rank} D = r$, and system (1) is nondegenerate, then each of its system zeros is in $\sigma(A-BD^+C)$. Moreover, $\lambda$ is an invariant zero of (1) iff $\lambda \in \sigma(A-BD^+C)$.

When system (1) is taken in its Kalman canonical form $[A,B,C$ as in (3)], then

$$A - BD^+C = \begin{bmatrix} A_{11} & A_{12} - B_1D^+C_2 & A_{13} & A_{14} - B_1D^+C_4 \\ 0 & A_{22} - B_2D^+C_2 & 0 & A_{24} - B_2D^+C_4 \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}.\tag{8}$$

*Corollary 6:* $\sigma(A_{11}),\ \sigma(A_{33}),\ \sigma(A_{44})$ consist of o.d., i.o.d., and i.d. zeros of (1), respectively.

*Corollary 7:* At the assumptions of Corollary 5 $\lambda \in C$ is a transmission zero of (1) iff $\lambda \in \sigma(A_{22} - B_2D^+C_2)$ and an associated eigenvector is in $\operatorname{Ker}(I_r - DD^+)C_2$.

## III. Transmission Zeros

The transmission zeros of a minimal system (1) with the first nonzero Markov parameter not of full rank may be characterized as invariant zeros of an auxiliary strictly proper system with a smaller number of inputs and outputs.

### *A. Strictly Proper Systems* $(D = 0)$

In (1) let the first Markov parameter $CA^kB$ be of $\operatorname{rank} p < \min\{m,r\}$. Applying SVD [2] to $CA^kB$ we can write

$$CA^kB = U\Lambda V^T, \qquad \text{where } \Lambda = \begin{bmatrix} M_p & 0 \\ 0 & 0 \end{bmatrix}\tag{9}$$

where $r \times m$ is dimensional, $M_p$ is $p \times p$ diagonal with singular values of $CA^kB$, and $U$ and $V$ are orthogonal matrices. Introducing into (1) $V$ and $U^T$ as pre- and postcompensator, one obtains a system $S(A,\bar{B},\bar{C})$

$$\dot{x} = Ax + \bar{B}\bar{u}, \qquad \bar{y} = \bar{C}x\tag{10}$$

where $\bar{B} = BV$, $\bar{C} = U^TC$, $\bar{u} = V^Tu$, $\bar{y} = U^Ty$, and $\bar{C}A^k\bar{B}$ is its first nonzero Markov parameter. Decomposing

$$\bar{B} = [\underbrace{\bar{B}_p}_{p}\ \ \underbrace{\bar{B}_{m-p}}_{m-p}], \qquad \bar{C} = \begin{bmatrix} \bar{C}_p \\ \bar{C}_{r-p} \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} \bar{u}_p \\ \bar{u}_{m-p} \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} \bar{y}_p \\ \bar{y}_{r-p} \end{bmatrix}$$

and taking into account (9), we have

$$\bar{C}A^k\bar{B} = \begin{bmatrix} M_p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{C}_pA^k\bar{B}_p & \bar{C}_pA^k\bar{B}_{m-p} \\ \bar{C}_{r-p}A^k\bar{B}_p & \bar{C}_{r-p}A^k\bar{B}_{m-p} \end{bmatrix}$$

which yields $M_p = \bar{C}_pA^k\bar{B}_p$. For $S(A,\bar{B},\bar{C})$ in (10) we form $\bar{K}_k := I - \bar{B}(\bar{C}A^k\bar{B})^+\bar{C}A^k$. From (9) and the decomposition of $\bar{B},\bar{C}$ it follows that $\bar{K}_k = I - \bar{B}_pM_p^{-1}\bar{C}_pA^k$. It is clear that a triple $\lambda,\ x_o,\ g$ satisfies (1a) iff triple $\lambda,\ x_o,\ \bar{g} = V^Tg$ satisfies

$$\bar{P}(\lambda)\begin{bmatrix} x_o \\ \bar{g} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \text{where } \bar{P}(\lambda) = \begin{bmatrix} \lambda I - A & -\bar{B} \\ \bar{C} & 0 \end{bmatrix}.\tag{10a}$$

Thus, $\lambda$ is an invariant zero of (1) iff $\lambda$ is an invariant zero of (10). On the other hand, the transformation of (1) into (10) does not change ranks of controllability and observability matrices and $\lambda \in C$ is an o.d., i.o.d., or an i.d. zero of (1) iff $\lambda$ is an o.d., i.o.d., or an i.d. zero of (10), respectively. Now, applying Lemma 4 to (10) we infer that if $\lambda,\ x_o,\ \bar{g}$ satisfies (10a), then $x_o \in \cap_{l=0}^{k} \operatorname{Ker} \bar{C}A^l$ and $\bar{C}A^k\bar{B}\bar{g} = -\bar{C}A^{k+1}x_o$. This gives $\cap_{l=0}^{k} \operatorname{Ker} \bar{C}A^l \subset \cap_{l=0}^{k} \operatorname{Ker} \bar{C}_pA^l \subset \bar{\Sigma}_k = \{x\colon \bar{K}_kx = x\} = \operatorname{Ker} \bar{C}_pA^k$ (which means that $x_o \in \bar{\Sigma}_k$) and $\bar{C}A^k\bar{B}\bar{g} = [\begin{smallmatrix} M_p & 0 \\ 0 & 0 \end{smallmatrix}][\begin{smallmatrix} \bar{g}_p \\ \bar{g}_{m-p} \end{smallmatrix}] = -[\begin{smallmatrix} \bar{C}_pA^{k+1}x_o \\ \bar{C}_{r-p}A^{k+1}x_o \end{smallmatrix}]$ (which follows that $\bar{g}_p = -M_p^{-1}\bar{C}_pA^{k+1}x_o$ and $\bar{C}_{r-p}A^{k+1}x_o = 0$). From the above we infer that if $\lambda$ is an invariant zero of (10) [i.e., $\lambda,\ x_o,\ \bar{g}$ satisfies (10a)], then

$$\lambda x_o - \bar{K}_kAx_o = \bar{B}_{m-p}\bar{g}_{m-p}, \qquad \bar{C}_{r-p}x_o = 0\tag{11}$$

$$\bar{C}_px_o = 0\tag{11a}$$

i.e., $\lambda$ is an invariant zero of $S(\bar{K}_kA,\bar{B}_{m-p},\bar{C}_{r-p})$ $(\dot{x} = \bar{K}_kAx + \bar{B}_{m-p}\bar{u}_{m-p},\bar{y}_{r-p} = \bar{C}_{r-p}x)$ and $\bar{C}_px_o = 0$. Inversely, if $\lambda,\ x_o,\ \bar{g}_{m-p}$ satisfies (11) and $\bar{C}_px_o = 0$, then $\lambda,\ x_o,\ \bar{g}$, with $\bar{g}_p = -M_p^{-1}\bar{C}_pA^{k+1}x_o$, satisfies (10a). In this way, we have the following.

*Proposition 7:* A triple $\lambda,\ x_o,\ g$ satisfies (1a) iff $\lambda,\ x_o,\ \bar{g}_{m-p}$ satisfies (11) and $\bar{C}_px_o = 0$.

This result enables one to replace the computation of invariant zeros of (1) by the task of computing those invariant zeros of $S(\bar{K}_kA,\bar{B}_{m-p},\bar{C}_{r-p})$ (Fig. 1) which satisfy $\bar{C}_px_o = 0$. In particular, from Proposition 7 we obtain the following.

*Corollary 8:* $\lambda \in C$ is a transmission zero of a minimal system (1) with $p < \min\{m,r\}$ iff $\lambda$ is an invariant zero of $S(\bar{K}_kA,\bar{B}_{m-p},\bar{C})$ (Fig. 1).

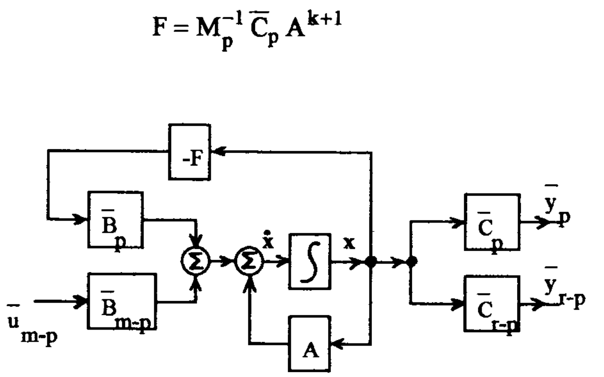


Fig. 1. $S(\bar{K}_k A, \bar{B}_{m-p}, \bar{C}_{r-p})$.

*B. Proper Systems* $(D \neq 0)$

Consider system (1) with $D$ of $\operatorname{rank} p$, $0 < p < \min\{m, r\}$, and write SVD of $D$ as $D = U\Lambda V^T$, $\Lambda = \left[\begin{smallmatrix} D_p & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Introducing to (1) $V$ and $U^T$ as pre- and postcompensators we transform (1) to the form

$$\dot{x} = Ax + \bar{B}\bar{u}, \qquad \bar{y} = \bar{C}x + \Lambda\bar{u}. \tag{12}$$

In (12) we use the same decomposition of $\bar{B}$, $\bar{C}$, $\bar{u}$, $\bar{y}$ as in III.1. If $\lambda$, $x_o$,

$$\bar{g} = \begin{bmatrix} \bar{g}_p \\ \bar{g}_{m-p} \end{bmatrix}$$

is an invariant zero of (12), then (via Proposition 5) $\lambda x_o - (A - \bar{B}\Lambda^+\bar{C})x_o = \bar{B}\bar{g}_1$, $x_o \in \operatorname{Ker}(I_r - \Lambda\Lambda^+)\bar{C}$, and $\bar{g} = \bar{g}_1 + \bar{g}_2$, with $\bar{g}_1 \in \operatorname{Ker}\Lambda$, $\bar{g}_2 = -\Lambda^+\bar{C}x_o \in \operatorname{Im}\Lambda^T$. However, $A - \bar{B}\Lambda^+\bar{C} = A - \bar{B}_p D_p^{-1}\bar{C}_p$, $\operatorname{Ker}(I_r - \Lambda\Lambda^+)\bar{C} = \operatorname{Ker}\bar{C}_{r-p}$ and

$$\bar{g}_2 = -\Lambda^+\bar{C}x_o = \begin{bmatrix} -D_p^{-1}\bar{C}_p x_o \\ 0_{m-p} \end{bmatrix}.$$

Now, since $\bar{g}_1 \in \operatorname{Ker}\Lambda$ and $\bar{g} = \bar{g}_1 + \bar{g}_2$, thus $\bar{g}_p = -D_p^{-1}\bar{C}_p x_o$,

$$\bar{g}_1 = \begin{bmatrix} 0_p \\ \bar{g}_{m-p} \end{bmatrix}$$

and $\bar{B}\bar{g}_1 = \bar{B}_{m-p}\bar{g}_{m-p}$. Thus, if a triple $\lambda$, $x_o$,

$$\bar{g} = \begin{bmatrix} \bar{g}_p \\ \bar{g}_{m-p} \end{bmatrix}$$

is an invariant zero of (12), then

$$\lambda x_o - (A - \bar{B}_p D_p^{-1}\bar{C}_p)x_o = \bar{B}_{m-p}\bar{g}_{m-p}, \qquad \bar{C}_{r-p}x_o = 0 \tag{13}$$

and $\bar{g}_p = -D_p^{-1}\bar{C}_p x_o$. Inversely, if $\lambda$, $x_o$, $\bar{g}_{m-p}$ satisfies (13) and we define

$$\bar{g} := \begin{bmatrix} \bar{g}_p \\ \bar{g}_{m-p} \end{bmatrix}$$

with $\bar{g}_p := -D_p^{-1}\bar{C}_p x_o$, then $\lambda x_o - Ax_o = \bar{B}\bar{g}$, $\bar{C}x_o + \Lambda\bar{g} = 0$. In this way, $\lambda \in C$ is an invariant zero of $S(A, \bar{B}, \bar{C}, \Lambda)$ (12) iff $\lambda$ is an invariant zero of $S(A - \bar{B}_p D_p^{-1}\bar{C}_p, \bar{B}_{m-p}, \bar{C}_{r-p})$ (13). Since (1) and (12) have the same set of invariant zeros, we have proved the following proposition.

*Proposition 8:* $\lambda \in C$ is an invariant zero of (1) with $p < \min\{m, r\}$ iff $\lambda$ is an invariant zero of $S(A - \bar{B}_p D_p^{-1}\bar{C}_p, \bar{B}_{m-p}, \bar{C}_{r-p})$. Moreover, $\lambda$, $x_o$, $\bar{g}_{m-p}$ satisfies (13) iff $\lambda$, $x_o$,

$$g = V\begin{bmatrix} -D_p^{-1}\bar{C}_p x_o \\ \bar{g}_{m-p} \end{bmatrix}$$

satisfies (1a).

In particular, Proposition 8 characterizes transmission zeros of any minimal system $S(A, B, C, D)$ with $p < \min\{m, r\}$ as invariant zeros of $S(A - \bar{B}_p D_p^{-1}\bar{C}_p, \bar{B}_{m-p}, \bar{C}_{r-p})$.

## IV. Examples

*Example 1 [10]:* In (1) let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

We have $k = 0$, $p = 1$ and $(CB)^+ = [1/10 \;\; -3/10]$. Via Corollary 2 all system zeros are in $\sigma(K_o A)$. The system is observable and not controllable. In the Kalman form $K_o A$, $A, B, C$ are equal

$$\begin{bmatrix} -24/10 & 1 & 0 \\ 0 & 0 & -1/10 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} -24/10 & 1 & 0 \\ 84/100 & -16/10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 6/10 & 1 & 1 \\ 2/10 & -3 & 0 \end{bmatrix}$$

with $A_{44} = 1$. The number zero on the diagonal of $K_o A$ represents zero at infinity; $-24/10$ is not a transmission zero of (1) [it is a transmission zero of $S(A, B_D, C_D)$]. The system possesses one i.d. zero (which is not invariant zero) at one.

*Example 2:* Consider a minimal nondegenerate system (1) with

$$A = \begin{bmatrix} -1 & 0 & -3 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where $k = 0$, $p = 2$. We have

$$(CB)^+ = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}, \quad K_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad K_o A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The system has two zeros at infinity represented by two zero eigenvalues of $K_o A$ and, by virtue of Corollary 4, the single transmission zero at zero.

*Example 3 [23]:* Let in a minimal system (1)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

The zero polynomial of $G(s)$ equals $(s^3+s+1)(s-1)$, whereas $\sigma(A-BD^+C)=\{s\colon s^3(s^3+\frac{1}{2}s+\frac{1}{2})=0\}$. From Proposition 8 we obtain

$$\Lambda=\begin{bmatrix}\sqrt{2} & 0\\ 0 & 0\end{bmatrix},\qquad U=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\qquad V=I_2$$

$$\bar{C}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 1 & -1 & 0\\ 1 & 1 & 0 & -1 & 1 & 0\end{bmatrix},\qquad \bar{B}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$$

and

$$A-\bar{B}_pD_p^{-1}\bar{C}_p=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -1/2 & -1/2 & 0 & -1/2 & 1/2 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

To the single-input/single-output (SISO) system $S(A-\bar{B}_pD_p^{-1}\bar{C}_p, \bar{B}_{m-p}, \bar{C}_{r-p})$ we use Corollary 2. This system is minimal and its first nonzero Markov parameter $\bar{C}_{r-p}(A-\bar{B}_pD_p^{-1}\bar{C}_p)\bar{B}_{m-p}=1/\sqrt{2}$, i.e., $k=1$, $p=1$. We form a $6\times 6$ projective matrix $K_1=I-\bar{B}_{m-p}[\bar{C}_{r-p}(A-\bar{B}_pD_p^{-1}\bar{C}_p)\bar{B}_{m-p}]^{-1}\bar{C}_{r-p}(A-\bar{B}_pD_p^{-1}\bar{C}_p)$ and compute $K_1(A-\bar{B}_pD_p^{-1}\bar{C}_p)$. Finally, $\det(sI-K_1(A-\bar{B}_pD_p^{-1}\bar{C}_p))=s^2(s-1)(s^3+s+1)$, which means that the SISO system has zero at infinity of order two, while the remaining zeros are its transmission zeros. Via Proposition 8 these transmission zeros are all transmission zeros of (1).

## V. Conclusion

In the presented approach $K_kA$ and $A-BD^+C$ are treated merely as auxiliary matrices associated with (1). Each of them may be treated as state matrix of an appropriate state feedback system (with $F=-(CA^kB)^+CA^{k+1}$ and $F=-D^+C$). Such a treatment leads, however, to another characterization of individual invariant zeros. As a set of numbers invariant zeros remain unchanged under constant state feedback. Since state feedback may change observability, it may also change the kind of individual invariant zeros, i.e., a number which is an invariant zero of some kind (transmission, o.d., i.o.d., or i.d.) of (1) does not need to be invariant zero of the same kind in the closed-loop system. In fact, consider (at the notation of Section II-A) the feedback system $S(K_kA,B,C)$. The first Markov parameter of $S(K_kA,B,C)$ is $C(K_kA)^kB=CA^kB$ (for $C(K_kA)^l=CA^l$ at $0\le l\le k$). For $S(K_kA,B,C)$ we form matrices $K_{k,cl}$ (via Definition 1) and $K_{k,cl}(K_kA)$. Since $K_{k,cl}=K_k$, we have $K_{k,cl}(K_kA)=K_kA$. Thus $S(A,B,C)$ and $S(K_kA,B,C)$ have the same $K_kA$ as matrix-characterizing zeros. Since these systems have in general different Kalman forms, $K_kA$ associated with (1) may have a Kalman form other than $K_kA$ associated with $S(K_kA,B,C)$. For instance, the system of Example 2 has transmission zero at 0 while this 0 is an o.d. (but not i.d.) zero of the closed-loop system. An analogous situation takes place for proper systems. The presented approach extends results of [18], [19], [22], and [25] on arbitrary systems. It delivers more information about the structure of zeros and their action in a system than the approaches of [1] and [23] as well as being simpler than the method of [14] based on scb.

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## References

[1] M. H. Amin and M. H. Hassan, "Determination of invariant zeros and zero directions of the system $S(A,B,C,E)$," *Int. J. Contr.*, vol. 47, pp. 1011–1041, 1988.
[2] F. M. Callier and C. A. Desoer, *Multivariable Feedback Systems*. New York: Springer Verlag, 1982.
[3] C. T. Chen, *Linear System Theory and Design*. HRW, 1984.
[4] O. El-Ghezawi, A. Zinober, D. Owens, and S. Billings, "Computation of the zeros and zero directions of linear multivariable systems," *Int. J. Contr.*, vol. 36, pp. 833–843, 1982.
[5] F. R. Gantmacher, *Theory of Matrices* (in Russian). Nauka, 1988.
[6] G. A. Hewer and J. M. Martin, "An exact characterization of linear systems with zeros everywhere in the complex plane," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 728–730, Aug. 1984.
[7] B. Kouvaritakis and A. G. J. MacFarlane, "Geometric approach to analysis and synthesis of system zeros, Part 1: Square systems," *Int. J. Contr.*, vol. 23, pp. 149–166, 1976.
[8] B. Kouvaritakis and A. G. J. MacFarlane, "Geometric approach to analysis and synthesis of system zeros, Part 2: Non-square systems," *Int. J. Contr.*, vol. 23, pp. 167–181, 1976.
[9] V. Lovass-Nagy and D. L. Powers, "Determination of zeros and zero directions of linear time-invariant systems by matrix generalized inverses," *Int. J. Contr.*, vol. 31, pp. 1161–1170, 1980.
[10] A. G. J. MacFarlane and N. Karcanias, "Poles and zeros of linear multivariable systems: A survey of the algebraic, geometric and complex variable theory," *Int. J. Contr.*, vol. 24, pp. 33–74, 1976.
[11] T. Kaczorek, *Theory of Control and Systems* (in Polish). PWN, 1993.
[12] T. Mita, "On maximal unobservable subspace, zeros and their applications," *Int. J. Contr.*, vol. 25, pp. 885–899, 1977.
[13] D. H. Owens, "Invariant zeros of multivariable systems: A geometric analysis," *Int. J. Contr.*, vol. 26, pp. 537–548, 1977.
[14] P. Sannuti and A. Saberi, "Special coordinate basis for multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling," *Int. J. Contr.*, vol. 45, pp. 1655–1704, 1987.
[15] C. B. Schrader and M. K. Sain, "Research on system zeros: A survey," *Int. J. Contr.*, vol. 50, pp. 1407–1433, 1989.
[16] V. Sinswat, R. V. Patel, and F. Fallside, "A method for computing invariant zeros and transmission zeros of invertible systems," *Int. J. Contr.*, vol. 23, pp. 183–196, 1976.
[17] E. Soroka and V. Shaked, "On the geometry of the inverse system," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 751–754, Aug. 1986.
[18] J. Tokarzewski, "On a geometric characterization of system zeros," in *Proc. Int. Symp. MMAR*, Miedzyzdroje, Poland, Aug. 30–Sept. 2, 1995, pp. 187–192.
[19] ——, "On the output-zeroing problem and pole-zero cancellations in MIMO systems satisfying uniform rank condition," in *Proc. Int. Conf. ICCI*, Hong Kong, June 5–9, 1995, pp. 265–269.
[20] P. Van Dooren, "The generalized eigenstructure problem in linear system theory," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 111–129, Jan. 1981.
[21] K. S. Yeung and Y. J. Huang, "On determination of the zeros of a SISO system," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 475–476, Apr. 1989.
[22] J. Tokarzewski, "Zeros and Kalman canonical form of MIMO LTI systems," in *Proc. European Control Conf., ECC*, Brussels, Belgium, July 1–4, 1997.
[23] A. Emami-Naeini and P. Van Dooren, "Computation of zeros of linear multivariable systems," *Automatica*, vol. 18, no. 4, pp. 415–430, 1982.
[24] P. Misra, P. Van Dooren, and A. Vargas, "Computation of structural invariants of generalized state-space systems," *Automatica*, vol. 30, no. 12, pp. 1921–1936, 1994.
[25] J. Tokarzewski, "A note on determination of zeros and zero directions by the Moore–Penrose pseudoinverse of the first nonzero Markov parameter," in *Proc. UKACC Int. Conf. CONTROL*, Exeter, U.K., Sept. 2–5, 1996, pp. 42–47.