Submerged cylinder wave energy device: theory and experiment

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Linearized water wave theory is used to show that a submerged long circular cylinder suitably constrained by springs and dampers to make small harmonic oscillations, can be extremely efficient in absorbing the energy in an incident regular wave whose crests are parallel to the axis of the cylinder. Experimental results are described which confirm the theory for small amplitude waves and which suggest that the device can still be fairly efficient in waves of moderate amplitude.

INTRODUCTION

In a recent paper[1] one of the authors has shown how the efficiency of power absorption of various two- and three-dimensional bodies may be expressed in terms of fundamental hydrodynamical properties of the bodies. In the present paper attention is focused on one particular body which was mentioned briefly in the previous work. This is a completely submerged long circular cylinder having its axis parallel to the incident wave-crests. The cylinder is assumed to be capable of making small oscillations both vertically and horizontally in the incident wave direction, being restrained by a system of vertical and horizontal springs and dampers which provide resistive forces proportional to displacement and velocity respectively.

The theory for the *forced* oscillation of a submerged cylinder has been derived by Ursell[2] and extended by Ogilvie[3] who showed that if the centre of the cylinder moves uniformly in a circle of small radius, the waves produced on the free surface above the cylinder travel away from the cylinder in one direction only, and this is true for all wave frequencies. This effect can be achieved by any shape for a given frequency, by suitably combining horizontal and vertical motions so as to cancel the waves at one infinity, but it seems likely that only for the submerged circular cylinder is the resultant motion circular, regardless of frequency. It follows by reversing the time coordinate that a circular motion of the cylinder exists which completely absorbs an incident regular wave-train of any frequency. It is this possibility that makes the submerged cylinder attractive as an efficient wave-energy device.

In the next section it is shown how the efficiency of the cylinder as a wave-energy absorber depends on the solution to the so-called radiation problem whereby the cylinder makes small forced sinusoidal horizontal or vertical oscillation. For completeness the solution of the radiation problem is presented in the Appendix and the Ogilvie results rederived.

The results of experiments carried out to test the submerged cylinder devive are presented later and compared to the linearized theory.

THEORY

The motion is two-dimensional with Cartesian coordinates (x, y) chosen such that $y=0$ is the undisturbed free surface with y measured vertically downwards and x to the right. The cylinder, of radius a, has its centre at $(0, +h)$, $h>a$, and is constrained to make small horizontal and vertical oscillations under springs of stiffness k_1 and k_2 and dampers of damping constants d_1 and d_2 respectively in response to an incident wave of amplitude A from $x=+\infty$. Let $\zeta_i(t)$ be the displacement of the centre of the cylinder from its equilibrium position, where $i=1$ and 2 refers to horizontal and vertical displacements respectively. Then the equations of motion of the cylinder can be written:

$$m\ddot{\zeta}_i = -d_i\dot{\zeta}_i - k_i\zeta_i + F_i \qquad (i=1, 2) \qquad (1)$$

where m is the mass per unit length of the cylinder, which is assumed to be neutrally buoyant and F_i is the total hydrodynamic force on the cylinder, per unit length. For small motions it is possible to write:

$$F_i = F_{is} + \sum_{j=1}^{2} F_{ij}$$

where F_{is} is the force acting on the cylinder in the ith direction when it is held fixed in the incident wave, and F_{ij} is the force on the body in the ith direction due to oscillation of the cylinder in the jth direction. Because of the symmetry of the circular cylinder there is clearly no horizontal (vertical) force induced on the cylinder by its vertical (horizontal) motion so that $F_{12}=F_{21}=0$. Now

$$F_{ii} = -a_{ii}\ddot{\zeta}_i - b_{ii}\dot{\zeta}_i \qquad (2)$$

where a_{ii} is the added mass of the cylinder describing the apparent increase in inertia of the cylinder due to the fluid, and b_{ii} is the damping coefficient, being a measure of the energy radiated away from the cylinder due to its forced oscillation in the ith direction, each per unit length of the cylinder.

The exciting force F_{is} is given by:

$$F_{is} = Re\{\rho g A A_i^+ e^{i\omega t}\} \tag{3}$$

a result due to Haskind[4] where $Re\ A \exp(i\omega^2 x/g)$ is the incident wave elevation in deep water, and A_i^+ is the wave amplitude at $x = +\infty$ due to a *forced* oscillation of unit amplitude of the cylinder in the ith direction. We also have the relation[5]:

$$b_{ii} = \rho\omega|A_i^+|^2 \tag{4}$$

for horizontally symmetric bodies.

It follows from equations (1), (2) and (3) that:

$$\xi_i = i\omega^{-1} Z_i^{-1} \rho g A A_i^+ \qquad (i = 1, 2) \tag{5}$$

where

$$\zeta_i = Re\,\xi_i e^{i\omega t} \qquad (i = 1, 2)$$

and

$$Z_i = (d_i + b_{ii}) + i[(m + a_{ii})\omega - k_i\omega^{-1}]$$
$$= |Z_i| \exp(i\chi_i) \tag{6}$$

and if we write

$$A_i^+ = |A_i^+| \exp(i\delta_i^+) \tag{7}$$

then
$$\zeta_i = C_i \sin(\omega t + \alpha_i) \tag{8}$$

where
$$C_i = \rho g A \omega^{-1} |Z_i^{-1}|\,|A_i^+| \tag{9}$$

and
$$\alpha_i = \delta_i^+ - \chi_i \qquad (i = 1, 2) \tag{10}$$

The total hydrodynamic pressure force acting on the cylinder is, from equations (1) and (8):

$$F_i = m\ddot{\zeta}_i + d_i\dot{\zeta}_i + k_i\zeta_i$$
$$= C_i[(k_i - m\omega^2)\sin(\omega t + \alpha_i) + d_i\omega\cos(\omega t + \alpha_i)] \tag{11}$$

and the instantaneous rate of doing work on the cylinder by the waves is:

$$W(t) = \sum_{i=1}^{2} F_i\dot{\zeta}_i$$
$$= \sum_{i=1}^{2} C_i^2\omega\cos(\omega t + \alpha_i)[(k_i - m\omega^2) \times \sin(\omega t + \alpha_i) + d_i\omega\cos(\omega t + \alpha_i)] \tag{12}$$

Notice if $d_i = 0$, the hydrodynamic pressure force is in phase with the displacement.

It follows that the mean power absorbed by the cylinder over a period is:

$$P_c = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} W(t)\mathrm{d}t = \tfrac{1}{2}\sum_{i=1}^{2} C_i^2\omega^2 d_i$$

The mean power in the incident wave is $\frac{1}{4}\rho g^2 A^2/\omega$ so the efficiency of the device is:

$$\eta = 2\sum_{i=1}^{2} d_i b_{ii}/|Z_i|^2 \tag{13}$$

where equation (4) has been used.

The result (13) applies generally to symmetric bodies oscillating in a combination of two uncoupled modes. This expression η is maximized by choosing:

$$k_i = (m + a_{ii})\omega^2 \quad \text{and} \quad d_i = b_{ii} \quad (i = 1, 2) \tag{14}$$

when $\eta_{max} = 1$, and complete absorption of the incident wave energy occurs. This result was derived in ref 1 and also independently by Mei[6]. Now Ogilvie[3] has shown that for the submerged circular cylinder

$$A_2^+ = iA_1^+ \tag{15}$$

indicating that the waves radiated away from the cylinder due to horizontal and vertical oscillations of unit amplitude are equal in amplitude and 90° out of phase with each other. It is this fact which gives rise to the unidirectional generation of waves when the cylinder centre describes a uniform circular motion. As a consequence of this result it follows that there is no reflection from a fixed submerged circular cylinder and further, that the added mass and damping coefficients in heave and surge are the same, i.e.

$$a_{11} = a_{22}, \qquad b_{11} = b_{12}$$

Indeed this last result follows directly from equations (4) and (15). These results for the submerged circular cylinder can all be found in ref. 3 but for completeness they are rederived in the Appendix.

So choosing $k_1 = k_2 = k$ and $d_1 = d_2 = d$ the total hydrodynamic pressure force given by equation (11) then reduces to:

$$F_1 = C\omega|Z_0|\cos(\omega t + \alpha_1 + \chi_0)$$
$$F_2 = -C\omega|Z_0|\sin(\omega t + \alpha_1 + \chi_0) \tag{16}$$

where

$$Z_0 = |Z_0|e^{i\chi_0} = d + i(m\omega - k\omega^{-1}), \quad C_1 = C_2 = C, \text{ and } \alpha_2 = \frac{\pi}{2} + \alpha_1 \tag{17}$$

Since the velocity components are given by:

$$\dot{\zeta}_1 = C\omega\cos(\omega t + \alpha_1)$$
$$\dot{\zeta}_2 = -C\omega\sin(\omega t + \alpha_1) \tag{18}$$

it follows that the instantaneous rate at which the waves do work on the cylinder is:

$$W(t) = \sum_{i=1}^{2} \dot{\zeta}_i F_i = C^2\omega^2|Z_0|\cos\chi_0$$
$$= \sum_{i=1}^{2} d\dot{\zeta}_i^2 = C^2\omega^2 d = \rho^2 g^2 A^2 |Z^{-2}|\,|A_i^+|^2 d \tag{19}$$

where $$Z_i = Z \quad (i=1, 2)$$

Thus the instantaneous and mean powers are identical and the efficiency is:

$$\eta = 4b_{ii}d|Z|^{-2}$$

If we now choose, at frequency ω_0,

$$k = k_0 = [m + a_{ii}(\omega)]\omega_0^2, \qquad d = d_0 = b_{ii}(\omega_0),$$

then

$$\eta = \frac{4b_{ii}(\omega)b_{ii}(\omega_0)}{(k_0 - (m+a_{ii})\omega^2)^2 + (d_0 + b_{ii})^2} \tag{20}$$

$$= 1 \text{ for } \omega = \omega_0$$

and $$\chi_1 = 0$$

Notice from equations (4) and (18) that the cylinder describes a circle of radius C where

$$C^2 = \rho g^2 A^2 |Z|^{-2} b_{ii}/\omega^3$$

with a phase angle relative to the incident wave in the absence of the cylinder given by equation (10). When optimally tuned:

$$C^2 = \rho g^2 A^2 / 4\omega^3 b_{ii}$$

and $\alpha_1 = \delta_1^+$.

Since the induced motion of the cylinder is circular, it follows that there is no reflection of the incident wave for any frequency, and at ω_0, the tuning frequency, no transmission either.

EXPERIMENTAL

The preceding theory was tested using the experimental facilities developed at the Department of Mechanical Engineering, University of Edinburgh, as part of the Edinburgh Wave Power Project.

Wave measurements

The wave tank was 10 m long, 30 cm wide and 60 cm deep, with a Salter absorbing wave-maker at one end capable of producing accurately controlled waves even in the presence of reflections. The waves were absorbed at the far end of the tank using a 'beach' consisting of vertical Expamet sheets which reflected about 5% at all wavelengths and steepnesses in the test range.

Wave amplitudes were measured using a heaving float gauge. This is a light cylindrical float, 11 in. wide and 1.25 in. in diameter. It is held by a linkage which constrains it to move in a vertical line. Both cylinder and linkage are made of rolled paper. The float is water-proofed with polyurethane while the linkage is impregnated with epoxy resin. The float velocity was sensed by a pair of microammeter movements, integrated and calibrated to give a direct reading on a transfer function analyser (t.f.a.) of wave amplitude in centimetres.

The method of obtaining the incident wave amplitude from measurements contaminated by the reflected wave was as follows. The wave-maker was driven by the t.f.a. at the required frequency $\omega/2\pi$, generating waves of a certain wavelength L satisfying the dispersion relation:

$$K = k_0 \tanh k_0 H \tag{21}$$

where $k_0 = 2\pi/L$, $K = \omega^2/g$, and H is water depth. Two heaving float wave gauges were clamped $\frac{1}{4}$ or $\frac{3}{4}$ wavelengths apart and moved together so that the difference in wave amplitude measured by the gauges was as large as possible. The mean of the two readings then gave the incident wave amplitude while half the difference gave the reflected amplitude. The validity of this procedure can be shown by the following argument.

Let the wave elevation at the first wavegauge positioned at x_1 say, be

$$f(x_1 t) = A \sin(k_0 x_1 - \omega t) + B \sin(k_0 x_1 + \omega t + \delta)$$

where the second term describes the reflected wave.

This may be rewritten:

$$f(x_1, t) = C(x_1) \sin(\omega t + \alpha)$$

where

$$C^2(x_1) = A^2 + B^2 - 2AB\cos(2k_0 x_1 + \delta)$$

The second wavegauge is at x_2, say, where:

$$|x_2 - x_1| = \frac{L}{4} \text{ or } \frac{3L}{4}$$

so that $|2k_0 x_2 - 2k_0 x_1| = \pi$ or 3π.

It follows that:

$$f(x_2, t) = C(x_2) \sin(\omega t + \alpha)$$

where

$$C^2(x_2) = A^2 + B^2 + 2AB\cos(2k_0 x_1 + \delta)$$

Now max $\{|C(x_1) - C(x_2)|\} = 2B$ while max$\{|C(x_1) + C(X_2)|\} = 2A$. Thus the incident wave power at a given point in the tank was determined from:

$$P_{\text{inc}} = \frac{\rho g^2 A^2}{4\omega} \tanh k_0 H \left(1 + \frac{2k_0 H}{\sinh 2k_0 H}\right) W \tag{22}$$

where W is the width of the tank. Over most of the range of wavelengths considered the simpler deep water expression:

$$P_{\text{inc}} = \frac{\rho g^2 A^2}{4\omega} W \tag{23}$$

was sufficient, so that comparison with a deep-water theory was justified. Because of attenuation of wave amplitude along the tank the power reaching the model was less than that measured by the wave gauges 2 m away. To allow for this a correction factor was used for each frequency based on measurements of amplitude attenuation at points along the tank in the absence of the model.

The surge-heave-pitch rig

The model, a neutrally buoyant circular cylinder of diameter 10 cm and length slightly less than the width of the tank, was positioned in the surge-heave-pitch rig. This is a rig originally designed for the Salter 'duck' wave energy device[6], which allows independent horizontal and vertical motions, and in the case of the duck, pitch motion also. The heave (vertical) and surge (horizontal) motions are transmitted to two horizontal spindles. Heave is constrained by a straight-line linkage but surge is a short arc of a circle which is straight enough for the small displacements which occur. Each spindle can have its inertia increased by the addition of pairs of weights. Its stiffness can be controlled by the length and thickness of torsion springs which are contained in rotatable housings used for adjusting axis depth and surge position. A set of strain gauge transducers measure heave and surge force close to the cylinder. The surge-heave-pitch rig also enables arbitrary opposing forces to be applied to the cylinder electronically. The input is a velocity signal measured by 50 μA meter movements whose needles are mechanically connected to the heave and surge axes of the rig. This velocity signal is conditioned to a reasonable voltage level near its source. An integrator turns the velocity signal into a displacement signal calibrated in convenient units. The displacement and velocity signals each pass through an attenuator, a polarity switch and gain switch to a summing junction. This summed signal is taken through a power amplifier and used to drive a torque motor which is coupled to the motion whose velocity was originally measured. By this means the motion of the cylinder can be opposed (or assisted) in heave or surge by forces proportional to displacement (spring forces) or velocity (damping forces). This device is referred to as the Dynabox.

The mean power absorbed in surge was obtained by electronically multiplying surge force and surge velocity and averaging. This was also done for heave power, the total power absorbed being the sum of surge power and heave power.

Procedure

Because of the support structure of the surge-heave-pitch rig the theory outlined above is not directly applicable and the equations of motion must be modified to:

$$(m+m_i)\ddot{\zeta}_i + d_i\dot{\zeta}_i + k_i\zeta_i = F_i$$

where m_i $(i=1, 2)$ is the additional structural mass per unit length of the cylinder in heave and surge. Then the efficiency

$$\eta = 2b_{ii}\sum_{i=1}^{2} d_i|Z_i|^{-2} \qquad (24)$$

where

$$|Z_i|^2 = (d_i+b_{ii})^2 + [(m+m_i+a_{ii})\omega - k\omega^{-1}]^2$$

and $b_{11}=b_{22}$, $a_{11}=a_{22}$ for the submerged cylinder.

For a given frequency $\omega/2\pi$, the cylinder is tuned to maximum efficiency if:

$$k_i = (m+m_i+a_{ii})\omega^2 \qquad (i=1, 2) \qquad (25)$$

and

$$d_i = b_{ii} \qquad (i=1, 2) \qquad (26)$$

The method of tuning the cylinder in a particular frequency was as follows. The cylinder axis was first positioned at a given depth. An oscillating external force was applied to the rig causing the cylinder to heave or surge and radiate waves outwards. The equation of motion is:

$$(m+m_i+a_{ii})\ddot{\zeta}_i + (d_i+b_{ii})\dot{\zeta}_i + k_i\zeta_i = F_{ie}/W \qquad (27)$$

where F_{ie} is the external force in the ith direction, and the hydrodynamic force due to the motion has been incorporated into the left-hand side. Then F_{ie} and $\dot{\zeta}_i$ were displayed as X and Y coordinates on an oscilloscope and in general took the shape of an ellipse. By suitably adjusting k_i using the Dynabox the ellipse could be squashed into a straight line showing that F_{ie} and $\dot{\zeta}_i$ were exactly in phase so that from equation (27) the first tuning equation (25) was satisfied.

To satisfy the second tuning equation (26), the polarity of the damping force d_i was reversed so that $d_i = -d_i$ and the rig set in motion impulsively in heave and surge in turn. The values of d_i were adjusted on the Dynabox so that the ensuing motion was purely simple harmonic. Then from equation (27) $d_i = b_{ii} = 0$ and the cylinder was correctly tuned as soon as the polarity of d_i was reversed again. The wave-maker was now switched on at the tuned frequency and the trajectory of the centre of the cylinder observed on the oscilloscope. It was always close to circular as expected. Fine adjustments to the values of the spring and damping forces k_i and d_i were then made while observing the mean power output on a digital meter. The power in heave and surge could not be observed simultaneously so that care had to be taken not to adjust k_1 and d_1, say, in such a way that an increase in surge power resulted in a loss in total power in surge *and* heave. In general the maximum total power output occurred when the trajectory of the centre of the cylinder was most circular in shape, especially for small amplitude waves. The power was then noted together with the incident and reflected wave amplitude as described above.

With the fixed settings for k_i and d_i the wave-maker was driven at different frequencies, each time adjusting the input to keep the wave amplitude as nearly the same as possible. The process was repeated for a range of different incident amplitudes from 0.15 cm to about 1.0 cm. The range of frequencies covered was from 1 to 2 Hz. Two cylinder depths were considered: $a/h = 0.8, 0.67$ where h is the depth of the axis of the cylinder, and a the radius of the cylinder. This corresponded to the top of the cylinder being submerged to a depth of 1.25 cm and 2.5 cm respectively.

Results

The values of m_i were determined by applying a known external force to the rig at high frequency *without* the cylinder in position and measuring the maximum displacement. Then equation (27) gives $\omega^2 m_i$ as the ratio of force to maximum displacement. It was found that: $m_1W = 0.85$ kg, $m_2W = 2.96$ kg whilst $mW = 2.36$ kg. The expression (24) for the efficiency can be non-dimensionalized as in Evans[1] to become:

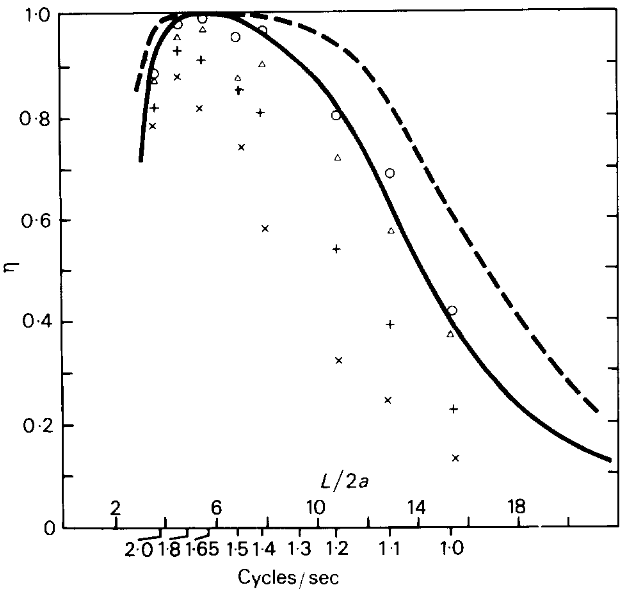

Figure 1. Variation of efficiency of power absorption η vs. *wavelength to cylinder diameter ratio,* L/2a, *for different incident wave amplitudes* A. *Cylinder axis submerged to depth* h = 6.25 *cm so that* a/h = 4/5. *Cylinder tuned to* 1.65 Hz *or* L/2a = 5.7. A/a ≑ ○, 0.033; △, 0.054; +, 0.096; ×, 0.18. ———, $m_1' = 1.36$; $m_2' = 2.26$; - - - -, $m_1' = m_2' = 1.0$

$$= 2\nu^{3/2}\nu_0^{1/2}\lambda\lambda_0 \sum_{i=1}^{2} [\{(m_i' + \mu_0)\nu_0 - (m_i' + \mu)\nu\}^2 + \nu(\nu^{1/2}\lambda + \nu_0^{1/2}\lambda_0)^2]^{-2} \tag{28}$$

where $a_{ii} = \pi\rho a^2 \mu$, $\omega b_{ii} = \pi\rho a^2 \lambda$ and

$$m_1' = 1 + \frac{m_1}{m} = 1.36, \quad m_2' = 1 + \frac{m_2}{m} = 2.26$$

Here $\nu = 2\pi a/L = \omega^2 a/g$ is the non-dimensional wave-number while the suffix zero denotes the value of that parameter at the tuned wave-number.

Values of μ and λ computed from equations (A26) and (A27) for $a/h = 0.8$, 0.67 were used in equation (28) to compute η for different values of frequency $\omega/2\pi$ and hence ν, since $a = 5$ cm. The cylinder was tuned to frequencies of 1.65 Hz corresponding to $\nu = 0.55$ or $L/2a = 5.7$, and 1.5 Hz, corresponding to $\nu = 0.45$ or $L/2a = 7.1$.

Comparison of theory and experiment is shown in Fig. 1, where η is plotted against $L/2a$ for a range of different amplitudes. It can be seen that for extremely small amplitude incident waves agreement is good, 98.2% efficiency being achieved at the theoretical 100% optimal point. The broad bandwidth predicted by the theory is also confirmed by the experiments for small waves, the cylinder being over 80% efficient for wavelengths between 4 and 11 times the cylinder diameter. As the size of the waves increases it is clear that the linear theory is not adequate and lower efficiencies and narrower bandwidths occur. It was also apparent in the experiments that higher-order effects were important since except for very small incident wave amplitudes, a train of higher harmonic waves could be seen downstream of the cylinder, generally of double the frequency of the incident waves. This is because the submerged cylinder, being close to the surface, acts as a sloping beach to the incident wave, causing the wave to steepen as it passes over the cylinder and to degenerate into its higher harmonic components. A more complete discussion of the physical mechanism involved is given in Longuet-Higgins[7].

Nevertheless even for steep waves the cylinder is a fairly good energy absorber. Thus it can be seen from Fig. 1 that for waves of amplitude 0.9 cm the cylinder absorbs over 55% of the energy from waves of wavelengths from 4 up to 8 times the cylinder diameter, and over 30% for waves up to 11 times the cylinder diameter. Also shown on Fig. 1 is the theoretical curve of efficiency against wavelength for a neutrally buoyant cylinder when $m_i' = 1$ $(i = 1, 2)$. It can be seen that the effect of the increased inertia due to the rig is to narrow the efficiency bandwidth. It is reasonable to suppose that experiments with a genuinely neutrally buoyant cylinder would show a corresponding improvement in bandwidth.

Figure 2 illustrates the variation of efficiency with wave amplitude, for different values of $L/2a$. Also shown are the theoretical efficiencies based on a 'zero amplitude' theory. For $L/2a = 5.7$ corresponding to 1.65 Hz, the tuning frequency, the variation is linear and as the amplitude decreases the efficiency tends to the theoretical efficiency of 100%. For other values of $L/2a$ the efficiency still decreases with increasing amplitude but is no longer linear.

Figure 3 shows the effect of increasing the depth of submergence of the cylinder from $a/h = 0.8$ to $a/h = 0.67$. Although the maximum theoretical efficiency is 100% at *all* of submergence, equation (28) shows that for frequencies away from the tuning frequency the efficiency depends crucially on the damping coefficient which is known to decrease rapidly with increasing depth of submergence. The predicted narrower bandwidth in the efficiency curve is borne out by the experimental points which are close to the theory except near the theoretical maximum of 100%. This discrepancy is due to the larger wave amplitudes considered in this case.

It is noticeable that for the steep waves (A/a ≑ 0.18) and

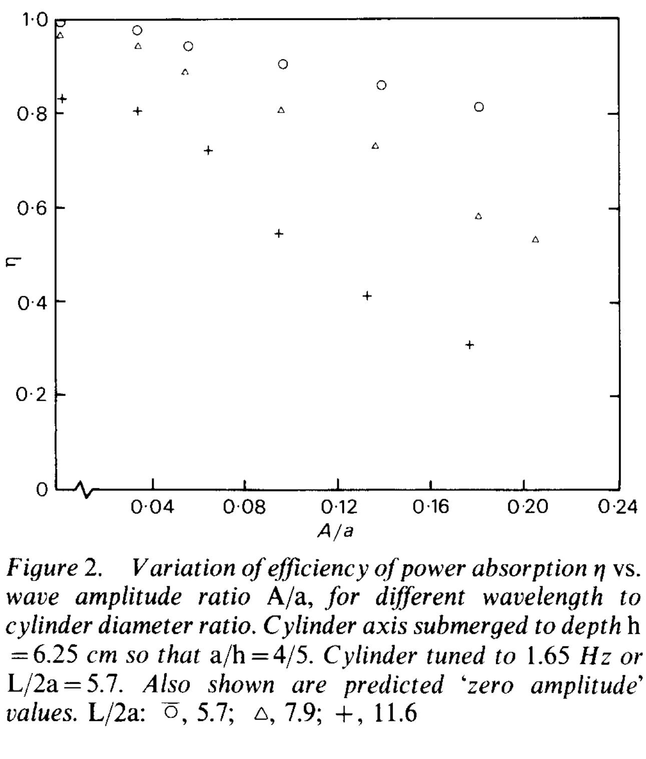

Figure 2. Variation of efficiency of power absorption η vs. *wave amplitude ratio* A/a, *for different wavelength to cylinder diameter ratio. Cylinder axis submerged to depth* h = 6.25 *cm so that* a/h = 4/5. *Cylinder tuned to* 1.65 *Hz or* L/2a = 5.7. *Also shown are predicted 'zero amplitude' values.* L/2a: ○̄, 5.7; △, 7.9; +, 11.6

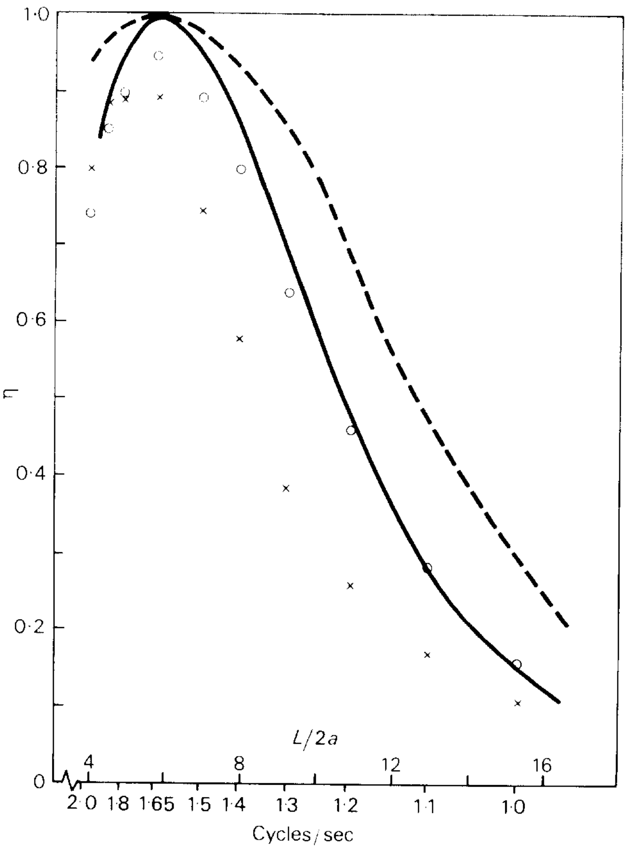

Figure 3. Variation of efficiency of power absorption η vs. *wavelength to cylinder diameter ratio,* L/2a, *for different incident wave amplitudes* A. *Cylinder axis submerged to depth* h = 7.5 *cm so that* a/h = 2/3. *Cylinder tuned to* 1.65 *Hz or* L/2a = 5.7. A/a ≑ ○, 0.052; ×, 0.18; ———, m'_m = 1.36; m'_2 = 2.26; – – –, $m'_1 = m'_2 = 1.0$

for frequencies close to the tuning frequency, the efficiency curve is comparable to the corresponding curve in Fig. 1 where $a/h = 0.8$. This is confirmed in Fig. 4 where the cylinder is tuned to 1.5 Hz corresponding to incident wavelengths about 7 times the cylinder diameter. At this fixed frequency the variation of efficiency with wave amplitude is compared for the two depths of submergence.

This suggests that near the tuning frequency, more deeply submerged cylinders cope better with steep waves than cylinders closer to the surface. A possible explanation for this is that for cylinders closer to the surface there is a greater tendency for waves to steepen and to degenerate into higher harmonics whose energy cannot be absorbed. For more deeply submerged cylinders this is less likely to happen. Away from the tuning frequency this advantage is offset by the adverse bandwidth effect already referred to. Also, for increasingly deep submergence, even at the tuning frequency the efficiency can be expected to fall due to the non-linear effects induced by the increasingly large oscillations of the cylinder as it attempts to couple with the waves passing overhead.

Also shown in Fig. 3 is the theoretical curve for a neutrally buoyant cylinder illustrating the improved bandwidth, which is, however, narrower than the corresponding curve in Fig. 1.

The linearized theory predicts that the reflection from the cylinder is zero at all frequencies. The experiments

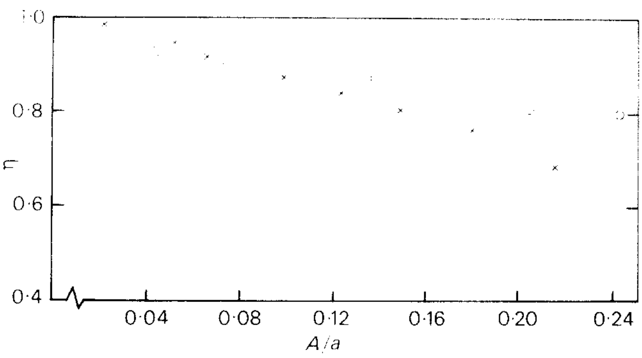

Figure 4. Variation of maximum efficiency of power absorption η vs. *wave amplitude ratio* A/a, *for two different depth of submergence ratios* a/h. *Cylinder tuned to* 1.5 *Hz or* L/2a = 7.1. a/h: ×, 0.8; ○, 0.67

showed that the reflected wave amplitude measured by the two wave gauges in front of the cylinder never exceeded 10% of the incident wave amplitude and in most cases was much lower. Since some of this reflection was probably from the beach anyway, it follows that the amount of incident power lost by reflection from the cylinder was at most 1% of the total, and that the main source of power loss is in the transmitted wave. This is an attractive feature of the submerged cylinder wave-energy device since the mean horizontal second-order forces on the cylinder are that much smaller than for a surface device which reflects more of the incident wave. This point has been discussed more fully in ref. 8.

CONCLUSION

It has been shown that a submerged circular cylinder can be very efficient in absorbing energy from a regular wave-train. The experiments confirm the theoretical prediction that for infinitesimal waves the maximum efficiency possible is 100%, and they also confirm the broad bandwidth in the efficiency curves. For finite-amplitude waves the cylinder is less efficient, as expected, and the linear theory is no longer applicable. The device differs in this respect from other surface wave-power devices being highly sensitive to wave amplitude. This could be advantageous in a real situation since power varies with the amplitude squared and too much power can be an embarrassment. A submerged device could also have advantages as far as survival in storm seas is concerned but this must be offset by the extra problems involved in positioning the device beneath the surface. However, progress in this direction has already been made and a research programme is underway at the University of Bristol, England to explore the full potentiality of the submerged cylinder wave energy device.

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APPENDIX

The expression for η contains the added mass and damping coefficients for the submerged cylinder in heave and surge, and the motion of the centre given by equation (5) contains A_i^+. In this Appendix we rederive Ogilvie's results for the submerged circular cylinder making simple harmonic heave and surge motions of small amplitude. Under the usual assumptions of linearized potential theory it can be shown that there exists a velocity potential ($\Phi(x, y, t)$ satisfying Laplace's equation in the fluid condition:

$$\frac{\partial^2\Phi}{\partial t^2}-g\frac{\partial\Phi}{\partial y}=0 \text{ on } y=0, \text{ the linearized free surface.}$$

On the mean position of the cylinder, $r=a$

$$\frac{\partial\Phi}{\partial n}=\sum_{i=1}^{2}\zeta_i n_i \tag{A1}$$

where $\mathbf{n}=(n_1, n_2)$ is the outward normal to the cylinder.

It is assumed that the simple harmonic oscillations of frequency $\omega/2\pi$ give rise to waves of the same frequency. Thus we write:

$$\Phi(x, y, t)=Re\{\varphi(x, y)e^{i\omega t}\}$$

and

$$\varphi(x, y)=i\omega\sum_{i=1}^{2}\varphi_i\xi_i$$

so that φ has been separated into asymmetric (φ_1) and symmetric (φ_2) potentials. It follows that φ_i satisfy Laplace's equation (A1)

$$K\varphi_i+\frac{\partial\varphi_i}{\partial y}=0 \text{ on } y=0 \qquad (i=1, 2) \tag{A2}$$

$$\frac{\partial\varphi_i}{\partial r}=n_i \text{ on } r=a \qquad (i=1, 2) \tag{A3}$$

where $r^2=x^2+(y-h)^2$. Note that $n_1=\sin\theta$, $n_2=\cos\theta$ for the circular cylinder, where $r\cos\theta=y-h$, $r\sin\theta=x$. At large distances from the cylinder we assume that:

$$\begin{aligned}\varphi_i &\sim A_i^+e^{iKx-Ky}, \quad x\to+\infty \qquad (i=1, 2)\\ &\sim A_i^-e^{iKx-Ky}, \quad x\to-\infty \qquad (i=1, 2)\end{aligned} \tag{A4}$$

these asymptotic forms describing waves travelling away from the cylinder. It is clear that $A_1^+=-A_1^-$ and $A_2^+=A_2^-$ from symmetry.

We require potentials φ_1 and φ_2 satisfying equations (A1), (A2), (A3) and (A4). If the fluid extended to infinity in all directions it would be natural to look for a solution in terms of the basic singular potentials $\log r$, $r^{-n}\cos n\theta$, $r^{-n}\sin n\theta$, $(n=1, 2, \ldots)$ each of which satisfies Laplace's equation for $r\neq0$. Clearly the solution would be $\varphi_1=-a^2r^{-1}\sin\theta$, $\varphi_2=-a^2r^{-1}\cos\theta$.

The presence of the free surface suggests that the basic singular potentials need to be modified by an image set which will not introduce singularities into the flow but will ensure that the free surface condition (A2) is satisfied. We consider the asymmetric potentials forst and write:

$$\varphi_{1n}(x, y)=\frac{\sin n\theta}{r^n}+\int_0^\infty a(k)e^{-ky}\sin kx\,dk+Ce^{-Ky}\sin Kx$$

where the last two terms are both harmonic functions odd in x, and $a(k)$, C are to be determined from the free surface condition and the behaviour for large $|x|$ respectively.

Now it is easily verified that:

$$\frac{e^{ni\theta}}{r^n}=\frac{(-1)^n}{(n-1)!}\int_0^\infty k^{n-1}e^{-k(h-y)}e^{-ikx}dk, \qquad h>y$$

so that on $y=0$

$$K\varphi_{1n}+\frac{\partial\varphi_{1n}}{\partial y}=\frac{(-1)^{n-1}}{(n-1)!}\int_0^\infty(K+k)k^{n-1}e^{-kh}\sin kx\,dk+$$

$$\int_0^\infty(K-k)a(k)\sin kx\,dk=0$$

whence

$$a(k)=\frac{(-1)^n}{(n-1)!}\frac{K+k}{K-k}k^{n-1}e^{-kh}$$

and so

$$\varphi_{1n}(x, y)=\frac{\sin n\theta}{r^n}+\frac{(-1)^n}{(n-1)!}$$

$$\int_0^\infty\frac{K+k}{K-k}k^{n-1}e^{-k(y+h)}\sin kx\,dk+Ce^{+Ky}\sin Kx \tag{A5}$$

where the principal value of the integral is assumed. To determine C we write the integral in equation (A5) in the form:

$$\mathscr{I}m\int_0^{\infty}\frac{K+k}{K-k}k^{n-1}e^{-k(y+h)+ikx}dk$$

$$=\mathscr{I}m\left\{\int \ldots -2K^n\pi ie^{-K(y+h)+iKx}\right\}$$

$$=\mathscr{I}m\int_0^{\infty}\frac{K+ik}{K-ik}i^nk^{n-1}e^{-k(y+h)-kx}dk-2\pi K^ne^{-K(y+h)}\cos Kx$$

for $x>0$, where the path of integration has been rotated through 90°. Thus

$$\varphi_{1n}(x,y)\sim\left(-\frac{2\pi(-1)^n}{(n-1)!}K^ne^{-Kh}\cos Kx+C\sin Kx\right)e^{-Ky}$$

$$\text{as } |x|\to+\infty$$

$$=(-1)^{n-1}\frac{2\pi K^n}{(n-1)!}e^{-K(y+h)}e^{-iKx}\text{ as required}$$

if we choose

$$C=(-1)^n\frac{2\pi iK^n}{(n-1)!}e^{-Kh}\tag{A6}$$

and generally

$$\varphi_{1n}(x,y)\sim sgnx\frac{(-1)^{n-1}2\pi K^n}{(n-1)!}e^{-K(y+h)}e^{-iK|x|}$$

$$\text{as } |x|\to\infty\tag{A7}$$

In a similar fashion the even singular potentials can be constructed. Thus

$$\varphi_{2n}(x,y)=\frac{\cos n\theta}{r^n}+$$

$$\frac{(-1)^{n-1}}{(n-1)!}\int_0^{\infty}\frac{K+k}{K-k}k^{n-1}e^{-k(y+h)}\cos kx dk+$$

$$\frac{(-1)^{n-1}2\pi iK^n}{(n-1)!}e^{-K(y+h)}\cos Kx\tag{A8}$$

$$\sim\frac{(-1)^{n-1}2\pi iK^n}{(n-1)!}e^{-K(y+h)}e^{-iK|x|}\tag{A9}$$

The above derivation is essentially that of Thorne[8]. A solution for $\varphi_i(x,y)$ is now posed in the form

$$\varphi_i(x,y)=\sum_{n=1}^{\infty}\frac{a^{n+1}}{n}P_{in}\varphi_{in}(x,y)\qquad(i=1,2)\tag{A10}$$

Since each of the functions φ_{in} satisfies Laplace's equation, the free surface condition, and has the required behaviour for large $|x|$, it remains to satisfy the condition on the cylinder and so determine the unknown constants.

In order to apply condition (A3) it is necessary to obtain series expansions of φ_{in} in terms of $r^n\cos n\theta$ and $r^n\sin n\theta$. Consider

$$I=\int_0^{\infty}b_n(k)e^{-ky-ikx}dk$$

where

$$b_n(k)=\frac{K+k}{K-k}k^{n-1}e^{-kh}$$

Then

$$I=\int_0^{\infty}b_n(k)e^{-kh}e^{-kr\exp(i\theta)}dk$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^mr^me^{mi\theta}}{m!}\int_0^{\infty}b_n(k)k^me^{-kd}dk$$

It follows that:

$$\varphi_{1n}(x,y)=\frac{\sin n\theta}{r^n}+\sum_{m=0}^{\infty}A_{mn}r^m\sin m\theta\tag{A11}$$

$$\varphi_{2n}(x,y)=\frac{\cos n\theta}{r^n}+\sum_{m=0}^{\infty}A_{mn}r^m\cos m\theta\tag{A12}$$

valid for $0<r<2h$ where

$$A_{mn}=\frac{(-1)^{m+n-1}}{(n-1)!m!}\int_0^{\infty}\frac{K+k}{K-k}k^{m+n-1}e^{-2kh}dk-$$

$$\frac{2\pi i(-1)^{m+n}}{(n-1)!m!}K^{m+n}e^{-2Kh}\tag{A13}$$

and it is noticeable that the A_{mn} are the same for both φ_{1n} and φ_{2n}. Applying condition (A3) to $\varphi_1(x,y)$ we obtain:

$$\left.\frac{\partial\varphi_1}{\partial r}\right|_{r=a}=\sin\theta=-\sum_{n=1}^{\infty}P_{1n}x\sin n\theta-$$

$$\left.-\sum_{m=1}^{\infty}\frac{m}{n}a^{m+n}A_{mn}\sin m\theta\right\}$$

and equating coefficients of $\sin m\theta$, $m=1,2,\ldots$ gives the infinite system of equations

$$P_{1m}-\sum_{n=1}^{\infty}\frac{m}{n}a^{m+n}A_{mn}p_{1n}=-\delta_{1m}\qquad(m=1,2)\tag{A14}$$

An identical system is obtained for p_{1m} by applying condition (A3) to $\varphi_2(x,y)$.

It follows that if equation (A14) has a solution, then $p_{1m}=p_{2m}\equiv p_m$ say $(m=1,2,\ldots)$. In fact Ursell[2] has shown that equation (A14) has a unique solution such that equation

(A10) converges absolutely and uniformly in the fluid and on the cylinder. Now from equations (A7), (A9) and (A10):

$$\varphi_1(x, y) \sim -2\pi a e^{-K(y+h)-iK|x|} \operatorname{sgn} x \sum_{n=1}^{\infty} \frac{(Ka)^n(-1)^n}{n!} p_n$$

and

$$\varphi_2(x, y) \sim -2\pi i a e^{-K(y+h)-iK|x|} \sum_{n=1}^{\infty} \frac{(Ka)^n(-1)^n}{n!} p_n \quad \text{as} \quad |x| \to \infty$$

so that, by comparison with equation (A4)

$$A_1^+ = -2\pi a e^{-Kh_s}$$
$$A_2^+ = -2\pi i a e^{-Kh_s}$$

where

$$S = \sum_{n=1}^{\infty} \frac{(Ka)^n(-1)^n}{n!} p_n \qquad (A15)$$

Thus $\qquad A_2^+ = iA_1^+ \qquad$ (A16)

Now

$$\Phi(x, y, t) = Re\, i\omega \sum_{i=1}^{2} \varphi_i \xi_i e^{i\omega t}$$

$$\sim Re\, i\omega (A_1^+(\xi_i + i\xi_2) e^{-i(Kx-\omega t)} e^{-Ky}, \qquad x \to +\infty$$

$$\sim Re\, i\omega A_1^+(-\xi_i + i\xi_2) e^{\to i(Kx+\omega t)} e^{-Ky} \quad x \to -\infty \qquad (A17)$$

from equations (A4) and (A16).

The motion of the centre is $\zeta_i = Re\xi_i e^{i\omega t}$. Suppose that $\zeta_1 = \delta \cos \omega t$, $\zeta_2 = \delta \sin \omega t$ so that $\xi_1 = \delta$, $\xi_2 = i\delta$ and the centre of the cylinder describes a circle of radius δ in a clockwise direction. It follows from equation (A17) that:

$$\Phi \to 0 \text{ as } x \to -\infty \text{ and}$$

$$\Phi \sim Re 2\delta\omega |A_1^+| e^{-i(Kx-\omega t)} e^{-Ky}, \qquad x \to +\infty \qquad (A18)$$

showing that waves are radiated away from the cylinder on one side only. Similarly if $\xi_1 = \delta$, $\xi_2 = i\delta$ so that the cylinder rotates counterclockwise, waves are radiated to $x = -\infty$ only.

To derive the result originally due to Dean[9] that an incident wave undergoes no reflection when encountering a submerged circular cylinder, only a phase change, we use the Newman relations[10] connecting the reflection and transmission coefficients to the A_i^+. Thus

$$A_i^+ + \bar{A}_i^+ R + \bar{A}_i^- T = 0 \qquad (i = 1, 2)$$

whence

$$R + T = -(A_1^+/\bar{A}_1^+), \quad R - T = -(A_2^+/\bar{A}_2^+)$$

since $A_i^+ = (-1)^i A_i^-$ from symmetry. But we have shown that $A_2^+ = iA_1^+$ and so $R = 0$, $T = A_1^+/\bar{A}_1^+ = S/\bar{S}$, $|T| = 1$ where

$$S = \sum_{n=1}^{\infty} \frac{(Ka)^n(-1)^n}{n!} p_n$$

To obtain the added mass and damping coefficients consider the infinite system (A14) we write:

$$\frac{(-1)^m (Ka)^m p_m}{m} = q_m + ir_m, \quad q_m, r_m \text{ real}$$

and obtain:

$$q_m + \sum_{n=1}^{\infty} \alpha_{mn} q_n = -\frac{(-1)^m (Ka)^m}{m} \delta_{m1} + \frac{(Ka)^{2m}}{m!} S_r$$

$$r_m + \sum_{n=1}^{\infty} \alpha_{mn} r_n = -\frac{(Ka)^{2m}}{m!} S_q$$

where

$$S_x = 2\pi e^{-2Kh} \sum_{n=1}^{\infty} \frac{x_n}{(n-1)!}$$

and

$$\alpha_{mn} = \frac{(Ka)^{2m}}{(n-1)!m!} \int_0^{\infty} \frac{1+t}{1-t} t^{m+n-1} e^{-2Kht} dt \qquad (A20)$$

Following Ursell[2] let ε_n, γ_n satisfy

$$\varepsilon_m + \sum_{n=1}^{\infty} \alpha_{mn} \varepsilon_n = \frac{(Ka)^{2m}}{m!} \qquad (A21)$$

$$\gamma_m + \sum_{n=1}^{\infty} \alpha_{mn} \gamma_n = \delta_{m1} \qquad (A22)$$

Then, after some algebra, it follows that:

$$r_m = -S_q \varepsilon_m \qquad (A23)$$

and $\qquad q_m = Ka\gamma_m + S_r \varepsilon_m \qquad$ (A24)

where $\qquad S_r = -KaS_\gamma S_\varepsilon/(1 + S_\varepsilon^2)$

$$S_q = KaS_\gamma/(1 + S_\varepsilon^2)$$

It can be seen that the solution for q^m and r^m, and hence for the complex p_m, is given in terms of the real infinite systems (A21) and (A22), where for computational purposes it is convenient to rewrite α_{mn} as:

$$\alpha_{mn} = \frac{(Ka)^{2m}}{(n-1)!m!} \left\{ 2e^{-2Kh} Ei(2Kh) + \frac{(m+n-1)!}{(2Kh)^{m+n}} - 2 \sum_{k=0}^{m+n-1} \frac{k!}{(2Kh)^{k+1}} \right\} \qquad (A25)$$

Returning to $\varphi_i(x, y)$ we have for the even potential:

$$\varphi_2(x, y) = \sum_{n=1}^{\infty} \frac{a^{n+1}}{n} p_n \left\{ \frac{\cos n\theta}{r^n} + \sum_{m=0}^{\infty} A_{mn} r^m \cos m\theta \right\}$$

$$= \sum_{n=1}^{\infty} ap_n \frac{\cos n\theta}{n} \left\{ \left(\frac{a}{r}\right)^n + \left(\frac{r}{a}\right)^n \right\} + r\cos\theta$$

from equation (A14).

Thus

$$\int_{-\pi}^{\pi} \varphi_2|_{r=a} \cos\theta \mathrm{d}\theta = \pi a(1+2p_1)$$

$$= -\pi a(\mu - i\lambda)$$

where

$$\mu = 2\gamma_1 - 1 + \frac{2S_q S_\varepsilon \varepsilon_1}{1+S_\varepsilon^2} \tag{A26}$$

from equations (A19), (A23) and (A24).

The vertical hydrodynamical force on the cylinder is

$$F_{22} = \rho \int_{-\pi}^{\pi} \Phi_t|_{r=a} \cos\theta \mathrm{d}\theta$$

$$= Re a\rho i\omega \int_{-\pi}^{\pi} i\omega \sum_{i=1}^{2} \xi_i \varphi_i \cos\theta \mathrm{d}\theta e^{i\omega t}$$

$$= Re \pi\rho a^2 \omega^2 \xi_2(\mu - i\lambda) e^{i\omega t}$$

But the added mass and damping coefficients are given from equation (2) by:

$$F_{22} = -a_{22}\ddot{\xi}_2 - b_{22}\dot{\xi}_2$$

$$= Re\omega^2 \xi_2(a_{22} - i\omega^{-1} b_{22}) e^{i\omega t}$$

whence

$$a_{22} = \pi\rho a^2 \mu, \qquad \omega b_{22} = \pi\rho a^2 \lambda \tag{A28}$$

Similarly

$$F_{11} = a\rho \int_{-\pi}^{\pi} \Phi_t|_{r=a} \sin\theta \mathrm{d}\theta = F_{22}$$

showing that $a_{22} = a_{11}$, $b_{22} = b_{11}$.