Derivation of Positive Recurrence of Brownian Motion with Drift

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1 Introduction

This note is dedicated to deriving the first moment of the hitting time for a geometric Brownian motion with drift, defined by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \text{where} \quad \mu \leq 0, S_0 > a,$$

when it first hits a given lower boundary a.

2 Mathematical Derivation

According to Itô's Lemma, we can derive the analytical solution of the GBM:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

The stopping time τ is defined by:

$$\begin{aligned}\tau &= \inf\{t \geq 0 : S_t \leq a\} \\ &= \inf\{t \geq 0 : \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right) t + W_t \leq \frac{1}{\sigma} \ln \frac{a}{S_0}\} \\ &= \inf\{t \geq 0 : \left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right) t - W_t \geq \frac{1}{\sigma} \ln \frac{S_0}{a}\}.\end{aligned}$$

Let $\hat{\tau} = \inf\{t \geq 0 : \left(\frac{\sigma}{2} - \frac{\mu}{\sigma}\right) t + W_t \geq \frac{1}{\sigma} \ln \frac{S_0}{a}\}$. By virtue of the symmetry of the Brownian motion, we can deduce that τ and $\hat{\tau}$ are identically distributed.

Let $\alpha = \frac{\sigma}{2} - \frac{\mu}{\sigma}$, $m = \frac{1}{\sigma} \ln \frac{S_0}{a}$ and $W_{2,t} = W_{1,t} + \alpha t$. Let

$$\tau_m = \inf\{t \geq 0 : W_{2,t} = m\}, \quad m > 0.$$

Our objective can be transformed into solving $\mathbb{E}\tau_m$. Here, $W_{1,t}$ denotes the Brownian motion under measure P. According to Theorem 8.6.3 (The Girsanov theorem I) in Øksendal and Øksendal (2003), put

$$M_t = \exp\left(-\int_0^t \alpha dW_{1,s} - \frac{1}{2}\int_0^t \alpha^2 ds\right); \quad t \leq \tau_m.$$

Firstly, we assume that the drift term of the Itô process satisfies the Novikov's condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{\tau_m}\alpha^2 ds\right)\right] < \infty$$

where $\mathbb{E} = \mathbb{E}_P$ is the expectation w.r.t P. Define the measure Q on $(\Omega, \mathscr{F}_{\tau_m})$ by

$$\begin{aligned}\frac{dQ(\omega)}{dP(\omega)}\Big|_{\mathscr{F}_{\tau_m}} = M_{\tau_m} &= \exp\left(-\int_0^{\tau_m}\alpha dW_{1,s} - \frac{1}{2}\int_0^{\tau_m}\alpha^2 ds\right)\\ &= \exp\Big(-\alpha W_{1,\tau_m} - \frac{\alpha^2}{2}\tau_m\Big).\end{aligned}$$

Then $W_{2,t}$ is a Brownian motion w.r.t Q and

$$\mathbb{E}_{\tau_m} = \mathbb{E}^Q\left[\tau_m \exp\left(\alpha W_{1,\tau_m} + \frac{\alpha^2}{2}\tau_m\right)\right]. \tag{1}$$

Because $W_{2,\tau_m} = W_{1,\tau_m} + \alpha\tau_m$, $W_{1,\tau_m} = m - \alpha\tau_m$. Equation (2) can be written as

$$\begin{aligned}\mathbb{E}_{\tau_m} &= \mathbb{E}^Q\left[\tau_m \exp\left(m\alpha - \frac{\alpha^2}{2}\tau_m\right)\right]\\ &= e^{\alpha m}\mathbb{E}^Q\left[\tau_m \exp\left(-\frac{\alpha^2}{2}\tau_m\right)\right].\end{aligned} \tag{2}$$

Accoring to Theorem 3.7.1 in Shreve (2004), for all $m \neq 0$, the random variable τ_m follows a probability density function given by

$$f_{\tau_m}(t) = \frac{d}{dt}P\{\tau_m \leq t\} = \frac{|m|}{t\sqrt{2\pi t}}\exp\left(-\frac{m^2}{2t}\right), \quad t \geq 0.$$

So we have

$$\begin{aligned}&\mathbb{E}^Q\left[\tau_m \exp\left(-\frac{\alpha^2}{2}\tau_m\right)\right]\\ &= \int_0^\infty \frac{m}{\sqrt{2\pi t}}\exp\left(-\frac{\alpha^2}{2}t - \frac{m^2}{2t}\right)dt\\ &\leq \int_0^1 \frac{m}{\sqrt{2\pi t}}dt + \int_1^\infty m\exp\left(-\frac{\alpha^2}{2}t\right)dt, \quad \text{which converges.}\end{aligned} \tag{3}$$

Therefore, we conclude that $\mathbb{E}\tau_m < \infty$. At the same time, it also satisfies the Novikov's condition mentioned above.

By Theorem 8.3.2 in Shreve (2004), we have the Laplace transform for the passage time of difted Brownian motion:

$$\mathbb{E}e^{-\lambda\tau_m} = e^{-m(-\alpha+\sqrt{\alpha^2+2\lambda})}, \quad \text{for all } \lambda > 0,$$

and

$$\left|\frac{de^{-\lambda\tau_m}}{d\lambda}\right| = \tau_m e^{-\lambda\tau_m} \leq \tau_m.$$

According to Theorem 2.27 in Folland (1999),

$$\frac{d\mathbb{E}e^{-\lambda\tau_m}}{d\lambda} = -\mathbb{E}[\tau_m e^{-\lambda\tau_m}].$$

So

$$\mathbb{E}[\tau_m e^{-\lambda\tau_m}] = \frac{m}{\sqrt{\alpha^2+2\lambda}} \exp\left(m\alpha - m\sqrt{\alpha^2+2\lambda}\right).$$

Letting $\lambda \to 0$, we obtain $\mathbb{E}\tau_m = \frac{m}{\alpha}$.

Indeed, Equation (3) can be reduced directly by applying the substitution method.

$$\begin{aligned}&\int_0^\infty \frac{m}{\sqrt{2\pi t}} \exp\left(-\frac{\alpha^2}{2}t - \frac{m^2}{2t}\right) dt \\ &= m\sqrt{\frac{2}{\pi}} \int_0^\infty \exp\left(-\frac{\alpha^2}{2}t - \frac{m^2}{2t}\right) d\sqrt{t}.\end{aligned}$$

Let $x = \sqrt{t}$, then we have

$$\begin{aligned}&m\sqrt{\frac{2}{\pi}} \int_0^\infty \exp\left(-\frac{\alpha^2}{2}x^2 - \frac{m^2}{2x^2}\right) dx \\ &= e^{-\alpha m} m\sqrt{\frac{2}{\pi}} \int_0^\infty \exp\left[-\left(\frac{\alpha x}{\sqrt{2}} - \frac{m}{\sqrt{2}x}\right)^2\right] dx.\end{aligned}$$

Setting $y = \frac{x}{\sqrt{2}}$, it follows that

$$= e^{-\alpha m}\frac{2m}{\sqrt{\pi}} \int_0^\infty \exp\left[-\left(\alpha y - \frac{m}{2y}\right)^2\right] dy. \tag{4}$$

Because

$$\begin{aligned}&\int_0^\infty \exp\left[-\left(\alpha y - \frac{m}{2y}\right)^2\right] d\left(\alpha y - \frac{m}{2y}\right) \\ &= \alpha\int_0^\infty \exp\left[-\left(\alpha y - \frac{m}{2y}\right)^2\right] dy - \alpha\int_0^\infty \exp\left[-\left(\alpha y - \frac{m}{2y}\right)^2\right] d\frac{m}{2\alpha y}\end{aligned}$$

and

$$\int_0^\infty \exp\left[-\left(\alpha y - \frac{m}{2y}\right)^2\right] d\frac{m}{2\alpha y} \overset{u=\frac{m}{2\alpha y}}{=} -\int_0^\infty \exp\left[-\left(\frac{m}{2u} - \alpha u\right)^2\right] du,$$

we have

$$\begin{aligned}&2\alpha\int_0^\infty \exp\left[-\left(\alpha y - \frac{m}{2y}\right)^2\right] dy \\ &= \int_0^\infty \exp\left[-\left(\alpha y - \frac{m}{2y}\right)^2\right] d\left(\alpha y - \frac{m}{2y}\right) \\ &\overset{g=\alpha y - \frac{m}{2y}}{=} \int_{-\infty}^{+\infty} e^{-g^2} dg = \sqrt{\pi}.\end{aligned}$$

So Equation (4) equals to $e^{-\alpha m}\frac{m}{\alpha}$. According to Equation (2), we obtain $\mathbb{E}\tau_m = \frac{m}{\alpha}$.

References

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