Information Measures and Criticality in the Ising Model on Scale Free Network Topology

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1 Abstract

The Ising Model is a minimalistic mathematical model designed to explain the macroscopic transition from order to disorder in terms of local interactions at the microscopic level [1]. While initially the model was used to describe spatially localized interactions, it has been adapted to describe networks with arbitrary topologies by replacing spatial correlations with topological ones [2]. The behavior of an Ising Model is determined in large part by its critical temperature, which is a measure of how much noise correlations in the system can withstand [3]. Here, we quantify these correlations as a function of network temperature in terms of two information theoretic measures: Transfer Entropy and Active Information [4]. We find that near criticality, the Transfer Entropy of the network is maximized. Similarly, the Active Information undergoes a major transition at this temperature. This implies that networks poised for major phase transitions are also optimized for information processing. Due to the close relationship between information processing and living systems [5], this connection is especially relevant when thinking of the origin and evolution of biological complexity.

2 Introduction

Ising Models have been used to study phase transitions in a wide variety of complex network topologies including scale free,[6], small world [7], and neural networks [8]. Similarly, information measures such as Transfer Entropy and Active Information are commonly applied to these same topologies. However, to the best of our knowledge, these two mathematical analyses have only been united in the case of spatially localized networks [8]. The reason for this is likely due to the limitations of the Ising Model in situations where information theory is typically applied. The canonical Ising Model works only for networks with nodes taking binary values. Furthermore, the updating rule for the network must match that of the Ising Model in order for the results to be physically relevant. Thus, where the Ising Model is practical is in a stochastic network where the state of a node is determined by the nodes its connected to, which is a standard assumption of spatially localized networks.

As a simple example, one can imagine how an opinion spreads in a room full of people. Each person in the room can be treated as a node and the opinion he or she holds represents its state. An interaction between two people establishes an edge between nodes, and the opinion of a person at any given time can change as a result of his or her interactions. The randomness in the system is the chance that a person will change opinion for reasons unrelated to interactions, and therefore, the temperature of the system is the average robustness of the individuals to such a change. If the people in the room are constantly changing their minds for reasons unrelated to interactions, the temperature of the system is too high to establish order; however, if the people in the room only change opinion when the majority of the people they interact with share the opposite opinion, the temperature of the system is low and large regions of the network will share a common opinion. In this manner, one can think of other biological, social, and physical networks for which Ising Models may be an accurate description of the phenomena at hand.

3 Model Description

As a general starting point, we consider a scale free network topology with a scale free index of 3. This network, known as a Barabasi-Albert network, has been well studied using the Ising Model [9] and has a critical temperature that has been computed theoretically [2]. Due to the computational requirements necessary to compute Transfer Entropy and Active Information, we limit our network to 10 nodes for this study. To initialize the network, we generate a Barabasi-Albert network and align the initial state of all nodes. For historical reasons, we will call the initial state of the nodes "spin up" and assign them all an energy value of +1; the opposite state is "spin down" and is assigned an energy value of -1. We then evolve the network through 10^5 time steps using a Metropolis updating method [10] and calculate the average energy of the system as a function of temperature - discarding the first 50,000 time steps to ensure equilibrium has been reached.

To calculate the Transfer Entropy and Active Information of the system, we must capture the evolution of the network as it transitions from order to disorder. We begin with the same initial setup (i.e. Barabasi-Albert network on 10 nodes initialized in spin up configuration) but no longer discard the beginning of the time series. Since calculating the Transfer Entropy and Active Information within the system is computationally intensive, we shorten the length of our simulation from 10^5 time steps to 10^3 time steps. For a 10 node system, this should still give us enough time to adequately capture the information architecture of the network, but for larger systems, more time steps may be required. The total Transfer Entropy and Active information at each temperature is then calculated by summing over all nodes.

4 Results

For low temperature systems, order dominates and the final energy is a superposition of macroscopic domains. This results in a non-zero average energy. For high temperature systems, randomness destroys macroscopic order, and the average energy is close to zero. One can numerically find the critical temperature of the network by examining the temperature at which the average energy departs from zero. Due to finite size effects, the critical temperature found in the simulation is an approximation to the theoretical critical temperature [10]. However, for long enough time series, the simulated critical temperature approaches that of the theoretical. For the 10 node Barabasi-Albert network, we see that the network does indeed undergo a phase transition very near its theoretical critical temperature (Figure 1).

As for the information measures, we see that the critical temperature plays an important role in both the total Transfer Entropy and the total Active Information in the system. For Transfer Entopy, *the critical temperature marks the maximum amount of information processed by the network* (Figure 2). This implies that at criticality, the network is most interconnected and correlations between nodes enable signal to propogate through the system in a maximally efficient way. As for the Active Information, there is a sharp increase at the critical temperature (Figure 3). This signifies the change in the amount of information a single node processes in steady state versus a quasi-random state. Neither case is able to store a large amount of information, but the drastic change in the informational architecture of the system on either side of the critical temperature is of interest.

5 Discussion

The peak in transfer entropy at the location of the critical temperature provides empirical evidence for a deep connection between information processing and phase transitions. For the case of a Barabasi-Albert network, being near a phase transition implies optimal information processing. This idea is likely much more general than just Barabasi-Albert networks, but needs to be explored in more detail. In terms of biological systems, there is good reason to believe that information processing plays a pivotal role in evolution [5]. This may imply that origin of complexity in biological systems is really a result of the systems being optimized for information processing and therefore consistently near major phase transitions.

References

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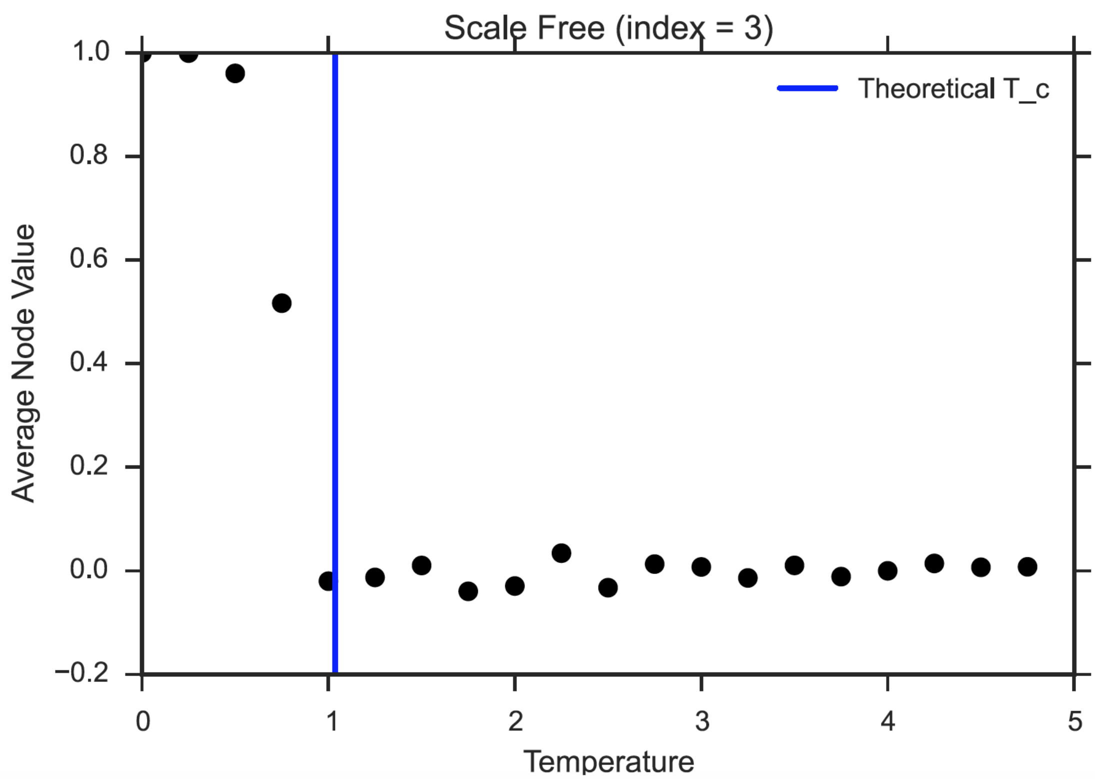

Figure 1: Node value as a function of temperature for the 10 node Barabasi-Albert network evolving according to a Metropolis update method. Average node value of zero corresponds to chaotic network, where thermal fluctuations destroy meaningful correlations. Non-zero average values imply the establishment of macroscopic domains where nodes have aligned. The theoretical temperature at which these two behaviors should be distinguishable is plotted in blue.

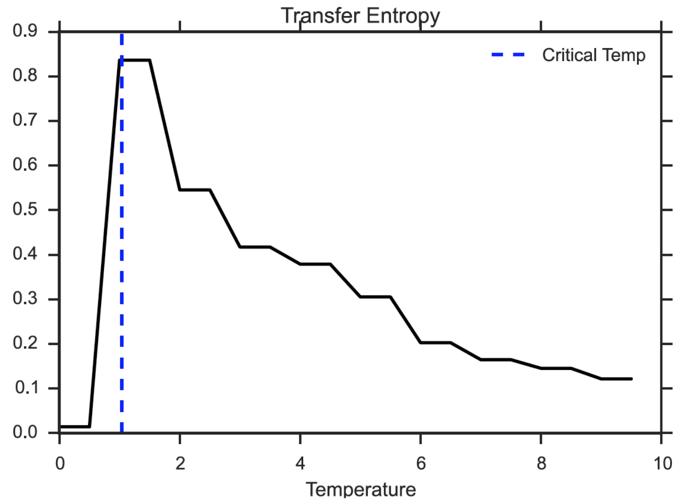

Figure 2: Total Transfer Entropy as a function of network temperature. As the plot shows, the maximum amount of Transfer Entropy occurs at the location of the theoretical critical temperature (blue dashed line). Results are averaged over 10 runs for each temperature with a history length of 4 and a maximum step size of 1000.

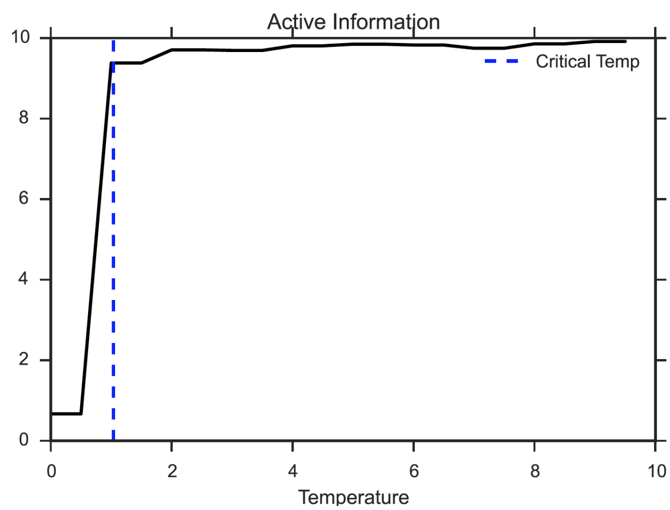

Figure 3: Total Active Information as a function of network temperature. Above critical temperature, the Active Information plateaus due to thermal fluctuations within the system, whereas below the critical temperature the Active Information is close to zero due to the network nodes being in fixed states. Results are averaged over 10 runs for each temperature with a history length of 4 and a maximum step size of 1000.