

Computational Environments and Toolchains

Topic 03 : Computational Problems involving Probability

Lecture 01 : Review of Probability Concepts

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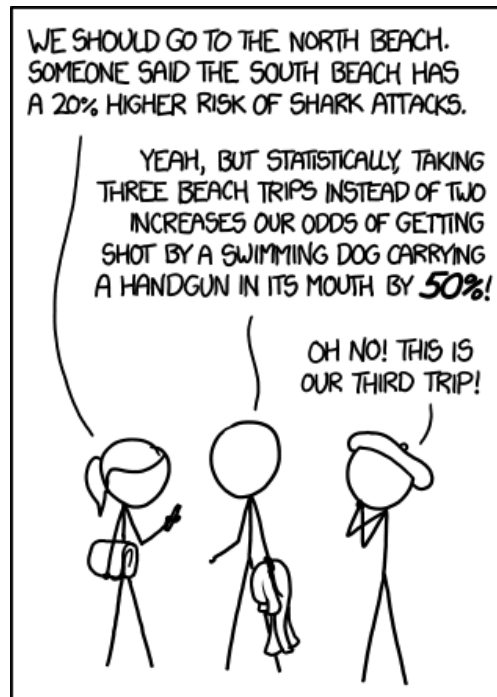
Outline

- Fundamental concepts in probability
- Laws of probability

Probability — What? Why? How?



Source: <http://dilbert.com/strip/2001-10-25>



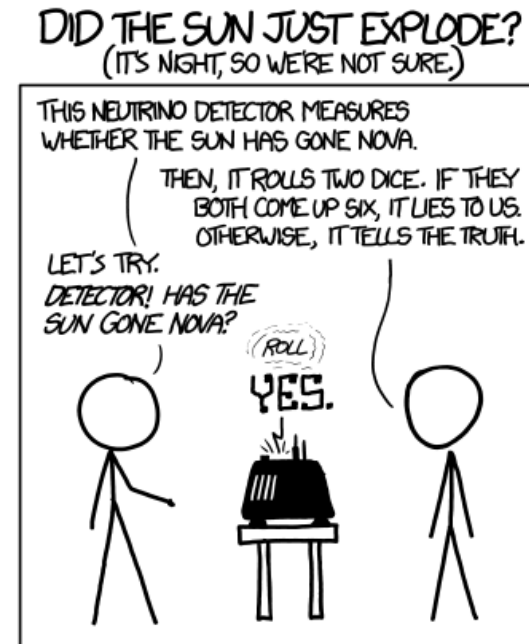
REMINDER: A 50% INCREASE IN A TINY RISK IS *STILL TINY*.

Source: <https://xkcd.com/1152/>



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

Source: <https://xkcd.com/795/>



FREQUENTIST STATISTICIAN:

THE PROBABILITY OF THIS RESULT HAPPENING BY CHANCE IS $\frac{1}{36} = 0.027$. SINCE $p < 0.05$, I CONCLUDE THAT THE SUN HAS EXPLODED.

BAYESIAN STATISTICIAN:

BET YOU \$50 IT HASN'T.


Source: <https://xkcd.com/1132/>

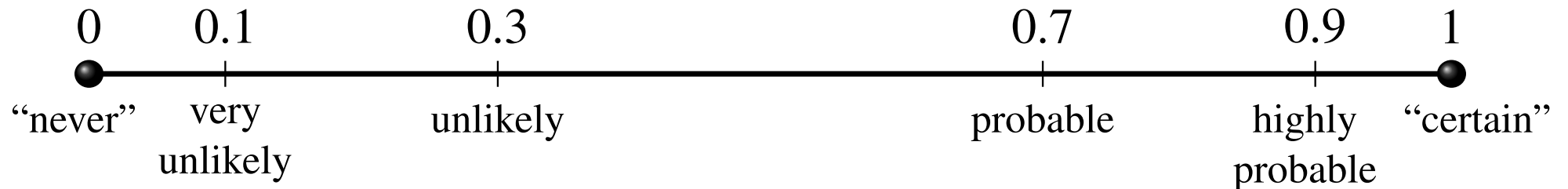
Concept: Probability

Definition 1 (Probability)

Probability is a measure of the likelihood or chance that a particular event would occur. It is given as a number in the range 0 to 1.

- Informal interpretation ...

 human intuition only good in 0.1 to 0.9 range.



- Other scales can be used:
 - odds for (betting, medicine)

$$\text{odds for} = \frac{\text{probability}}{1 - \text{probability}} \iff \text{probability} = \frac{(\text{odds for})}{(\text{odds for}) + 1}$$

- entropy (in communication theory) uses scale $[0, \infty)$

$$\text{entropy} = (\text{probability}) \times \log_2 \left(\frac{1}{\text{probability}} \right)$$

Concept: A Probability Experiment

Definition 2 (A Probability Experiment)

A **probability experiment** is any process with a well-defined set of possible **outcomes**.

Experiment	Outcomes
Toss a coin	Head, Tail
Roll a die	1, 2, 3, 4, 5, or 6
Walk to college blindfolded	Dead, Alive

- It is important to state what the possible outcomes are in an experiment before the experiment is performed.
- The list of possible outcomes of a random experiment must be **exhaustive*** and **mutually exclusive**† At most .

*Listed outcomes covers every possibility \implies at least one
 †No two listed outcomes can occur as same time \implies at most one } \implies exactly one

Concept: Sample Space

Definition 3 (Sample Space)

Sample Space, S , is the set of all possible outcomes in the experiment,

Experiment	Sample Space
Toss a coin	$S = \{\text{Head}, \text{Tail}\}$
Roll a die	$S = \{1, 2, 3, 4, 5, 6\}$
Height of last person to join class today	$S = \{x \in \mathbb{R} 1 \leq x \leq 2.5\}$

- The sample space can be finite or infinite, and can be represented by categorical data, or (discrete or continuous) numerical data.
- When the sample space consists of continuous numerical data there are a few technical issues that mean we will need to tweak what we mean by having a probability of zero or a probability of one. We will worry about this later and for now in all our examples the sample space will contain only categorical, or discrete numerical data.

Concept: Events

Definition 4 (Event)

An **event** is a collection (set) of some possible outcomes.

- An event may consist of a single outcome (a **simple** event) or consist of a number of possible outcomes (a **compound** event).
- The probability of any event is equal to the sum of the probabilities of the individual outcomes in the event.
- Events are sets \implies More complicated events can be constructed in terms of simpler events using set operations of intersection, union, complement (set difference between sample space and a set).
- A set of events are **mutually exclusive** (**disjoint**) if they at most one can occur.
- A set of events are **exhaustive** if at least one must occur.

Example — Mutually Exclusive vs Exhaustive

Example 5

- When a card is drawn from a deck at random, the four suits (hearts, diamonds, clubs and spades) are at the same time disjoint and exhaustive.
- Any event, A , and its complement, \bar{A} are mutually exclusive and exhaustive pair of events.
- In the experiment of attempting this module, the events “getting a honour”, or “getting a pass” are mutually exclusive but they are not exhaustive. Why?
- In my experiment of walking to college blindfolded the events “alive”, “hospitalised”, and “dead” are exhaustive but are not mutually exclusive. Why?

Example 6— Set Notation Applied to Probability

Example 6

Consider the experiment of rolling a single die, and we are interested in the event of getting an even number.

- The sample space is

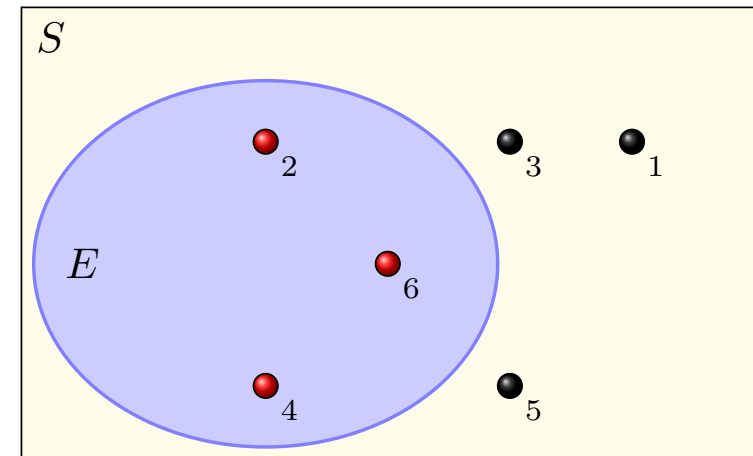
$$S = \{1, 2, 3, 4, 5, 6\}$$

and is represented as a set using a rectangular box. (In set theory this was called the **universal set**.)

- The event of interest is

$$E = \{2, 4, 6\}$$

and is represented as a set using an oval.



Notation

The notation in probability theory can get a bit intense[‡], so I will try to

- Keep notation to a minimum.
- Be consistent (within this module).
- Define notation as the need arises.

Probability Law:

$$\Pr(E) = \text{“the probability that event ‘}E\text{’ occurs”} \quad (1)$$

English	Set Theory	java	python	pandas
OR	\cup	<code> </code>	<code>or</code>	<code> </code>
AND	\cap	<code>&&</code>	<code>and</code>	<code>&</code>
NOT	$S \setminus \bullet$ or \bullet^c or $\bar{\bullet}$	<code>!</code>	<code>not</code>	<code>!</code>

[‡]For example it is not unusual to see all of the symbols x_k , x , X , \mathcal{X} to represent particular aspects of the random variable x .

Assigning Probabilities

There are three main methods used in calculating probabilities. Regardless of the method used, the probability values satisfy:

- 1 The probability value must be in the range 0 to 1, i.e.,

Probability Law:

If an event E is a particular outcome then $\Pr(E)$ represents the probability that E will occur and

$$0 \leq \Pr(E) \leq 1 \quad (2)$$

- 2 The sum of the probabilities of all the possible outcomes of an experiment must equal 1, i.e.,

Probability Law: (Total Law of Probability)

If E_1, E_2, \dots, E_n are all the mutually exclusive outcomes of an experiment then

$$\Pr(E_1) + \Pr(E_2) + \dots + \Pr(E_n) = 1 \quad (3)$$

In English — “Some outcome must happen.”

Method 1 — Classical Probability

I

Classical probability theory is very much influenced by games of chance — cards, dice, roulette wheel, etc. — in such games there is a clear set of elementary outcomes which can be combined to get more complicated outcomes. Also games can be repeated ad infinitum.

Probability Law: (Classical Probability 1)

If an experiment has n possible outcomes, with all equally likely, then the probability of any one of these outcomes occurring is $1/n$.

- 1 Tossing a coin

$$\Pr(\text{Head}) = \Pr(\text{Tail}) = \frac{1}{2}$$

- 2 Rolling a die

$$\Pr(1) = \Pr(2) = \dots = \Pr(6) = \frac{1}{6}$$

- 3 Picking a card from a deck of 52 cards

$$\Pr(\text{Ace of Hearts}) = \Pr(\text{Two of Hearts}) = \dots = \frac{1}{52}$$

Method 1 — Classical Probability

II

Sometimes the result of the experiment that we want can occur a number of ways, e.g.,

Get an even number when rolling a die.

Can occur 3 ways: $\{2, 4, 6\}$.

Probability Law: (Classical Probability 2))

Given an experiment with each possible outcome equally likely and

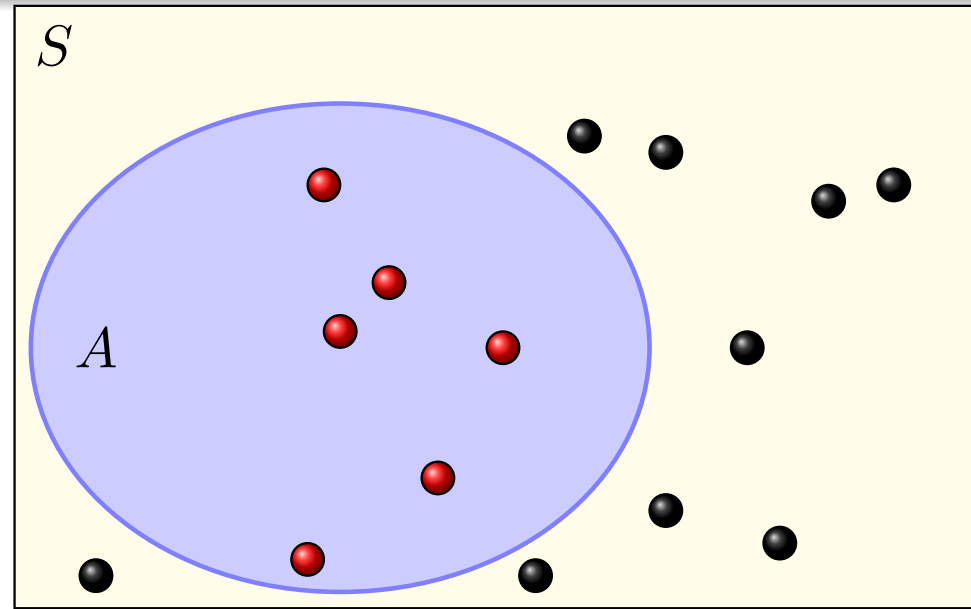
S = Sample Space — Set of possible outcomes

E = Desired result — Set of outcomes that give the desired result

then

$$\Pr(E) = \frac{\#E}{\#S} = \frac{\text{Number of desired outcomes}}{\text{Number of possible outcomes}} \quad (4)$$

Connection to Set Notation



- The sample space, S , has size $\#S$.
This is the number of different outcomes to the probability experiment, all of which are assumed to be equally likely.
- The event A , has size $\#A$. This is the event of interest.
- The probability that event A occurs is given by

$$\Pr(A) = \frac{\#A}{\#S}$$

Example 7

I

Example 7

On rolling a die what is the probability of getting:

- ① an even number?
- ② a number that is divisible by 3?
- ③ a number less than 5?

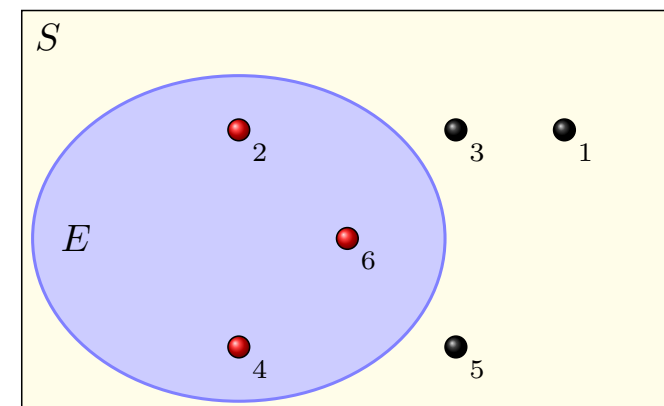
The Sample space is the same for each part, i.e.,

$$S = \{1, 2, 3, 4, 5, 6\} \quad (\#S = 6)$$

- ① *an even number ...*

$$E = \{2, 4, 6\}$$

$$\Pr(E) = \frac{\#E}{\#S} = \frac{3}{6} = 0.5 = 50\%.$$



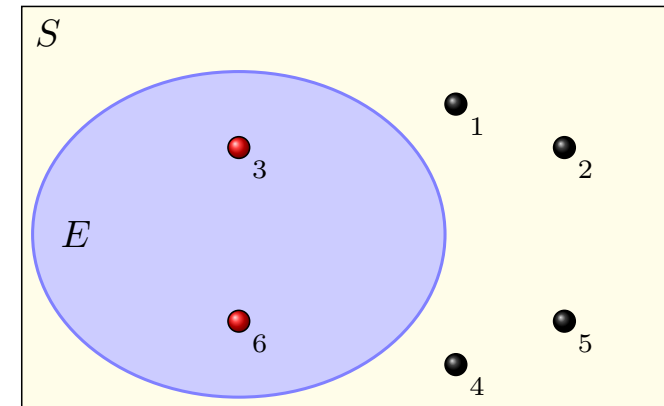
Example 7

II

- 2 a number that is divisible by 3 ...

$$E = \{3, 6\}$$

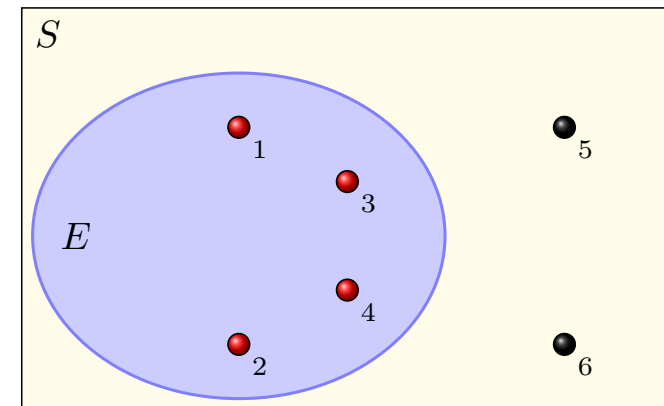
$$\Pr(E) = \frac{\#E}{\#S} = \frac{2}{6} = 0.33333 \approx 33\%.$$



- 3 a number less than 5 ...

$$E = \{1, 2, 3, 4\}$$

$$\Pr(E) = \frac{\#E}{\#S} = \frac{4}{6} = 0.66667 \approx 67\%.$$



Method 2 — Frequentist Method / Historical Method

I

Suppose that in a survey of 400 customers 100 were happy with the price of a particular product and 300 were unhappy with the price. The relative frequency distribution for this data is

Satisfaction	Frequency	Relative Frequency
Happy	100	0.25
Unhappy	300	0.75
	400	1.00

Using the relative frequency as probabilities we have

$$\Pr(\text{Picking a customer who is happy}) = 0.25$$

$$\Pr(\text{Picking a customer who is unhappy}) = 0.75$$

- Probabilities are assigned on the basis of experimentation or historical data.
- Classical probability would not work in this case as it is unreasonable to assume, a priori, that the two outcomes are equally likely.
- Also called the **frequentist view** or the **empirical view** of probability.

Method 2 — Frequentist Method / Historical Method

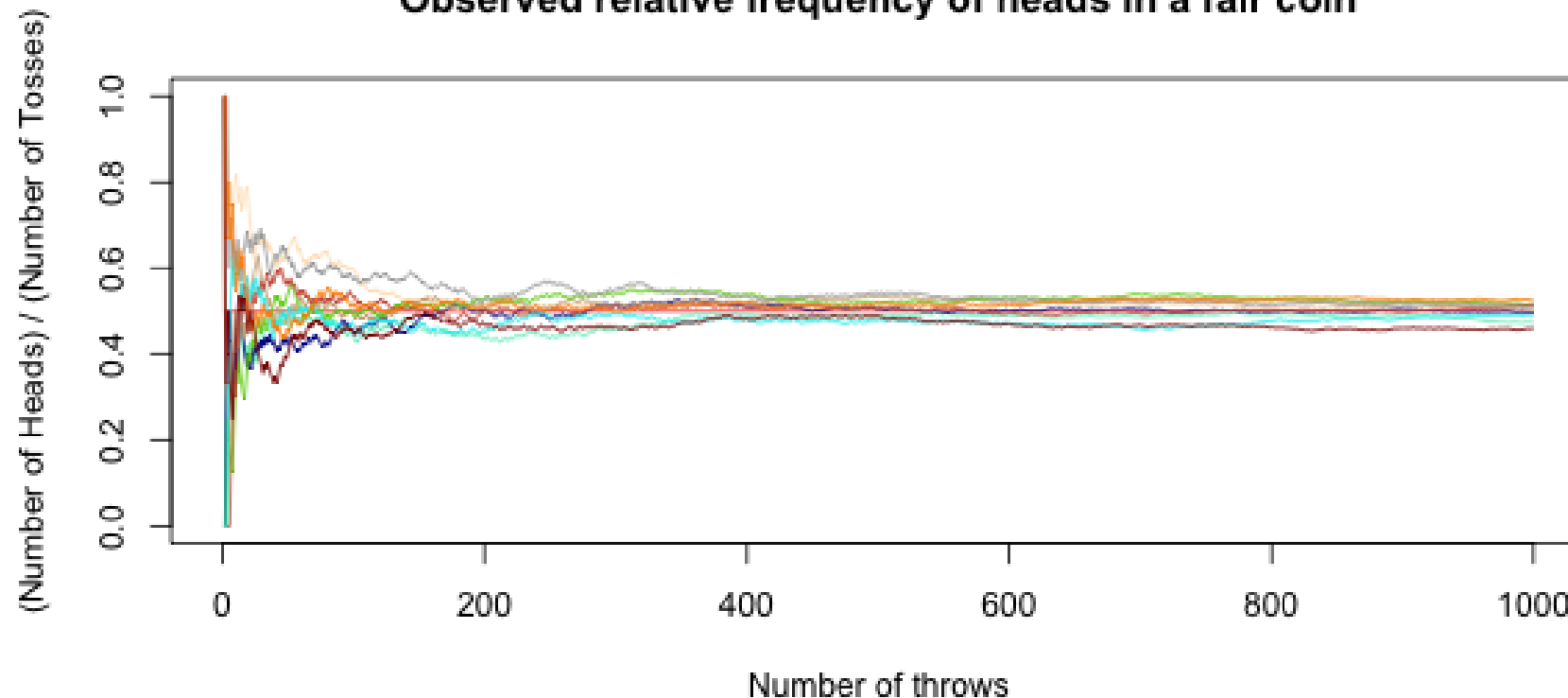
II

A **Frequentist** assigns probability based on experience.

Definition 8 (Frequentist Probability)

Probability of a certain outcome to occur in a random experiment is the proportion of times that this outcome would occur in a very long series of repetitions of the random experiment.

Observed relative frequency of heads in a fair coin



Python Implementation

```

1 • import numpy as np
import numpy.random as rnd

def trial():
    coin = rnd.choice(['Heads', 'Tails'])
    success = coin=='Heads'
    return success

def run_trials(n):
    return sum([trial() for _ in range(n)]) / n

print('%7s \t %s\n' % ('n', 'Pr(H)') + "="*25)
for n in [1, 10, 100, 1_000, 10_000]:
    print('%7d \t %.5f' % (n, run_trials(n)))
  
```

n	Pr(H)
1	0.00000
10	0.60000
100	0.46000
1000	0.48800
10000	0.49660

Python Implementation (fast)

2 ●

```

import numpy as np
import numpy.random as rnd

def run_trials(n):
    return rnd.choice([0,1], size=n).sum() / n

# output
print('%7s \t %s\n' % ('n', 'Pr(H)' ) + "="*25)
for n in [1, 10, 100, 1_000, 10_000, 100_000]:
    print('%7d \t %.5f' % (n, run_trials(n)))

```

n	Pr(H)
1	1.00000
10	0.80000
100	0.45000
1000	0.48400
10000	0.50590
100000	0.50174

For simple problems it is possible to reduce the number of calls to the random number generator by requesting multiple return values.