Combinatorial Games

Maths Week
(14–18 October 2024)

Dr Aoife Hennessy, Narciso Gonzaga, Dr Kieran Murphy

Department of Computing and Mathematics, SETU (Waterford).
(kieran.murphy@setu.ie)

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South East Technological University

Part I

Combinatorial Games

What is a Combinatorial Game?

A **Combinatorial Game** satisfies the following conditions:

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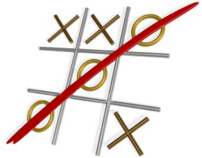

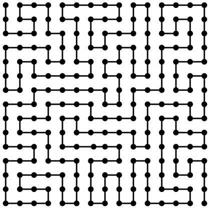

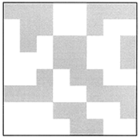

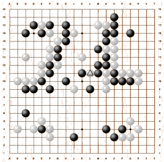

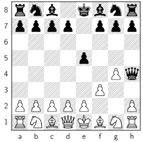

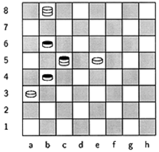

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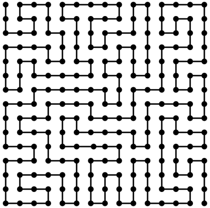

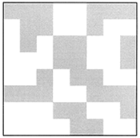

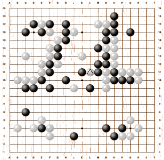

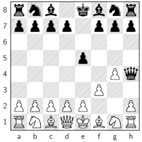

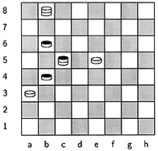

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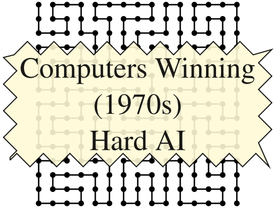

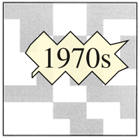

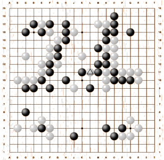

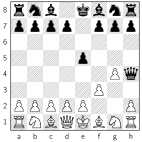

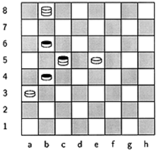

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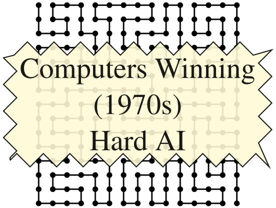

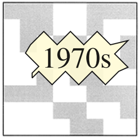

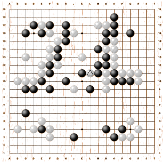

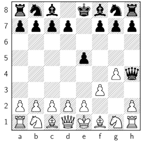

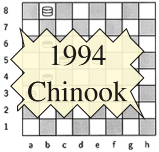

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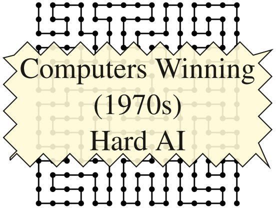

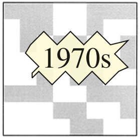

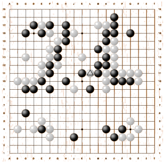

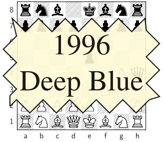

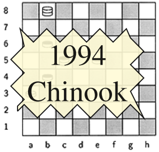

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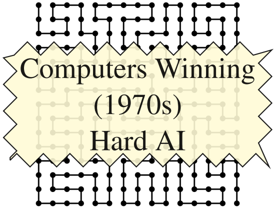

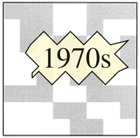

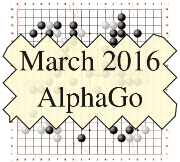

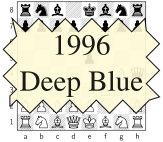

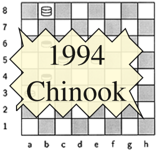

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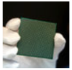

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What Games are Not Combinatorial Games?

- No random moves such as the rolling of dice or the dealing of cards are allowed.

This rules out games like poker, snakes and ladders, and backgammon.

- A combinatorial game is a game of **perfect information**: simultaneous moves and hidden moves are not allowed.

This rules out games like rock–paper–scissors, snap, and battleship.

Why Study Combinatorial Games?

A Mathematician's Answer

- Its fun.

A Game Developer's Answer

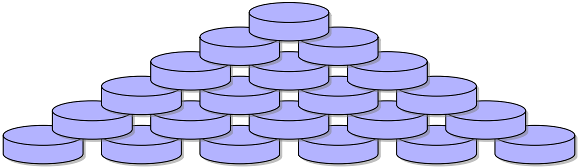
- Combinatorial are relatively easy to program and are often the first game created by new game developers.
  - While there are over twenty implementations of *X and O* on the app store there are many combinatorial games which have not yet been implemented (for example see *Numerical X and O*).
  - Most combinatorial games (with exception of chess, go and similar games) can be created by a single game developer (even a beginner).

Part II

The Simple Take-Away Game

A Simple Take-Away Game

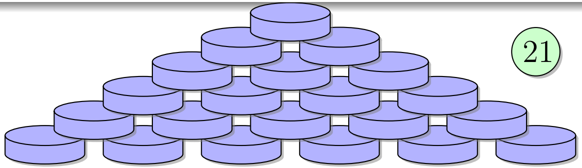
Let's start with the following game of removing chips from a pile of chips.



Game 1: Simple Take-Away Game

1. There are two players. We label them *Player A* and *Player B*.
2. There is a pile of 21 chips in the centre of a table.
3. A move consists of removing 1, 2, or 3 chips from the pile.
4. Players alternate moves with *Player A* starting.
5. The player that removes the last chip wins.

Sample Game



- Game starts with 21 chips.

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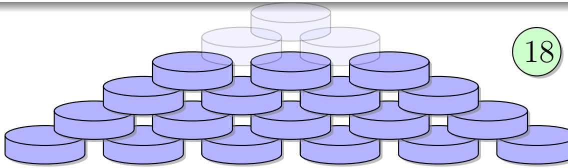
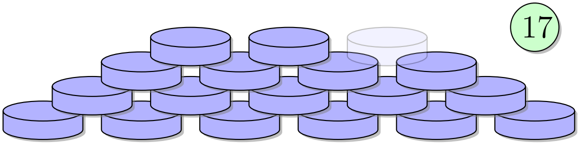


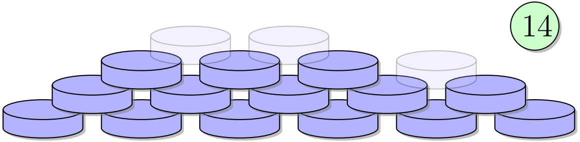
- Game starts with 21 chips.
- *Player A* takes 3 chips $\Longrightarrow$ number of chips drops from 21 to 18.

Sample Game



- Game starts with 21 chips.
- *Player A* takes 3 chips $\Longrightarrow$ number of chips drops from 21 to 18.
- *Player B* takes 1 chip $\Longrightarrow$ number of chips drops from 18 to 17.

Sample Game



- Game starts with 21 chips.
- *Player A* takes 3 chips $\Longrightarrow$ number of chips drops from 21 to 18.
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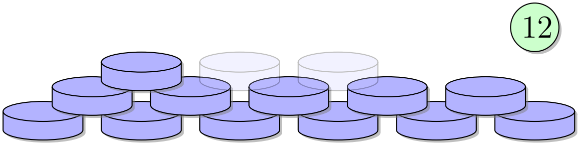
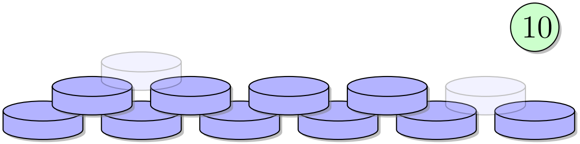
- *Player A* takes 3 chips $\Longrightarrow$ number of chips drops from 17 to 14.

Sample Game



- Game starts with 21 chips.
- *Player A* takes 3 chips $\Longrightarrow$ number of chips drops from 21 to 18.
- *Player B* takes 1 chip $\Longrightarrow$ number of chips drops from 18 to 17.
- *Player A* takes 3 chips $\Longrightarrow$ number of chips drops from 17 to 14.
- *Player B* takes 2 chips $\Longrightarrow$ number of chips drops from 14 to 12.

Sample Game



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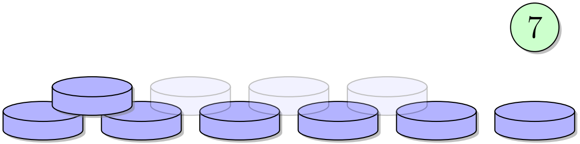
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 12 to 10.

Sample Game



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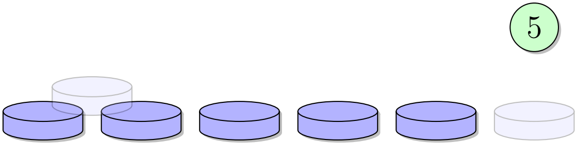
- *Player B* takes 2 chips $\Longrightarrow$ number of chips drops from 14 to 12.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 12 to 10.
- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 10 to 7.

Sample Game



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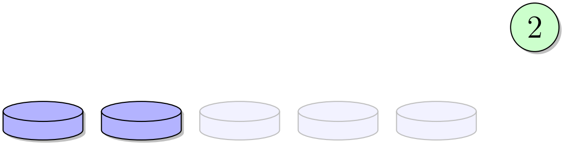
- *Player B* takes 2 chips $\Longrightarrow$ number of chips drops from 14 to 12.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 12 to 10.
- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 10 to 7.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 7 to 5.

Sample Game



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- *Player B* takes 2 chips $\Longrightarrow$ number of chips drops from 14 to 12.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 12 to 10.
- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 10 to 7.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 7 to 5.
- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 5 to 2.

Sample Game

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- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 10 to 7.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 7 to 5.
- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 5 to 2.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 2 to 0.

Sample Game

0

- Game starts with 21 chips.
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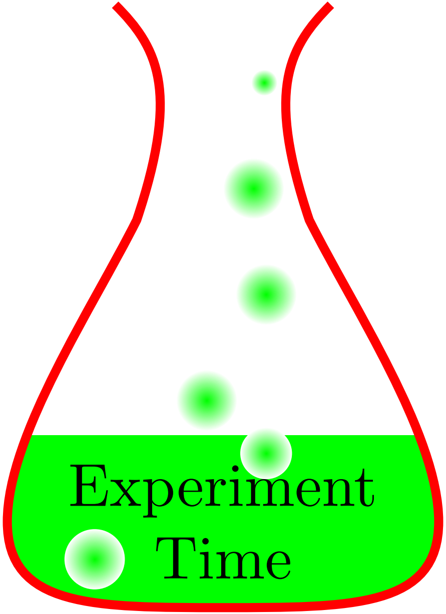
- *Player B* takes 2 chips $\Longrightarrow$ number of chips drops from 14 to 12.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 12 to 10.
- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 10 to 7.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 7 to 5.
- *Player B* takes 3 chips $\Longrightarrow$ number of chips drops from 5 to 2.
- *Player A* takes 2 chips $\Longrightarrow$ number of chips drops from 2 to 0.
- ***Player B* can't move** $\Longrightarrow$ ***Player A* wins.**


Experiment
Time

Analysis of the Simple Take-Away Game I

When analysing a game we try to answer the following questions:

Q1: Is there a simple approach to **unambiguously** describe this game?

- We don't want to keep drawing piles of chips every time ...

Q2: Can one of the players force a win in this game?

- Even if both players are playing optimally can a player still guarantee a win?

Q3: Does any player have an advantage?

- Which player would you rather be, the player who starts or the player who goes second?

Q4: What is a good strategy?

Analysis of the Simple Take-Away Game II

Q5: What happens if the game parameters are modified?

- For this game we could change the starting number of chips or the number of chips that the players can remove …

  Changing the starting amount:

  Consider the variation of starting with 35 chips and players allowed to take 1, 2, or 3 chips.

  Changing the take away amounts:

  Consider the variation of starting with 21 chips and players allowed to take 1, 2, 3, 4, or 5 chips.

  Gaps in the take away amounts:

  Consider the variation of starting with 21 chips and players allowed to take 1, 2, or 4 chips.

Game Analysis I

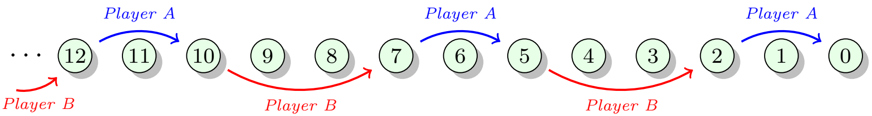
Representing Game Positions/States

Rather than drawing the chips remaining during the game it is simpler to display a counter . . .

. . . 12 11 10 9 8 7 6 5 4 3 2 1 0

Note that the counter does not go up as far as 21 because we are interested in what happens near the end of the game, i.e., when the counter approaches zero.

The sample game described earlier ends as follows . . .



Now starting at position 0 and working upwards we analyse the game . . .

Game Analysis II

Let us consider the game in terms of the next player to move …

- If there are no chips, then the player who moves next has lost the game.
- If there are just 1, 2, or 3 chips left, the player who moves next wins simply by taking all the chips.
- Suppose there are 4 chips left. Then the player who moves next must leave either 1, 2 or 3 chips in the pile and his opponent will be able to force a win. So four chips left is a loss for the next player to move.
- With 5, 6, or 7 chips left, the player who moves next force a win by moving to the position with four chips left.
- With 8 chips left, the next player to move must leave 5, 6, or 7 chips, and from there his opponent will be able to force a win. So 8 chips is a loss for the next player … and so on … and on …

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↑ L (under 0)

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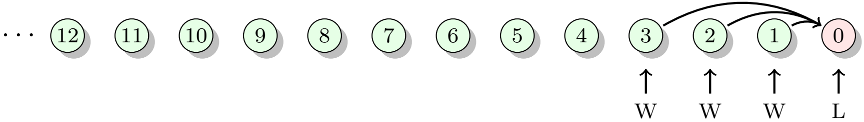
Let us consider the game in terms of the next player to move . . .



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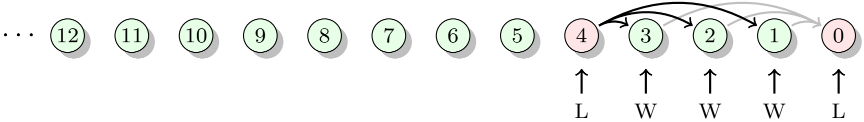

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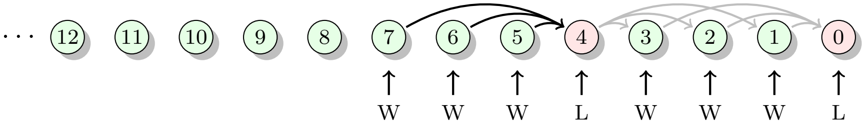

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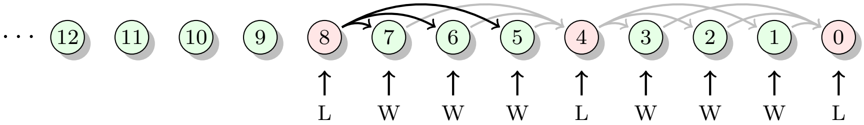

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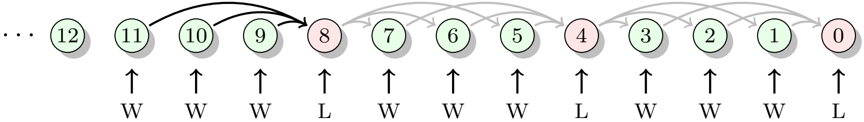

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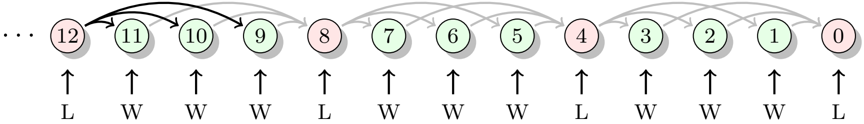

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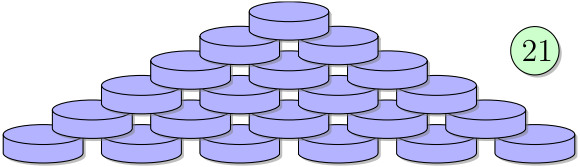
- Positions 0, 4, 8, 12, . . . are losing positions for the next player — these are called **target positions**.
- From all other positions the current player can move to a target position and ultimately force the next player to lose.

Game Analysis III

We may now analyise the game with 21 chips . . .



Since 21 is not divisible by 4, the first player to move can win. The unique optimal strategy is to:

- Take one chip and leave 20 chips which is a target position.
- In subsequent moves, aim for the next target position.

Extensions to the Simple Take-Away Game I

Question 1. *(Changing the starting amount and gaps in take away amounts)*

Consider the variation of starting with 11 chips and players allowed to take 1, 3, or 4 chips.
Which player would you rather be?

11 10 9 8 7 6 5 4 3 2 1 0

Label the terminal positions as L-positions. (STEP 1)

Label all positions that can reach an L-position as W-positions. (STEP 2)

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(STEP 4)

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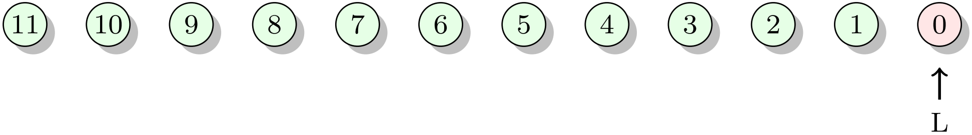

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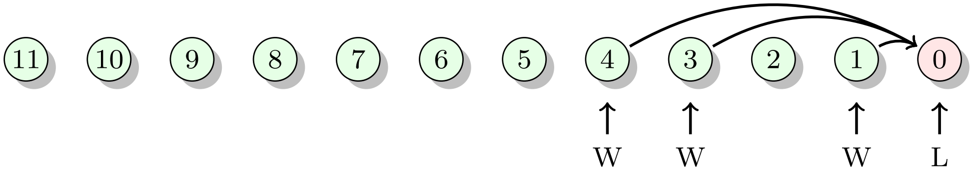

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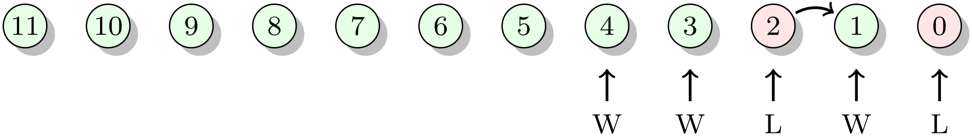

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11	10	9	8	7	6	5	4	3	2	1	0
							↑	↑	↑	↑	↑
							W	W	L	W	L

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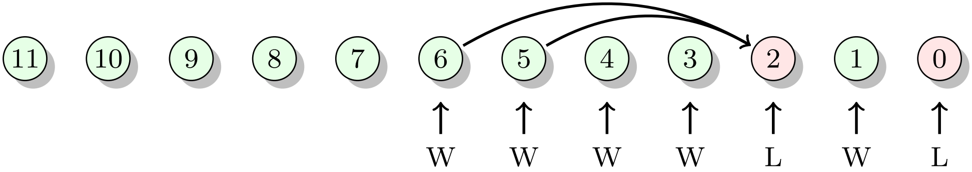

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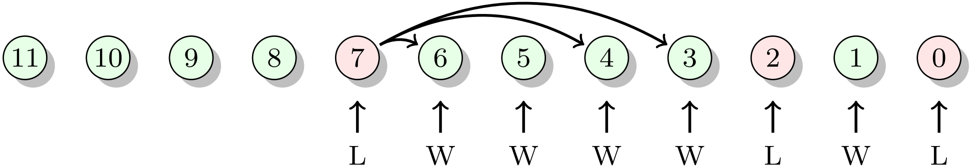

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				↑	↑	↑	↑	↑	↑	↑	↑
				L	W	W	W	W	L	W	L

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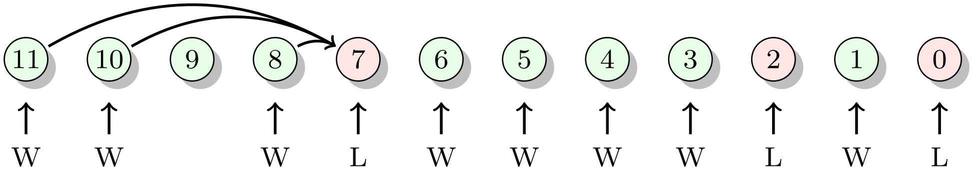

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W	W	L	W	L	W	W	W	W	L	W	L

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W	W	L	W	L	W	W	W	W	L	W	L

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☞ Everything is now labeled so STOP (STEP 4)

Part III

The SOS Game

The SOS Game

Game 2: The SOS Game

1. The board consists of a row of n squares, initially empty (here $n = 12$).
2. Players take turns selecting an empty square and writing either an **S** or an **O** in it.
3. The player who first succeeds in completing **SOS** in consecutive squares wins the game.
4. If the whole board gets filled up without an **SOS** appearing consecutively anywhere, the game is a draw.

The SOS Game

Game 2: The SOS Game

1. The board consists of a row of n squares, initially empty (here $n = 12$).
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The SOS Game

	S	**S**									

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The SOS Game

	S	**S**			**S**				**O**	**O**	

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The SOS Game

	S	**S**			**S**				**O**	**O**	**S**

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The SOS Game

	S	**S**	**O**	**S**	**S**				**O**	**O**	**S**

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The SOS Game

O	**S**	**S**	**O**	**O**	**S**	**S**	**O**	**O**	**O**	**O**	**S**

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The SOS Game

O	**S**	**S**	**O**	**O**	**S**	**S**	**O**	**O**	**O**	**O**	**S**

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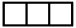
Who will win for $n = 3, 4, 5, 6, 7, \ldots$ squares?

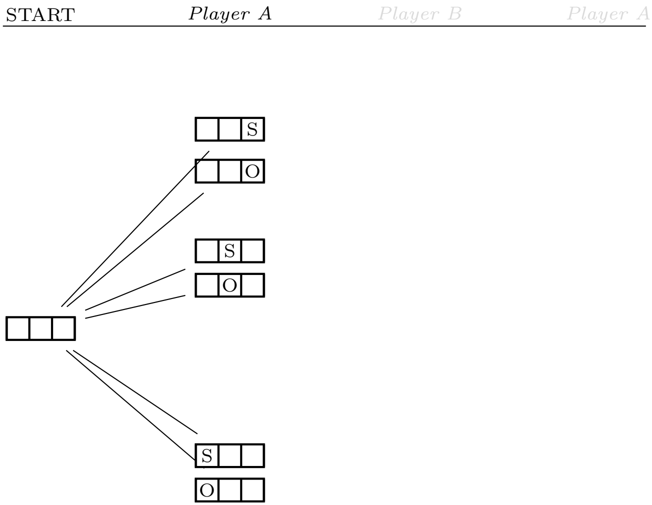
SOS Game Tree ($n = 3$)

START *Player A* *Player B* *Player A*



- We start with an empty board.
- *Player A* has six possible first moves.
- Three moves are possible winners.
- *Player B* has 12 possible first moves.
- Six moves are possible winners.
- *Player A* has 12 possible first moves.
- Six moves result in a win for *Player A*, otherwise it is a draw

SOS Game Tree ($n = 3$)



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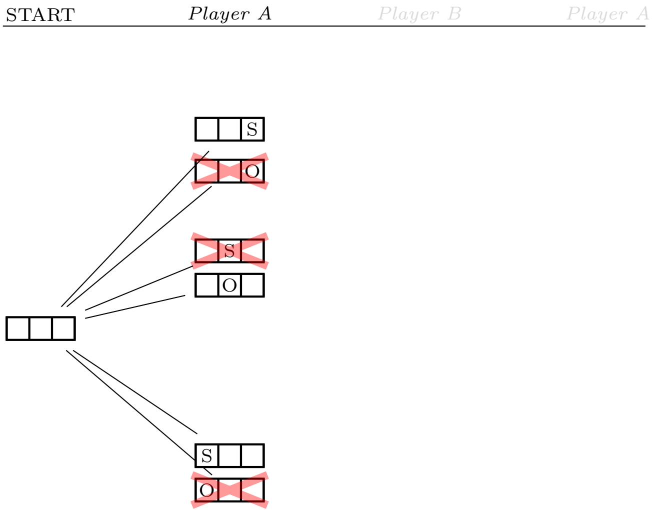

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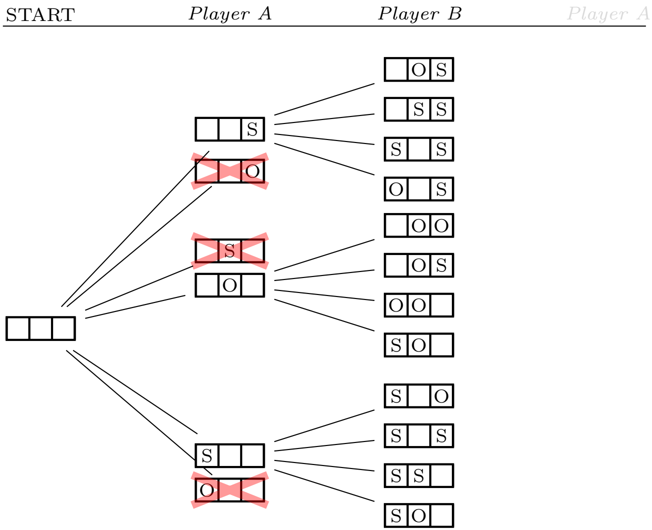

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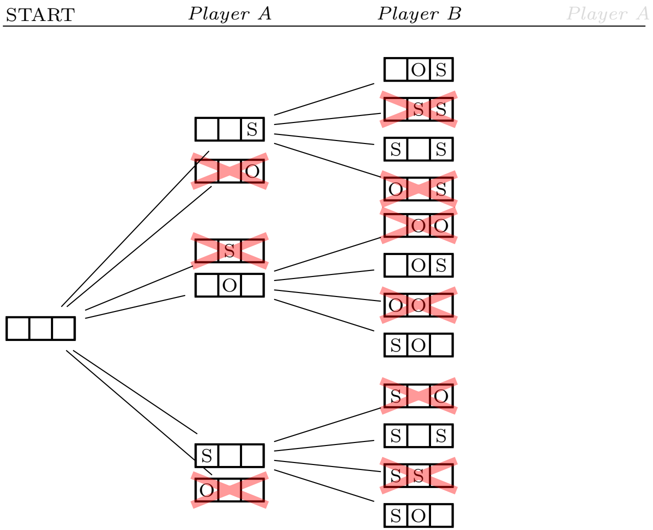

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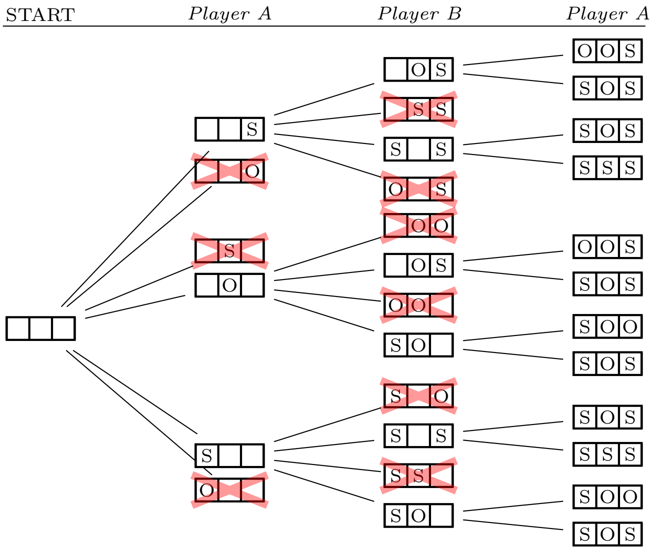

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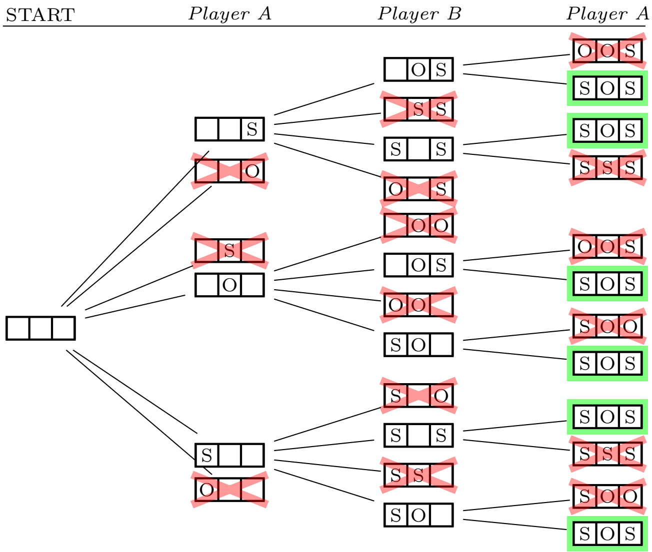

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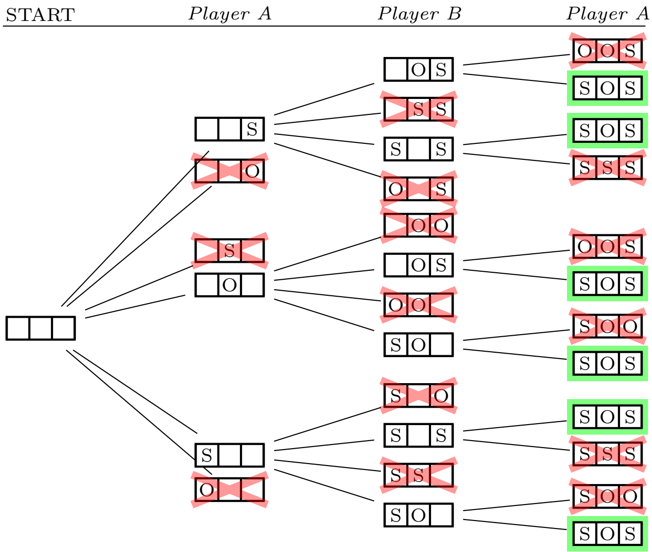

Observations

- *Player B* can never win $\implies$ best hope is a draw.
- For every first move by *Player A* that is a possible winner, there is a move for *Player B* that forces a draw.
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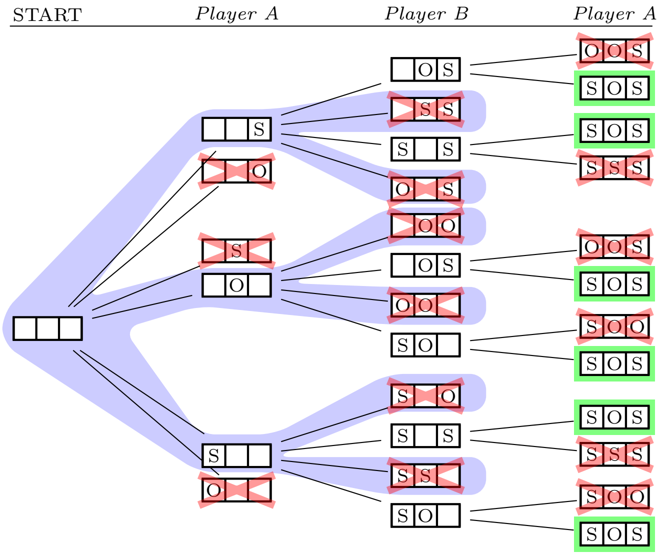

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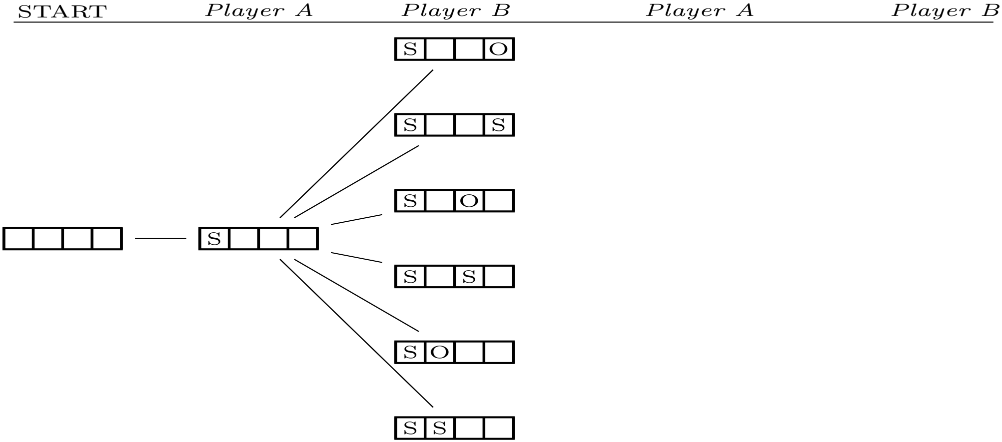
The SOS Game with $n = 3$ is a draw.

Poor First Move for *Player A* when $n = 4$

Question 2.

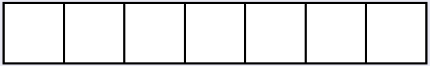
Suppose n = 4 and *Player A* puts an **S** in the first square. Show the second player can win.



Player A wins for $n = 7$

Question 3.

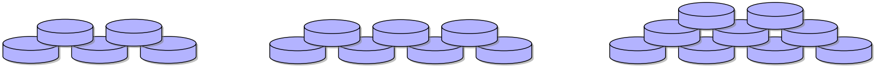
Show that if $n = 7$, then *Player A* can force a win regardless of the actions of *Player B*.



Part IV

Other Combinatorial Games

Game of Nim

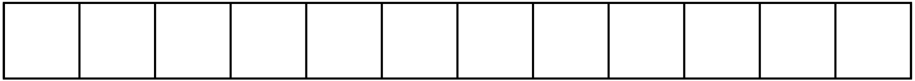


$(n_1 = 5)$ $(n_2 = 7)$ $(n_3 = 9)$

Game 3: Game of Nim

1. The game starts with three piles of chips. (Piles of sizes 5, 7, and 9 make a good game.)
2. Two players take turns moving.
3. A move consists of selecting one of the piles and removing at least one chip from it (up to all of the chips in the selected pile can be removed.)
4. The winner is the player who removes the last chip.

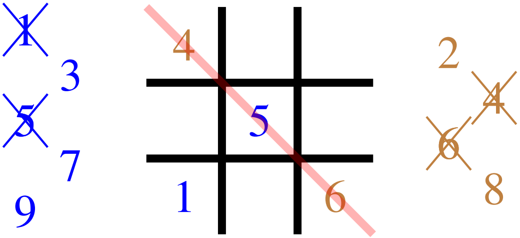
Dawson's Chess



Game 4: Dawson's Chess

1. The board consists of a row of n squares, initially empty (here $n = 12$).
2. Players take turns selecting an empty square and writing an **X**, subject to the restriction that an **X** may not be placed in a square adjacent to another **X**.
3. The last player to move wins.

Numerical X and O



Game 5: Numerical X and O

1. Using a standard X and O board, the first player plays using odd integers (1, 3, 5, 7, and 9), while the second player plays using the even integers (2, 4, 6, and 8). Each integer is used at most once.
2. The first player to complete a row, column, or diagonal containing three integers that total 15 wins. (note you can use your opponent's numbers to get the total.)