Discrete Mathematics

Topic 05 : Enumeration

Lecture 01 : Basic Counting Principles

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Outline

- Overview of enumeration
- Addition, multiplication and subtraction principles
- Pigeonhole principle

Outline

Enumeration — What? Why? How? I

What?

Counting : 0,1, 2, 3, 4, 5, . . .

Why?

- Need to count execution paths or the number of times a particular statement is executed in an algorithm in order to estimate the running time. *(algorithm complexity)*
- Need to count number of outcomes to an experiment in order to measure the likelihood of events. *(probability)*
- Number of distinct edges/nodes in a network. *(reliability)**
- In social networks, need to quantify global metrics such as size and complexity, and local metrics such as the effect of influencers, early adopters, . . . epidemics, fake news, etc *(social networks)*†

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*Canada/USA, Northeast blackout of 2003

†Networks, Crowds, and Markets — Reasoning About a Highly Connected World

Enumeration — What? Why? How? II

How?

Basic Counting Principles

Five fundamental principles used in counting that are the building blocks for the subsequent more advanced techniques.

Binomial Coefficients

In here we look a problems that appear different but all can be treated as repeated choices each of which have two options (a repeated binomial experiment).

Combinations and Permutations

- Count the number of ordered arrangements or ordered selections of objects without repeating any object.
- Count the number of unordered arrangements or unordered selections of objects without repeating any object.

Stars and Bars

A handy technique to count arrangements when we have a mixture of distinct and identical objects.

Terminology — English

If, like me, you come from the Capital of Ireland[‡], you might want to learn/remember …

choice | tʃɔɪs |

noun

an act of choosing between two or more possibilities: *the **choice between** good and evil.*

• [mass noun] the right or ability to choose: *I had to do it, I had no choi*

• a range of possibilities from which one or more may be chosen: *you have a sofa made in a choice of forty fabrics.*

• a thing or person which is chosen

computer

option | ˈɒpʃ(ə)n |

noun

a thing that is or may be chosen: *choose the cheapest options for supplying e*

• [in sing.] the freedom or right to choose something: *she was given th*

So in this topic we are attempting to answer the question

$$\text{How many} \left\{\begin{array}{c}\text{options}\\ \text{outcomes}\\ \text{objects}\end{array}\right\} \text{exists in this} \left\{\begin{array}{c}\text{choice}\\ \text{experiment}\\ \text{collection}\end{array}\right\} ?$$

[‡]“Limerick was, Dublin is, and Cork will be.”

Addition Principle

In Week 3 we discussed the following property of the union of sets ...

Addition Principle (for Sets)

Given two sets A and B, if $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B| \,.$$

Sets are said to be disjoint (mutually exclusive) if their intersection is the empty set.

Where $|A|$ is the size (number of elements) of A, etc.

If we think of A, as the set of all the different outcomes from event A and similarly for B, then we then have

Addition Principle (for counting)

The addition principle states that if event A can occur in m ways, and event B can occur in n ways, with A and B *disjoint*, then the event "A or B" can occur in $m + n$ ways.

- The *disjoint* condition is important[§], it must be impossible for both events to occur.
- This property can be extended to three or more events.

[§]Later, we will deal with the non-disjoint case using the Inclusion-Exclusion principle.

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Example 1

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How many two letter “words” start with either A or B? (A word is just a string of letters; it doesn’t have to be English, or even pronounceable.)

Solution. First, how many two letter words start with A? We just need to select the second letter, which can be accomplished in 26 ways. So there are 26 words starting with A. There are also 26 words that start with B. To select a word which starts with either A or B, we can pick the word from the first 26 or the second 26, for a total of 52 words.

$$\underbrace{\text{26 two letter words starting with `A'}}_{\text{number of ways event } A \text{ can occur}} + \underbrace{\text{26 two letter words starting with `B'}}_{\text{number of ways event } B \text{ can occur}} = 52$$

Note that event A and event B cannot both occur, i.e., distinct events.

$\boxed{\text{exclusive OR} \leftrightsquigarrow \text{addition}}$

A Non-example

Example 2

A standard deck of 52 cards contains 26 red cards[a] and 12 face cards[b]. However, the number of ways to select a card which is either red or a face card is not $26 + 12 = 38$. This is because the two events

- $A =$ *drawing a red card*
- $B =$ *drawing a face card*

are not disjoint (mutually exclusive). There are 6 cards which are both red and face cards.

The correct answer for the number of ways is 32. But that is not the important point here.

---

[a] desk is divided into four suits: clubs ♣, spades ♠, diamonds ♦, and hearts ♥.

[b] The face cards are the 4 kings, 4 queens and the 4 jacks.

not exclusive OR $\leftrightsquigarrow$ not addition

Multiplication Principle

We also discussed the following property of the Cartesian product of sets …

Multiplication Principle (for Sets)

Given two sets A and B, then the Cartesian product, $A \times B$, satisfies

$$|A \times B| = |A|\,|B|\,.$$

Size of Cartesian product $A \times B$ is size of A times size of B.

If we think of an event, A, as a set of all the ways that an event can occur, and similarly for B, then we then have

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The multiplication principle states that if event A can occur in m ways, and each possibility for A allows for exactly n ways for event B, then the event "A and B" can occur in $m \cdot n$ ways.

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Example 3

I

Example 3

A snack bar serves five different sandwiches (Beef, Cheese, Chicken, Ham, and Bologna) and three different beverages (Milk, Juice, and Coffee). How many different lunches can a person order?

Solution. Our first choice

$$A = \{\text{Beef}, \text{Cheese}, \text{Chicken}, \text{Ham}, \text{Bologna}\}$$

has 5 options, i.e., $|A| = 5$. Our second choice

$$B = \{\text{Milk}, \text{Juice}, \text{Coffee}\}$$

has 3 options, i.e., $|B| = 3$.

Hence, when we make a choice of "A and B" we have $5 \times 3 = 15$ options, i.e.,

$$\{(Beef, Milk), (Beef, Juice), (Beef, Coffee), (Cheese, Milk), (Cheese, Juice), \ldots (Bologna, Coffee)\}$$

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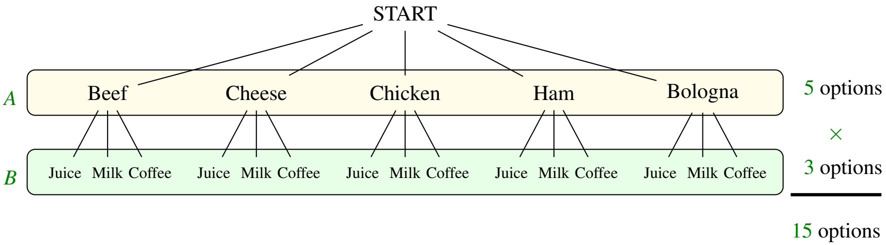
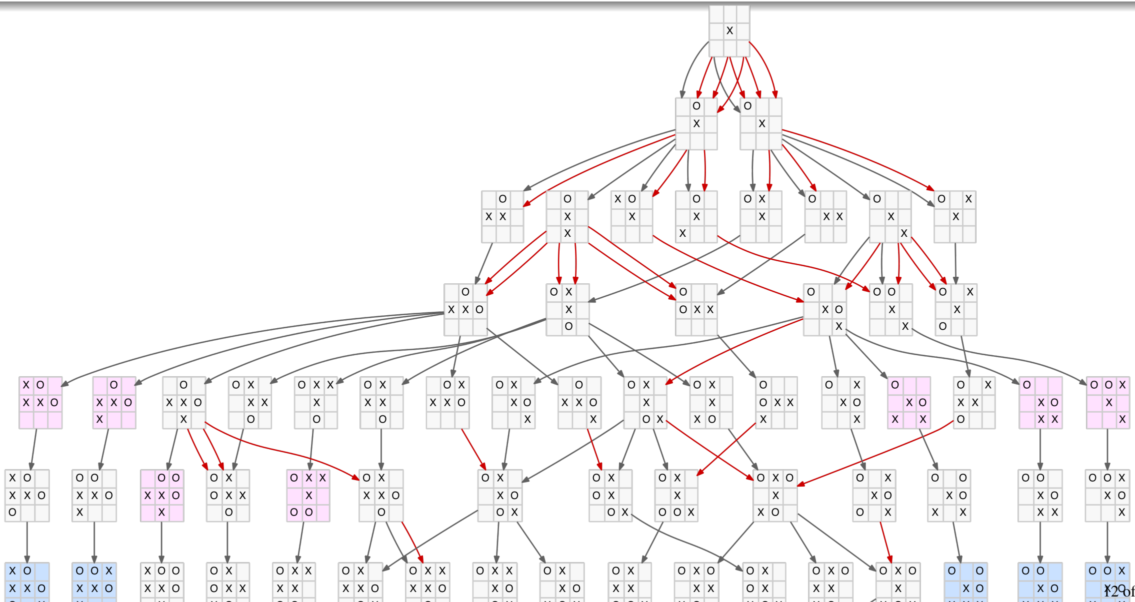
Example 3 II

Alternatively we could determine the number of possible lunches by listing or enumerating all the possibilities. One systematic way of doing this is by means of a tree[¶]



[¶] A tree is a graph in which there are no cycles or self-loops (so only one way to get from a node to any other node). Tree are important in data structures (maintaining a mutable, sorted collection), and in machine learning (combinatorial games, such a tic-tac-toe, chess, go, etc.)

A (Partial) Tree Representation of X and O



Non-Examples

The events must represent separate choices

For example, how many playing cards are both red and a face card?

There are 26 red cards in a standard deck and 12 face cards in a standard, but this does not mean that there $26 \times 12 = 312$ red and face cards in a standard deck. (The answer is 6.)

The events must be independent

How many ways can you select two cards, so that the first one is a red card and the second one is a face card?

This looks more like the multiplicative principle (you are counting two separate events) but the answer is not 26×12 here either.

The problem is that while there are 26 ways for the first card to be selected, it is not the case that *for each* of those there are 12 ways to select the second card. If the first card was both red and a face card, then there would be only 11 choices for the second card.

Example

Example 4 (Counting functions)

How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ are there?

Solution. Remember that a function sends each element of the source/domain to exactly one element of the target.

To determine a function, we just need to specify the image of each element in the domain:

- Where can we send 1? There are 4 options.
- Where can we send 2? Again, 4 options.
- ...

What we have here is 5 "events" (picking the image of an element in the domain) each of which can happen in 4 ways (the options for that image). Thus there are

$$4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$$

functions.

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Example 5

A multiple choice quiz[a] contains four questions that have two possible answers and three questions with five possible answers. Since the answer to each question is independent of the answers to the other questions, the extended multiplication principle and there are

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different ways to answer the quiz.

[a] If the English language was consistent, this would be "multiple option quiz".

- A call-forward question: Given that there is 100 students enrolled on the *Discrete Mathematics* module, how many times will I have to give the quiz in order to guarantee that at least two students will have identical answers to the quiz?

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How big is the Power Set?

In Week 3 we introduced the power set of a set A, $\mathcal{P}(A)$, which is the set of all subsets of A. Can we compute how many elements are in $\mathcal{P}(A)$ for a given finite set A?

Theorem 6 (Power Set Cardinality Theorem)

If A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

Proof: (for simplicity let $n = |A|$)

Consider how we might determine any element of $\mathcal{P}(A)$ ...

- If $B \in \mathcal{P}(A)$ then B is a subset of A.
- For each element $x \in A$ there are two options, either $x \in B$ or $x \notin B$.
- Since there are n elements of A we have, by the multiplication principle,

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ choices each with 2 options}} = 2^n$$

different subsets of A. Therefore, $|\mathcal{P}(A)| = 2^n$.

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Example 7

We can combine the addition and the multiplication principles ...

Example 7 (My wardrobe)

If you have been paying attention (and I would be worried if you were) you will have noticed that I wear[a]

- 5 near identical chinos
- 3 near identical check red shirts
- 13 t-shirts

Given that I wear a shirt or a t-shirt with chinos, how many different outfits do I have?

Solution: From the additive principle, the number of options for the choice for top is $3 + 13 = 16$.

From the multiplication principle, the number of options overall is

$$5 \times 16 = 80$$

[a] I follow the David Attenborough has only one shirt school of thought.

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Example 8

In the multiplication principle the options for choice B may vary with the option selected for choice A. The only requirement for the multiplication principle is that there be the same number of options, not necessarily the same options.

Example 8

How many two-digit numbers have distinct and nonzero digits?

Solution:

- A two-digit number ab can be regarded as an ordered pair (a, b), where a is the tens digit and b is the units digit.
- Neither of these digits is allowed to be 0 in the problem, and the two digits are to be different.
- There are nine options for a, namely $1, 2 \ldots, 9$.
- Once a is chosen, there are eight options for b.
  - If $a = 1$, these eight options are $2, 3, \ldots, 9$, if $a = 2$, the eight options are $1, 3, \ldots, 9$, and so on.
- By the multiplication principle, we have answer $9 \times 8 = 72$.

Subtraction Principle

In Week 3, we also discussed the set minus operator in sets, which has property ...

Subtraction Principle (for Sets)

Given a set A and the universal set U, then the complement of A, defined by $\overline{A} = U \setminus A$, satisfies

$$|A| = |U| - |\overline{A}|.$$

The size of A is equal to the size of the universal set minus the size of the complement of A.

If we think of an event, A, as a set of all the ways that that event can occur, then we then have

Subtraction Principle (for counting)

The subtraction principle number of ways that event A can happen is equal to the total number of outcomes minus number of ways event A cannot happen.

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How many two-digit numbers have distinct and nonzero digits?

Solution:

We have already answered this using the multiplication principle. Now using the subtraction principle . . .

There are 100 two-digit numbers: $00, 01, 02, \ldots, 99$. Of these numbers,

- 1 has a double zero (namely 00)
- 9 have a leading zero followed by a non-zero digit (namely $01, 02, \ldots, 09$),
- 9 have a non-zero digit followed by 0, (namely, $10, 20, \ldots, 90$), and
- 9 have identical digits (namely, $11, 22, \ldots, 99$.

Thus the number of two-digit numbers with distinct and nonzero digits equals

$$100 - 1 - 9 - 9 - 9 = 72$$

Example

Example 10 (Poor passwords)

Computer passwords are to consist of a string of six symbols taken from the digits $0, 1, 2, \ldots, 9$ and the lowercase letters $a, b, c, \ldots, z$. How many computer passwords have a repeated symbol?

Solution: We want to count the number of objects in the set A of computer passwords with a repeated symbol. Let U be the set of all computer passwords of length 6 characters. Taking the complement of A in U we get the set $\overline{A}$ of computer passwords with no repeated symbol. By two applications of the multiplication principle, we get

$$|U| = 36^6 = 2,176,782,336$$

and (36 characters =26 letters +10 digits)

$$|\overline{A}| = 36 \times 35 \times 34 \times 33 \times 32 \times 31 = 1,402,410,240.$$

Therefore,

$$|A| = |U| - |\overline{A}| = 2,176,782,336 - 1,402,410,240 = 774,372,096.$$

To put this number in context, in 2012 a 25 node GPU machine was constructed that was capable of 350 billion passwords per second — use better passwords.

Example

Example 11

You wish to give your granny a basket of fruit. In your food store you have six identical oranges and nine identical apples. The only requirement is that there must be at least one piece of fruit in the basket (that is, an empty basket of fruit is not allowed). How many different baskets of fruit are possible?

Solution:

What distinguishes one basket of fruit from another is the number of oranges and number of apples in the basket.

- First, we ignore the requirement that the basket cannot be empty. We can compensate for that later.
- There are 7 options for the number of oranges $(0, 1, \ldots, 6)$ and 10 options for the number of apples $(0, 1, \ldots, 9)$.
- By the multiplication principle, the number of different baskets is $7 \times 10 = 70$.
- Subtracting the empty basket, the answer is 69.

The Pigeonhole Principle

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If more than n pigeons fly into n pigeon holes, then at least one pigeon hole will contain at least two pigeons.

- The pigeonhole principle is used to generate necessary conditions for collisions.

Corollary 12

If f is a function from a finite set A to a finite set B with $|A| > |B|$ then f cannot be injective.

Proof

Since f is a function from set A then there are $|A|$ outgoing arrows from A and incoming to B. However set B has only $|B|$ elements with $|A| > |B|$, so by the pigeonhole principle there must be at least one element in B which has more than one incoming arrow, i.e., f is not injective.

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Example 13

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Suppose you pick any 12 positive integers. Then you must be able to subtract one of the numbers from another one to get a multiple of 11.

Proof

- If you have a set of 12 numbers, if you divide the numbers in the set by 11, you must get a remainder of somewhere in between 0 and 10.
- That's 11 different remainders that are possible. But there are 12 numbers in the set and so the pigeonhole principle tells you that at least 2 of the numbers have the same remainder after division by 11.
- If you subtract one from the other, you'll get a number which is exactly divisible by 11, and hence is a multiple of 11.

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Question 1:
Your wardrobe consists of 5 shirts, 3 pairs of pants, and 17 bow ties. How many different outfits can you make?
Question 2:
For your college interview, you must wear a tie. You own 3 regular (boring) ties and 5 (cool) bow ties.

1. How many choices do you have for your neck-wear?
2. You realise that the interview is for clown college, so you should probably wear both a regular tie and a bow tie. How many choices do you have now?
3. For the rest of your outfit, you have 5 shirts, 4 skirts, 3 pants, and 7 dresses. You want to select either a shirt to wear with a skirt or pants, or just a dress. How many outfits do you have to choose from?

Question 3:
If $|A| = 8$ and $|B| = 5$, what is $|A \cup B| + |A \cap B|$?
Question 4:
Consider all 5 letter "words" made from the letters a through h. (Recall, words are just strings of letters, not necessarily actual English words.)

1. How many of these words are there total?
2. How many of these words contain no repeated letters?
3. How many of these words start with the sub-word "aha"?

4. How many of these words either start with “aha” or end with “bah” or both?
5. How many of the words containing no repeats also do not contain the sub-word “bad”?