Discrete Mathematics

Topic 06 — Graphs and Networks

Lecture 01 — Fundamental Concepts in Graph Theory

Dr Kieran Murphy

Department of Computing and Mathematics,
SETU (Waterford).
(kieran.murphy@setu.ie)

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Outline

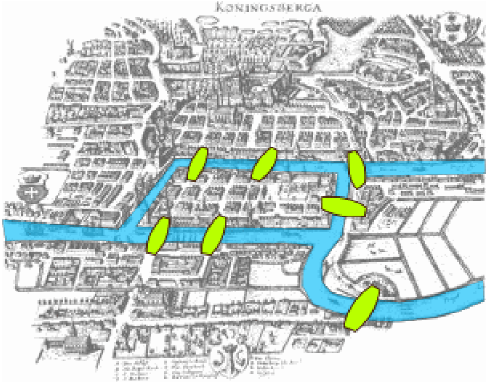
- Fundamental graph concepts and definitions
- A selection of common graphs

Outline

History of Graph Theory — Stage 1: Birth (circa 1735)

The Bridges of Konigsberg Problem

Is it possible to walk through the city of Konigsberg that would cross each of the seven bridges once and only once and return to one's starting point.



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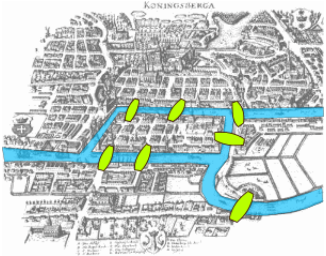
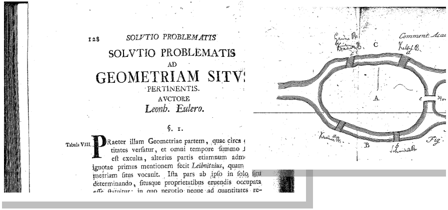
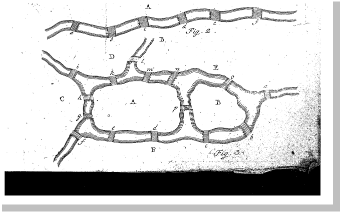


Solved by Euler

- Lead to the birth of graph theory.





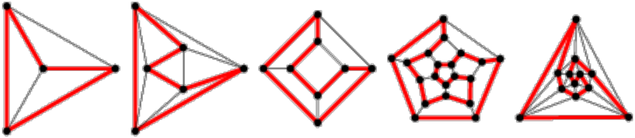
History of Graph Theory — Stage 2: Mathematical Interest

Problems were only of mathematical interest, for example

Cycles in Polyhedra

Thomas P. Kirkman

William R. Hamilton

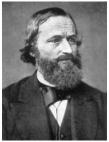


Hamiltonian cycles in Platonic graphs

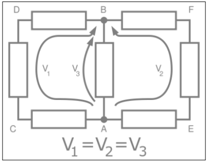
History of Graph Theory — Stage 3: Applications

In Physics . . .

Trees in Electric Circuits

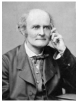


Gustav Kirchhoff



In Chemistry . . .

Enumeration of Chemical Isomers



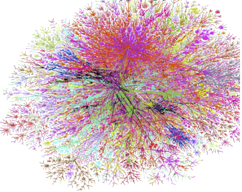
Arthur Cayley

James J. Sylvester

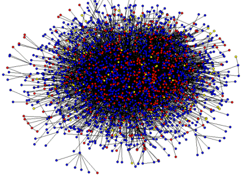
George Polya

In Computing . . . , in Biology . . . , in Social Science . . . , in . . .

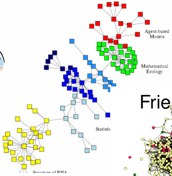
Internet



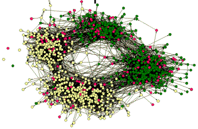
Protein-Protein Interaction Network



Scientific collaboration Network



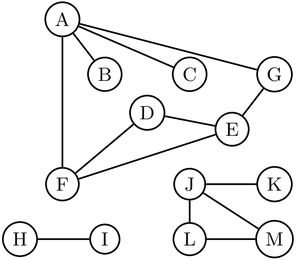
Friendship Network



Graphs — An Informal Definition

A graph is a set of objects with pairwise connections.

- The notion of a graph is deceptively simple: It is a collection of points (called vertices or nodes) that are joined by lines (called edges or arcs).
- All that matters about an edge is which two vertices it connects — and sometimes its length, capacity and/or cost — but not the layout of the edge.



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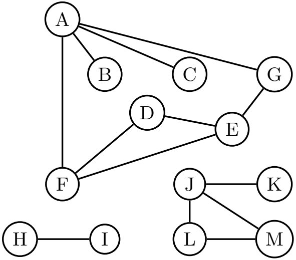

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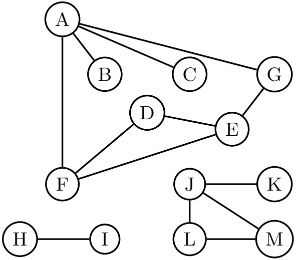


Graph Applications

Graph theory is one of the most widely applicable areas of mathematics.

Application	Vertices	Edges
communication	telephones, computers	cables, connections
circuits	gates, registers, processors	wires
mechanical	joints	rods, beams, springs
financial	stocks, currency	transactions
games	board positions	legal moves
transportation	road intersections, airports	roads, air routes
scheduling	tasks	precedence constraints
software systems	functions	function calls
social relationship	people, actors	friendships, movie casts
chemical compounds	molecules	bonds
hydraulic	reservoirs, pumping stations	pipelines
Internet	web pages	hyperlinks
neural networks	neurons	synapses

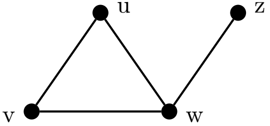
Outline

Graph — Formal Definition

Definition 1 (Simple Graph)

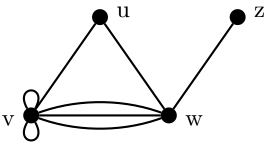
A simple graph, G, consists of a non-empty finite set, $V(G)$, of elements called vertices (or nodes), and a finite set, $E(G)$, of distinct unordered pairs of distinct elements of $V(G)$ called edges (or arcs).

- The set $V(G)$ is called the vertex set and has $n = |V(G)|$ elements (vertices).
- The set $E(G)$ the edge set of G.and has $m = |E(G)|$ elements (edges).
- An edge (v, w) is said to join the vertices v and w, and is usually abbreviated to vw or $v{-}w$.
- The simple graph on the right has vertex set $V(G) = \{u, v, w, z\}$ and edge set $E(G) = \{uv, uw, vw, wz\}$.



General Graph

In any simple graph there is at most one edge joining any given pair of vertices (no multiple edges), and all edges join distinct vertices (no loops). There are situations where these restrictions are not desirable. (e.g., adding redundancy in a communication network.)



Definition 2 (General Graph)

A general graph, G, consists of a non-empty finite set, $V(G)$, of elements called vertices, and a finite multi-set (or family), $E(G)$, of unordered, not necessarily distinct, elements of $V(G)$ called edges.

- In the theory component of this course we will try to prove results for general graphs but we will be more restrictive in our Python implementations and mainly concentrate on simple graphs.

Warning — No Standard Notation

Graph notation is not standard!

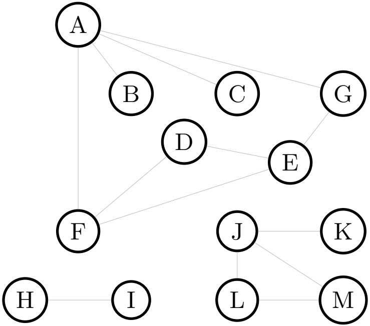
Some of the different variations that are popular are:

- **Wilson:1996**
  - simple graph — A graph with no multiple edges and no loops.
  - general graph or graph — A graph with multiple edges and/or loops.
- **Sedgwick:2003**
  - graph — A graph with no multiple edges and no loops.
  - multi-graph — A graph with multiple edges and/or loops.
- **Trudeau:1993**
  - graph — A graph with no multiple edges and no loops.
  - multi-graph — A graph with multiple edges.
  - pseudo-graph — A graph with multiple edges and loops.

Graph Jargon — Vertex

Terminology:

- Vertex
- Edge
- Parallel edges, self loop
- Directed graph
- Weighted graph
- Path, cycle
- Tree, forest
- Connected, connected components



This graph has 13 vertices:

$$\{A, B, C, D, E, F, G, H, I, J, K, L, M\}$$

Graph Jargon — Edge

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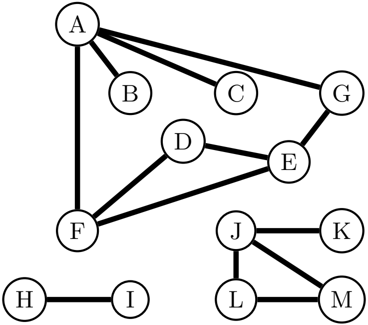

This graph has 13 edges:

$$\{(A, B), (A, C), (A, G), (A, F), (D, E), (D, F), (E, F), (E, G), (H, I), (J, K), (J, L), (J, M), (L, M)\}$$

Graph Jargon — Parallel Edges, Self Loop

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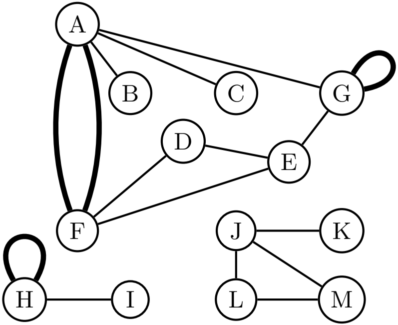

This graph has two parallel edges $\{(A, F), (A, F)\}$ and two self-loops $\{(H, H), (G, G)\}$.

Graph Jargon — Directed Graph

Terminology:

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- Edge
- Parallel edges, self loop
- **Directed graph**
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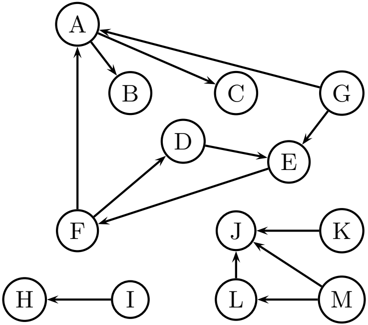

The edges in a directed graph (diagraph) are drawn with an arrow indicating the direction. So for example, this graph has edge (A, B) but not edge (B, A).

Graph Jargon — Weighted Graph

Terminology:

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- Directed graph
- **Weighted graph**
- Path, cycle
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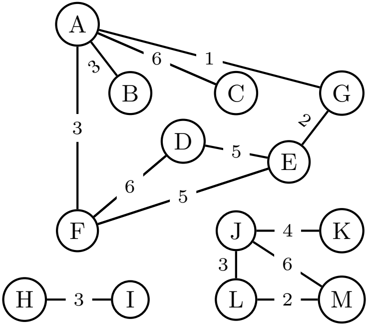

The edges in a weighted graph have weights representing length, cost, or delay in traversing that edge.

Graph Jargon — Path, Cycle

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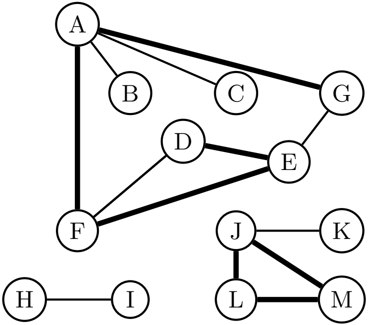

A path is a sequence of connected vertices, e.g., $\{D, E, F, A, G\}$.
A cycle is a path with same end vertices, e.g., $\{J, L, M, J\}$.

Graph Jargon — Tree, Forest

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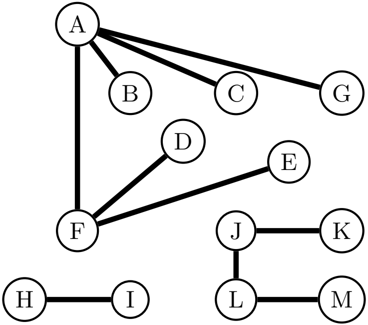

A tree is graph with no cycles.
A forest is a set of trees.

Graph Jargon — Connected Components

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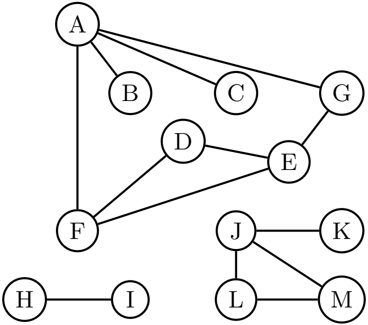

A graph is connected if there is a path between any two vertices.
Parts of a graph that are not connected to each other are called connected components.

Typical Graph Problems

Paths

Cycles and Tours

Connectivity

Typical Graph Problems

Paths

Path: Is there a path between between two nodes, u and v?

Shortest Path: What is the shortest path between u and v?

Longest Path: What is the longest path between u and v?

Cycles and Tours

Connectivity

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Shortest Path: What is the shortest path between u and v?

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Cycles and Tours

Cycle: Is there a cycle in the graph?

Euler Tour: Is there a cycle path that uses each edge exactly once?

Hamilton Tour: Is there a cycle path that uses each vertex exactly once?

Connectivity

Typical Graph Problems

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Cycles and Tours

Cycle: Is there a cycle in the graph?

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Hamilton Tour: Is there a cycle path that uses each vertex exactly once?

Connectivity

Connectivity: Is it possible to connect all of the vertices?

MST: What is the optimum way to connect all of the vertices?

Bi-connectivity: Is there a vertex whose removal disconnects the graph?

Typical Graph Problems II

Planarity

Graph Colouring

Typical Graph Problems II

Planarity

Planarity: Is it possible to draw the graph in the plane with no crossing edges?

Depth: What is the minimum number of crossing need to layout a non-planar graph.

Graph Colouring

Typical Graph Problems II

Planarity

Planarity: Is it possible to draw the graph in the plane with no crossing edges?

Depth: What is the minimum number of crossing need to layout a non-planar graph.

Graph Colouring

Vertex Colouring: What is the minimum number of colours needed to colour the graph vertices so that adjacent vertices have different colours?

Edge Colouring: What is the minimum number of colours needed to colour the graph edges so that adjacent edges have different colours?

Outline

Isomorphism

Given two graphs are they equal* or distinct?
— Remember a graph is not changed by rearanging its drawing layout.

Definition 3 (Isomorphism)

Two graphs G_1 and G_2 are isomorphic if there is a one-to-one correspondence between the vertices of G_1 and G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2.

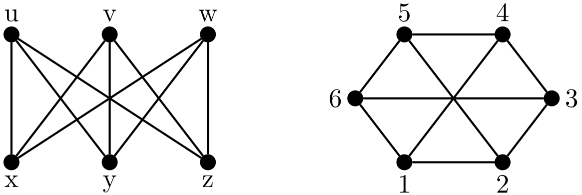
- For many problems, the labels on the vertices are unnecessary and can be disregarded. In this case we say that the two unlabelled graphs are isomorphic if we can assign labels to both graphs so that the resulting labelled graphs are isomorphic.

*Isomorphic = Greek “equal” + “shape”

Example — Isomorphic Graphs

Example 4

The two graphs shown bellow are isomorphic under the correspondence $u \leftrightarrow 5$, $v \leftrightarrow 3$, $w \leftrightarrow 1$, $x \leftrightarrow 4$, $y \leftrightarrow 2$, $z \leftrightarrow 6$.



It is easier to check this by building the one-to-one correspondence a step at a time: $u \leftrightarrow 5$, $v \leftrightarrow 3$, $w \leftrightarrow 1$, $x \leftrightarrow 4$, $y \leftrightarrow 2$, $z \leftrightarrow 6$.

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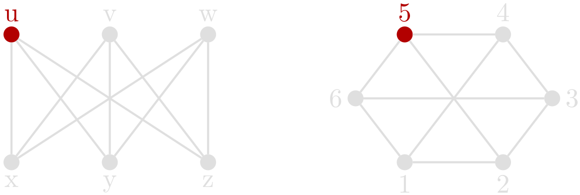

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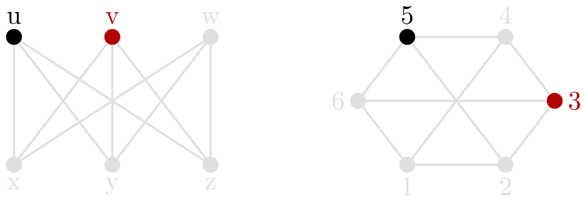

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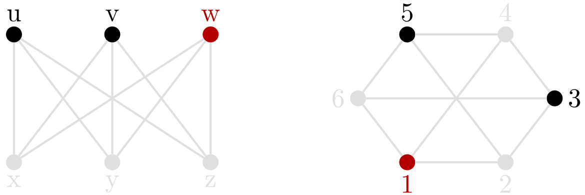

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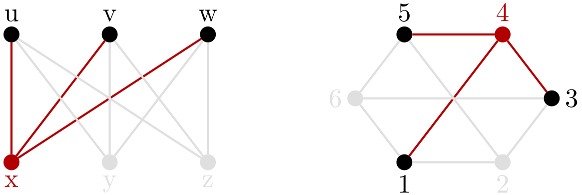

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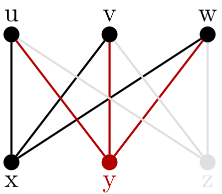

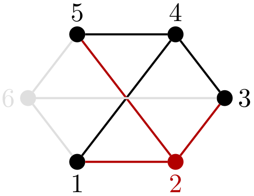

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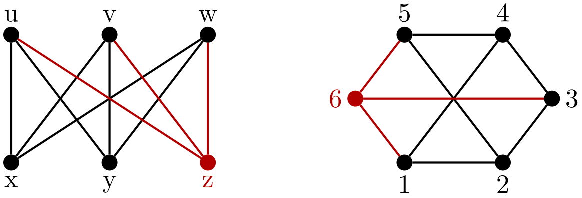

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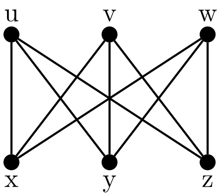

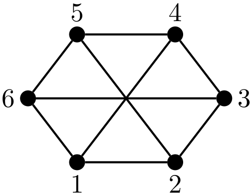

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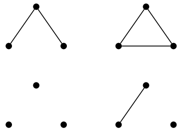
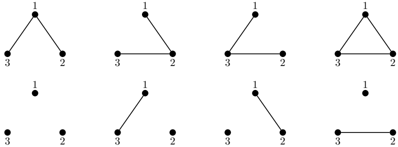
Labelled and Unlabelled Graphs

The number of permutations of vertex labels grows large rapidly as the number of vertices in the graph increases. For example, for simple graphs with three vertices, there are only four distinct unlabelled simple graphs while there are eight distinct labelled simple graphs.



Unlabelled simple graphs



Labelled simple graphs

This difference increases as the number of vertices increase, for example, consider graphs with four vertices, here there are 6 distinct unlabelled simple graphs and 64 distinct labelled simple graphs with four vertices, (see Wilson, page 10).

Connectness

Graphs can be combined to make a larger graph.

Definition 5 (Graph union)

If the two graphs are $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are disjoint, then their union, $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge multi-set $E(G_1) \cup E(G_2)$.

Definition 6 (Connected)

A graph is connected if it cannot be expressed as the union of two graphs, and disconnected otherwise.

- Any disconnected graph, G, can be expressed as the union of connected graphs, each of which is a component of G.

Connectness

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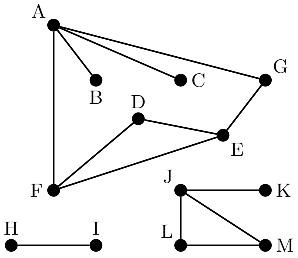
Example

The graph, G, depicted bellow has three components



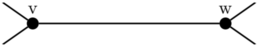
Hence we have $G = G_1 \cup G_2 \cup G_3$ where

$$G_1 = \left(\{A, B, C, D, E, F, G\}, \{A–B, A–C, A–F, A–G, D–E, D–F, E–F, E–G\}\right)$$
$$G_2 = \left(\{H, I\}, \{H–I\}\right)$$
$$G_3 = \left(\{J, K, L, M\}, \{J–K, J–L, J–M, L–M\}\right)$$

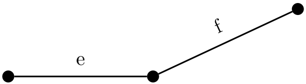
Adjacency

Definition 7 (Adjacent/Incident)

Two vertices, v and w, are adjacent if there is an edge, $v{-}w$, joining them, and the vertices v and w are then incident with such an edge.
Similarly, two distinct edges, e and f, are adjacent if they have a vertex in common.



adjacent vertices



adjacent edges

Degree and Related Concepts

Definition 8 (Degree)

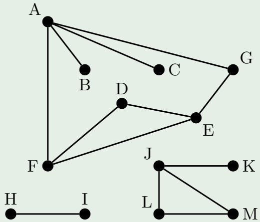
The degree of a vertex, v, is the number of edges incident with v, denoted by $\deg(v)$.

- Normal convention is that a loop at v contributes 2 (rather than 1) to the degree of v.
- A vertex of degree 0 is an isolated vertex, and a vertex of degree 1 is an end-vertex.
- A graph in which all vertices have degree r is said to be regular of degree r.
- The degree sequence of a graph, G, consists of degree of all of the vertices of G sorted in non-decreasing order.

Example

Example 9

Consider the following graph



- Vertices B, C, H, I, and K are end-vertices.
- Vertex A has degree 4.
- The degree sequence of the graph is $1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4$.

Review Exercises 1 (Fundamental Concepts)

Question 1:
Draw a graph with degree sequence $(3, 3, 5, 5, 5, 5)$. Does there exist a *simple* graph with this degree sequence? Justify your answer.

Question 2:
State, with an explanation, which of the following sequences are the degree sequences of a simple graph. For those sequence that are degree sequences, draw a simple graph with that degree sequence.

(a) $(1, 1, 2, 2, 3, 4, 4, 5, 5,)$ **(b)** $(2, 2, 2, 3, 5, 6, 6, 6)$ **(c)** $(1, 2, 3, 4, 5, 6, 7)$

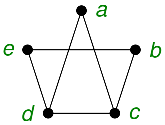
Question 3:
If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree.

Question 4:
(Hard) Is it possible for two *different* (non-isomorphic) graphs to have the same number of vertices and the same number of edges? What if the degrees of the vertices in the two graphs are the same (so both graphs have vertices with degrees 1, 2, 2, 3, and 4, for example)? Draw two such graphs or explain why not.

Question 5:
Are the two graphs below equal? Are they isomorphic? If they are isomorphic, give the isomorphism. If not, explain.

Graph 1:
$V = \{a, b, c, d, e\}$
$E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}\}$.

Graph 2:



Outline

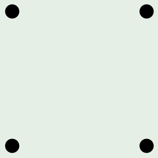
Null Graphs

Definition 10 (Null graph, N_n)

A graph whose edge-set is empty is a null graph. A null graph with n vertices is denoted by N_n. $(n \geq 1)$

- The vertices of a null graph are all isolated.

Example 11 (The null graph, N_4)



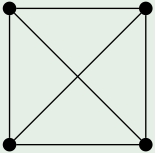
Complete Graphs

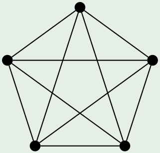
Definition 12 (Complete graph, K_n)

A simple graph in which every pair of vertices are adjacent is a complete graph. A complete graph with n vertices is denoted by K_n. $(n \geq 1)$

- A complete graph of n vertices has $n(n-1)/2$ edges.

Example 13 (The complete graphs, K_4 and K_5)





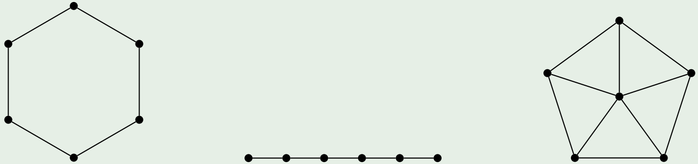
Cycle Graphs

Definition 14 (Cycle graph, C_n)

A connected graph that is regular of degree 2 is a cycle graph. A cycle graph with n vertices is denoted by C_n. $(n \geq 3)$

- The graph obtain from C_n by removing an edge is the path graph on n vertices, denoted by P_n. $(n \geq 3)$
- The graph obtain from C_{n-1} by joining each vertex to a new vertex v is called the wheel graph on n vertices, denoted by W_n. $(n \geq 4)$

Example 15 (Graphs C_6, P_6 and W_6)



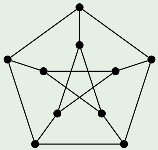
Regular Graphs

Definition 16 (Regular graph)

A graph in which all of the vertices are of degree r is a regular graph of degree r or r-regular.

- A cubic graph is a regular of degree 3 graph.
    - A important example of a cubic graph is the Petersen graph.
- The null graph, N_n, is a regular graph of degree 0.
- The complete graph, K_n, is a regular graph of degree $n-1$.

Example 17 (The Petersen graph)



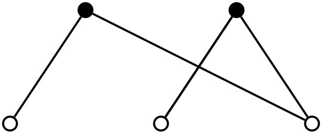
Bipartite Graphs

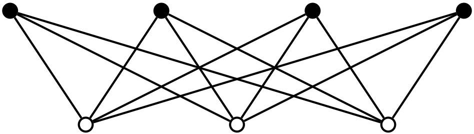
Definition 18 (Bipartite graph)

A graph, G, in which its vertex set can be split into two disjoint sets, A and B, so that each edge of G joins a vertex of A and a vertex of B is a bipartite graph.

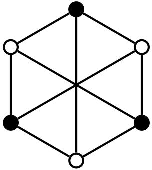
- An alternative definition: A bipartite graph is a graph whose vertices can be coloured black and white so that each edge joins a black vertex and a white vertex.
- A complete bipartite graph is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge, i.e., all black vertices are joined to all white vertices, and all white vertices are joined to all black vertices.
- A complete bipartite graph with r black vertices and s white vertices is denoted by $K_{r,s}$.

Examples of Bipartite Graphs





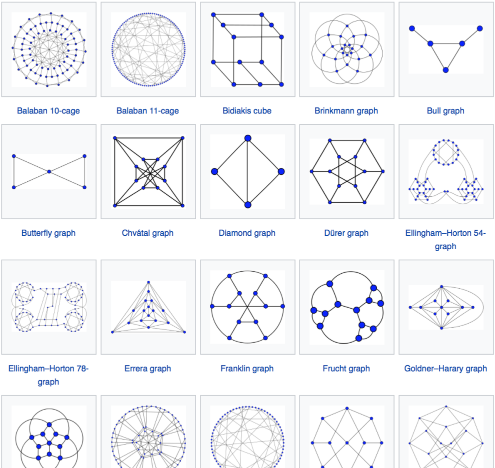
$K_{4,3}$



$K_{3,3}$

Other Graphs

- Gallery of named graphs
  http://en.wikipedia.org/wiki/Gallery_of_named_graphs
- ISEM's MATtours:
  http:
  //www.hamline.edu/~lcopes/SciMathMN/gallery.html



Question 1:
Give an example of, or explain why it doesn't exist, each of the following.

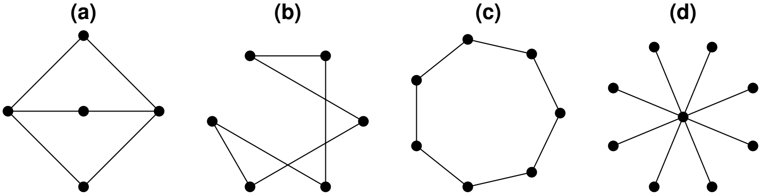
(a) A bipartite graph that is regular of degree 5.

(b) A complete graph that is a wheel.

(c) A cubic graph with 11 vertices.

(d) A graph (other than K_5, $K_{4,4}$, or Q_4) that is regular of degree 4.

Question 2:
Which of the graphs below are bipartite? Justify your answers.



Question 3:
For which $n \geq 3$ is the graph C_n bipartite?