Discrete Mathematics

Topic 04 — Relations and Functions

Lecture 01 — Relation Concepts and Definitions

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Outline

- Defining a relation via Cartesian product
- Relation Terminology

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1. Relation Definition 2
1.1. Cartesian product and Relations 3
1.2. Graphical Representation of Relations using Venn Diagrams 9

Cartesian product

Recall that the Cartesian product of two sets, A and B, is the set of all ordered pairs of all elements where the first element is from set A and the second element is from B.

Definition 1 (Cartesian product)

The Cartesian product of two sets A and B, denoted by $A \times B$ is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

- The order within the pair matters, so $(a, b) \neq (b, a)$.
- But, since $A \times B$ is a set, the order between the pairs is not important.

$$\{(a, b), (c, d)\} = \{(c, d), (a, b)\}$$

- The set $A \times B$ has $|A||B|$ elements.

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Example

Example 2

The Cartesian product of $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 4\}$ is

Or in Python*...

cartesian_product.py

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A = {0,1,2,3}
B = {0,1,4}

C = {(a,b) for a in A for b in B}

print (C)
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The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

Definition 3 (Relation)

Given two sets A and B. **Any** subset of the Cartesian product between A and B is called a relation.

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The Cartesian product of the sets $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 4\}$ is

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So possible relations between A and B include

- $R = \{(0,0),(1,1),(2,4)\}$ (relation is based on $x \mapsto x^2, x < 3$)
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Relation II

You should spend some time thinking about the consequences of the definition that we have just covered …

- *Given two sets, A and B, how many distinct relations can we construct?*

- *Relation vs. Cartesian product vs. power set of the Cartesian product*

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- *Given two sets, A and B, how many distinct relations can we construct?*
  - Set A and B have $|A|$ and $|B|$ elements respectively.
  - The Cartesian product, $A \times B$, has $|A||B|$ elements.
  - Relation between A and B is **any** subset of $A \times B$.
  - Sets of size n have 2^n subsets.
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  - R is an element of the power set of the Cartesian product of A and B.

    $$R \in \mathcal{P}(A \times B)$$

Example

Example 5

Let $A = \{2, 3, 5, 6\}$ and define a relation R from A to A by $(a, b) \in R$ if and only if a divides evenly into b.

The relation R is defined by

$$R = \{(a, b) \mid a \in A, b \in A, a \text{ divided evenly into } b\}$$

The set of pairs that qualify for membership of R is

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

Definition 6 (Relation on a Set)

A relation from set A to A is called a relation on A.

Notation Warning —- Divisibility

When explaining relations we will often use (as in the previous example) the idea of “divides”. Lets make sure we all agree on what this means …

Definition 7 (Divides)

Let $a, b \in \mathbb{Z}$. We say that a divides b, denoted $a \mid b$, if and only if there exists an integer k such that $ak = b$.

- Be careful in writing about the relation “divides.” The vertical line symbol use for this relation, if written carelessly, can look like division. While $a \mid b$ is either **True** or **False**, a/b is a number[†].
- Even worse. We, mathematicians, use the same symbol “$|$” for "such that" in set builder notation and for “divides”.
  - Usually this is not a problem as the intended meaning for “$|$” will be clear from the context.
  - Use alternative symbols: “$|$” is replaced by “:” in set builder notation.

[†] Also the direction is different. “$a \mid b$” means “a divides (evenly) into b”, while “a/b” means “the value of a divided by b”.

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Graphical Representation of Relations I — Venn Diagrams

Representing relations graphically can help in identifying its properties …

Consider the relation R from A into A, where $A = \{2, 3, 5, 6\}$ and $(a, b) \in R$ if and only if a divides evenly into b.

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

- Draw set A.
- This relation is from A to A, so we make a copy of set A and called it B.
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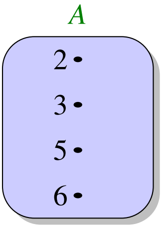

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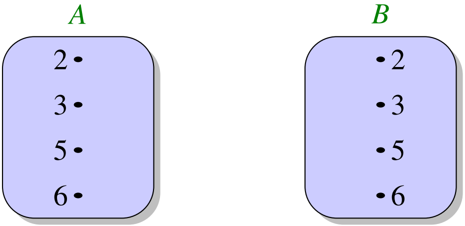

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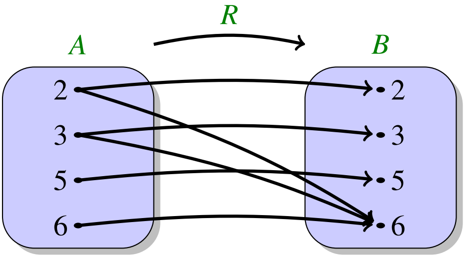

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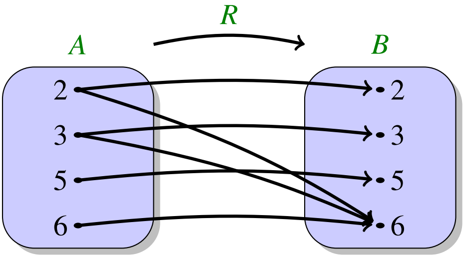

Things we are interested in seeing …

- Is there an arrow from every element in the first set?
- Is there an arrow to every element in the second set?
- Are there multiple arrows from some elements?
- Are there multiple arrows into some elements?

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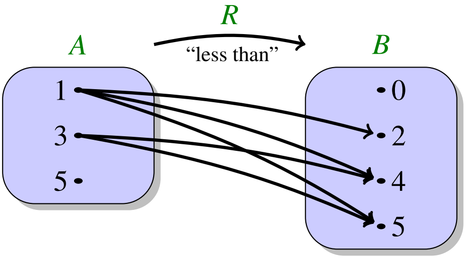

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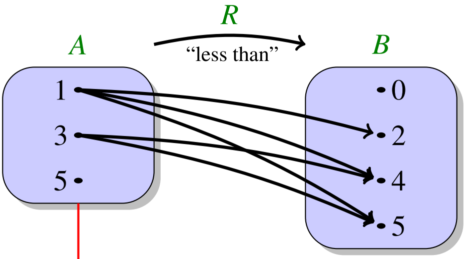

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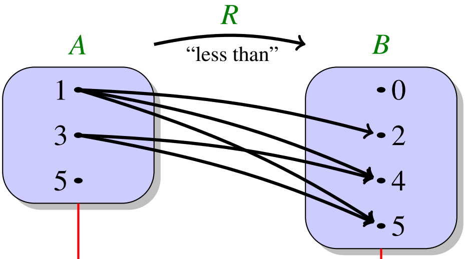

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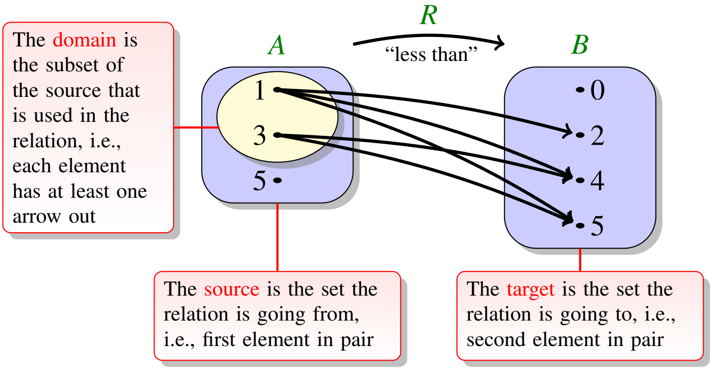

Relation Terminology I

Consider the relation "is less than" from set $A = \{1, 3, 5\}$ to set $B = \{0, 2, 4, 5\}$. We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

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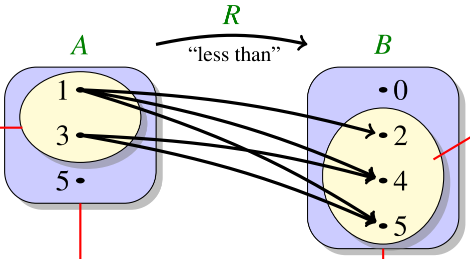
The domain is the subset of the source that is used in the relation, i.e., each element has at least one arrow out



The image is the subset of the target that is used in the relation, i.e., each element has at least one arrow in

The source is the set the relation is going from, i.e., first element in pair

The target is the set the relation is going to, i.e., second element in pair

Relation Terminology II

Given relation R from set S to set T we have:

- The source, S, is the set that the relation is going from.
- The target, T, is the set that the relation is going to.
- The domain of R, denoted by $\mathrm{Dom}(R)$, is the subset of the source for which there is at least one arrow leaving each element.

$$\mathrm{Dom}(R) = \{s \mid s \in S, \underbrace{\exists t \in T((s,t) \in R)}_{\text{exists at least one arrow leaving each element}}\} \subseteq S$$

- The image of R, denoted by $\mathrm{Im}(R)$, is the subset of the target for which there is at least one arrow entering each element.

$$\mathrm{Im}(R) = \{t \mid t \in T, \underbrace{\exists s \in S((s,t) \in R)}_{\text{exists at least one arrow entering each element}}\} \subseteq T$$

Relation Terminology — Into vs. Onto III

From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\mathrm{Im}(R) \subseteq T$$

This gives us two possibilities …

or

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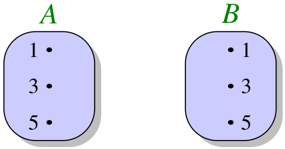

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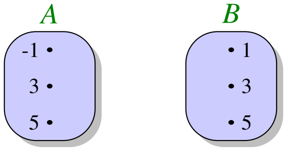

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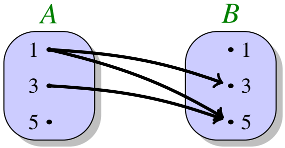

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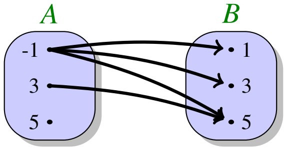

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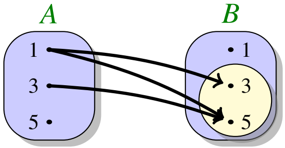

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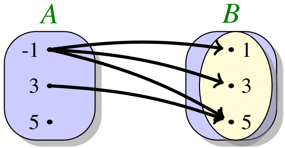

Relation Terminology — Into vs. Onto III

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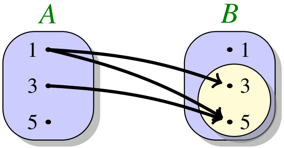

A relation, R, in which the image is a proper subset of the target is said to be an into relation.

or

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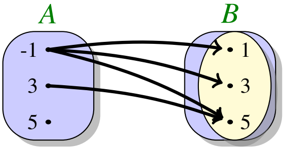

A relation, R, in which the image is equal to the target is said to be an onto relation.

Relation Terminology — Injective (one-to-one) III

Definition 8 (Injective)

A relation is said to be injective (or one-to-one) if there is at most one arrow into every element in the target set.

or

Relation Terminology — Injective (one-to-one) III

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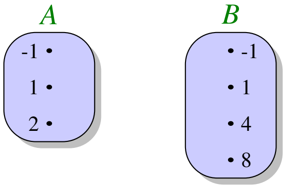
A relation is said to be injective (or one-to-one) if there is at most one arrow into every element in the target set.

Consider the relation “is square root of” from set $A = \{-1, 1, 2\}$ to set $B = \{-1, 1, 4, 8\}$.



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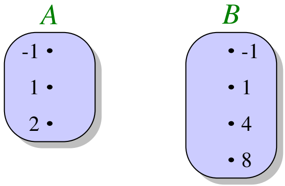

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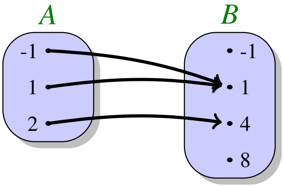

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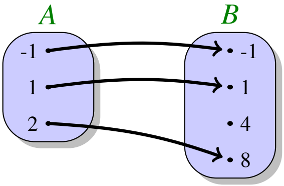

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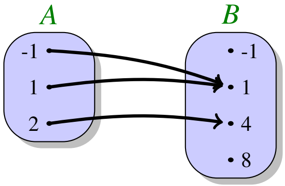

Not injective, since there exists at least one element in the target, (1), which has more than one incoming arrows.

or

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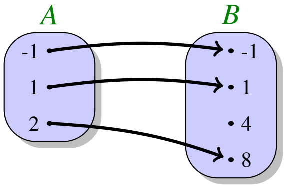

Is injective, since there is at most one arrow into each element in the target.

Review Exercises 1 (Relation Definition)

Question 1:
Consider the sets $A = \{0, 1, \dots, 6\}$ and $B = \{0, 1, \dots, 12\}$. Draw each of the following relations, and specify the domain and image of R from A to B and whether it is into or onto, and injective or not.

a) $(a, b) \in R$ iff $a \mid b$
b) $(a, b) \in R$ iff $a > b$
c) $(a, b) \in R$ iff number of primes less than a is equal to number of primes less than b
d) $(a, b) \in R$ iff number of factors of a is equal to number of factors of b.
e) $(a, b) \in R$ iff number of letters in writing a in English is equal number of letters in writing b in English.

Question 2:
Let R be the relation from $\mathbb{N}$ to $\mathbb{N}$ where $(a, b) \in R$ iff $b = a + 2$. Is R onto?
Question 3:
Let R be the relation from $\mathbb{Z}$ to $\mathbb{Z}$ where $(a, b) \in R$ iff $b = a + 2$. Is R onto?
Question 4:
Let R be the relation from $\mathbb{N}$ to $\mathbb{N}$ where $(a, b) \in R$ iff $b = a^2$. Is R one-to-one?
Question 5:
Let R be the relation from $\mathbb{Z}$ to $\mathbb{Z}$ where $(a, b) \in R$ iff $b = a^2$. Is R one-to-one?