WIT

BACHELOR OF SCIENCE (HONS) IN
- APPLIED COMPUTING
- COMPUTER FORENSICS & SECURITY
- ENTERTAINMENT SYSTEMS
- THE INTERNET OF THINGS

EXAMINATION:

DISCRETE MATHEMATICS (COMMON MODULE)

SEMESTER 1 - YEAR 1

DECEMBER 2021 **DURATION: 2 HOURS**

INTERNAL EXAMINERS:	DR DENIS FLYNN DR KIERAN MURPHY	DATE: TIME: VENUE:	20 DEC 2021. 14.15 PM MAIN HALL
EXTERNAL EXAMINER:	MS MARGARET FINNEGAN		

INSTRUCTIONS TO CANDIDATES

1. ANSWER ALL QUESTIONS.
2. TOTAL MARKS = 100.
3. EXAM PAPER (5 PAGES) AND FORMULA SHEET (1 PAGE)

MATERIALS REQUIRED

1. NEW MATHEMATICS TABLES.
2. GRAPH PAPER

WATERFORD INSTITUTE OF TECHNOLOGY

Question 1

(a) Alice, Bob and Carol are three students that took last year's *Discrete Mathematics* exam. Given propositions

- A = "*Alice passed the exam.*"
- B = "*Bob passed the exam.*"
- C = "*Carol passed the exam.*"

Formalise (i.e., express using logical operators) the following statements

(i) "*Alice is the only one passing the exam.*"

(ii) "*Bob passed the exam.*"

(iii) "*All three students passed the exam.*"

(iv) "*At least one student, among Alice, Bob and Carol, passed the exam.*"

(v) "*Exactly one student passed the exam.*"

(10 marks)

(b) Which of the following are well formed propositional formulas? Justify your answers.

(i) $\vee pq$ **(ii)** $p\neg\neg r$

(iii) $p\neg \rightarrow (q \wedge q)$ **(iv)** $(p \wedge \neg q) \vee (q\neg \rightarrow q)$

(4 marks)

(c) Consider the functions defined by the following Python code:
*(recall that // is integer division, and ** is exponentiation (powers).)*

```python
def f(x):
    return x ** 2

def g(x):
    return x + 1

def h(x):
    return x // 2
```

Evaluate the following:

(i) f(5) **(iii)** h(3) **(v)** f(5) + g(7)

(ii) f(g(2)) **(iv)** f(g(h(1))) **(vi)** f(f(f(f(2))))

(6 marks)

(Total 20 marks)

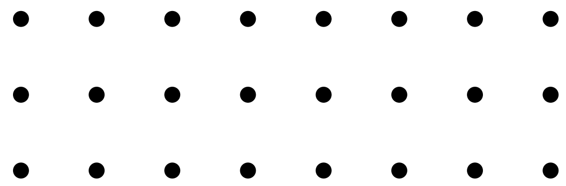
Question 2

(a) Use truth tables to determine whether the proposition

$$(\neg p \vee q) \wedge (q \rightarrow (\neg r \wedge \neg p)) \wedge (p \vee r)$$

is satisfiable. **(5 marks)**

(b) Consider the follow diagram, consisting of three rows of eight dots.



How many

(i) Squares (*Hint: Count* 1×1 *and* 2×2 *sized squares separately.*)

(ii) Rectangles (*Hint: Count cases based on top row of rectangle.*)

(iii) Right angled triangles

can be drawn using the dots as vertices (corners). **(9 marks)**

(c) Draw the following graphs. Classify each as Eulerian, semi-Eulerian, or neither.

(i) The wheel graph, W_4.

(ii) The complete bipartite graph, $K_{4,2}$.

(iii) The Peterson graph.

(6 marks)

(Total 20 marks)

Question 3

(a) Construct sets A and B satisfying the following three properties:

$$A \setminus B = \{4, 9\}, \qquad B \setminus A = \{8\}, \qquad A \cap B = \{1\}.$$

(2 marks)

(b) Consider the *Collatz function* with definition and Python implementation

$$f(n) = \begin{cases} 3n + 1 & n \text{ odd} \\ n/2 & n \text{ even} \end{cases}$$

```python
def f(n):
  if n%2==1:
    return 3 * n + 1
  else:
    return n // 2
```

Also implemented in python are the following functions

```python
def g(n):
  return f(f(f(n)))

def h(n):
  result = [n]
  while n!=1:
    n = f(n)
    result.append(n)
  return result
```

(i) What is the output of `f(7)`? and `f(10)`?

(ii) What is the output of `h(12)`?

(iii) What is the output of `g(10)`?

(iv) For which values of `n` does the expression `g(n)==n` compute to `True` ?

(8 marks)

(c) Let $S = \{1, 2, 3, 4, 5, 6\}$

(i) How many subsets are there of cardinality 4?

(ii) How many subsets of S are there? That is, find $|\mathcal{P}(S)|$.

(iii) How many subsets of cardinality 4 have $\{2, 4, 6\}$ as a subset?

(iv) How many subsets of cardinality 4 contain at least one prime (2, 3, or 5)?

(v) How many subsets of cardinality 4 contain exactly one prime?

(10 marks)

(Total 20 marks)

Question 4

(a) Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$, and $C = \{1, 5, 9\}$. Determine which of the following statements are true. Give reasons for your answers.

(i) $3 \in A$

(ii) $\{3\} \in A$

(iii) $\{3\} \subseteq A$

(iv) $\emptyset \subseteq C$

(v) $\emptyset \in A$

(vi) $A \subseteq A$

(6 marks)

(b) Let R be the relation on the set $A = \{0, 1, 2, 3\}$ given by

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$

(i) Represent R using a digraph.

(ii) Is R reflexive? symmetric? transitive? Justify your answers.

(iii) Is R irreflexive? Justify your answer.

(2 + 6 + 1 = 9 marks)

(c) How many shortest lattice paths start at (2,4) and

(i) end at (20,15)?

(ii) end at (20,15) and pass through (10,6)?

(iii) end at (20,15) and avoid (10,6)?

(1 + 2 + 2 = 5 marks)

(Total 20 marks)

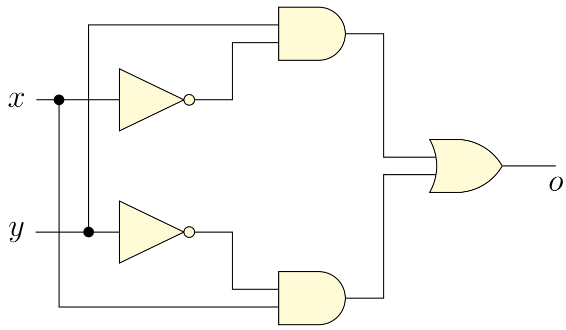
Question 5

(a) Consider the following logical circuit with two inputs, and single output.

(i) Construct a logical expression to represent this circuit.

(ii) Is there an input case for which the output is on?

(iii) Hence, construct a logical circuit or equivalent expression, containing three inputs, x, y, and z, for which the output is on when exactly one input is on.



(2 + 2 + 2 = 6 marks)

(b) Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ given by the table below:

x	1	2	3	4	5
$f(x)$	3	2	4	1	2

(i) Is f injective? Explain.

(ii) Is f surjective? Explain.

(4 marks)

(c) How many 9-bit strings (that is, bit strings of length 9) are there which satisfy each of the following criteria? Explain your answers.

(i) Start with the sub-string 101.

(ii) Have weight 5 (i.e., contain exactly five 1's) and start with the sub-string 101.

(iii) Either start with 101 or end with 11 (or both).

(iv) Have weight 5, and starts with 101 and ends with 11.

(2 + 2 + 3 + 3 = 10 marks)

(Total 20 marks)

Laws of Logic

Logical Connective	Symbol	Python Operator	Precedence	Logic Gate
Negation (NOT)	$\neg$	**not**	Highest	
Conjunctive (AND)	$\wedge$	**and**	Medium	
Disjunctive (OR)	$\vee$	**or**	Lowest	

Basic Rules of Logic

Commutative Laws

$$p \vee q \Leftrightarrow q \vee p \qquad p \wedge q \Leftrightarrow q \wedge p$$

Associative Laws

$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r) \qquad (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Distributive Laws

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \qquad p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

Identity Laws

$$p \vee \mathbf{F} \Leftrightarrow p \qquad p \wedge \mathbf{T} \Leftrightarrow p$$

Negation Laws

$$p \wedge (\neg p) \Leftrightarrow \mathbf{F} \qquad p \vee (\neg p) \Leftrightarrow \mathbf{T}$$

Idempotent Laws

$$p \vee p \Leftrightarrow p \qquad p \wedge p \Leftrightarrow p$$

Null Laws

$$p \wedge \mathbf{F} \Leftrightarrow \mathbf{F} \qquad p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$$

Absorption Laws

$$p \wedge (p \vee q) \Leftrightarrow p \qquad p \vee (p \wedge q) \Leftrightarrow p$$

DeMorgan's Laws

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \qquad \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

Involution Law

$$\neg(\neg p) \Leftrightarrow p$$

Implications and Equivalences

Detachment (Modus Ponens)

$$(p \to q) \wedge p \Rightarrow q$$

Indirect Reasoning (Modus Tollens)

$$(p \to q) \wedge \neg q \Rightarrow \neg p$$

Disjunctive Addition

$$p \Rightarrow (p \vee q)$$

Conjunctive Simplification

$$(p \wedge q) \Rightarrow p \qquad (p \wedge q) \Rightarrow q$$

Disjunctive Simplification

$$(p \vee q) \wedge \neg p \Rightarrow q \qquad (p \vee q) \wedge \neg q \Rightarrow p$$

Chain Rule

$$(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$$

Resolution

$$(\neg p \vee r) \wedge (p \vee q) \Rightarrow (q \vee r)$$

Conditional Equivalence

$$p \to q \Leftrightarrow \neg p \vee q$$

Biconditional Equivalences

$$\begin{aligned}(p \leftrightarrow q) &\Leftrightarrow (p \to q) \wedge (q \to p)\\ &\Leftrightarrow (p \wedge q) \vee (\neg q \wedge \neg q)\end{aligned}$$

Contrapositive

$$p \to q \Leftrightarrow \neg q \to \neg p$$

Question 1

(a)

(i) "*Alice is the only one passing the exam.*"

$$\neg C \land A \land \neg B$$

(ii) "*Bob passed the exam.*"

$$B$$

(iii) "*All three students passed the exam.*"

$$A \land B \land C$$

(iv) "*At least one student, among Alice, Bob and Carol, passed the exam.*"

$$A \lor B \lor C$$

(v) "*Exactly one student passed the exam.*"

$$(A \land \neg B \land \neg C) \lor (\neg A \land B \land \neg C) \lor (\neg A \land \neg B \land C)$$

10 marks $= 5 \times 2$ **marks**

(b)

(i) $\lor pq$
Not well formed. Logical and, $\lor$ is a binary operator, expected infix notation.

(ii) $p\neg\neg r$
Not well formed. Logical not operator, $\neg$ is a unary operator.

(iii) $p\neg \rightarrow (q \land q)$
Not well formed. $\neg$ applied to conditional operator

(iv) $(p \land \neg q) \lor (q\neg \rightarrow q)$
Not well formed. Conditional operator, $\rightarrow$ is a binary operator.

4 marks $= 4 \times 1$**, justification required**

(c)

(i) 25

(ii) 9

(iii) 1

(iv) 1

(v) 3

(vi) $2^{16} = 65,536$

6 marks $= 6 \times 1$

Question 2

(a)

Partial marks for correct parsing of expression, demonstrating ability to compute logical expression, implication of satisfiability definition.

$$\underbrace{\underbrace{(\neg p \vee q)}_{1} \wedge \underbrace{(q \to \underbrace{(\neg r \wedge \neg p)}_{2})}_{3} \wedge \underbrace{(p \vee r)}_{4}}_{E}$$

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \to (\neg r \wedge \neg p)$	$p \vee r$	E
F	F	F	**T**	F	F	**T**	F
F	F	**T**	**T**	F	F	**T**	F
F	**T**	F	F	F	**T**	**T**	F
F	**T**	**T**	F	F	**T**	**T**	F
T	F	F	**T**	F	F	**T**	F
T	F	**T**	**T**	**T**	**T**	F	F
T	**T**	F	**T**	F	**T**	**T**	**T**
T	**T**	**T**	**T**	**T**	**T**	F	F

Since there is (at least) one row in which the output is **True**, the expression is satisfiable.

5 marks

(b)

(i) *Squares*

1×1 Square: $7 \times 2 = 14$. Pick a dot for the top left corner, leaving room for 1×1 square, the other three dots are determined. 2×2 Square: $6 \times 1 = 6$. Pick a dot for the top left corner, leaving room for 2×2 square, the other three dots are determined.

Ans: 20 Squares.

(ii) *Rectangles*

Starting at top row: $2 \times \binom{8}{2} = 56$ rectangles. Pick any two of the eight dots from the top row, order does not matter, and then rectnagle height is either 1 or 2.

Starting at middle row: $1 \times \binom{8}{2} = 28$ rectangles. Pick any two of the eight dots from the middle row, order does not matter, and then rectnagle height is 1.

Ans: 84 Rectangles.

(iii) Right angled triangles

We could count triangles directly but easier to think of any right angled triangle is half of a rectangle (cut diagonally). There are two ways to cut a rectangle diagonally so each rectangle gives rise to four distinct right angled triangles.

Ans: $84 \times 4 = 336$ Right angled triangles.

9 marks $= 3 \times 3$ **marks**

(c)

(i) The wheel graph, W_4.
Is non-Eulerian as it has 4 nodes with odd degree.

(ii) The complete bipartite graph, $K_{4,2}$.
Is Eulerian as is connected and all vertices are of even degree.

(iii) The Peterson graph.
Is non-Eulerian as all 10 vertices are of odd degree.

6 marks

Question 3

(a)

$$A = \{1, 4, 9\}, \qquad B = \{1, 8\}$$

2 marks

(b)

(i) What is the output of f(7)? and f(10)?

f(7) = 22, f(10)=5

(ii) What is the output of h(12)?

$$[12, 6, 3, 10, 5, 16, 8, 4, 2, 1]$$

(iii) What is the output of g(10)?

g(10) = 8

(iv) For which values of n does the expression g(n)==n compute to True ?

4,2, or 1 since this sequence forms a cycle in f.

8 marks

(c)

(i) *How many subsets are there of cardinality 4?*

$\binom{6}{4} = 15$ subsets.

(ii) *How many subsets of S are there? That is, find $|\mathcal{P}(A)|$.*

$2^6 = 64$ subsets.

(iii) *How many subsets of cardinality 4 have $\{2, 4, 6\}$ as a subset?*

$\binom{3}{1} = 3$ subsets. We need to select 1 of the 3 remaining elements to be in the subset.

(iv) *How many subsets of cardinality 4 contain at least one prime number (2, 3, or 5)?*

$\binom{6}{4} = 15$ subsets. All subsets of cardinality 4 must contain at least one prime.

(v) *How many subsets of cardinality 4 contain exactly one prime?*

$\binom{3}{1} = 3$ subsets. Select 1 of the 3 primes. The three non-primes (composite numbers) of S must all be in the set.

10 marks $= 5 \times 2$ **marks**

Question 4

(a)

(reason = any correct, relevant statement)

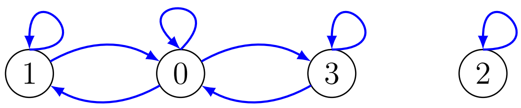
(i) True **(iii)** True **(v)** False

(ii) False **(iv)** True **(vi)** True

6 marks $= 6 \times 1$

(b)

(i) *Represent R using a digraph.*



(ii) *Is R reflexive? symmetric? transitive?*

R is reflexive, i.e. there is a loop at each vertex.

R is symmetric, i.e. the arrows joining a pair of different vertices always appear in a pair with opposite arrow directions.

R is not transitive. This is because otherwise the arrow from 1 to 0 and arrow from 0 to 3 would imply the existence of an arrow from 1 to 3 (which doesn't exist).

(iii) *Is R irreflexive?*

No (+ justification)

Mark breakdown: 2 (digraph) + 6 (3×2)**+1=9**

(c)

(i) *start at (2,4) and end at (20,15)?*

$\binom{18+11}{11} = 34,597,290$ paths. The paths all have length 29 (18 steps right and 11 steps up), we just select which 11 of those 29 should be up.

(ii) *start at (2,4) and end at (20,15) and pass through (10,6)?*

$\binom{8+2}{2}\binom{10+9}{9} = 45 \times 92,378 = 4,157,010$ paths. First travel to (10,6), and then continue on to (20,15).

(iii) *start at (2,4) and end at (20,15) and avoid (10,6)?*

$\binom{29}{11} - \binom{10}{2}\binom{19}{9} = 34,597,290 - 4,157,010 = 30,440,280$ paths.

Remove all the paths found in preceding question.

5= 1+2+2

Question 5

(a)

(i) *Construct a logical expression to represent this circuit.*

$$(y \wedge \neg x) \vee (\neg y \wedge x)$$

(ii) *Is there an input case for which output is on?*

x	y	$y \wedge \neg x$	$\neg y \wedge x$	$(y \wedge \neg x) \vee (\neg y \wedge x)$
F	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**
T	**F**	**F**	**T**	**T**
T	**T**	**F**	**F**	**F**

Output is on when exactly one input is one.

(iii) *Hence, construct a logical circuit or equivalent expression, containing three inputs that will have output on when exactly one input is on.*

Define logical operator $x \uparrow y = (y \wedge \neg x) \vee (\neg y \wedge x)$ then required expression is

$$x \uparrow y \uparrow z$$

Note $x \uparrow y$ is the exclusive or operator.

6 marks = 2 + 2 + 2

(b)

(i) No, element 2 in target set has two incoming arrows.

(ii) Yes, every element in target set has at least one incoming arrow.

4 marks

(c)

(i) *Start with the sub-string 101.*

No constraints on remaining 6 bits, so 2^6.

(ii) *Have weight 5 (i.e., contain exactly five 1's) and start with the sub-string 101.*

In the remaining 6 bit, three of which must be 1, so $\binom{6}{3} = 20$

(iii) *Either start with* 101 *or end with* 11 *(or both).*

Start with 101: No constraints on remaining 6 bits, so 2^6.

Ends with 11: No constraints on preceeding 7 bits, so 2^7.

Start with 101 and ends with 11: No constraints on middle 4 bits, so 2^4.

Ans (remove double counting): $2^6 + 2^7 - 2^4 = 64 + 128 - 16 = 176$

(iv) *Have weight 5, and starts with* 101 *and ends with* 11.

The middle 4 digits must have weight 1 so that the entire string has weight 5. Hence have $\binom{4}{1} = 4$ possibilities.

10 marks = 2 + 2 + 3 + 3