Discrete Mathematics

Topic 03 — Sets

Lecture 01 — Sets

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Outline

- Definition of a set
- Relationships between sets
- Set operations

Outline

Sets

Sets are fundamental discrete structures that form the basis of more complex discrete structures such as graphs. relational data bases, etc.

Definition 1 (Set)

A set is an unordered collection of distinct well-defined objects (called elements).

The Set definition is a perfect example of how all definitions need to be read.

Take care to read and parse the definition of sets carefully:

- "unordered" means order is not important.
  - So two sets with the same elements but in different order are equal.
- "collection" means zero or more items.
- "distinct" means elements are unique.
  - So adding an element more than once has no effect.
- "well-defined" means that we have a clear rule for deciding what is in the set and what is not in the set.
  - So the "set of all healthy foods" is not a set.

Notation

- We use braces "$\{$" and "$\}$" to enclose the elements of a set.
- We write $\underbrace{x \in A}_{\text{"}x\text{ is an element of }A\text{"}}$ if set A contains element x, and $\underbrace{x \notin A}_{\text{"}x\text{ is not an element of }A\text{"}}$ otherwise.
- The empty set, or null set, is denoted by $\{\}$ or $\emptyset$.

Enumeration

We can define a set by enumerating (listing) its elements:

- Set of decimal digits

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \qquad 1 \in D \qquad 15 \notin D$$

- Set of vowels

$$V = \{\text{"}a\text{"}, \text{"}e\text{"}, \text{"}i\text{"}, \text{"}o\text{"}, \text{"}u\text{"}\}$$

- Set of letters in the English alphabet

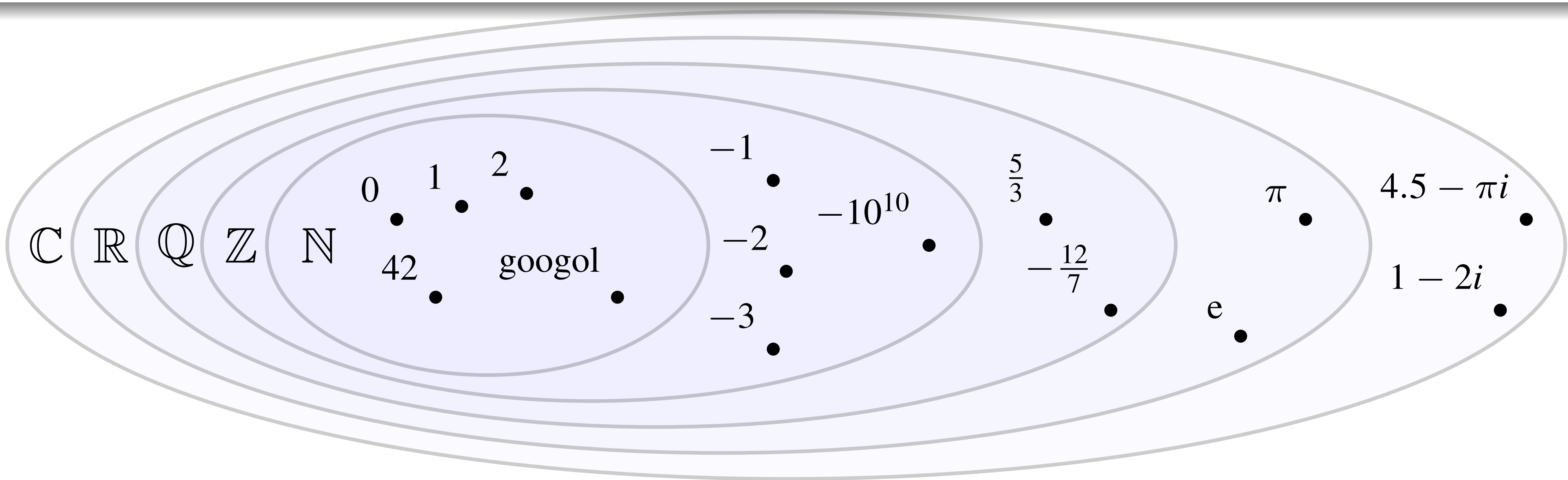
$$L = \{\text{"}a\text{"}, \text{"}b\text{"}, \text{"}c\text{"}, \ldots, \text{"}z\text{"}\}$$

- Set of positive integers

$$\mathbb{P} = \{1, 2, 3, 4, \ldots\}$$

The three consecutive "dots" are called an ellipsis. We use them when it is clear what elements are included but not listed.

Number Sets



- $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$ (Natural Numbers)
  - Contains zero and the positive integers.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (Integers)
- $\mathbb{Q}$ (Rational Numbers)
  - Any number that can be expressed as a fraction.
- $\mathbb{R}$ (Real Numbers)
  - Rational and irrational numbers
- $\mathbb{C}$ (Complex)

Set Builder Notation

Another way of describing sets is to use set builder notation.
For example, we could define the set of rational numbers as

$$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$$

where

- a/b indicates that a typical element of the set is a "fraction."
- The vertical line, "|", is read as "such that" or "where".
  Note: Many authors use a colon, ":" instead of the vertical line "|".
- $a, b \in \mathbb{Z}$ is an abbreviated way of saying a and b are integers.
- Commas are usually read as "and."

The above mathematical statement can be read as

$\mathbb{Q}$ is the set of things that can be expressed as a/b where a and b are integers and $b \neq 0$.

Review Exercises 1 (Builder Notation)

Question 1:
List any four elements of each of the following sets:

a) $\{k \in \mathbb{N} \mid k - 1 \text{ is a multiple of } 7\}$
b) $\{x \mid x \text{ is a fruit and its skin is normally eaten}\}$
c) $\{x \in \mathbb{Q} \mid \frac{1}{x} \in \mathbb{Z}\}$
d) $\{2n \mid n \in \mathbb{Z}, n < 0\}$
e) $\{s \mid s = 1 + 2 + \cdots + n, n \in \mathbb{N}, n \geq 1\}$

Question 2:
List all elements of the following sets:

a) $\{\frac{1}{n} \mid n \in \{3, 4, 5, 6\}\}$
b) $\{\alpha \in \text{ the alphabet } \mid \alpha \text{ precedes F}\}$
c) $\{-k \mid k \in \mathbb{N}\}$
d) $\{n^2 \mid n = -2, -1, 0, 1, 2\}$
e) $\{n \in \mathbb{P} \mid n \text{is a factor of } 24\}$

Question 3:
Describe the following sets using set-builder notation.

a) $\{5, 7, 9, \ldots, 77, 79\}$
b) the rational numbers that are strictly between -1 and 1
c) the even integers
d) $\{-18, -9, 0, 9, 18, 27, \ldots\}$

Cardinality

Definition 2 (Cardinality)

Let A be a finite set. The number of different elements in A is called its cardinality and is denoted by $|A|$.
If $|A|$ is finite then A is said to be a finite set, otherwise it is an infinite set.

- The empty set, $\emptyset$, has cardinality zero, i.e.,

$$\boxed{|\emptyset| \quad = \quad 0}$$

- A singleton set is a set that has only one element.
- Note the difference between $\{a\}$ and a. The braces indicate that the object is a set, while a without the brace is an element.
- This difference also applies to the empty set, in that*

$$\boxed{\emptyset \quad \neq \quad \{\emptyset\}}$$

*If this is confusing, think of a bag containing a empty bag. Is the first bag empty?

Examples (Membership and Cardinality)

- Let $A = \{1, \{2\}, \{\{3\}\}\}$. Then

$$1 \in A \qquad \begin{matrix} 2 \notin A \\ \{2\} \in A \end{matrix} \qquad \begin{matrix} 3 \notin A \\ \{3\} \notin A \\ \{\{3\}\} \in A \end{matrix}$$

- Let $A = \{23, 24, \ldots, 37, 38\}$.
  Then $|A| = 38 - 23 + 1 = 16$

Note the "+1".

- Let $B = \{1, \{2, 3, 4\}, \emptyset\}$.
  Then $|B| = 3$

B has three elements, the number 1, the set $\{2, 3, 4\}$, the empty set $\emptyset$.

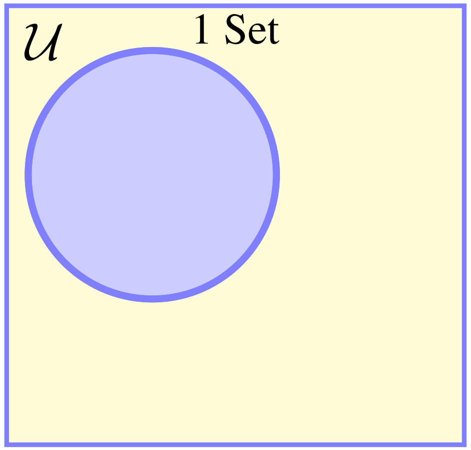
- Let $C = \{n \mid n < 100, n \text{ is prime}\}$.
  Then $|C| = 25$

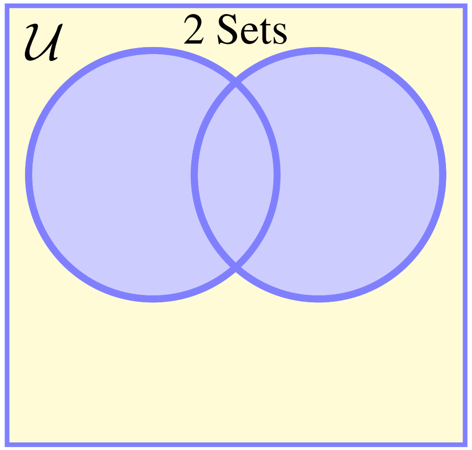
List and count the primes $\{2, 3, 5, 7, \ldots, 97\}$.

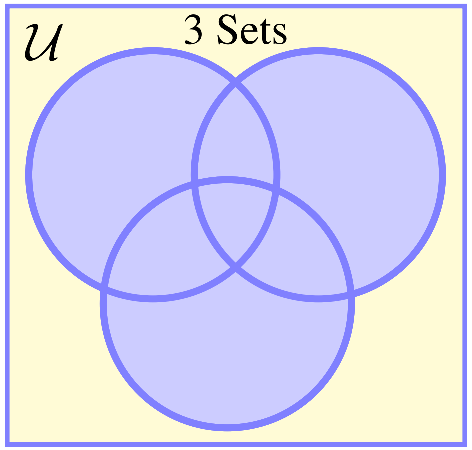
Venn Diagrams

A Venn diagram is a graphical representation of sets that is effective when dealing with the relationship between a few sets. We use

- (overlapping) ovals to represent the individual sets.
- a rectangle to represent the universal set — a set
- an element is placed in exactly one region, based on which set, if any, it is a member of,







Python Implementation

defining_sets .py

```
A = set()
print(A)

A.add(3)
print(A)

A.add(3)
print(A)

A.add(2)
print(A)

A.add("Hello")
print(A)

A.add("All")
print(A)

B = {"This", "is", "a", "set"}
print(B)
```

- Create an empty set using **set**()
- Note: braces {} are used to represent a dictionary.
- Add individual items using add
- Items are unique so repeated add has no effect.
- Sets can contain ‘anything’. Need quotes around strings
- Note order is not important!
- Can also define a set with elements

```
set()
{3}
{3}
{2, 3}
{2, 3, 'Hello'}
{2, 3, 'Hello', 'All'}
{'This', 'set', 'a', 'is'}
```

Outline

Equal Sets

Definition 3 (Equal Sets)

Two sets A, and B are equal iff they contain the same elements.

Expressing this in predicate logic terms we have

$$\forall x \left[\underbrace{x \in A \leftrightarrow x \in B}_{x \text{ is in } A \text{ iff } x \text{ is in } B} \right]$$

which in terms of the IFTHEN operator is

$$\forall x \left[\underbrace{(x \in A \rightarrow x \in B)}_{\text{if } x \text{ is in } A \text{ then } x \text{ is in } B} \wedge \underbrace{(x \in B \rightarrow x \in A)}_{\text{if } x \text{ is in } B \text{ then } x \text{ is in } A} \right]$$

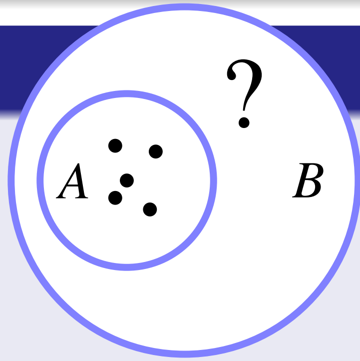
Example

The sets

- $A = \{1, 3, 5, 7, \ldots\}$
- $B = \{n \mid n \in \mathbb{N}, \exists k(n = 2k + 1, k \in \mathbb{N})\}$
- $C = \{n \mid n \in \mathbb{N}, \text{remainder of } n \div 6 \in \{1, 3, 5\}\}$

are equal, despite the apparent difference in their definitions.

Subsets and Proper Subsets



Definition 4 (Subset)

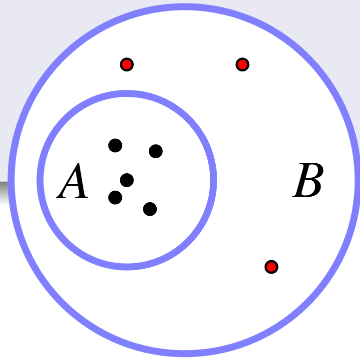
Set A is said to be a subset of B and we write

$$A \subseteq B$$

if and only if every element of A is also an element of B.

If, in addition, B contains **at least one** element not in A we say that A is a proper subset of B, and write

$$A \subset B$$



- In terms of predicate logic we have
  - A is a subset of B

$$A \subseteq B \quad \Longleftrightarrow \quad \forall x \big[x \in A \to x \in B \big]$$

  - A is a proper subset of B

$$A \subset B \quad \Longleftrightarrow \quad \forall x \big[x \in A \to x \in B \big] \wedge \exists x \big[x \in B, x \notin A \big]$$

- Note that the operators $\subset$ and $\subseteq$ play a similar role to $<$ and $\leq$.

The Empty Set is a Subset of Every Set

Example

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3\}$ and $D = \{7, 8, 9\}$. Determine which of the following are true, false, or meaningless.

a) $A \subset B$.
b) $B \subset A$.
c) $B \in C$.
d) $C \in A$.

e) $\emptyset \in A$.
f) $\emptyset \subset A$.
g) $A < D$.
h) $3 \in C$.
i) $3 \subset C$.
j) $\{3\} \subset C$.

Power Set

If you collect all the subsets of set S into a new set, we get a set of sets …

Definition 5 (Power Set)

The power set of a set S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S.

The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set.

Theorem 6 (Size of the power set)

Let S be a set such that $|S| = n$, then

$$|\mathcal{P}(S)| = 2^n$$

Example

Example 7

Let $A = \{a, b, c\}$, then the power set is

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

- Note that the empty set and the set itself are elements of the power set.
  - The empty set is a subset of every set . . .

    $$\boxed{\emptyset \quad \subseteq \quad A}$$
  - Every set is a subset of itself . . .

    $$\boxed{A \quad \subseteq \quad A}$$
- Note that while $b \in A$, it is wrong to say $b \in \mathcal{P}(A)$. $b \neq \{b\}$
- However $\{b\} \in \mathcal{P}(A)$ since $\{b\} \subseteq A$.
- $\mathcal{P}(\emptyset) = \{\emptyset\}$

Python Implementation

- Create two sets, A and B
- Test for subset using command issubset
- Test for superset using command issuperset
- Every set is a subset of itself
- The empty set is a subset of every set

```
A = {4,2}
B = {1,2,3,4,5}
print("A =", A)
print("B =", B)

print("Is A a subset of B?", A.issubset(B))
print("Is B a subset of A?", B.issubset(A))

print("Is A a superset of B?", A.issuperset(B))
print("Is B a superset of A?", B.issuperset(A))

print(A.issubset(A))

A = set()
print(A.issubset(A))
```

```
A = {2, 4}
B = {1, 2, 3, 4, 5}
Is A a subset of B? True
Is B a subset of A? False
Is A a superset of B? False
Is B a superset of A? True
True
True
```

Outline

Intersection

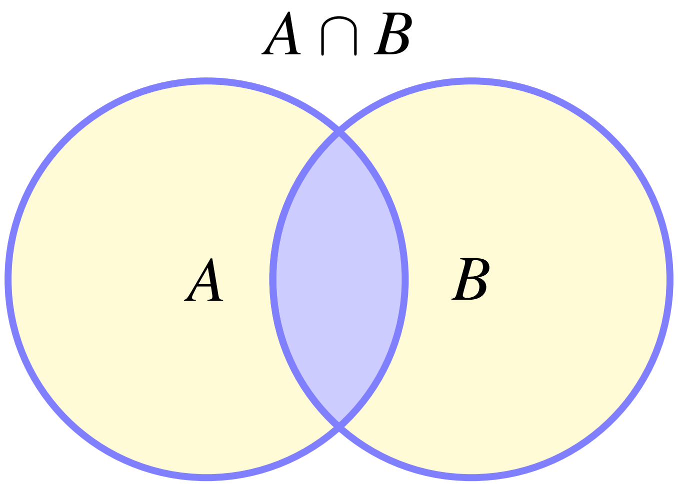
Definition 8 (Intersection)

The intersection of two sets, A and B, denoted by $A \cap B$, is the set that contains all elements that are elements of both A and B. We write

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Properties:

- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap B = A \Rightarrow A \subseteq B$
- $A \cap \emptyset = \emptyset$
- Acts like the logical AND



Two sets are said to be disjoint if their intersection is empty

$$A \cap B = \emptyset$$

Union

Definition 9 (Union)

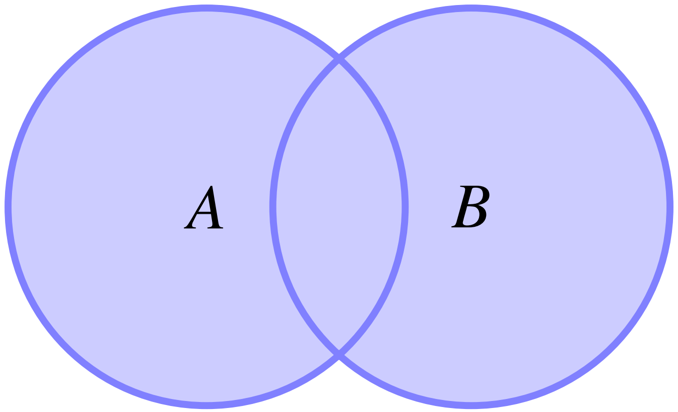
The union of two sets, A and B, denoted by $A \cup B$, is the set that contains all elements that are elements of A or B or both. We write

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Properties:

- $A \cup B = B \cup A$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $A \cup B = A \Rightarrow B \subseteq A$
- $A \cup \emptyset = A$
- Acts like the logical OR
- $|A \cup B| = |A| + |B| - |A \cap B|$

$A \cup B$



Set Difference

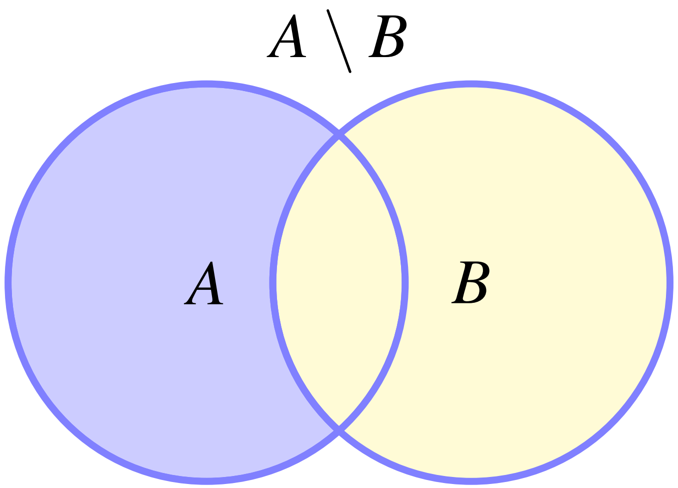
Definition 10 (Set Difference)

The set difference of two sets, A and B, denoted by $A \setminus B$, is the set that contains all elements that are in A but not in B. We write

$$A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\}$$

Properties:

- $A \setminus B \neq B \setminus A$
- $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$
- $A \setminus B = A \Rightarrow B \cap A = \emptyset$
- $A \setminus B = \emptyset \not\Rightarrow A = B$
- $A \setminus B = \emptyset \Rightarrow A \subseteq B$



Symmetric Difference

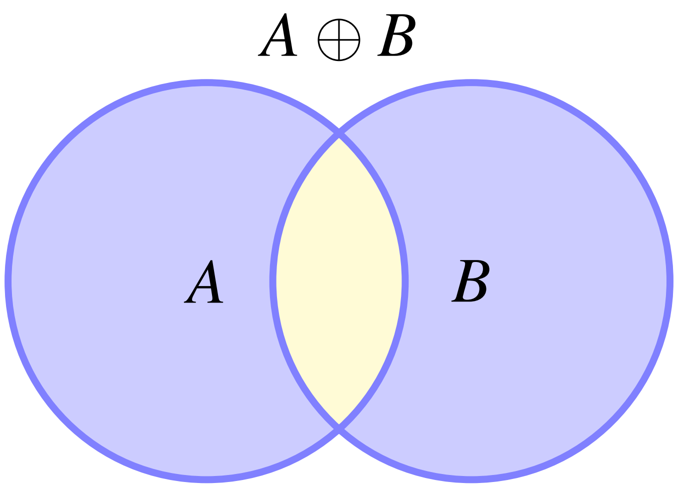
Definition 11 (Symmetric Difference)

The symmetric difference of two sets, A and B, denoted by $A \oplus B$, is the set that contains all elements that are in A or in B but not both. We write

$$A \oplus B = \{x \mid (x \in A \cup B) \wedge (x \notin A \cap B)\}$$

Properties:

- $A \oplus B = B \oplus A$
- $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
- $A \oplus B = A \Rightarrow B = \emptyset$
- $A \oplus B = \emptyset \Rightarrow A = B$



Set Complement

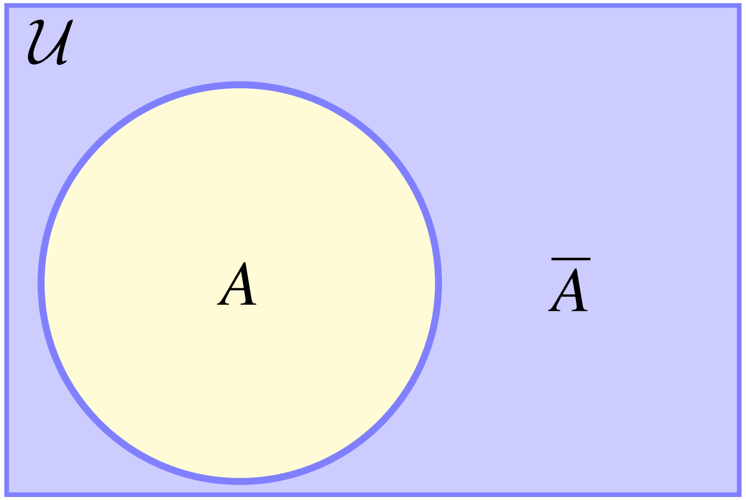
Often when dealing with sets, we will have some understanding as to what "everything" is. Perhaps we are only concerned with natural numbers. In this case we would say that our universe is $\mathbb{N}$. We denote this universe by $\mathcal{U}$. Given this context, we might wish to speak of all the elements which are not in a particular set.

Definition 12 (Complement)

The complement of a set A, denoted by $\overline{A}$, is the set containing all elements not in A.

Properties:

- $\overline{A} \cap A = \emptyset$
- $\overline{A} \cup A = \mathcal{U}$
- $A \cap \overline{B} = A \setminus B$



Example

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3\}$ and $D = \{7, 8, 9\}$. If the universe is $\mathcal{U} = \{1, 2, \ldots, 10\}$, find:

a) $A \cup B$.

b) $A \cap B$.

c) $B \cap C$.

d) $A \cap D$.

e) $\overline{B \cup C}$.

f) $A \setminus B$.

g) $(D \cap \overline{C}) \cup \overline{A \cap B}$.

h) $\emptyset \cup C$.

i) $\emptyset \cap C$.

Cartesian Product

Given two sets we often need to construct a set of all possible pairing of elements from both sets.

Definition 13 (Cartesian Product)

The Cartesian product or two sets A and B, denoted by $A \times B$ is the set of ordered pairs where the first member is an element of the first set and the second member is an element of the second.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

- When we take the Cartesian product of a set, say A, by itself, we write A^2.
- The 2D plane is the Cartesian product of the set of real numbers ($\mathbb{R}$) with itself.

Example

Example 14

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$.

a) Find $A \times B$ and $A \times A$.
b) How many elements do you expect to be in $B \times B$?
c) Is $A \times B = B \times A$?

Solution.

a) $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$.
   $A \times A = A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
b) $|B \times B| = 9$. There will be 3 pairs with first coordinate 3, three more with first coordinate 4, and a final three with first coordinate 5.
c) No. Cartesian product generates ordered pairs.

Python Implementation

set_operations.py

```python
A = {1,2,3,4,5,6}
B = {2,4,6}
C = {1,2,3}
D = {7,8,9}
U = set(range(1,11))

ans = A.union(B)
print("A u B = ", ans)

ans = A.intersection(B)
print("A n B = ", ans)

ans = A.difference(B)
print("A \ B = ", ans)

ans = U.difference(A)
print("A complement", ans)

N = set()
print(N.union(C)==C)
print(N.intersection(C)==N)
```

- Create A, B, C, D and universal set U
- Use union for set union
- Use intersection for set intersection
- Use difference for set difference. Also have symmetric_difference
- Complement is implemented using set difference.
- Properties of operations hold as expected.

```
A u B =  {1, 2, 3, 4, 5, 6}
A n B =  {2, 4, 6}
A \ B =  {1, 3, 5}
A complement {8, 9, 10, 7}
True
True
```

Outline

Proving Equivalence

- We have mentioned a number of properties of set operations. Hopefully, some are obvious. How can we prove the others?
- We want some technique/process that will allow us to
  - Determine whether one set is a subset, a proper subset of another set.
  - Determine whether two sets are equal.
- We will use two† approaches:
  - on predicate logic based on the subset relationships

$$A \subseteq B \iff \forall x\big[x \in A \rightarrow x \in B\big]$$

$$A \subset B \iff \forall x\big[x \in A \rightarrow x \in B\big] \wedge \exists x\big[x \in B, x \notin A\big]$$

and

$$A = B \iff (A \subseteq B) \wedge (B \subseteq A)$$

  - constructing a membership table.

†Well three approaches if we count using IPython to do the donkey work for us.

Example

Example 15

Let $A = \{x \mid x \text{ is even}\}$ and $B = \{x \mid x \text{ is a multiple of } 3\}$ and $C = \{x \mid x \text{ is a multiple of } 6\}$. Prove

$$A \cap B = C$$

Proof.

<u>Proving $A \cap B \subseteq C$...</u>

Let $x \in A \cap B$. Then x is a multiple of 2 and x is a multiple of 3. Therefore we can write

$$x = (2)(3)k \qquad \text{for some integer } k$$

Hence x is a multiple of 6 and therefore $x \in C$.

<u>Proving $C \subseteq A \cap B$...</u>

Let $x \in C$. Then x is a multiple of 6 and so $x = 6k$ for some integer k, i.e.,

$$x = 6k = (2)(3)k$$

Therefore x is a multiple of 2 and a multiple of 3, and so $x \in A \cap B$. □

Example

Example 16 (Using Membership Tables)

Prove

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

Proof.

A membership table lists all possibilities of whether an element is in some sets or not . . .

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
F	**F**	**F**	**F**	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**F**	**T**	**T**	**T**	**F**	**T**
F	**T**	**F**	**F**	**T**	**T**	**F**	**T**	**T**
F	**T**	**T**	**F**	**T**	**T**	**F**	**F**	**T**
T	**F**	**F**	**F**	**T**	**F**	**T**	**T**	**T**
T	**F**	**T**	**F**	**T**	**F**	**T**	**F**	**T**
T	**T**	**F**	**F**	**T**	**F**	**F**	**T**	**T**
T	**T**	**T**	**T**	**F**	**F**	**F**	**F**	**F**

Columns are identical, so given identity is **True**. □