Discrete Mathematics

Topic 02 — Methods of Mathematical Proof

Lecture 02 — Proof by Contrapositive and by Cases

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Outline

- Proof by Contrapositive
- Proof by Cases

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Proof by Contrapositive

Proof by Contrapositive

In a proof by contrapositive argument you prove the contrapositive of the claim rather than the claim itself.

Proof by Contrapositive (Formal Structure)

Given claim

$$P \implies Q$$

the contrapositive (and logically equivalent claim) is

$$\neg Q \implies \neg P$$

1. Assume $\neg Q$.
2. Demonstrate that $\neg P$ must follow from $\neg Q$.

Please, please, …, pretty please don't confuse this with proof by contradiction (covered later).

Example

Example 1

If x^2 is odd then x must be odd.

(by contrapositive)*.

The contrapositive is

If x is even, then x^2 is even.

We assume x is even. Hence we can write $x = 2k$ for some integer k. Now

$$x^2 = (2k)^2 = 4k^2 = \underbrace{2\,\underbrace{(2k^2)}_{\text{integer}}}_{\text{even integer}}$$

Hence the contrapositive is true, and so is the original statement. □

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Proof by Cases

Proof by Cases

In a Proof by cases argument you

- List of of the possible cases and analyse each separately.
- Need to ensure that the cases are exhaustive — cover all possibilities

Proof by Cases (Formal Structure)

Given claim

$$P \implies Q$$

1. Show that there exist a number of distinct cases $C_1, C_2, \ldots$ such that whenever P is true then at least one of the cases must be true.
2. Then, for each case, C, in $C_1, C_2, \ldots$,
   1. Assume case C.
   2. Demonstrate that Q must follow from C.

Example 2 I

Example 2

In a cave you find three boxes. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are:

Only one message is true; the other two are false. Which box has the gold?

- Notice that I changed the question to “Which box has the gold?”. I could have left it as “Prove that the gold is in box A.” since, for this problem the two versions are equivalent.

Example 2 II

In a proof by cases, there are three cases based on where the gold is located. In each case we check the truth value of the three messages[†]

Gold is in box A

Gold is in box B

Gold is in box C

[†] You might complain that in the direct proof we did earlier building a truth table is really a proof by cases. You would be correct.

Example 2 II

In a proof by cases, there are three cases based on where the gold is located. In each case we check the truth value of the three messages[†]

Gold is in box A

A: "*The gold is not here*"
B: "*The gold is not here*"
C: "*The gold is in box B*"

Gold is in box B

A: "*The gold is not here*"
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In a proof by cases, there are three cases based on where the gold is located. In each case we check the truth value of the three messages[†]

Gold is in box A

A: "*The gold is not here*" **F**
B: "*The gold is not here*" **T**
C: "*The gold is in box B*" **F**

Gold is in box B

A: "*The gold is not here*"
B: "*The gold is not here*"
C: "*The gold is in box B*"

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C: "*The gold is in box B*" **F**

Exactly one message true? ✔

Gold is in box B

A: "*The gold is not here*"
B: "*The gold is not here*"
C: "*The gold is in box B*"

Gold is in box C

A: "*The gold is not here*"
B: "*The gold is not here*"
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C: "*The gold is in box B*"	**F**	

Gold is in box B

A: "*The gold is not here*"	**T**
B: "*The gold is not here*"	**F**
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A: "*The gold is not here*"
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Exactly one message true? ✔

Gold is in box B

A: "*The gold is not here*" **T**
B: "*The gold is not here*" **F**
C: "*The gold is in box B*" **T**

Exactly one message true? ✘

Gold is in box C

A: "*The gold is not here*"
B: "*The gold is not here*"
C: "*The gold is in box B*"

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Exactly one message true? ✔

Gold is in box B

A: "*The gold is not here*" **T**
B: "*The gold is not here*" **F**
C: "*The gold is in box B*" **T**

Exactly one message true? ✘

Gold is in box C

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Exactly one message true? ✔

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So in order that exactly one message is true, the gold must be in box A.

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Example 3 I

Example 3

Every group of 6 minions includes a group of 3 minions who all know each other or a group of 3 minions who are mutual strangers.

Call one of the minions Bob. There are five others. Either Bob knows three of them, or he does not know three of them.

CASE 1: *Bob knows three of the five others …*

Say that Bob knows three of the five others. Of those five minions either there exists two minions who know each other or no two know each other.

CASE 1.1: *Within the three minions, there exists two who know each other …*

Then those two and Bob form a mutually acquainted threesome.

CASE 1.2: *No two of the three minions know each other …*

Then any three of the five minions are a mutually unacquainted threesome.

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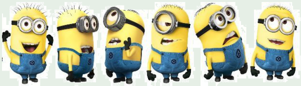

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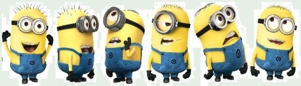

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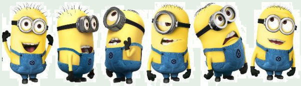

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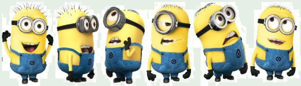

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Example 3 II

CASE 2: *Bob does not know three of the five others ...*

CASE 2.1: *No two of the three minions know each other ...*

Then those two and Bob form a mutually unacquainted threesome.

CASE 2.2: *All pairs within the three minions know each other ...*

Then any three of the five minions are a mutually acquainted threesome.

We have covered all possibilities, and in every instance come up either with a mutually acquainted threesome or a mutually unacquainted threesome.

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Examples

a) Prove that for any integer n, the number $(n^3 - n)$ is even.

b) Prove that every prime number greater than 3 is either one more or one less than a multiple of 6. Hint. Prove the contrapositive by cases.

c) Let a, b, c, d be integers. If $a > c$ and $b > c$, then $\max(a, b) - c$ is always positive.