Enhanced Vector Perturbation Precoding based on Adaptive Query Points

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***Abstract*—In current vector perturbation (VP) precoding architecture, the optimum perturbation vector is found with a closest lattice vector search for a given query point. In this work, we show that the query point should be judiciously chosen such that the effective noise power is minimized. The reduced noise power results in a better error rate performance for VP. The crux in the design is to decode an integer-multiple of lattice point within a modulo lattice architecture, where the integer-multiple can be a prime or a product of primes. Simulations show that around 2 dBs' performance gain can be observed even in the small-scale systems.**

***Index Terms*—vector perturbation, lattice, closest point**

I. Introduction

In view of the growing demand for high data-rate wireless communications, precoding techniques in multiple-input multiple-output (MIMO) systems have been widely investigated in the past two decades. Although "dirty paper coding" (DPC) is now known to achieve the whole capacity region of a MIMO broadcast channel, its high processing complexity motivates the search for other alternatives. These alternatives are categorized into two types: linear and non-linear precoding. Popular linear precoding techniques include zero-forcing (ZF) and minimum mean square error (MMSE), though these channel inversion-based methods perform poorly at all singal-to-noise ratios (SNRs). Non-linear techniques are mainly vector perturbation (VP) precoding [1], and Tomlinson-Harashima precoding (THP) [2]. Since THP follows the inherent principle of successive interference cancellation, the diversity order remains one. Hence it suffices to investigate VP for understanding the best performance of non-linear techniques.

Much work has been done on the development of VP. I) Rate analysis: [3] shows the achievable information rates based on the modulo-lattice-additive-noise technique, while [4] lower bounds the rates using the second moment of a convex body. II) Low-complexity alternatives with trade-offs: The optimization objective of VP requires to solve a closest vector problem (CVP) in a random lattice, which incurs prohibitive computational complexity. To address this issue, [5] exploits bounds on the optimization space, [6] considers lattice reduction-aided methods that maintains the diversity order, while [7] combines both and exploits approximate message passing. III) Improving the fundamental performance of VP: This type of work is mainly based on a minimum mean square error (MMSE) criterion [8]. Note that such an MMSE approach involving effective noise can be applied to tackle the imperfect channel state information (CSI) scenario.

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In this paper, we aim to further improve the fundamental performance of VP by optimizing the strategy in transceiver designs. The new scheme differs from conventional adaptive schemes (e.g., [5]) which control the search space, here it is the query point that becomes adaptive. The crux of our method is that the adaptive scaling factor has to be invertible in finite fields. Based on this, it would induce multiple CVP instances. By choosing the instance that has the minimal Euclidean norm, the effective noise in the received-equations of users can be reduced to a large extent. We also show that the scaling factor can be a product of primes, and the implementation can be associated with MMSE-based VP (actually our scheme can be combined with most existing VP variants). The developed adaptive MMSE-based VP, to our knowledge, serves as the best precoder under the VP architecture.

The rest of the paper is organized as follows. Section II consists of system model and introduction to VP precoding. In Section III, we describe the new precoding scheme and analyze its design criterion and feasible models. Simulation results are presented in Section IV and the last section concludes this paper.

II. Preliminaries

A. System Model

Consider a MIMO broadcast channel where the base station is equipped with m transmitting antennas to broadcast messages to n individual users, each with only one antenna. For

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notation convenience we only consider symbols and vectors from the real space. The observed signals at users 1 to n are collectively expressed as as a vector:

$$\mathbf{r} = \mathbf{B}\mathbf{t} + \mathbf{w}. \tag{1}$$

Here $\mathbf{t} \in \mathbb{R}^m$ is the transmitted signal vector, $\mathbf{w} \in \mathbb{R}^n$ is the additive noise vector with its entries admitting a Gaussian distribution $\mathrm{N}(0, \sigma_w^2)$, and $\mathbf{B} \in \mathbb{R}^{n\times m}$ denotes a channel matrix whose entries admit $\mathrm{N}(0,1)$.

B. Vector Perturbation and CVP

Vector perturbation is a non-linear precoding technique that aims to minimize the transmitted power that is associated with the transmission of a certain data vector [1], [9]. With perfect channel knowledge at the base station, the transmitted signal $\mathbf{t}$ is designed to be a truncation of the channel inversion precoding :

$$\mathbf{t} = \mathbf{B}^{\dagger}(\mathbf{s} - p\mathbf{x}), \tag{2}$$

where $\mathbf{x} \in \mathbb{Z}^n$ is an integer vector to be optimized, $\mathbf{s} \in \mathcal{M}^n$ is the symbol vector, and $\mathbf{B}^{\dagger}$ denotes the inverse/pseudoinverse of $\mathbf{B}$. With scaling and shifting, the constellation can be written as $\mathcal{M} = \mathbb{F}_p = \{0, \ldots, p-1\}$ where $p > 1$ is a positive integer. Note that when extending the scheme to the complex space, the constellation can be similarly written as the quotient set of either Gaussian integers or Eisensten integers.

Assume that the transmitted signal has a unit power, and let $E \triangleq \|\mathbf{t}\|$ be a normalization factor. Then the signal vector at users is represented by

$$\mathbf{r} = (\mathbf{s} - p\mathbf{x})/E + \mathbf{w}. \tag{3}$$

Let $\mathbf{r}' = E\mathbf{r}$, $\mathbf{w}' = E\mathbf{w}$, since $p\mathbf{x} \mod p = \mathbf{0}$, the signal vector $\mathbf{r}'$ is then quantized to integers and then modulo p:

$$\lfloor \mathbf{r}' \rceil \mod p = \lfloor \mathbf{s} + \mathbf{w}' \rceil \mod p, \tag{4}$$

where $\lfloor\cdot\rceil$ denotes rounding to the nearest integer. From (4), we can see that if $|\bar{w}_i'| < \frac{1}{2}$ for all i, where $\mathbf{w}' \in \mathrm{N}(\mathbf{0}, \sigma_w^2 E\mathbf{I}_n)$, then $\mathbf{s}$ can be faithfully recovered.

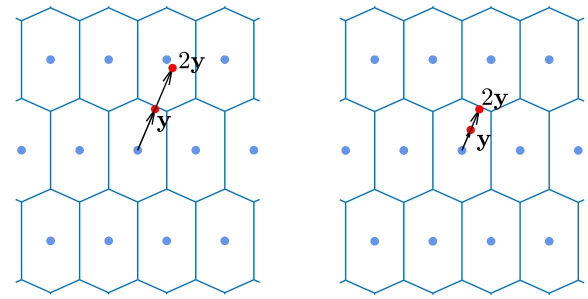
To decrease the decoding error probability which is dominated by E, the transmitter has to address the following optimization problem:

$$\min_{\mathbf{x}\in\mathbb{Z}^n} \left\|\mathbf{B}^{\dagger}(\mathbf{s} - p\mathbf{x})\right\|^2. \tag{5}$$

Define $\mathbf{y} = \mathbf{B}^{\dagger}\mathbf{s} \in \mathbb{R}^m$, $\mathbf{H} = p\mathbf{B}^{\dagger} \in \mathbb{R}^{m\times n}$, then (5) represents a CVP instance:

$$\mathbf{x}^{\mathrm{cvp}} = \arg\min_{\mathbf{x}\in\mathbb{Z}^n} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2, \tag{6}$$

in which $\mathcal{L}(\mathbf{H}) \triangleq \{\mathbf{v} \mid \mathbf{v} = \sum_{i=1}^{n} x_i \mathbf{h}_i\,;\, x_i \in \mathbb{Z}\}$ is the associated lattice. This CVP generally cannot be solved in polynomial time because: the data symbols are optimized over $\mathbb{Z}^n$ rather than over a small finite constellation, the distance distribution from $\mathbf{y}$ to lattice $\mathcal{L}(\mathbf{H})$ is unknown, and the lattice basis $\mathbf{H}$ is sometimes far from being orthogonal.



(a) The query $2\mathbf{y}$ to closer to a lattice point. (b) The query $\mathbf{y}$ to closer to a lattice point.

Fig. 1: The residue distance based on scaled query point.

III. Adaptive Framework

Here we present two motivating examples. The first one shows the distance from $\mathbf{y}$ to $\mathcal{L}(\mathbf{H})$ can sometimes be significantly larger than that of $k\mathbf{y}$ to $\mathcal{L}(\mathbf{H})$ where k is an integer greater than 1, while the second one emphasizes the optimal k depends on the given instance.

Example 1. Consider the following basis $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4]$ with

$$\begin{aligned}
\mathbf{h}_1 &= \begin{bmatrix} k & 0 & 0 & 0 & 0 \end{bmatrix}^{\top} \\
\mathbf{h}_2 &= \begin{bmatrix} 0 & k & 0 & 0 & 0 \end{bmatrix}^{\top} \\
\mathbf{h}_3 &= \begin{bmatrix} 0 & 0 & k & 0 & 0 \end{bmatrix}^{\top} \\
\mathbf{h}_4 &= \begin{bmatrix} 0 & 0 & 0 & k & 0 \end{bmatrix}^{\top}
\end{aligned}$$

and the query point $\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & \varepsilon \end{bmatrix}^{\top}$, $\varepsilon \to 0$. For any $\mathbf{x} \in \mathbb{Z}^4$, the distance function is lower bounded by

$$\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 \geq \|\mathbf{y}\|^2 = 4 + \varepsilon^2.$$

However, if we scale the query point to get $k\mathbf{y}$, then the residue vector becomes

$$k\mathbf{y} - \mathbf{h}_1 - \mathbf{h}_2 - \mathbf{h}_3 - \mathbf{h}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & k\varepsilon \end{bmatrix}^{\top},$$

whose Euclidean norm $k^2\varepsilon^2 \to 0$ can be arbitrarily small.

Example 2. The second example is explained using Fig. 1. In the two sub-figures, we observe that either the scaled or nonscaled query point $\mathbf{y}$ can be closer to the lattice, where $2\mathbf{y}$ is closer in Fig 1-(a) while $\mathbf{y}$ is closer in Fig 1-(b).

Note that reducing the residue distance is critical in VP, as the distance from the query point to the lattice basis controls the effective noise power in the decoding step. As a result, if we judiciously design the perturbation operation based on another query point $k\mathbf{y}$ when its residue distance is smaller, the error rate performance of the whole scheme can be improved.

Now we present a scheme that incorporate the above idea while ensuring the messages are decodable.

A. The New Vector Perturbation Scheme

Consider the same problem model as in Eq. (2). The constellation is still set as $\mathcal{M} = \mathbb{F}_p$, but the difference here is that p should be a prime, such that any element in $\mathbb{F}_p \backslash 0$ has a multiplicative inverse.

In the new scheme, rather than transmitting $\mathbf{B}^{\dagger}(\mathbf{s} - p\mathbf{x})$, we adopt $k'\mathbf{B}^{\dagger}\mathbf{s} - \mathbf{B}^{\dagger}p\mathbf{x}$ as the signal vector. Specifically, it first solves the following CVP instances:

$$\min_{\mathbf{x} \in \mathbb{Z}^n} \left\| k\mathbf{B}^{\dagger}\mathbf{s} - \mathbf{B}^{\dagger}p\mathbf{x} \right\|^2, \tag{7}$$

with k chosen from a sequence of numbers $\{1, \ldots, p-1\}$, which are all coprime to p. Denote $E_{(k)} = \left\| k\mathbf{B}^{\dagger}\mathbf{s} - \mathbf{B}^{\dagger}p\mathbf{x} \right\|$; assuming k' corresponds the index such that $E_{(k)}$ has the minimum energy, then the transmitted signal is set as $k'\mathbf{B}^{\dagger}\mathbf{s} - \mathbf{B}^{\dagger}p\mathbf{x}$.

Subsequently, the received signal vector is written as

$$\mathbf{r} = (k'\mathbf{s} - p\mathbf{x})/E_{(k')} + \mathbf{w}. \tag{8}$$

Similar to Sec. II-B, the effective noise in (8) is $\mathbf{w}' \in \mathrm{N}(\mathbf{0}, \sigma_w^2 E_{(k')}\mathbf{I}_n)$, whose variance is no larger than that in the classic case.

Regarding the decoding part of the new scheme, only one additional multiplication step is needed. We first decode

$$\lfloor E_{(k')}\mathbf{r} \rceil \mod p = \lfloor k'\mathbf{s} - p\mathbf{x} + E_{(k')}\mathbf{w} \rceil \mod p.$$

When the effective noise is small, assume we have $\lfloor E_{(k')}\mathbf{w} \rceil = \mathbf{0}$, then it yields

$$\lfloor E_{(k')}\mathbf{r} \rceil \mod p = k'\mathbf{s} \mod p. \tag{9}$$

Since k' is coprime to p, denote its multiplicative inverse as $\tilde{k}'$. Finally we estimate $\mathbf{s}$ by

$$\left(\tilde{k}' \lfloor E_{(k')}\mathbf{r} \rceil \mod p \right) \mod p.$$

By inspecting the properties of a modulo operation, the whole decoding procedure can be simply written as

$$\tilde{k}' \lfloor E_{(k')}\mathbf{r} \rceil \mod p.$$

A worked out example on the flow of the symbol space is given in Fig. 2. Suppose $s \in \mathbb{F}_5$, then after VP precoding, the effective symbol space observed by the receiver is $k's = 2s$, as shown in Eq. (8). To decode, note that the inverse of k' is $\tilde{k}' = 3$, so we can get $s = \tilde{k}'(k's)$ if given a correct $k's$.

Remark 3. I) The adaptive skills can still be applied even if we only approximately solve Eq. (7) by using ZF, MMSE or other alternatives. This becomes especially tempting as the exponential complexity of sphere precoding prevents its usage for massive MIMO systems with hundreds of antennas. II) The worst-case performance bounds in the adaptive scheme would still use that of the classical one, which is

$$\mathrm{dist}(k\mathbf{y}, \mathcal{L}(\mathbf{H})) \leq 1/2 \sqrt{\sum_{i=1}^{n} \lambda_i^2(\mathbf{H})},$$

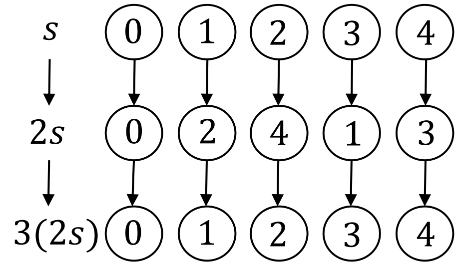


Fig. 2: The effective symbol space of $s \in \mathbb{F}_5$ in the flow of messages.

where $\lambda_i(\mathbf{H})$ is the ith successive minimum [10] of lattice $\mathcal{L}(\mathbf{H})$. The expected residue distance should however be smaller, as it is no longer uniformly distributed in the Voronoi region of $\mathcal{L}(\mathbf{H})$ but rather closer to a lattice point.

B. Discussions

1) Choosing the optimal k: Given a finite constellation size of p, the search for the optimal k in (7) needs not be enumerated from the integer set $\mathbb{Z}$ in view of the following reason:

Proposition 4. *The parameter k in Eq. (7) is both symmetric and periodic w.r.t. p, i.e., $E_{(k)} = E_{(\pm k + p\mathbb{Z})}$.*

Proof: The norm function is independent of reflection, one has $\min_{\mathbf{x}} \left\| k\mathbf{B}^{\dagger}\mathbf{s} - \mathbf{B}^{\dagger}p\mathbf{x} \right\| = \min_{\mathbf{x}} \left\| -k\mathbf{B}^{\dagger}\mathbf{s} + \mathbf{B}^{\dagger}p\mathbf{x} \right\|$ and thus $E_{(k)} = E_{(-k)}$. As for the periodic relation, we have for all $t \in \mathbb{Z}$ that

$$\begin{aligned} \min_{\mathbf{x}} \left\| k\mathbf{B}^{\dagger}\mathbf{s} - \left(\mathbf{B}^{\dagger}p\right)\mathbf{x} \right\| &= \min_{\mathbf{x}} \left\| k\mathbf{B}^{\dagger}\mathbf{s} + \left(\mathbf{B}^{\dagger}p\right)(t\mathbf{s} - \mathbf{x}) \right\| \\ &= \min_{\mathbf{x}} \left\| (k + pt)\,\mathbf{B}^{\dagger}\mathbf{s} - \left(\mathbf{B}^{\dagger}p\right)\mathbf{x} \right\|, \end{aligned}$$

where in the first equation the optimizations over $\mathbf{x}$ and $t\mathbf{s} - \mathbf{x}$ are the same. Therefore the periodic relation also holds. ∎

The above proposition also shows that we must have $p \geq 5$ to make queries other than $k = 1$. For instance, the adaptive range is $k = 1, 2$ for $\mathbb{F}_5$, and is $k = 1, 2, 3$ for $\mathbb{F}_7$.

2) Choosing p beyond a prime: Here we show that p does not have to be a prime, but can be the product of distinct primes. The main enabler of this choice is the Construction π_A technique [11]. Let $p_1, \ldots, p_L$ be distinct primes, $p = p_1 p_2 \cdots p_L$ and $q_l = p/p_l$. Consider the message space

$$\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_L} \cong \mathbb{F}_{p_1} \times \mathbb{F}_{p_2} \times \cdots \times \mathbb{F}_{p_L}.$$

Based on Chinese remainder theorem for commutative rings, we have the following isomorphism:

$$\mathcal{M}(v_1, \ldots, v_L) = s_1 q_1 v_1 + \cdots + s_L q_L v_L \bmod p,$$

where $s_1, \ldots, s_L$ are solutions to Bezout's identity $s_1 q_1 + \cdots + s_L q_L = 1$.

In the following example, consider $\mathbb{Z}/15\mathbb{Z} \cong \mathbb{F}_3 \times \mathbb{F}_5$. The isomorphism is given by $\mathcal{M} = -5v_1 + 6v_2 \bmod 15$. We therefore obtain the isomorphism in Table I. Suppose $s = 7 = (1, 2), p = 15, k' = 2$. After quantization over the fine lattice and modulo over the coarse lattice, for small enough effective noise such that the first decoding step is correct, we have

$$k's \mod p = 14 = (2, 4). \tag{10}$$

Further we know that the inverses of k' over $\mathbb{F}_3$ and $\mathbb{F}_5$ are respectively 2 and 3. Therefore, we can recover s from (10) by

$$(2 \times 2 \mod 3, 3 \times 4 \mod 5) = (1, 2).$$

3) MMSE-based VP: The aforementioned implementations were only based on a ZF principle. In [8], Schmidt et al. have proposed an MMSE-based VP (noted as MMSE-VP) technique which minimizes the mean square error of the received signal. The perturbing vector $p\mathbf{x}$ in this scheme is chosen from

$$\mathbf{x} = \arg\min_{\mathbf{x}\in\mathbb{Z}^n} ||\mathbf{L}(\mathbf{s} + p\mathbf{x})||, \tag{11}$$

where $\mathbf{L}$ is obtained through Cholesky factorization of the following matrix:

$$\left(\mathbf{H}\mathbf{H}^\top + \frac{n}{\rho}\mathbf{I}_n\right)^{-1} = \mathbf{L}^\top\mathbf{L}.$$

The front-end precoding matrix in this case is given as $\mathbf{F} = \frac{1}{E}\mathbf{H}^\top\left(\mathbf{H}\mathbf{H}^\top + \frac{n}{\rho}\mathbf{I}_n\right)^{-1}$, where E is again a power scaling factor enforcing a unit power constraint, and ρ denotes SNR. By tuning the query point in Eq. (11) from $\mathbf{Ls}$ to $k\mathbf{Ls}$, we similarly arrive at an adaptive MMSE-VP scheme.

C. Complexity and Limitations

Based on Proposition 4, the number of CVP instances in ad-VP is at most $(p-1)/2$. For the complexity analysis of sphere-decoding-based or MCMC-based algorithms used to solve CVP, we refer the interested readers to [10], [12], [13].

The optimal precoding factor k' is unknown in practice, while implementing multiple sphere decoding algorithms is computationally intensive. We argue that the following alternative optimization approach is possible. First initialize the algorithm with the precoding factor obtained from ZF/MMSE/THP, where ZF/MMSE/THP is used for each k-scaled query point and find the local optimum k'. Then solve one sphere decoding problem based on the query point provided by ZF/MMSE/THP to complete the whole precoding procedure.

If the optimal k is chosen from the entire domain of s, then s has to be from a prime field or a product of distinct prime fields. However, in order to be compatible with channel codes, the modulation size is usually a power of 2. In this regard, we argue that the scheme may only need one "multiplicative factor" in the domain of s (except the unit 1) to be invertible.

As the scheme needs to feed-forward the adaptive parameter to all users, it is more practical to design the system with a large message space while minimizing the adaptive space.

IV. Simulation Results

In this section we show some numerical results obtained by computer simulations. The results are obtained by averaging over 2×10^4 channel realizations. The constellation is chosen from a finite field $\mathbb{F}_p$. Note that our simulation results using $\mathbb{F}_p$ can be easily extended to those using complex constellations.

Figure 3 depicts the symbol-error rate (SER) of the proposed adaptive VP and conventional VP schemes. SNR values specified in the figure are in decibels (dBs) per channel. We denote the zero-forcing (i.e., channel inversion) precoding by "ZF", the regularized channel inversion precoding by "MMSE", the conventional VP by "Sphere precoding (ZF)", and the VP based on a MMSE criterion by "Sphere precoding (MMSE)". Their adaptive versions proposed in this paper are denoted with a prefix "ad", e.g., "ad-MMSE". In Fig. 3-(a), where $n = m = 4$, $p = 7$, and k is only adaptively chosen from $1, 2$, all the proposed schemes have much better performance than their respective counterparts. For instance, we observe a gain of 2 dBs of ad-Sphere precoding (MMSE) over Sphere precoding (MMSE) in the entire SNR range. These gains rise to around 4 dBs for ZF/MMSE-based adaptive implementations, where we associate the adaptive scheme with linear ZF/MMSE detectors. We repeat this simulation to a larger system size of $n = m = 6$ in Fig. 3-(b). Gains of the adaptive schemes are similar, but larger diversities in the SERs are observed compared to Fig. 3-(b). In the figure, we emphasize that only one bit of information ($k = 1, 2$) needs to feedforward to users to obtain much better SER performance, while the size of the constellation can be larger.

V. Conclusions

We have proposed an improved vector perturbation scheme, which adaptively changes the query point based on the channel state information. This technique can be attached to conventional VP schemes to observe certain performance gains. To enjoy this improvement, the message space should be a finite field, or a ring that consists of the product of primes. One possible drawback of the scheme is that the adaptive parameter should feed-forward to all users, though we can choose a large message space and minimize the adaptive space.

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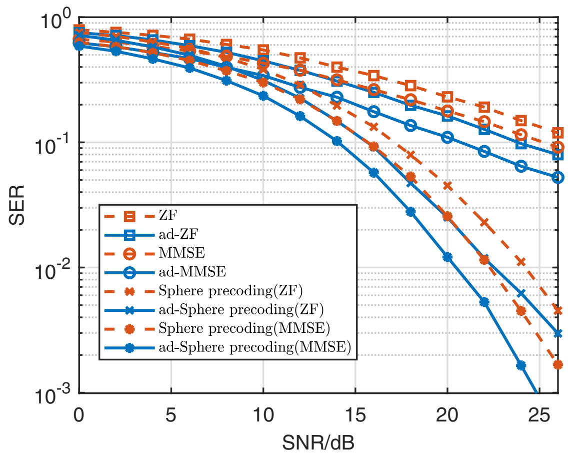
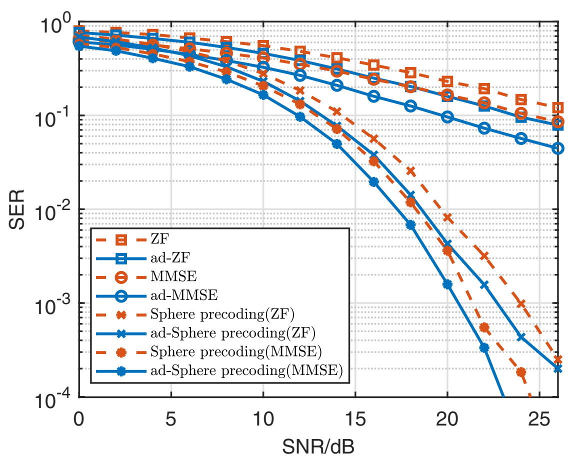
TABLE I: One example of the isomorphism table when choosing the symbol space from the product of primes.

(v_1, v_2)	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(2,0)$	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
$\mathcal{M}(v_1, v_2)$	0	6	12	3	9	10	1	7	13	4	5	11	2	8	14



(a) $n = 4$, $p = 7$.



(b) $n = 6$, $p = 7$.

Fig. 3: The SER performance of different algorithms.

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