Assignment 9

Consider to solving (stable) the following LTI, DT dynamical system using the model predictive control method:

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} u(k), y(k) = \begin{bmatrix} 2 & 0 \end{bmatrix} x(k)$$

1 MPC Formulation

For the given LTI system:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \end{bmatrix}$.

1.1 Prediction Model

Let N be the prediction horizon. The predicted states over the horizon are:

$$X = \Phi x(k) + \Gamma U \tag{3}$$

where $X = [x(k+1|k)^T, \ldots, x(k+N|k)^T]^T$ and $U = [u(k|k)^T, \ldots, u(k+N-1|k)^T]^T$. The matrices Φ and Γ are defined as:

$$\Phi = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Gamma = \begin{bmatrix} B & 0 & \ldots & 0 \\ AB & B & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \tag{4}$$

1.2 Cost Function

We define the quadratic cost function to minimize:

$$J = \sum_{i=1}^{N} x(k+i|k)^T Q x(k+i|k) + \sum_{i=0}^{N-1} u(k+i|k)^T R u(k+i|k) \tag{5}$$

In matrix form:

$$J = X^T \mathcal{Q} X + U^T \mathcal{R} U \tag{6}$$

where $\mathcal{Q} = \text{diag}(Q, \ldots, Q)$ and $\mathcal{R} = \text{diag}(R, \ldots, R)$. Substituting $X = \Phi x(k) + \Gamma U$:

$$J = (\Phi x(k) + \Gamma U)^T \mathcal{Q}(\Phi x(k) + \Gamma U) + U^T \mathcal{R} U \tag{7}$$

1.3 Optimal Control

The optimal control sequence U^* is found by setting $\frac{\partial J}{\partial U} = 0$:

$$U^* = -(\Gamma^T \mathcal{Q} \Gamma + \mathcal{R})^{-1} \Gamma^T \mathcal{Q} \Phi x(k) \tag{8}$$

The MPC law takes only the first element of U^*:

$$u(k) = [I \quad 0 \ldots 0] U^* \tag{9}$$

1.4 Receding Horizon Control

At each time step k, we solve the optimization problem to find U^*, apply the first control action $u(k)$, and then repeat the process at time $k+1$ with the new state $x(k+1)$.

2 Numerical Solution

To provide a concrete solution, we assume the following parameters: prediction horizon $N = 2$, state weight $Q = I_2$, and control weight $R = 1$.

2.1 Matrix Calculation

The prediction matrices are:

$$\Phi = \begin{bmatrix} A \\ A^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} B & 0 \\ AB & B \end{bmatrix} = \begin{bmatrix} 0.8 & 0 \\ 1 & 0 \\ 1.8 & 0.8 \\ 1 & 1 \end{bmatrix} \tag{10}$$

The MPC gain matrix K is derived from:

$$K = -[I \quad 0](\Gamma^T \mathcal{Q} \Gamma + \mathcal{R})^{-1} \Gamma^T \mathcal{Q} \Phi \tag{11}$$

Substituting the values:

$$\Gamma^T \mathcal{Q} \Gamma + \mathcal{R} = \begin{bmatrix} 6.88 & 2.44 \\ 2.44 & 2.64 \end{bmatrix} \tag{12}$$

$$K \approx \begin{bmatrix} -0.621 & -1.134 \end{bmatrix} \tag{13}$$

2.2 Final Law

The optimal control law for the system is:

$$u(k) = \begin{bmatrix} -0.621 & -1.134 \end{bmatrix} x(k) \tag{14}$$

The eigenvalues of the closed-loop system $A + BK$ are approximately $0.183 \pm 0.354i$, which lie within the unit circle, confirming that the MPC controller stabilizes the system.