Assignment 10

Solve the following nonlinear programming problem using the dynamic programming method (in a backward recursive way):

$$\begin{aligned} \max \quad & z = x_1^2 x_2 \\ \text{s.t.} \quad & x_1^2 + x_2 = 2 \end{aligned}$$

1 formuation

We can define the states of the DP problem.

$$s_2 = x_2, s_1 = s_2 + x_1^2 = 2, x_1^2 \leq s_1 \tag{1}$$

2 Solution

Period 1:

$$f_2(s_2) = max(x_2) = s_2, x_2^* = s_2 \tag{2}$$

Period 2:

$$f_1(s_1) = max(x_1^2 x_2) = max(x_1^2(s_1 - x_1^2)) \tag{3}$$

Taking its derivative yields, we can get:

$$\frac{d}{dx_1}(x_1^2 s_1 - x_1^4) = 2x_1 s_1 - 4x_1^3 = 0 \tag{4}$$

Taking the second derivative yields, we can get:

$$\frac{d^2}{dx_1^2}(x_1^2 s_1 - x_1^4) = 2s_1 - 12x_1^2 \tag{5}$$

Setting the first derivative to zero, we can get:

$$x_1^{*2} = \frac{s_1}{2}, \frac{d^2}{dx_1^2}(x_1^2 s_1 - x_1^4) < 0 \tag{6}$$

So it is a maximum point.
Thus, because $s_1 = 2$, we can get:

$$x_1^{*2} = 1, x_2^* = s_1 - x_1^{*2} = 1 \tag{7}$$

Therefore, the optimal solution is:

$$x_1^* = \pm 1, x_2^* = 1, z_{max} = 1 \tag{8}$$