Assignment 6

Please judge the following problem whether it is convex function or not, and then prove it.

$$(1)\mathbf{f(x,y)=\sqrt{xy}}$$
$$(2)\mathbf{f(x)=\sum_{i=1}^{m}\omega_i f_i(x), \omega_i \geq 0, f_i \text{ is convex}}$$

If a function is twice differentiable, we can use this condition to judge whether it is a convex function:

$$\nabla^2 f \geq 0$$

For function(1), take the second derivative yields,we can get

$$\nabla^2 f = \begin{pmatrix} -\frac{y^2(xy)^{-3/2}}{4} & -\frac{(xy)^{-1/2}}{4} \\ -\frac{(xy)^{-1/2}}{4} & -\frac{x^2(xy)^{-3/2}}{4} \end{pmatrix}$$

The eigenvalues of this matrix are:

$$\lambda_1 = 0, \lambda_2 = -\frac{(x^2+y^2)(xy)^{-3/2}}{4}$$

So the matrix is not positive semi definite, the function is not a convex function.

For function(2), if $f_i(x)$are all twice differentiable, there will be

$$\nabla^2 f_i(x) \geq 0$$

Because $\omega_i \geq 0$, we can get

$$\nabla^2 f(x) \geq 0$$

In that case, $f(x)$ is a convex function.

If there exist some $f_i(x)$ are not twice differentiable, we need to use the definition of convex function to solve and prove it.

Definition

If a $f(x)$ is a convex function, for $\forall x_1 \leq x_2$:

$$\forall \lambda \in [0,1], f[\lambda x_1 + (1-\lambda x_2)] \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Using this definition, we can easily prove that $f(x)$ is a convex function.

Firstly, for $\omega \geq 0$, we can get

$$(1)\forall\lambda \in [0,1], \omega f[\lambda x_1 + (1-\lambda x_2)] \leq \lambda\omega f(x_1) + \omega(1-\lambda)f(x_2)$$

So $\omega_i f_i(x)$ are convex functions.

for several convex functions $(f_1(x), f_2(x)......, f_n(x))$, using the sign-preserving property of inequality, we can get

$$(2)\forall\lambda \in [0,1], \sum_{i=1}^{n} f_i[\lambda x_1 + (1-\lambda x_2)] \leq \lambda f_i(x_1) + (1-\lambda)f_i(x_2)$$

Using formula (1),(2), we can prove that $f(x)$ is a convex function.