Instructions: To plot phase trajectories (asked in the exercises below) use any computer software capable of plotting vector fields or isoclines (for instance MatLab or Mathematica).

Problem 1 (Predator-Prey). Consider the system of Lotka-Volterra that monitors the interactions of sharks and fish:

$$\begin{aligned} \frac{dF}{dt} &= F\big(a - bF - cS\big) \\ \frac{dS}{dt} &= S\big(\lambda F - k\big). \end{aligned} \tag{1}$$

Let $\lambda = k = b = c = 1$ and $a = 2$. Use the method of isoclines to plot the phase trajectories and the nullclines of the system (1) in the region $F, S > 0$. Give an ecological interpretation of the trajectories.

Problem 2 (Competition). If N_1, N_2 are two species that live in 'separate' environments or simply do not interact with each other, the logistic equations

$$\frac{dN_i}{dt} = \rho_i N_i\Big(1 - \frac{N_i}{K_i}\Big), \quad i = 1, 2$$

would be appropriate for modeling the dynamics of these populations. Here the constant K_i is called the *carrying capacity* of the population N_i (recall that $N_i \to K_i$ as $t \to \infty$).

Suppose now that the two-species N_1 and N_2 live in the same environment and compete for the same food source. In that case, the growth rate of the population N_1 is affected negatively by N_2 and vice versa. The simplest system for competition then would be

$$\begin{aligned} \frac{dN_1}{dt} &= \rho_1 N_1\Big(1 - \frac{N_1}{K_1} - a_{21}N_2\Big) \\ \frac{dN_2}{dt} &= \rho_2 N_2\Big(1 - \frac{N_2}{K_2} - a_{12}N_1\Big). \end{aligned} \tag{2}$$

Objective: Consider the system (2) with $\rho_1 = \rho_2 = 2$ and $K_1 = K_2 = 1$. Use the method of isoclines to plot the phase trajectories and the nullclines for this system in the following cases:

(i) $a_{12} = a_{21} = 0.75$.
(ii) $a_{12} = a_{21} = 1.25$.
(iii) $a_{12} = 1.25, a_{21} = 0.75$.

For each case give an ecological interpretation of the trajectories.

Problem 3 (Mutualism). Suppose that two-species N_1 and N_2 live in the same environment and benefit from the presence of each other. In that case, the growth rate of the population N_1 is affected positively by N_2 and vice versa. The appropriate system for modeling of such interactions would be

$$\begin{aligned} \frac{dN_1}{dt} &= \rho_1 N_1\Big(1 - \frac{N_1}{K_1} + a_{21}N_2\Big) \\ \frac{dN_2}{dt} &= \rho_2 N_2\Big(1 - \frac{N_2}{K_2} + a_{12}N_1\Big). \end{aligned} \tag{3}$$

Note, that in the absence of mutualistic interactions ($a_{12} = a_{21} = 0$) each species grows to its respective carrying capacity K_i.

Objective: Consider the system (3) with $\rho_1 = \rho_2 = 2$ and $K_1 = K_2 = 1$. Using the method of isoclines plot the phase trajectories and the nullclines for this system in the following cases:

(i) $a_{12} = 0.4, a_{21} = 0.3$.

(ii) $a_{12} = 2, a_{21} = 1$.

For each case give an ecological interpretation of the trajectories.