Instructions: To plot phase trajectories (asked in the exercises below) use any computer software capable of plotting vector fields or isoclines (for instance MatLab or Mathematica).

Problem 1 (Predator-Prey). Consider the system of Lotka-Volterra that monitors the interactions of sharks and fish:

$$\begin{aligned} \frac{dF}{dt} &= F\big(a - bF - cS\big) \\ \frac{dS}{dt} &= S\big(\lambda F - k\big). \end{aligned} \tag{1}$$

Let $\lambda = k = b = c = 1$ and $a = 2$. Use the method of isoclines to plot the phase trajectories and the nullclines of the system (1) in the region $F, S > 0$. Give an ecological interpretation of the trajectories.

Problem 2 (Competition). If N_1, N_2 are two species that live in 'separate' environments or simply do not interact with each other, the logistic equations

$$\frac{dN_i}{dt} = \rho_i N_i\Big(1 - \frac{N_i}{K_i}\Big), \quad i = 1, 2$$

would be appropriate for modeling the dynamics of these populations. Here the constant K_i is called the *carrying capacity* of the population N_i (recall that $N_i \to K_i$ as $t \to \infty$).

Suppose now that the two-species N_1 and N_2 live in the same environment and compete for the same food source. In that case, the growth rate of the population N_1 is affected negatively by N_2 and vice versa. The simplest system for competition then would be

$$\begin{aligned} \frac{dN_1}{dt} &= \rho_1 N_1\Big(1 - \frac{N_1}{K_1} - a_{21}N_2\Big) \\ \frac{dN_2}{dt} &= \rho_2 N_2\Big(1 - \frac{N_2}{K_2} - a_{12}N_1\Big). \end{aligned} \tag{2}$$

Objective: Consider the system (2) with $\rho_1 = \rho_2 = 2$ and $K_1 = K_2 = 1$. Use the method of isoclines to plot the phase trajectories and the nullclines for this system in the following cases:
(i) $a_{12} = a_{21} = 0.75$.
(ii) $a_{12} = a_{21} = 1.25$.
(iii) $a_{12} = 1.25, a_{21} = 0.75$.
For each case give an ecological interpretation of the trajectories.

Problem 3 (Mutualism). Suppose that two-species N_1 and N_2 live in the same environment and benefit from the presence of each other. In that case, the growth rate of the population N_1 is affected positively by N_2 and vice versa. The appropriate system for modeling of such interactions would be

$$\begin{aligned} \frac{dN_1}{dt} &= \rho_1 N_1\Big(1 - \frac{N_1}{K_1} + a_{21}N_2\Big) \\ \frac{dN_2}{dt} &= \rho_2 N_2\Big(1 - \frac{N_2}{K_2} + a_{12}N_1\Big). \end{aligned} \tag{3}$$

Note, that in the absence of mutualistic interactions ($a_{12} = a_{21} = 0$) each species grows to its respective carrying capacity K_i.

Objective: Consider the system (3) with $\rho_1 = \rho_2 = 2$ and $K_1 = K_2 = 1$. Using the method of isoclines plot the phase trajectories and the nullclines for this system in the following cases:

(i) $a_{12} = 0.4, a_{21} = 0.3$.

(ii) $a_{12} = 2, a_{21} = 1$.

For each case give an ecological interpretation of the trajectories.

Predator-Prey

For the isocline, we first put

$$\frac{dS}{dF} = \frac{dS/dt}{dF/dt} = \frac{S(F-1)}{F(2-F-S)}$$

Isocline are the the curves for which

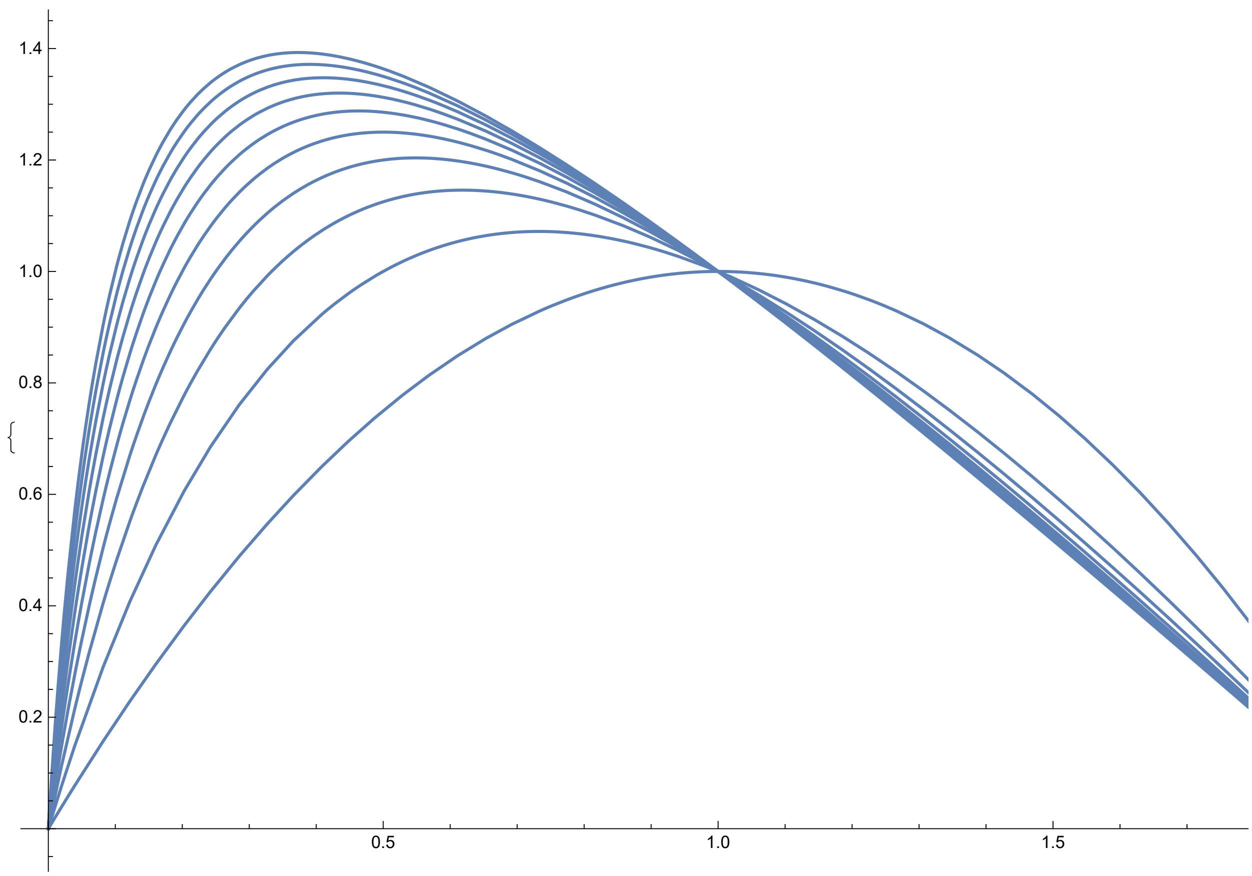
$$\frac{S(F-1)}{F(2-F-S)} = c$$

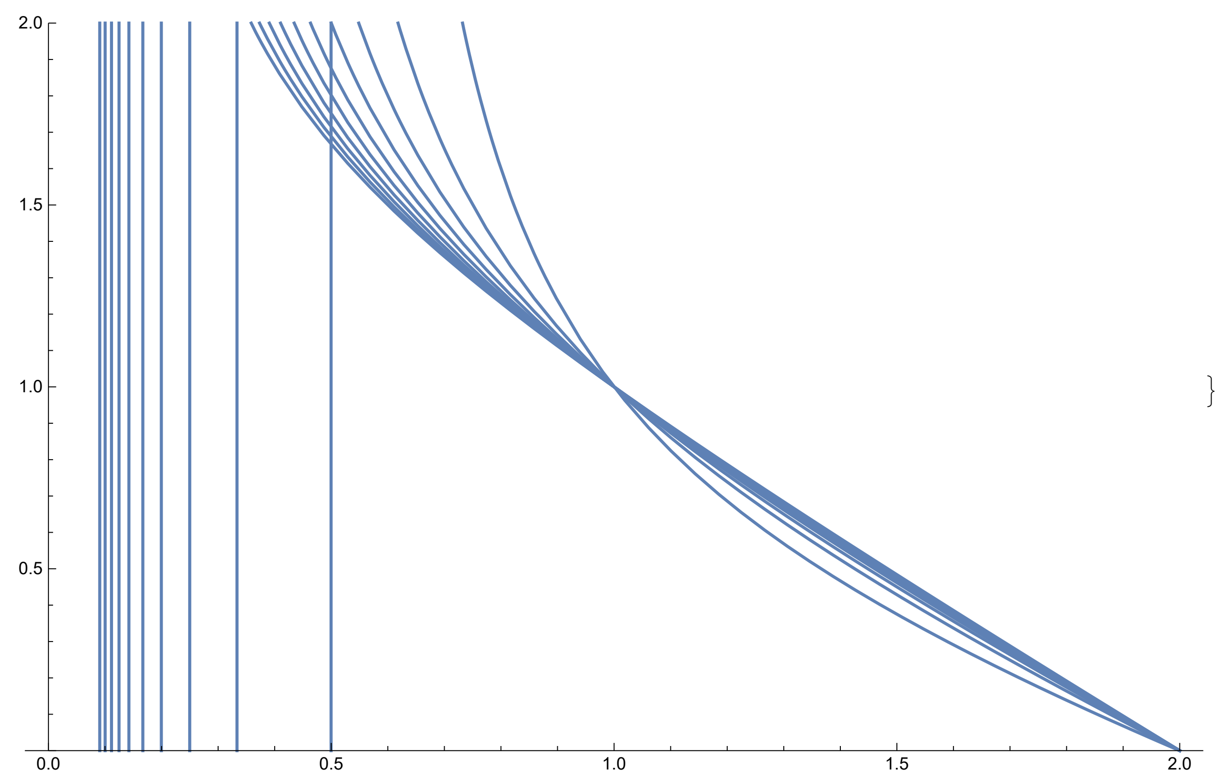
for some constant c. Solving for S gives

$$S = \frac{cF(F-2)}{1-(c+1)\,F}.$$

Here are the isocline for $c = -10, -9, \ldots, 9, 10$ (omitting that for $c = 0$ which is just the constant curve $S = 0$).

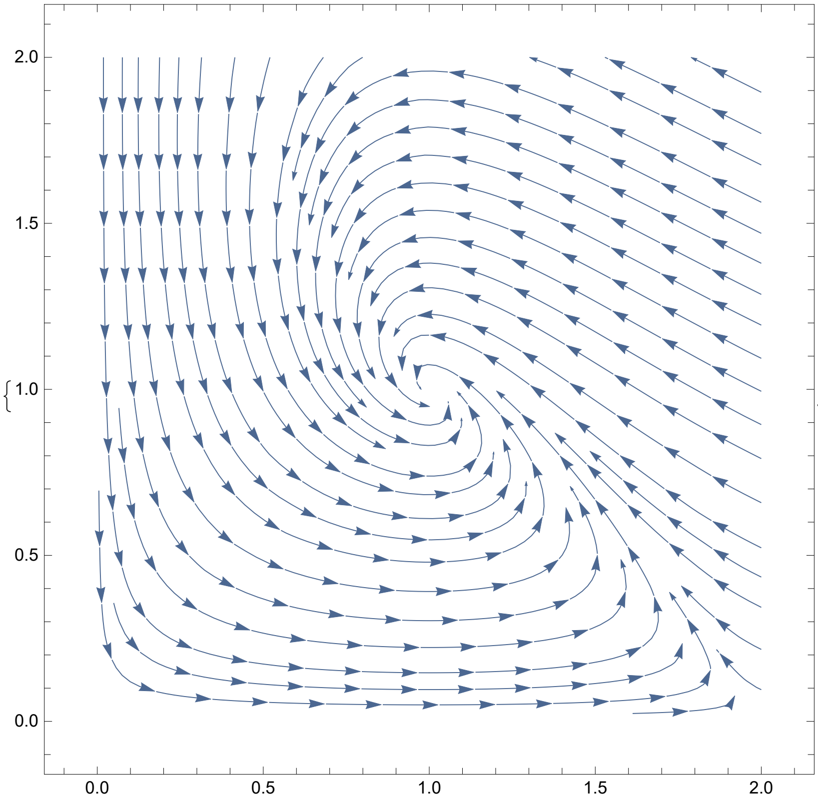
```
{Plot[Table[(2 c x - c x^2)/(-1 + x + c x), {c, -10, -1}], {x, 0, 2}],
 Plot[Table[(2 c x - c x^2)/(-1 + x + c x), {c, 1, 10}], {x, 0, 2}, PlotRange → {0, 2}]}
```

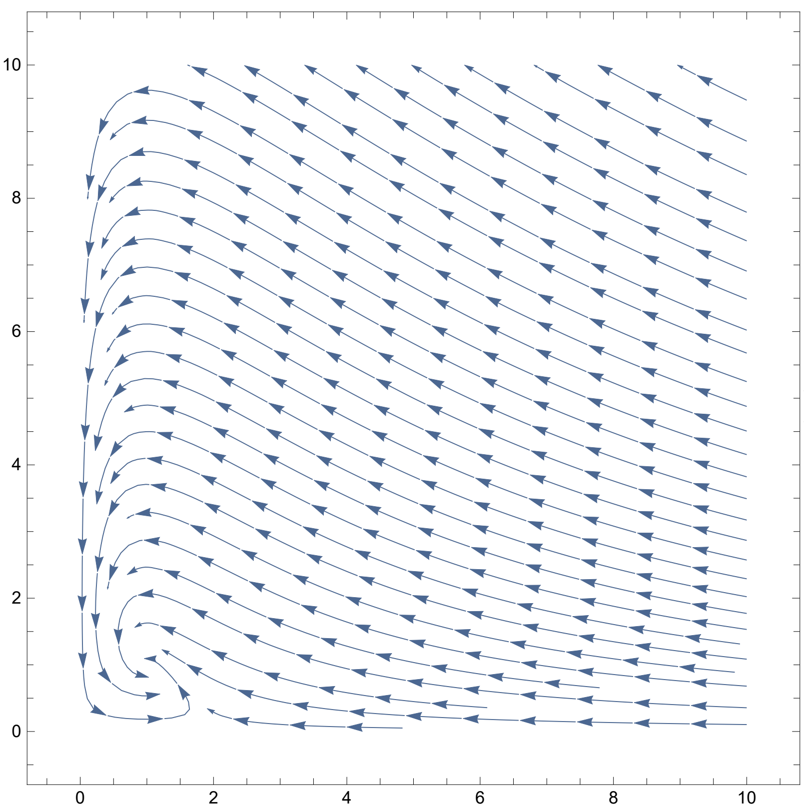

1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5


2.0
1.5
1.0
0.5
0.0
0.5
1.0
1.5
2.0

We note that the isocline cross at $(-1, 1)$. This is possible because there is a singularity in the vector field at that point; the isocline corresponding to given *c* may be better thought of as consisting of two curves than one. Here is the vector field with trajectories using StreamPlot, at two different scales.

```
{StreamPlot[{x (2 - x - y), y (x - 1)}, {x, 0, 2}, {y, 0, 2}],
 StreamPlot[{x (2 - x - y), y (x - 1)}, {x, 0, 10}, {y, 0, 10}]}
```





We may interpret this as follows: give some number of fish and sharks (both not zero), the sharks rapidly consume the fish, increasing the shark population while decreasing the fish population. Eventually collapse of the fish population leads to collapse of the sharks as well, until the numbers settle to an equilibrium or one fish and one shark (or perhaps “one” of some index number of fish and sharks).

Competition

For the isocline, we first put

$$\frac{dN_2}{dN_1} = \frac{N_1 (1-N_1-b N_2)}{N_2 (1-N_2-a N_1))}$$

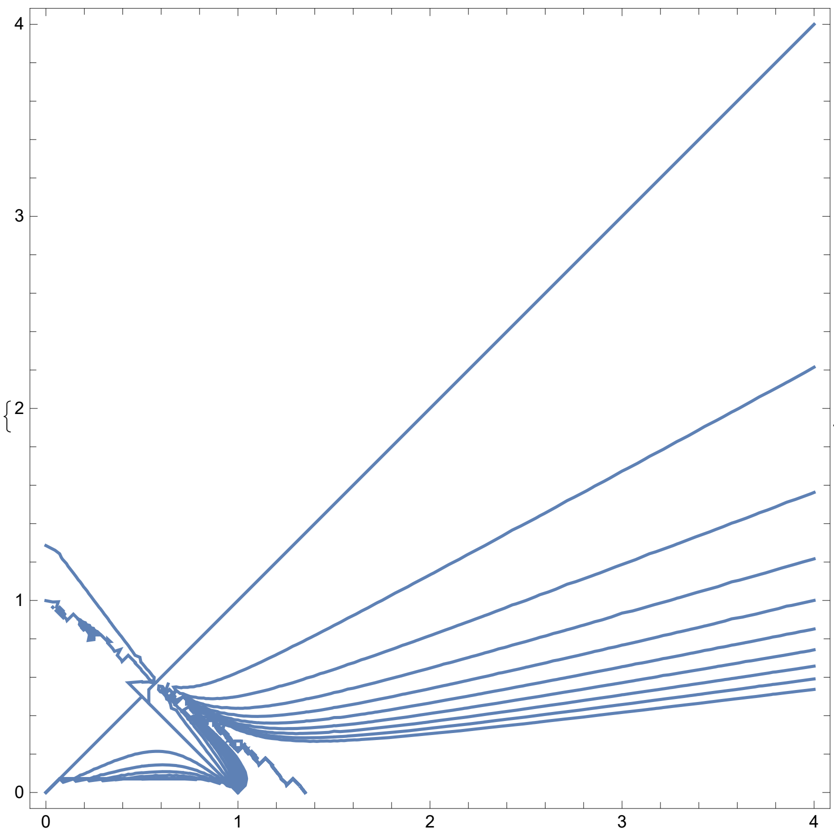
Isocline are the the curves for which

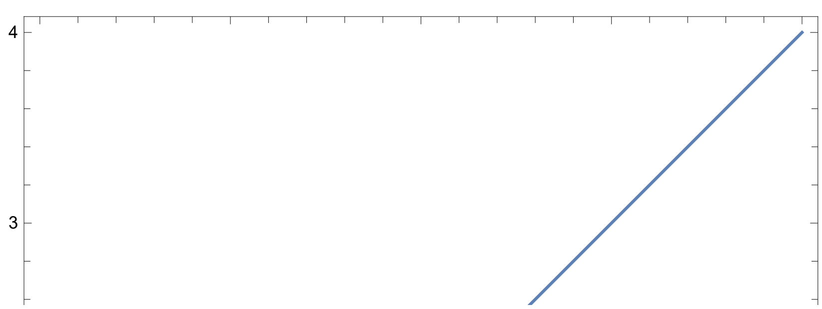
$$\frac{N_1 (1-N_1-b N_2)}{N_2 (1-N_2-a N_1))} = c$$

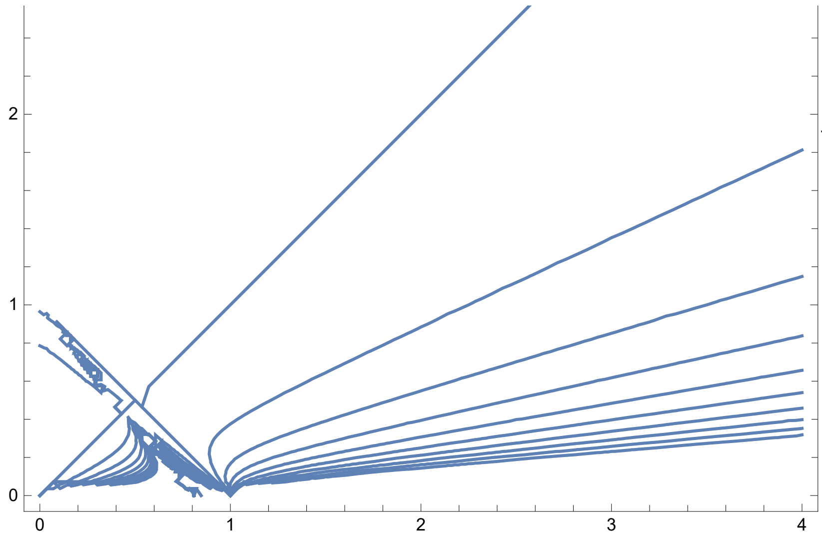
for some constant c. Here are the isocline for $c = -10, -9, \ldots, 9, 10$.

```
iso[a_, b_, c_] := ContourPlot[(x (1 - x - b y))/(y (1 - y - a x)) == c, {x, 0, 4}, {y, 0, 4}]
```

```
{Show[Table[iso[.75, .75, i], {i, -10, 10}]], Show[Table[iso[1.25, 1.25, i], {i, -10, 10}]],
 Show[Table[iso[1.25, .75, i], {i, -10, 10}]]}
```

{  ,





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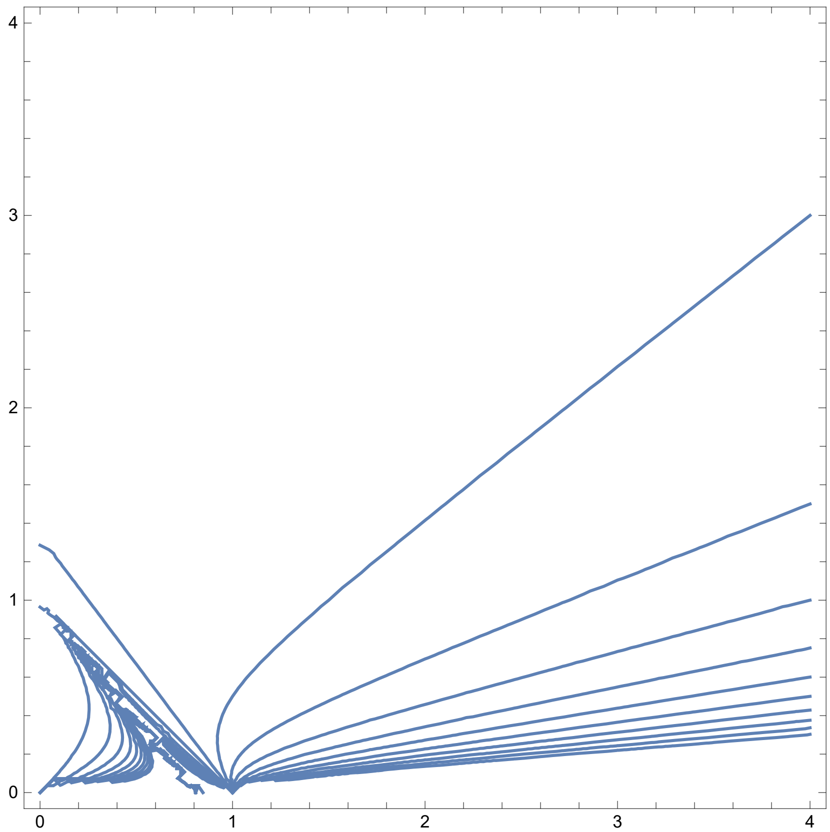
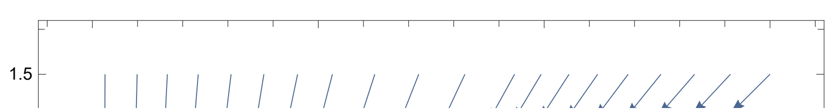


}

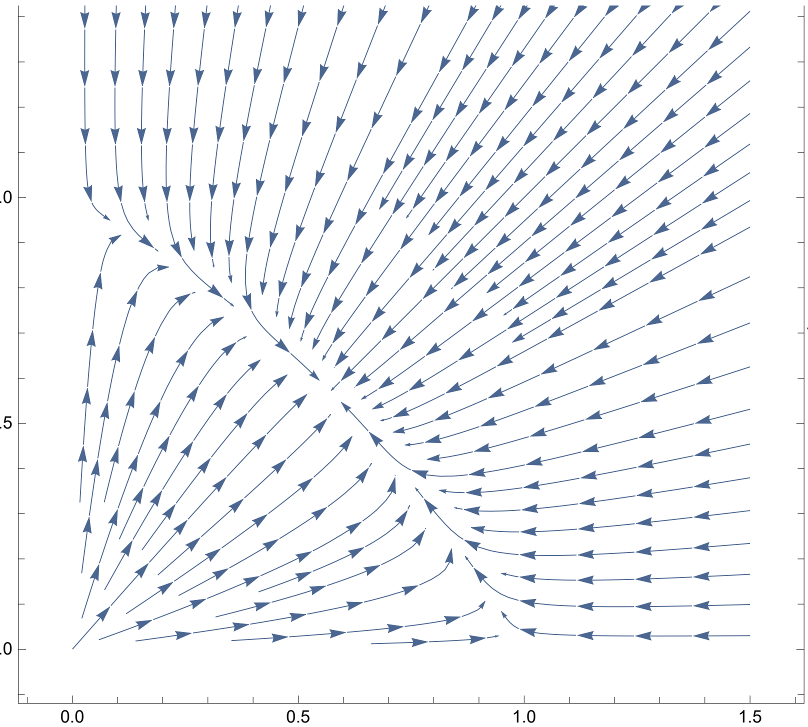
Some ugliness in the output is caused by the singular behavior of the vector fields around the line $N_2 = 1 - a_{21}\, N_1$. Below are the trajectory plots.

```
f[a_, b_] := StreamPlot[{2 x (1 - x - b y), 2 y (1 - y - a x)}, {x, 0, 1.5}, {y, 0, 1.5}]
```

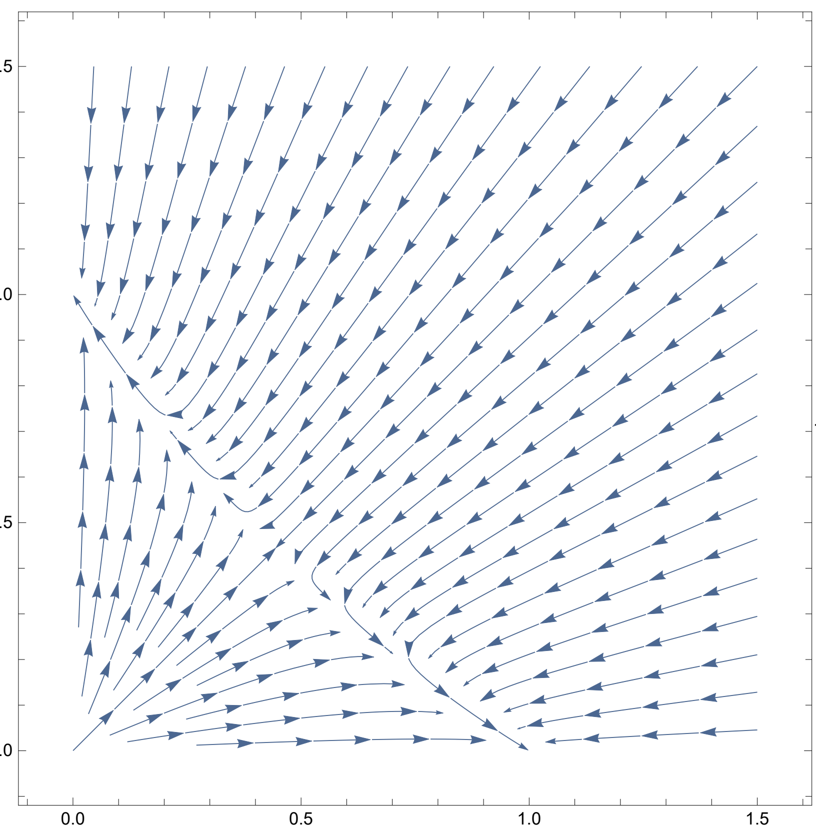
```
{f[.75, .75], f[1.25, 1.25], f[1.25, .75]}
```



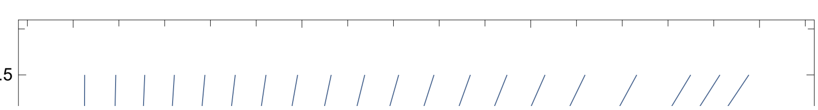
{

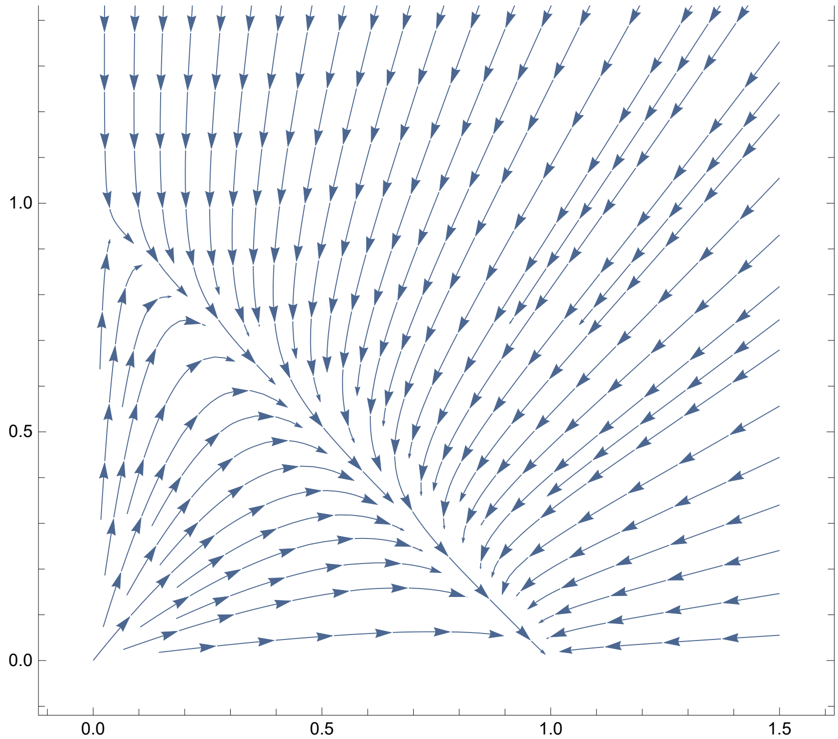

1.0
0.5
0.0
0.0
0.5
1.0
1.5

,


1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5

,


1.5



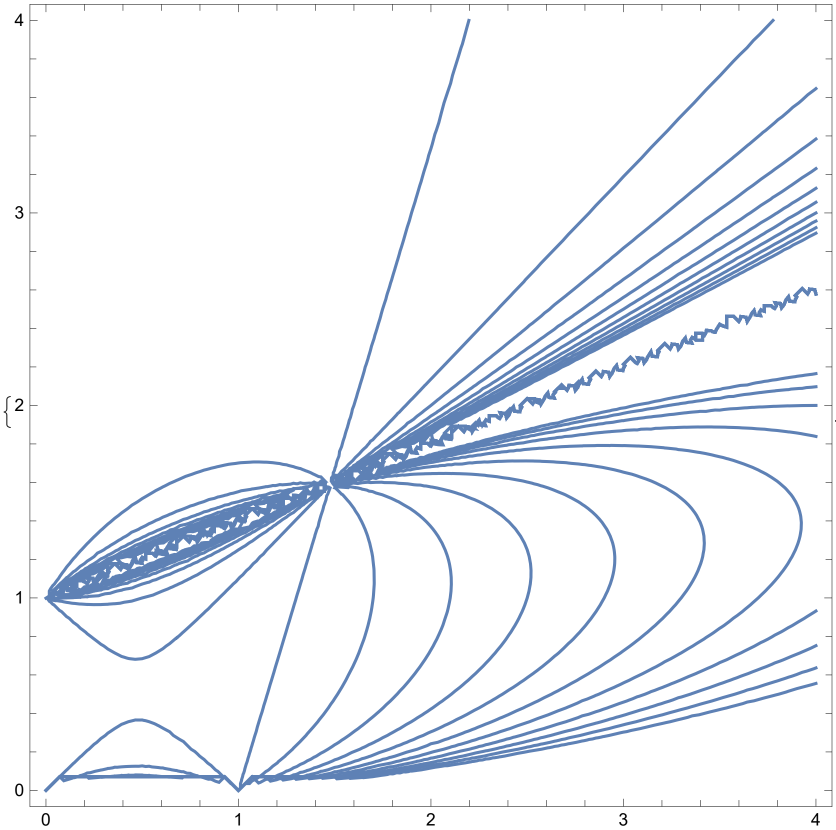
We see that both populations contract due to competition when each is “large.” In the first case, having both constants equal to .75 means that each species’ effect on the other is relatively small, leading to trajectories that eventually reach a stable equilibrium at (1,1). In the second case, both species are detrimental to one another, and for most trajectories we end with either one or the other species going extinct (i.e., the trajectories rush toward one of the axes). In the third case, species 1 is much more detrimental to species 2 than vice versa. Hence species 1 eventually causes species 2 to go extinct.

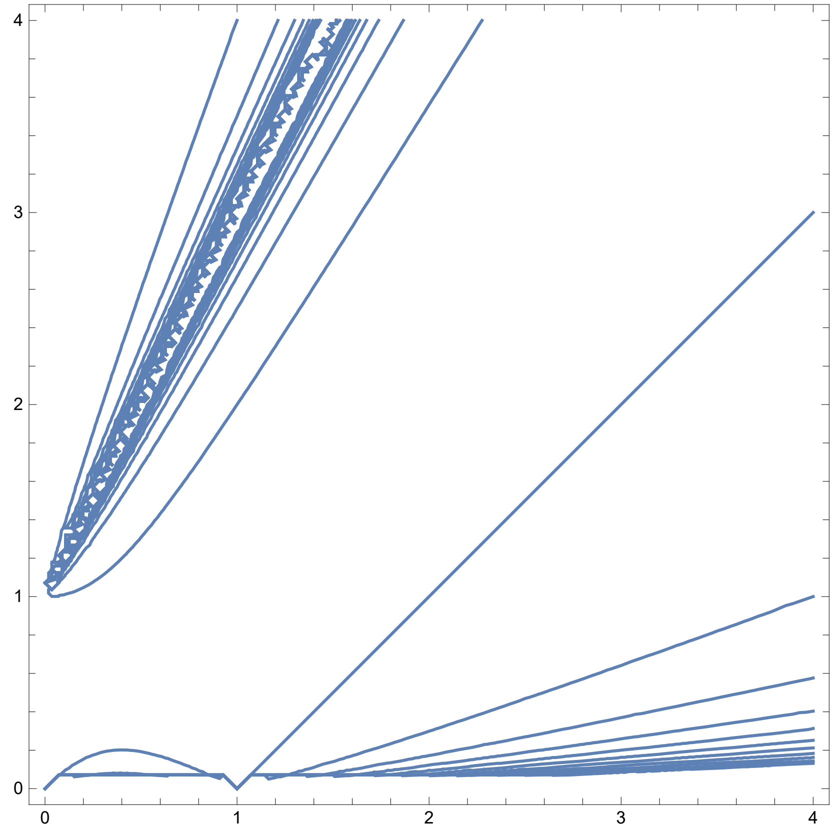
Mutualism

The process here is almost identical. Here are the isocline for $c = -10, -9, \ldots, 9, 10$.

```
iso2[a_, b_, c_] := ContourPlot[(x (1 - x + b y))/(y (1 - y + a x)) == c, {x, 0, 4}, {y, 0, 4}]
```

```
{Show[Table[iso2[.4, .3, i], {i, -10, 10}]], Show[Table[iso2[2, 1, i], {i, -10, 10}]]}
```

{ 
0
1
2
3
4
0
1
2
3
4
,



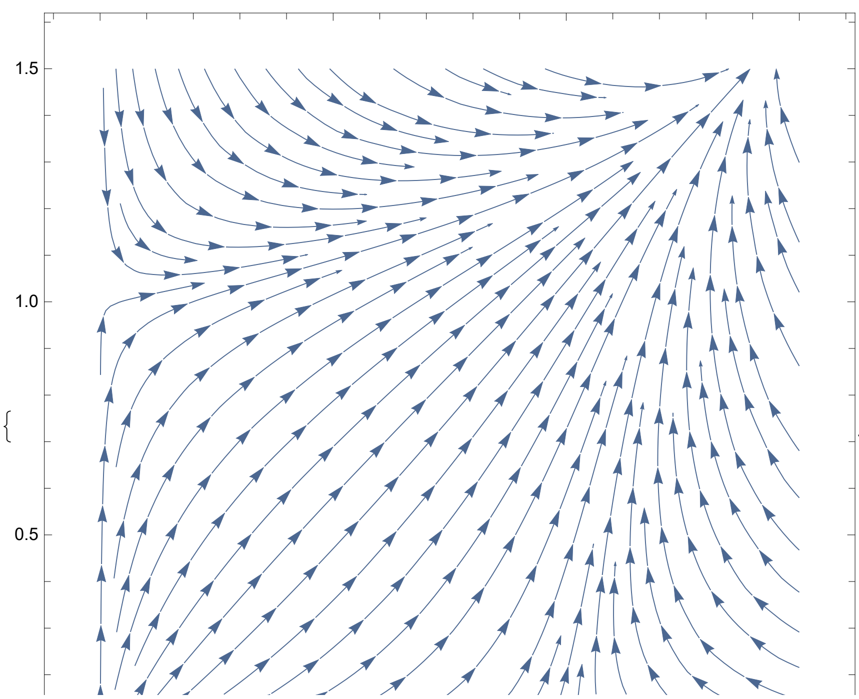
}

Once again, some ugliness, but also a general sense of several of the curves.

```
g[a_, b_] := StreamPlot[{2 x (1 - x + b y), 2 y (1 - y + a x)}, {x, 0, 1.5}, {y, 0, 1.5}]
```

```
{g[.4, .3], g[2, 1]}
```

{



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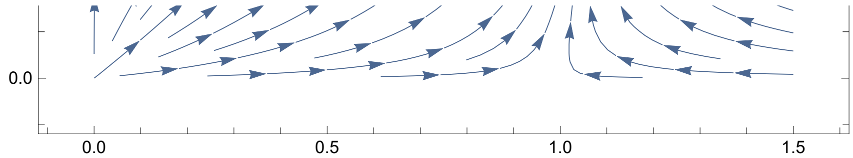

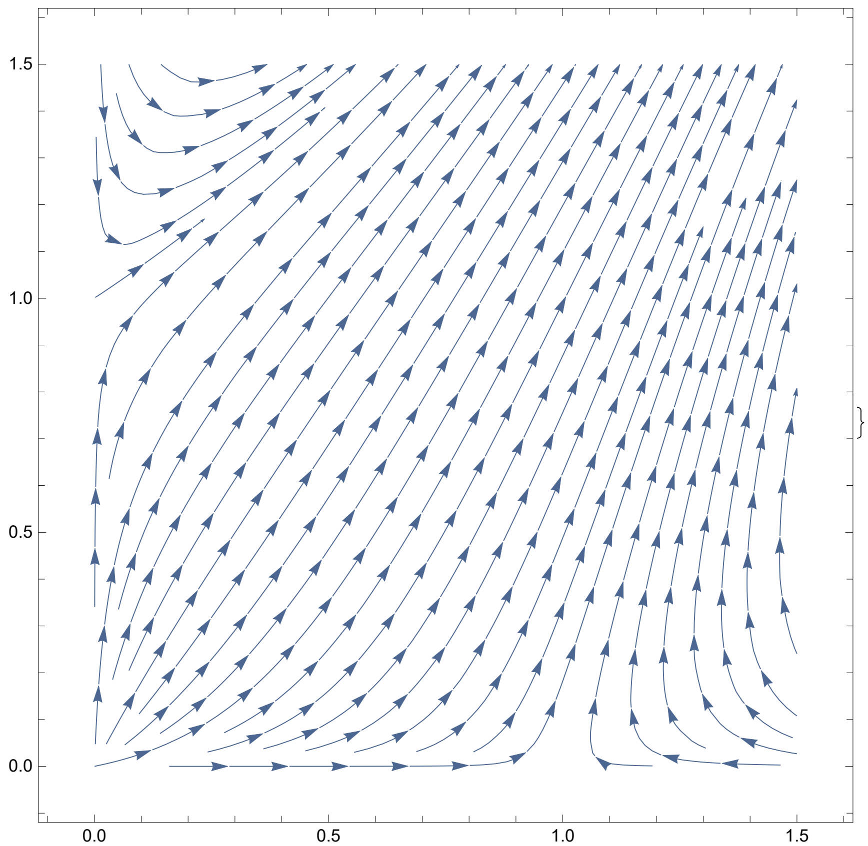

As we can see, the dynamics avoids extreme imbalances of one population versus another, since a high imbalance favoring one population leads to its contraction as it outstrips environmental capacity. However, as the ratio of species tends toward equality, they complement each other, leading to joint species increase in the direction of the line $y = x$.