homework1

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Theorem 2.5.4 (Hewitt-savage 0-1 law) *If* $X_1, X_2, \ldots$ *are i.i.d and* $A \in \mathcal{E}$ *then* $P(A) \in \{0, 1\}$.

Proof. Let $A \in \mathcal{E}$. As in the proof of Kolmogorov's $0 - 1$ law, we will show A is independent of itself, i.e., $P(A) = P(A \cap A) = P(A)P(A)$ so $P(A) \in \{0, 1\}$. Let $A_n \in \sigma(X_1, \ldots, X_n)$ so that

(a)

$$P(A_n \Delta A) \to 0$$

Here $A \Delta B = (A - B) \cup (B - A)$ is the symmetric difference. The existence of the A_n's is proved in part ii of Lemma A.2.1. A_n can be written as $\{\omega : (\omega_1, \ldots, \omega_n) \in B_n\}$ with $B_n \in \mathcal{S}^n$. Let

$$\pi(j) = \begin{cases} j + n & \text{if } 1 \le j \le n \\ j - n & \text{if } n + 1 \le j \le 2n \\ j & \text{if } j \ge 2n + 1 \end{cases}$$

Observing that π^2 is the identity (so we don't have to worry about whether to write π or π^{-1}) and the coordinates ard i.i.d. (so the ermuted coordinates are) gives

(b)

$$P(\omega : \omega \in A_n \Delta A) = P(\omega : \pi\omega \in A_n \Delta A)$$

Now $\{\omega : \pi\omega \in A\} = \{\omega : \omega \in A\}$, since A is permutable, and

$$\{\omega : \pi\omega \in A\} = \{\omega : (\omega_{n+1}, \ldots, \omega_{2n}) \in B\}$$

if we use A'_n to denote the last event then we have

(c)

$$\{\omega : \pi\omega \in A_n \Delta A\} = \{\omega : \omega \in A'_n \Delta A\}$$

Combining (b) and (c) gives

(d)
$$P(A_n \Delta A) = P(A'_n \Delta A)$$

It is easy to see that
$$|P(B) - P(C)| \leq |P(B \Delta C)|$$
so (d) implies $P(A_n), P(A'_n) \to P(A)$. Now $A - C \subset (A - B) \cup (B - C)$ and with a similar inequality for $C - A$ implies $A \Delta C \subset (A \Delta B) \cup (B \Delta C)$.
The last inequality, (d), and (a) imply

$$P(A_n \Delta A'_n) \leq P(A_n \Delta A) + P(A \Delta A'_n) \to 0$$

The last result implies

$$\begin{aligned} 0 &\leq P(A_n) - P(A_n \cap A'_n) \\ &\leq P(A_n \cup A'_n) - P(A_n \cap A'_n) = P(A_n \Delta A'_n) \to 0 \end{aligned}$$

so $P(A_n \cap A'_n) \to P(A)$. But A_n and $A'_n$are independent, so

$$P(A_n \cap A'_n) = P(A_n)P(A'_n) \to P(A)^2$$

This show $P(A) = P(A)^2$, and proves Theorem 2.5.4. $\square$