RECURRENCE RELATIONS

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1. Overview

Definition 1. A *recurrence relation* for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, ..., a_{n-1}$ where $\{n \in \mathbb{Z} | n \geq n_0\}$ and $n_0 \in \mathbb{Z}^+$. A sequence is called a *solution* of a recurrence relation if its terms satisfies the recurrence relation.

2. Linear Homogenous Recurrence Relations

One important class of recurrence relations can be solved systematically:

Definition 2. A *linear homogenous recurrence relation (LHRL) of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} \tag{1}$$

where $c_1, c_2, ..., c_k$ are real numbers, and $c_k \neq 0$.

A sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions: $a_0 = C_0, a_1 = C_1, ..., a_{k-1} = C_{k-1}$.

E.g. the Fibonacci recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogenous recurrence relation of degree two. On the other hand, the Hanoi recurrence relation $H_n = 2H_{n-1} + 1$ is not homogenous (cf. the $+1$).

$\diamond$

When solving recurrence relations we basically look for solutions of the form $a_n = r^n$, where r is a constant. It is easy to see that $a_n = r^n$ is a solution of the recurrence relation iff r is a solution of the so-called *characteristic equation*:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_{k-1} r - c_k = 0$$

which follows from direct substitution of $a_n = r^n$ into (1) and division by r^{n-k}.

Theorem 1. LHRL of degree two: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two *distinct* roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, 3, ...$ where α_1, α_2 are constants.

Proof. We shall prove the *if* part and the *only if* part separately.
The *if* part: We must show that if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ then the sequence $\{a_n\}$ solves our recurrence relation. Since r_1, r_2 are roots of $r^2 - c_1 r - c_2 = 0$ it follows that $r_1^2 = c_1 r_1 + c_2$, $r_2^2 = c_1 r_2 + c_2$.

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n_2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\ &= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n. \end{aligned}$$

Since we recover $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ we see that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

The *only if* part: To show that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, ...$ for some constants α_1, α_2, suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. It will be shown that there are constants α_1 and α_2 so that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions. I.e. we require

$$a_0 = C_0 = \alpha_1 + \alpha_2,$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

Eliminating α_1 and α_2 respectively from these equations we readily see that

$$\alpha_1 = \frac{(C_1 - C_0 r_2)}{r_1 - r_2}$$

and since $\alpha_2 = C_0 - \alpha_1$:

$$\alpha_2 = \frac{(C_0 r_1 - C_1)}{r_1 - r_2}$$

where these expressions for α_1 and α_2 depend on the fact that $r_1 \neq r_2$ (when $r_1 = r_2$ the theorem is not true). Hence, for these values of α_1 and α_2, the sequence $\{a_n\}$ with $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions. Since this recurrence relation and these initial conditions uniquely determine the sequence, it follows that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. □

E.g. in order to find an explicit formula for the Fibonacci sequence ($f_n = f_{n-1} + f_{n-2}$ where $f_0 = 0$ and $f_1 = 1$) we note that the characteristic equation is $r^2 - r - 1 = 0$. The characteristic roots happen to be $\phi = (1 + \sqrt{5})/2$ and $\Phi = (1 - \sqrt{5})/2$ (the golden ratio and its conjugate). Hence,

$$f_n = \alpha_1 \phi^n + \alpha_2 \Phi^n,$$

for some constants α_1 and α_2. Imposing the initial conditions $f_0 = 0$, $f_1 = 1$ we have $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 \phi + \alpha_2 \Phi = 1$ which we can solve simultaneously to get $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$. Consequently,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

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Theorem 2. Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only *one* root r_0 (rather than two roots as in Theorem 1). A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, ...$, where α_1 and α_2 are constants.

Proof. This can be demonstrated using a technique analogues to the one employing in proving Theorem 1. □

E.g. Consider the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$. The characteristic equation $r^2 - 6r + 9 = 0$ has only one root: $r = 3$, implying that the solution of the recurrence relation has the form

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n.$$

The initial conditions entail $a_0 = 1 = \alpha_1$ and $a_1 = 6 = 3\alpha_1 + 3\alpha_2$ i.e. solving simultaneously we get $\alpha_1 = 1$ and $\alpha_2 = 1$. Finally, we have the answer

$$a_n = 3^n + n3^n = 3^n(1 + n).$$

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Having considered the case where the degree k of our LHRL is two, let us now generalize to arbitrary values of k (under the assumption that all roots of the characteristic equation are distinct):

Theorem 3. Let $c_1, c_2, ..., c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - ... - c_k = 0$ has k distinct roots $r_1, r_2, ..., r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ iff

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + ... + \alpha_k r_k^n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for $n = 0, 1, 2, ...,$ where $\alpha_1, \alpha_2, ..., \alpha_k$ are constants.

E.g. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ if $a_0 = 2,_1 = 5$ and $a_2 = 15$. The characteristic equation is clearly $r^3 - 6r^2 + 11r - 6 = 0$, which can be factorized into $(r-1)(r-2)(r-3) = 0$ demonstrating that $r_1 = 1, r_2 = 2$ and $r_3 = 3$. Hence,

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$$

where the α_i are determined using the initial conditions. Explicitly, we have to solve three simultaneous equations viz.

$$\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3, \\ a_1 &= 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3, \\ a_2 &= 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3. \end{aligned}$$

whence $(\alpha_1, \alpha_2, \alpha_3) = (1, -1, 2)$. It follows that the unique solution to the recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

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We can now state the most general result about LHRL with constant coefficients, allowing the characteristic equation to have multiple roots. The key point is that for each root r of the characteristic equation, the general solution has a summand of the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m-1$, with m the multiplicity of the root.

Theorem 4. Let $c_1, c_2, ..., c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - ... - c_k = 0$ has t distinct roots $r_1, r_2, ..., r_t$ with multiplicity $m_1, m_2, ..., m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + ... + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ iff

$$\begin{aligned} a_n &= (\alpha_{1,0} + \alpha_{1,1} n + ... + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &\quad + (\alpha_{2,0} + \alpha_{2,1} n + ... + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &\quad + ... + (\alpha_{t,0} + \alpha_{t,1} n + ... + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \\ &= \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} \alpha_{i,j} n^j r_i^n \end{aligned}$$

for $n = 0, 1, 2, ...,$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

E.g. If we consider the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1, a_1 = -2, a_2 = -1$ then the characteristic equation turns out to be $r^3 + 3r^2 + 3r + 1 = 0$. This can be factorized as $(r+1)^3$ implying that we have a single root $r_1 = -1$ of multiplicity 3. By Theorem 4 the solutions of the recurrence relation are of the form

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n.$$

As always, we determine the values of the $\alpha_{1,i}$ using the initial conditions (forming three simultaneous equations).

$$a_0 = 1 = \alpha_{1,0},$$
$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2},$$
$$a_2 = -1 = \alpha_{1,0} + \alpha_{1,1}2 + \alpha_{1,2}4.$$

It turns out that

$$a_n = (1 + 3n - 2n^2)(-1)^n$$

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3. Linear Inhomogeneous Recurrence Relations

Definition 3. A *linear inhomogeneous recurrence relation with constant coefficients* (LIRL) is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

where $c_1, c_2, ..., c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n. The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ is known as the *associated homogeneous recurrence relation* and it plays an important role in the solution of the LIRL.

E.g. In $a_n = 3a_{n-1} - 2a_{n-2} + n3^n + n!$ we have $F(n) = n3^n + n!$ and the associated linear homogeneous recurrence relation is $a_n = 3a_{n-1} - 2a_{n-2}$.

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Theorem 5. If $\{a_n^{(p)}\}$ is a particular solution of the LIRL with constant coefficients $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$, then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated LHRL $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$.

Proof. We're given that $\{a_n^{(p)}\}$ solves the LIRL i.e.

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + ... + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose $\{b_n\}$ is a second solution of the LIRL so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + ... + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + ... + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated LHRL, say, $\{a_n^{(h)}\}$. Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n. □

E.g. Upon solving $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$, we see that $a_n^{(h)} = \alpha 3^n$ where α is a constant. Since $F(n) = 2n$ make an educated guess that the particular solution has the form $a_n^{(p)} = cn + d$, where c and d are constants. Substituting this into $a_n = 3a_{n-1} + 2n$ it follows that $cn + d$ is a solution iff $2 + 2c = 0$ and $2d - 3c = 0$ i.e. iff $c = -1$ and $d = -3/2$. Consequently $a_n^{(p)} = -n - 3/2$ and by Theorem 5 all solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha 3^n,$$

where α is a constant. By imposing $a_1 = 3$ we find that $\alpha = 11/6$.

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E.g. In finding all solutions of the LIRL $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ we readily see that the solutions of the associated LHRL $a_n = 5a_{n-1} - 6a_{n-2}$ are $a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n$, where α_1 and α_2 are constants. Now, since $F(n) = 7^n$ we posit the following trial solution $a_n^{(p)} = C \cdot 7^n$ which upon substitution into $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ yields $C = 49/20$ i.e. $a_n^{(p)} = (49/20) \cdot 7^n$ is a particular solution. Theorem 5 now tells us that all solutions are of the form

$$a_n = \alpha_1 3^n + \alpha_2 2^n + (49/20)7^n.$$

$\diamond$

The examples above are formally encoded in the following theorem

Theorem 6. Suppose that $\{a_n\}$ satisfies the LIRL $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$, where the c_i are real numbers and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0)s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is *not* a root of the characteristic equation of the associated LHRL, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n.$$

When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n.$$

E.g. Suppose $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$. What form does the particular solution of the LIRL have when (i) $F(n) = 3^n$, (ii) $F(n) = n3^n$, (iii) $F(n) = n^2 2^n$ and (iv) $F(n) = (n^2+1)3^n$?
The associated LHRL, $a_n = 6a_{n-1} - 9a_{n-2}$, has characteristic equation $r^2 - 6r + 9 = 0$ which in turn has a root, $r = 3$, of multiplicity two. Applying Theorem 6 we see that $s = 3^n$ in (i), (ii) and (iv) - the same as the characteristic root $r = 3$. Hence, (i): $p_0 n^2 3^n$, (ii): $n^2(p_1 n + p_0)3^n$ and (iv): $n^2(p_2 n^2 + p_1 n + p_0)3^n$. In (iii) $s = 2$ which is distinct from the characteristic root $r = 3$ so (iv): $(p_2 n^2 + p_1 n + p_0)2^n$.

$\diamond$

4. Generating Functions and Recurrence Relations

Definition 4. The *generating function for the sequence* $a_0, a_1, a_2, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k.$$

E.g. If the sequence is $1, 1, 1, 1, 1, 1$ then the generating function is $G(x) = 1 + x + x^2 + x^3 + x^4 + x^5$. This, in turn, is a geometric series with initial element 1 and common ration x, whence $G(x) = (x^6 - 1)/(x - 1)$.

$\diamond$

E.g. Suppose instead that $a_k = C_k^m$ (the binomial coefficient), where $m \in \mathbb{Z}^+$ and $k = 0, 1, 2, \ldots, m$. Then the generating function is clearly $G(x) = C_0^m + C_1^m x + C_2^m x^2 + \ldots + C_m^m x^m$, which by the Binomial Theorem is $G(x) = (1 + x)^n$.

$\diamond$

Theorem 7. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \qquad \text{and} \qquad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k.$$

Definition 5. Let $u \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)(u-2)...(u-(k-1))/k!, & \text{if } k > 0. \\ 1, & \text{if } k = 0. \end{cases}$$

E.g. Using Maclaurin expansion we can easily demonstrate the validity of the *extended binomial theorem*:

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k = 1 + ux + \frac{u(u-1)}{2!}x^2 + \frac{u(u-1)(u-2)}{3!}x^3 + ...$$

where $\{x \in \mathbb{R} | -1 < x < 1\}$ and $u \in \mathbb{R}$. E.g. when $u = -\frac{1}{2}$ then $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + ...$ $\diamond$

Notice that if we have $\binom{-n}{r}$ where $n \in \mathbb{Z}^+$, then we may express the coefficient as $(-1)^r \binom{n+r-1}{r}$ or simply $\binom{-n}{r} = (-1)^r C_r^{n+r-1}$.

4.1. **Recurrence relations.** We can find the solution to a recurrence relation and its initial conditions by finding an explicit formula for the associated generating function. More explicitly, suppose we have the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ with $a_0 = 1$. Now recall that we define the generating function of the sequence $a_0, a_1, a_2, ...$ as $G(x) = \sum_{n=0}^{\infty} a_n x^n$. For this reason, multiplying through by x^n in the recurrence relation and summing between 1 (NOT 0) and ∞ we get:

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \left(8a_{n-1}x^n + 10^{n-1}x^n\right).$$

Thus,

$$\begin{aligned} G(x) - 1 &= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\ &= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\ &= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\ &= 8xG(x) + x/(1-10x), \end{aligned}$$

since the last sum forms an infinite geometric series with initial element x and common ratio $10x$. Therefore, we have $G(x) - 1 = 8xG(x) + x/(1-10x)$. Solving for $G(x)$ shows that

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

which in terms of partial fractions becomes

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

Using the extended binomial theorem on both terms respectively we get

$$\begin{aligned} G(x) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n, \end{aligned}$$

from which we immediately conclude that $a_n = \frac{1}{2}(8^n + 10^n)$, as we defined $G(x) = \sum_{i=0}^{\infty} a_n x^n$. Thus, we have found an explicit expression for our recurrence relation.

E.g. Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n+1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$, to specify the order of multiplication (e.g. in $x_1 \cdot x_2 \cdot x_3$ we have $((x_1 \cdot x_2) \cdot x_3)$ and $(x_1 \cdot (x_2 \cdot x_3))$ so $C_2 = 2$). Solve the recurrence relation.

Consider the $n+1$ numbers:

$$x_0 \cdot x_1 \cdot \ldots \cdot x_k \cdot x_{k+1} \cdot x_{k+2} \cdot \ldots \cdot x_n$$

Clearly, there are C_k ways to parenthesize the first chunk $x_0 \cdot x_1 \cdot \ldots \cdot x_k$, and there are C_{n-k-1} ways to parenthesize the second chunk $x_{k+1} \cdot x_{k+2} \cdot \ldots \cdot x_n$. Thus, the total number of parenthesis combinations of chunks one and two is $C_k C_{n-k-1}$. However, we must sum over all possible two-chunks to find the total number of ways to parenthesize $x_0 \cdot x_1 \cdot \ldots \cdot x_n$:

$$\begin{aligned} C_n &= C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-2} C_1 + C_{n-1} C_0 \\ &= \sum_{k=0}^{n-1} C_k C_{n-k-1}. \end{aligned}$$

The initial conditions of this recurrence relation are $C_0 = C_1 = 1$.

In order to solve the recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ with $C_0 = C_1 = 1$ we use generating functions. First we will show that the generating function satisfies $x(G(x))^2 - G(x) + 1 = 0$. Take the recurrence relation $\sum_{k=0}^{n-1} C_k C_{n-k-1} - C_n = 0$ and multiply through by x^n and sum between 1 and ∞:

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n - \sum_{n=1}^{\infty} C_n x^n \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} C_k C_{j-k} x^{j+1} - \sum_{n=1}^{\infty} C_n x^n && (n = j+1) \\ &= x \sum_{j=0}^{\infty} \sum_{k=0}^{j} C_k C_{j-k} x^{j} - \sum_{n=0}^{\infty} C_n x^n + 1 && (C_0 x^0 = 1) \\ &= x \left(\sum_{i=0}^{\infty} C_i x^i \right) \left(\sum_{k=0}^{\infty} C_k x^k \right) - \sum_{n=0}^{\infty} C_n x^n + 1 && \text{(product rule)} \\ &= x(G(x))^2 - G(x) + 1. \end{aligned}$$

Clearly, $x(G(x))^2 - G(x) + 1 = 0$ is a hidden quadratic with solution

$$G(x) = \frac{-x - \sqrt{1-4x}}{2x}.$$

Using the extended binomial theorem we can rewrite this as

$$\begin{aligned} G(x) &= \frac{1 - \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k}{2x} \\ &= \frac{1}{2} x^{-1} - \sum_{k=0}^{\infty} \binom{1/2}{k} (-1)^k 2^{2k-1} x^{k-1} \\ &= \sum_{k=1}^{\infty} \binom{1/2}{k} (-1)^{k+1} 2^{2k-1} x^{k-1} && \text{(Separating } k=0\text{)} \\ &= \sum_{n=0}^{\infty} \binom{1/2}{n+1} (-1)^n 2^{2n+1} x^n && (k = n+1) \end{aligned}$$

A neat way of writing $\binom{1/2}{n+1}(-1)^n 2^{2n+1}$ when $n \in \mathbb{N}$ is $\frac{1}{n+1}\binom{2n}{n}$. To see this notice that

$$\begin{aligned}
\binom{1/2}{n+1}(-1)^n 2^{2n+1} &= \frac{\overbrace{\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}-1\right)\cdot\left(\frac{1}{2}-2\right)\cdot\ldots\cdot\left(\frac{1}{2}-n\right)}^{n+1\text{ factors}}}{(n+1)!}(-1)^n 2^{2n+1} \\
&= \frac{\left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\ldots\cdot\left(\frac{1-2n}{2}\right)}{(n+1)!}(-1)^n 2^{2n+1} \\
&= \frac{\frac{1}{2^{n+1}}(-1)^n(2n-1)!!}{(n+1)!}(-1)^n 2^{2n+1} \\
&= \frac{2^n(2n-1)!!}{(n+1)!} = \frac{2^n(2n)!}{(n+1)!(2n)!!} \\
&= \frac{2^n(2n)!}{(n+1)!n!2^n} = \frac{(2n)!}{(n+1)n!n!} \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

Hence, we conclude that

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n,$$

so that the solution to the recurrence relation is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

$\diamond$