

OPTIMAL HEDGE TRACKING PORTFOLIOS IN A LIMIT ORDER BOOK

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ABSTRACT. Derivative hedging under transaction costs has attracted considerable attention over the past three decades. Yet comparatively little effort has been made towards integrating this problem in the context of trading through a limit order book. In this paper we propose a simple model for a wealth-optimising option seller, who hedges his position using a combination of limit and market orders, whilst facing certain constraints as to how far he can deviate from a targeted (Bachelierian) delta strategy. By translating the control problem into a three-dimensional Hamilton-Jacobi-Bellman quasi-variational inequality and solving numerically, we are able to deduce optimal limit order quotes alongside the regions surrounding the targeted delta surface in which the option seller must place limit orders vis-à-vis the more aggressive market orders. Our scheme is shown to be monotone, stable, and consistent and thence, modulo a comparison principle, convergent in the viscosity sense.

Keywords: Delta Hedging and Limit Order Book and HJB QVI.

1. THE PROBLEM OF HEDGING WITH TRANSACTION COSTS

1.1. Introduction and Literature Survey. One of the core principles of prudent derivatives trading is the ability to place timely and accurate hedges of option portfolios. Reasons therefore extend beyond the curtailment of risk: by removing the delta-exposure of an option trade, a bet on implied versus realised volatility can be constructed, which in turn can be used to harvest volatility arbitrage [2] [27]. Yet, whilst hedging remains thoroughly understood in the axiomatic framework of Black-Scholes [13], the sheer empirical implausibility of this financial landscape in itself calls for an afterthought. Given that markets manifestly

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are *not* frictionless, any attempt at replicating a Black-Scholes hedge will theoretically result in infinite transaction costs [64]. It is precisely this uncomfortable fact which lies at the heart of much of the hedging literature produced over the past three decades. For example, Leland [47] Hoggard et al. [43] notably argue that for fixed-interval rebalancing of an underlying which incurs transaction costs in proportion to its price¹, the Black Scholes equation takes on a modified volatility in order to match expected costs between derivative and replicating portfolio. Whilst illuminating, this model nonetheless faces criticism from several directions. First, as shown by Kabanov and Safarian [44], Leland’s conjecture of a vanishing hedging error is erroneous; secondly, as pointed out by Hodges et al. [42] [21], Leland’s strategy clearly falls short of attaining any sense of optimality. In its stead, Hodges et al. ingeniously propose a utility indifference argument, in which the price of an option is determined as the amount of money required by a rational agent to make him indifferent to the prospect of being able to sell that option. Whilst this approach uniquely fixes the price and optimal hedging strategy in a market with friction, it comes at the cost of diluting the principle of pure no-arbitrage pricing by the subjective utility preferences of the agent. Furthermore, as the HJB equation of the problem in itself is not analytically tractable, one must invariably resort to tedious numerical procedures [42], or, more boldly, asymptotic solutions as championed by Whalley and Wilmott [62]. The latter duo, in particular, notes that for small proportional transaction costs, one effectively establishes a “no trade” region surrounding the original Black Scholes delta, Δ_{BS} , such that the option book is rebalanced just in case that the current hedge ratio pierces the surrounding boundary. Although the associated analytic boundary captures some intuitively pleasing features such as enclosing on the delta when transaction costs go to zero, or when the agent’s risk aversion increases, it is important to appreciate that the Whalley-Wilmott result does not codify the full richness of the numerical solution. Indeed, as argued by Zakamouline [63] [64], the asymptotic approximation performs notably worse than its numerical counterpart for realistic model parameters.

Meanwhile, following Ho and Stoll’s [41] trailblazing work from 1981 on optimal bid-ask quotes for wealth maximising dealers, recent years have been marked by a veritable boom in research on optimal market making in limit order books (LOBs). Avellaneda and Stoikov [7] have arguably played a pivotal role in facilitating this: by formalising the control problem in the context of electronic market making—and justifying the arrival rate of market orders using principles of econo-physics—they effectively brought Ho and Stoll into the 21st century. Much momentum has subsequently been garnered, in part due to the contributions of Gueant et al. [36], who transform the associated HJB equation into a system of linear ODEs, thereby enabling an actual solution, and Guilbaud and Pham [38], who explore the inherent trade-off between being able to place “slow, but cost-effective” limit-orders, and “quick, but costly” market orders. Indeed, one cannot neglect Cartea and Jaimungal who have made a small industry out of the field with numerous noteworthy papers including [16] [17] [18] [20]. Full justice to the rich literature on order-book prescriptivism cannot be done here; however, for readers interested in poignant surveys, alongside considerable original research, we refer to the excellent textbooks by Gueant [35] and Cartea et al. [19].

Surprisingly, comparatively little work has been done on trying to merge optimal hedging with the literature on dealings through a limit order book. To the best of our knowledge, only relatively few papers deal with this matter. Key citations here include Agliardi and Gencay’s [6] discrete time investigation, in which an explicit solution is found for an option hedger who aims to minimise illiquidity costs and the hedging error, and Rogers and Singh’s [53] determination of the optimal hedging policy at the face of nonlinear transaction costs incurred as a function of rate of change of the portfolio. Moreover, on the topic of pricing

¹Specifically, if S_t is the mid-price of underlying, and $N \in \mathbb{R}$ is the number of units acquired (sold) then $\lambda|N|S_t$ is the spread paid for the pleasure, where $\lambda \in \mathbb{R}_+$.

and hedging with temporary and permanent market impact, noteworthy papers include Li and Almgren [48] and Gueant and Pu [37]. Whilst superficially similar in nature, these papers employ fundamentally different philosophies in their interpretations of viable hedging: specifically, for the former, the “right” strategy is mean-reverting around the classical Black-Scholes delta (moving towards Δ_{BS} with a trading intensity proportional to the degree of mis-hedge and inversely proportional to illiquidity), while for the latter the hedging strategy in itself is changed to account for market micro-structure effects (ultimately resulting in smoother inventory paths, due to an in-built aversion towards round-trip trading).

In this paper we aim to further bridge the gap between delta hedging and the order book literature. We do so from the perspective of a rational option seller, who derives linear utility from the financial wealth generated from his dealings, and quadratic utility penalisation by deviating from some exogenously specified hedge target. Specifically, our concern here is the optimal policy the hedger should follow, insofar as the underlying asset can be traded using a combination of limit and market orders. The intuition here is that a hedger who is in close proximity to his delta target will be contended with placing slow and cost effective limit orders, while being significantly off-target will induce panic buying or selling through costly but instantaneous market orders. For us, the Δ -strategy we are tracking is for all practical purposes completely generic, although we for illustrative reasons shall restrict our attention to a Bachelierian hedge ratio [8]. Our framework thus finds its intellectual roots in the limit-market order duality developed by Guilbaud and Pham (mentioned above), and Cartea and Jaimungal [20], who consider an analogous set-up, albeit from the perspective of a trader who attempts to track the optimal execution algorithm developed by Almgren and Chriss [5]. However, while the Almgren Chriss strategy in itself codifies market-microstructure effects in virtue of being sensitive to market impact, we stress that the same thing cannot be said of our targeted Bachelier hedge. In this sense, our philosophy sides with the work of Li and Almgren, again mentioned above.

More interesting variations of the targeted Δ are conceivable: for example, it would be opportune to explore tracking the Whalley Wilmott boundary, thereby facilitating entire regions of time-price-inventory space in which the hedger posts no orders at all. Thus, arguing that we are tracking the wrong delta fundamentally misses the point of what we are trying to accomplish. Moreover, while we submit that it ultimately is desirable to obtain a hedge strategy endogenously within a model, we believe there is still room for a model like this. For example, it may be construed in the context of acting in compliance with externally enforced risk governance within a financial institution, or simply as stepping stones towards a better model, which is able to accommodate all of the opportunities that present themselves to option hedgers.

Remark 1. A plurality of sources provide highly readable accounts of the basic mechanisms and nomenclature of the limit order book, which prompts us to forgo a similar overview. Instead, we refer the reader to Gould et al. [34], or Foucault et al. [29].

1.2. Overview. The structure of the rest of this paper is as follows: in section 2 we explicate the fundamental assumptions of the market and the portfolio manager and state the associated control problem. Section 3 is a survey of the exogenously specified hedge ratio: a careful analysis shows that the dynamics of the order book is weakly convergent towards the dynamics associated with the targeted delta. Section 4 exposes the Hamilton-Jacobi-Bellman quasi-variational inequality (HJB QVI) associated with the control problem, and suggests a dimensional reduction. Finding ourselves unable to extract an analytic solution thereto, section 5 sets us up for a numerical scheme which is shown to have the desired properties of monotonicity, stability, and consistency as dictated by Barles [10]. Finally, section 6 provides concrete numerical results, comparing the performance of our algorithm with a more naïve approach to Δ -hedging. Section 7 concludes.

Remark 2. We formulate our problem through the employ of Poisson point processes (see e.g. Last and Brandt [46] for a survey thereof). These constructs find their natural habitat through the inherent discreteness of the order book, and are instances of Hawkes processes, which have successfully been applied in the high-frequency domain of financial modelling in recent years, see e.g. Bacry et al. [9].

2. AN IMPULSE-CONTROL APPROACH TO HEDGING IN THE LOB

2.1. Market Assumptions. We consider a financial market model carried by a filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$, where the augmented filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ is generated by the random measures J_1 , J_2 and J_M which will be defined shortly. For simplicity, the market is assumed interest rate free and of a singular non-dividend paying risky asset (a stock) which is traded in a limit order book. All prices in the order book are assumed to take values from the discrete set $\mathcal{S} := \{\sigma_m n : n \in \mathbb{Z}\}$ where $\sigma_m > 0$ defines the smallest permissible price interval for orders within the book. In particular, we assume that the dynamics of the mid-price follows

$$dS_t = \sigma_m \int_{z \in \mathbb{R}} z(J_1(dt \times dz) - J_2(dt \times dz)), \quad (1)$$

with initial value $s_0 = \sigma_m n$ for some positive integer n .² The mid-price is driven by two independent Poisson random measures J_1 and J_2 on $[0, T] \times \mathbb{R}$ with common intensity measure $\mu_i(dt \times dz) = \gamma_t F_t(dz)dt$, for $i = 1, 2$. Here γ_t is a positive deterministic function which encodes the intensity of upwards (resp. downwards) jumps of the stock price at time t . Furthermore, F_t is a discrete distribution function with support \mathbb{Z}_+ for each t (with subscript $+$ denoting subset of positive values) that captures the conditional distribution of the number of tick points the price jumps, given that a jump occurs at time t .³

The difference between the *best bid*, S_t^b , and the *best ask*, S_t^a , (the so-called *bid-ask spread*) is assumed time dependent and of magnitude

$$S_t^a - S_t^b = 2\Upsilon_t,$$

where Υ_t is a deterministic function taking values in \mathcal{S}_+ . Thus, investors who trade through market orders will minimally pay $S_t^a = S_t + \Upsilon_t$ per share if they wish to buy the stock, and maximally earn $S_t^b = S_t - \Upsilon_t$ per share if they wish to sell the stock, depending on the size of their orders vis-à-vis the number of shares available at the various price levels.

At the aggregate level, we assume that market orders arrive in the limit order book in a manner which can be modelled by an inhomogeneous Poisson process M_t , independent from S_t , with intensity rate

$$\lambda_t = \xi_t \exp\{\kappa_t \Upsilon_t\},$$

where ξ_t and κ_t are positive deterministic functions, and we denote the associated jump measure with J_M . Clearly, whilst market orders are guaranteed instantaneous execution, the liquidity taking fees they incur may be less than appealing. This is not to say that one optimally should opt for the limit order alternative: despite superior prices, there is no guarantee of execution at all.

²In practice, orders trading in the book are on a grid of integer multiples of the tick size, while the (non-tradable) “mid-price” is defined as the average of the best bid and best ask price, thus possibly taking values at half ticks where no orders are permitted. For simplicity, we treat all orders and prices to be taking values on the same set \mathcal{S} .

³A couple of comments are worth attaching to (1): first, the dynamics is form-invariant upon rewriting it in terms of the compensated Poisson random measures $\tilde{J}_i(dt \times dz) = J_i(dt \times dz) - \mu_i(dt \times dz)$. Hence, S_t is an \mathbb{F} -martingale. Secondly, there is a potential pathology nested in the dynamics in the sense that it admits negative price processes, i.e. $S_t \in \mathcal{S}$. Nonetheless, we shall assume that the probability of this occurring is sufficiently low to be ignored (an otherwise very reasonable assumption over short temporal horizons / for low intensity processes) or, at least, that all trading activity in the stock stops if the price reaches zero.

Finally we assume that the functions γ_t , F_t , Υ_t and λ_t are sufficiently well behaved for the dynamics of the prices and order arrivals to be well defined; for instance, Υ_t being bounded and $\gamma_t \int_{\mathbb{R}} z^2 F_t(dz) \leq K$ for a constant K for all t , which makes S_t a square-integrable martingale, and $\int_{[0,T]} (\gamma_t + \lambda_t) dt < \infty$ for the the jump rate intensities such that the inhomogeneous Poisson processes are well defined (requiring the intensity functions to be either bounded or continuous would suffice).

2.2. Portfolio Manager Assumptions. We consider the case of a portfolio manager who at time $t = 0$ sells off \mathfrak{N} European call options of strike K and maturity T due for physical delivery. Modulo an exogenously specified tolerance for risk, his aim is to keep his portfolio “delta neutral” throughout, whilst simultaneously maximising his terminal payoff at time T . Here, *delta neutrality* is to be understood as a pre-specified optimal inventory level $\Delta_t = \Delta(t, S_t)$ —the number of shares the portfolio manager *ideally* wishes to keep in his portfolio to hedge his short option position. If his inventory level at time t , denoted Q_t , is found to be in excess of Δ_t , the portfolio manager is incentivised to offload some of his shares. Conversely, if Q_t falls short of Δ_t , he is incentivised to acquire shares. To this end, we assume that the portfolio manager places unit-sized limit orders and (if necessary) integer-sized market orders in the order book, but only one of the two at any given time. From this, one is prompted to ask the following questions, which will form the backbone of this paper:

Question 1. *When should the portfolio manager trade in limit orders and when should he trade in market orders? In particular, regarding the former, which price quotes should he employ?*

Whilst market sell (buy) orders are assumed to take place at the best bid (ask), we assume the portfolio manager is at liberty to decide how deep in the limit order book he places his limit orders. Specifically, let $S_t^- = S_t + \delta_t^-$ be the price level at which the portfolio manager places his ask quote at time t , and let $S_t^+ = S_t - \delta_t^+$ be the price level at which the portfolio manager places his bid quote at time t , where $\delta_t^\pm \in \mathcal{S}_+$ is known as the *spread*. For mnemotechnical purposes we designate objects that give rise to a lower (higher) inventory by the superscript $- (+)$.

To capture the execution risk inherent to limit orders, we assume that the probability of a sell (buy) limit order being lifted, given that a market order arrives, is of the form $\exp\{-\kappa_t \delta_t^-\}$, $(\exp\{-\kappa_t \delta_t^+\})$. Thus, from the market assumptions, it follows from the thinning property of Poisson processes (of the market order arrivals M_t) that the portfolio manager’s successful limit sell and buy order executions are inhomogeneous Poisson processes L_t^- and L_t^+ , with intensities

$$\lambda_t \exp\{-\kappa_t \delta_t^-\} \quad \text{and} \quad \lambda_t \exp\{-\kappa_t \delta_t^+\} \quad (2)$$

respectively. The intuition here is clear: the further away from the mid-price the portfolio manager places his limit quotes, the less likely it is they will be executed. This disincentivises the portfolio manager from posting extreme limit order quotes, even though he obviously would benefit considerably from their execution. Insofar as a limit order placed at time t fails to be executed, we assume that the portfolio manager cancels it immediately, only to replace it with a (possibly) updated quote.

As the alternative to posting limit orders, let $0 \leq \tau_1^\pm \leq \tau_2^\pm \leq \dots < T$ denote the *stopping times* at which the investor chooses to place market orders. We have that the total *cash position* $B_t \in \mathcal{S}$ (bank holding) and inventory level $Q_t \in \mathbb{Z}$ of the portfolio manager obey the jump formulae

$$\begin{aligned} dB_t &= \mathbf{1}_{\{Q_t > \Delta_t\}} (S_t + \delta_t^-) dL_t^- - \mathbf{1}_{\{Q_t < \Delta_t\}} (S_t - \delta_t^+) dL_t^+, \\ dQ_t &= -\mathbf{1}_{\{Q_t > \Delta_t\}} dL_t^- + \mathbf{1}_{\{Q_t < \Delta_t\}} dL_t^+, \end{aligned} \quad (3)$$

for $\tau_{i-1}^\pm < t < \tau_i^\pm$, i.e. in the intermediate time interval between market orders. Note here that the portfolio manager's trading activity—as previously described in this section—is effectively constrained by the indicator functions: $\mathbf{1}_{\{Q_t > \Delta_t\}}$ codifies a sell initiative, potentially decreasing the inventory and increasing the cash position, while $\mathbf{1}_{\{Q_t < \Delta_t\}}$ codifies a buy initiative (potentially increasing inventory and decreasing cash). Further, a market order posted by the investor at time τ_i^\pm will bring the cash and inventory to a new state

$$\begin{aligned} B_{\tau_i^\pm} &= B_{\tau_i^\pm-} + \mathbf{1}_{\{Q_{\tau_i^\pm-} > \Delta_{\tau_i^\pm}\}}(S_{\tau_i^\pm} - \Upsilon_{\tau_i^\pm}) - \mathbf{1}_{\{Q_{\tau_i^\pm-} < \Delta_{\tau_i^\pm}\}}(S_{\tau_i^\pm} + \Upsilon_{\tau_i^\pm}), \\ Q_{\tau_i^\pm} &= Q_{\tau_i^\pm-} - \mathbf{1}_{\{Q_{\tau_i^\pm-} > \Delta_{\tau_i^\pm}\}} + \mathbf{1}_{\{Q_{\tau_i^\pm-} < \Delta_{\tau_i^\pm}\}}. \end{aligned} \quad (4)$$

A posting of two or more market orders at the time should be understood as a concatenation of the mapping $\Gamma : (B_{t-}, Q_{t-}) \mapsto (B_t, Q_t)$ defined by (4), that is, $(B_{\tau_{i+1}^\pm}, Q_{\tau_{i+1}^\pm}) = \Gamma(\Gamma(B_{\tau_i^\pm-}, Q_{\tau_i^\pm-}))$. Finally, as a result of the fact that $0 \leq \Delta_t \leq \mathfrak{N}$ note that Q_t takes values in the finite set $\mathcal{Q} := \{0, \dots, \mathfrak{N}\}$.

At maturity we imagine that one of the following scenarios obtains: if the options are out of the money, the portfolio manager immediately liquidates his inventory using market orders. On the other hand, if the options are at/in the money the option holders exercise their right to buy the stock for the strike price: insofar as there is a mismatch between the portfolio managers inventory and the \mathfrak{N} stocks due for delivery, he acquires the difference, again using market orders. Here we note that given that expiry and delivery can be temporally separated events, it is conceivable that the portfolio manager would wish to acquire the difference using, say, the execution algorithm proposed by Cartea and Jaimungal [20]. Furthermore, had the options been cash settled, the portfolio manager would naturally be incentivised to offload, rather than acquire shares, in order to finance the claim made by his counter-parties [37]. These considerations present obvious ways in which our work may be extended.

Finally, suppose the portfolio manager derives *linear utility* from his level of *financial wealth*, but incurs a quadratic lifetime penalisation from any deviation from the target hedge portfolio. The control problem to be solved can thus be stated as follows

$$\begin{aligned} V(t, b, s, q) &= \sup_{\{\delta^\pm, \tau^\pm\} \in \mathcal{A}(t, b, s, q)} \mathbb{E}_{t, b, s, q} \left[B_\theta + \mathbf{1}_{\{0 < S_\theta < K\}} Q_\theta (S_\theta - \Upsilon_\theta) \right. \\ &\quad \left. + \mathbf{1}_{\{S_\theta \geq K\}} [\mathfrak{N}K - (\mathfrak{N} - Q_\theta)(S_\theta + \Upsilon_\theta)] \right. \\ &\quad \left. - \eta \int_t^\theta (Q_u - \Delta_u)^2 du \right], \end{aligned} \quad (5)$$

where $\mathbb{E}_{t, b, s, q}[\cdot]$ is the conditional expectation given $(B_t, S_t, Q_t) = (b, s, q) \in \mathcal{S} \times \mathcal{S}_+ \times \mathcal{Q}$, and the supremum runs over all admissible controls, $\mathcal{A}(t, b, s, q)$, i.e. the set of all \mathbb{F} -predictable limit order spreads with values in \mathcal{S}_+ sufficiently integrable for (3) to be well behaved, and all \mathbb{F} -stopping times bounded above by T . Here we have introduced the stopping time $\theta := \theta_{\mathcal{S}} \wedge T$ where $\theta_{\mathcal{S}} = \inf\{u \geq t : S_u \notin \mathcal{S}_+\}$ is the first time S_t reaches zero. Hence, θ codifies that the investor may trade either until maturity, or until the game effectively is over, i.e. the event when the stock happens to reach zero and expires worthless. Finally, $\eta \in \mathbb{R}_+$ is a parameter which captures the portfolio managers “readiness” to depart from the desired hedge strategy Δ (clearly, the greater the η , the less prone the portfolio manager will be to depart from the prescribed strategy). In practical terms, η may be seen as a function of compliance with external (regulatory) risk measures, as well as internal (company specific) risk management. Notice though, that analogous to more traditional risk aversion parameters, there is an inexorable nebulosity wrapped around this construct: whether real world risk preferences

can be accurately mapped to this singular parameter, and if it can be done with a reasonable degree of empirical accuracy is arguably questionable.⁴

3. THE QUESTION OF THE Δ

The question of an optimal hedge strategy for jump processes in a market with friction is one of considerable complexity which we shall pass over in silence. As suggested above, our main concern here lies with the duality offered by the limit order book trading strategies; hence, the Δ we will consider is largely illustrative in nature. Specifically, in a slight variation of the work by Bachelier⁵, we shall suppose that the portfolio manager aims to track a hedge strategy as though he were trading in a driftless Arithmetic Brownian Motion economy with time dependent volatility, i.e. as though the market were frictionless with price dynamics

$$dS_t = \sigma_t dW_t, \quad (6)$$

where W_t is a Wiener process and $\sigma : [0, T] \mapsto \mathbb{R}_+$ is a deterministic function. Using a standard no-arbitrage argument one may readily show that strike K maturity T call options should be priced according to

$$C_t^{K,T} = (S_t - K)\Phi(\delta_t) + \Sigma_t \phi(\delta_t), \quad (7)$$

where $\delta_t \equiv \delta(t, S_t) = \Sigma_t^{-1}(S_t - K)$, and $\Sigma_t \equiv (\int_t^T \sigma_u^2 du)^{1/2}$, and we have introduced the usual functions: $\Phi(\cdot)$ as the standard normal cdf, and $\phi(\cdot)$ as the standard normal pdf. Hence, from the net portfolio position $B_t + \Delta_t S_t - \mathfrak{N}C_t$, to hedge a short position in \mathfrak{N} such call options, one should hold $\Delta_t = \mathfrak{N}\partial_S C_t$ or, equivalently,

$$\Delta_t = \mathfrak{N}\Phi(\delta_t), \quad (8)$$

units of the underlying asset, where we have used the standard identity $\phi'(x) = -x\phi(x)$. Viewed as a function of (t, S_t) we note that Δ_t has the obvious properties that $\Delta \rightarrow \mathfrak{N}$ when the options are deep in the money ($S_t \gg K$) and $\Delta \rightarrow 0$ when the options are deep out of the money ($S_t \ll K$). As $t \rightarrow T$ the transition between these two extremes becomes increasingly steep in its rise, ultimately converging towards the step function $\mathfrak{N}\mathbf{1}\{S_T \geq K\}$ at expiry. Thus, the Δ of at the money calls is exceedingly sensitive to fluctuations in the price process near expiry, potentially resulting in the acquisition or decumulation of a large amount of shares in a short span of time. An illustration of this Δ is provided in figure 1 for constant parameters.

Again, we emphasise that this choice largely is to get the ball rolling: the reader is encouraged to experiment with alternative specifications. However, forbye the neglected issue of market friction, we also note that the hedge ratio (8) is not altogether groundless; specifically, we have the following noteworthy result:

Proposition 1. *In the infinite intensity limit, the jump dynamics of the mid-price, (1), converges in distribution to the Bachelier dynamics, (6). In particular, for large $\bar{\gamma}$ we have that*

$$\sigma_t \approx \sigma_m \sqrt{\overline{2\mathbb{E}[Z^2]}\bar{\gamma}}, \quad (9)$$

where the over-line designates the function mean taken over the interval $[0, t]$.

Proof. We refer the reader to appendix A for a proof of this result. \square

⁴Tentatively, one might try to Monte Carlo simulate portfolio returns under various η specifications and study the associated portfolio return distributions.

⁵Louis Bachelier is widely credited as the father of mathematical finance. In his doctoral dissertation *The Theory of Speculation* [8] Bachelier introduced Brownian motion in the modelling of stock prices. Specifically, he assumed Arithmetic Brownian Motion with constant parameters: $dS_t = \mu dt + \sigma dW_t$.

Remark 3. The results presented here may be viewed as an abstract generalisation of the textbook result that $(N_t - \gamma t)/\sqrt{\gamma}$ converges in distribution to $N(0, t)$ as $\gamma \rightarrow \infty$, where $N_t \sim \text{Pois}(\gamma t)$ cf. e.g. equation (2.81) p. 53 of Cont and Tankov [23].

Remark 4. To accommodate the fact that only integer multiples of the underlying can be held (and thus, to avoid penalising the portfolio manager unnecessarily), we shall in practice track the integer equivalent of the Bachelier delta, i.e. $\Delta_t = \lfloor \mathfrak{N}\Phi(\delta_t) \rfloor$, where $\lfloor \cdot \rfloor$ designates the nearest integer function.

4. THE HAMILTON-JACOBI-BELLMAN FORMULATION

Following standard arguments in stochastic control theory, we postulate that the optimal value function (5) is the unique viscosity solution of the following Hamilton-Jacobi-Bellman quasi variational inequality (HJB QVI):⁶

$$\begin{aligned} \max \left\{ \partial_t V + \gamma_t \int_{\mathbb{R}} [V(t, b, s + \sigma_m z, q) - 2V(t, b, s, q) + V(t, b, s - \sigma_m z, q)] F_t(dz) \right. \\ + \sup_{\delta_t^- \in \mathcal{S}_+} \lambda_t e^{-\kappa_t \delta_t^-} [V(t, b + (s + \delta_t^-), s, q - 1) - V(t, b, s, q)] \mathbf{1}_{\{q > \Delta(t, s)\}} \\ + \sup_{\delta_t^+ \in \mathcal{S}_+} \lambda_t e^{-\kappa_t \delta_t^+} [V(t, b - (s - \delta_t^+), s, q + 1) - V(t, b, s, q)] \mathbf{1}_{\{q < \Delta(t, s)\}} \\ - \eta(q - \Delta(t, s))^2 \\ ; [V(t, b + (s - \Upsilon_t), s, q - 1) - V(t, b, s, q)] \mathbf{1}_{\{q > \Delta(t, s)\}} \\ \left. + [V(t, b - (s + \Upsilon_t), s, q + 1) - V(t, b, s, q)] \mathbf{1}_{\{q < \Delta(t, s)\}} \right\} = 0 \end{aligned}$$

in $[0, T) \times \mathcal{S} \times \mathcal{S}_+ \times \mathcal{Q}$, with the boundary conditions

$$V(T, b, s, q) = b + \mathbf{1}_{\{0 < s < K\}} q(s - \Upsilon_T) + \mathbf{1}_{\{s \geq K\}} [\mathfrak{N}K - (\mathfrak{N} - q)(s + \Upsilon_T)], \quad (10)$$

$$V(t, b, s, q) = b \text{ for } s = 0. \quad (11)$$

A graphical representation of the terminal condition $t = T$ is provided in figure 1 assuming $B_T = 0$. Notice that the step function nature of the terminal hedge, Δ_T , clearly is reflected in the pay-off here: specifically, the value surface is constant along $Q_T = 0$ for $S_T < K$ and along $Q_T = \mathfrak{N}$ for $S_T \geq K$.

Whilst searching for a closed form solution to (10) appears like an exercise in futility, it is immediately obvious that the problem at least offers a reduction in dimensionality. Specifically, since the money account B enters the terminal condition (10) linearly, and since the portfolio manager derives linear utility from his financial wealth, we have the ansatz

$$V(t, b, s, q) = b + \theta(t, s, q), \quad (12)$$

where θ is a real valued function. Hence, the HJB QVI reduces to

⁶Verifying that the value function of an impulse control problem is the (unique) solution of a particular HJB QVI is in general a delicate task. One method of doing so (along with sufficient conditions for the existence and uniqueness of a viscosity solution for HJB QVIs) is given by Seydel [57] for a general setting, where the state process is a jump-diffusion driven by Brownian motions and Poisson random measures. Based on these general results, Frey and Seydel [28] consider a problem in a setting very similar to ours where the conditions of [57] are certified to establish a solution for the value function. In particular, as noted in [28], the continuity conditions expire for discrete state variables, which is the case of (B_t, S_t, Q_t) in our setting. Thus, a thorough treatment of the necessary conditions in [57] following in the lines of [28] should eventually lead to guaranteeing the existence of a unique viscosity solution to (10), and that the solution is indeed the value function of our control problem (5). Since the level of technical detail of the conditions is beyond the scope of this paper we refrain from giving a full proof here and refer the interested reader to [57] and [28].

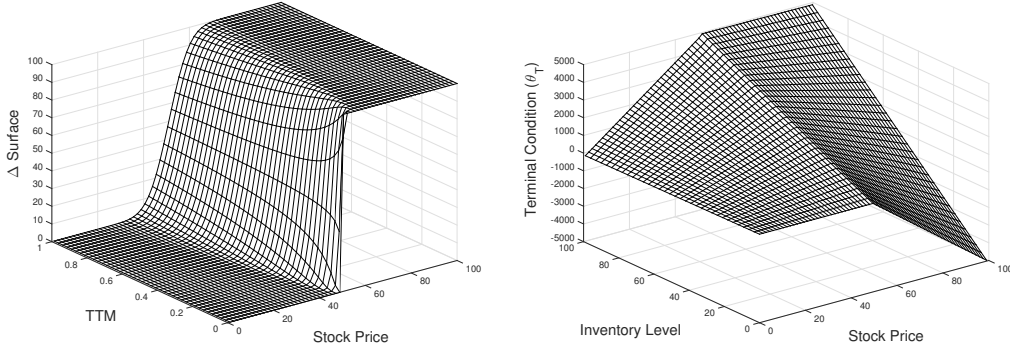


FIGURE 1. Assume $N = 100$ and $K = 50.5$, with constant values for σ and Υ . **(Left:)** The targeted Bachelierian hedge strategy $\Delta(t, S_t) = \mathfrak{N}\Phi(\delta_t)$. Notice that the surface converges towards a step function at the expiry of the call options. **(Right:)** The terminal condition of the control problem disregarding the bank account. Observe that at the zeroth level of inventory we have the classic “hockey stick” pay-off structure associated with a short option position. At the other end, with a full level of inventory $Q = \mathfrak{N}$, the short option position is perfectly hedged which is reflected in the constant in-the-money pay-off.

$$\begin{aligned}
0 = \max \Bigg\{ & \partial_t \theta(t, s, q) + \gamma_t \int_{\mathbb{R}} [\theta(t, s + \sigma_m z, q) - 2\theta(t, s, q) + \theta(t, s - \sigma_m z, q)] F_t(dz) \\
& + \sup_{\delta_t^- \in \mathcal{S}_+} \lambda_t e^{-\kappa_t \delta_t^-} [s + \delta_t^- + \theta(t, s, q - 1) - \theta(t, s, q)] \mathbf{1}_{\{q > \Delta(t, s)\}} \\
& + \sup_{\delta_t^+ \in \mathcal{S}_+} \lambda_t e^{-\kappa_t \delta_t^+} [-(s - \delta_t^+) + \theta(t, s, q + 1) - \theta(t, s, q)] \mathbf{1}_{\{q < \Delta(t, s)\}} \\
& - \eta(q - \Delta(t, s))^2 \\
& ; [s - \Upsilon_t + \theta(t, s, q - 1) - \theta(t, s, q)] \mathbf{1}_{\{q > \Delta(t, s)\}} \\
& + [-(s + \Upsilon_t) + \theta(t, s, q + 1) - \theta(t, s, q)] \mathbf{1}_{\{q < \Delta(t, s)\}} \Bigg\}, \tag{13}
\end{aligned}$$

subject to the boundary conditions

$$\begin{aligned}
\theta(T, s, q) &= \mathbf{1}_{\{0 < s < K\}} q(s - \Upsilon_T) + \mathbf{1}_{\{s \geq K\}} [\mathfrak{N}K - (\mathfrak{N} - q)(s + \Upsilon_T)], \\
\theta(t, s, q) &= 0 \text{ for } s = 0.
\end{aligned} \tag{14}$$

Upon solving the first order conditions

$$\delta_t^{*\pm} = \operatorname{argmax} \lambda_t e^{-\kappa_t \delta_t^\pm} [\pm s + \delta_t^\pm + \theta(t, s, q \pm 1) - \theta(t, s, q)], \tag{15}$$

(where the sign \pm is chosen uniformly) and substituting these back in to (13) we obtain the following key result

Proposition 2. Assume θ is a function which satisfies the following variational inequality

$$\begin{aligned}
0 = \max \Bigg\{ & \partial_t \theta(t, s, q) + \gamma_t \int_{\mathbb{R}} [\theta(t, s + \sigma_m z, q) - 2\theta(t, s, q) + \theta(t, s - \sigma_m z, q)] F_t(dz) \\
& + \frac{\lambda_t}{\kappa_t} \left[e^{-\kappa_t \delta_t^{*-}} \mathbf{1}_{\{q > \Delta(t, s)\}} + e^{-\kappa_t \delta_t^{*+}} \mathbf{1}_{\{q < \Delta(t, s)\}} \right] - \eta(q - \Delta(t, s))^2 \\
& ; [s - \Upsilon_t + \theta(t, s, q - 1) - \theta(t, s, q)] \mathbf{1}_{\{q > \Delta(t, s)\}}
\end{aligned}$$

$$+ [-(s + \Upsilon_t) + \theta(t, s, q + 1) - \theta(t, s, q)] \mathbf{1}_{\{q < \Delta(t, s)\}} \Big\}, \quad (16)$$

where the optimal limit order controls are given by

$$\delta_t^{*\pm} = \delta^{*\pm}(t, s, q) = \kappa_t^{-1} \pm s + \theta(t, s, q) - \theta(t, s, q \pm 1), \quad (17)$$

and the terminal condition is of the form (14), then the identity (12) yields a solution to the Hamilton-Jacobi-Bellman quasi variational inequality (10).

5. TOWARDS A NUMERICAL SOLUTION

Qua the dimensionality reduction offered by (12) we have a three-variable variational inequality, which with limited computational expenditure can be solved using an *explicit finite difference scheme*. Here we will elucidate how to set up the problem and demonstrate its convergence modulo a comparison principle (which would indeed be guaranteed subject to the conditions in [57] discussed in the previous section). To this end, we invoke a standard argument along the lines of Barles [10] [11] and Forsyth and Labahn [33], and show that our scheme is monotone, stable, and consistent. We remark that this is not the only way in which convergence of HJB numerical schemes may be demonstrated. For a probabilistic proof we refer the reader to Kushner and Dupuis [45], whose seminal work on Markov chain approximations of the dynamic programming principle indubitably will resonate with some readers. For a viscosity approach to Markov chain convergence see Fleming and Soner [30]. For highly readable accounts of finite difference methods in stochastic control theory we refer the reader to Tourin [61].

Now, following standard procedure, to approximate (13) in the finite difference sense, we introduce the finite mesh

$$\begin{aligned} \mathcal{T}^{\Delta t} \times \mathcal{S}^{\sigma_m} \times \mathcal{Q} &= \{0, \Delta t, \dots, n\Delta t, \dots, N\Delta t = T\} \times \{s_{\min} = 0, \sigma_m, \dots, i\sigma_m, \dots, I\sigma_m = s_{\max}\} \\ &\times \{0, \dots, j, \dots, J = \mathfrak{N}\}, \end{aligned}$$

in lieu of the full state space $[0, T] \times \mathcal{S}_+ \times \mathcal{Q}$. Here, s_{\max} is an artificially imposed upper boundary which encapsulate the solution region of interest. Notice that the only discretisation which takes place is along the time axis (the stock and inventory dimensions are manifestly already discrete in the original problem). Furthermore, to handle the integral in (13) suppose that the number of ticks that the mid-price can jump at any given time is bounded from above by a constant $K \in \mathbb{Z}_+$. Specifically, let $\int_{\mathbb{R}} F_s(dz) = \sum_{k=1}^K \mathbb{P}_s\{Z = k\} = 1$ and $K \ll I$ such that the price process surely stays in the mesh unless $S_t < K\sigma_m$ or $S_t > (I - K)\sigma_m$. To block the possibility that the price process jumps out of the mesh near the boundaries, a suitable rescaling of the probability weights are performed in those regions. Finally, the explicit approximation to the HJB QVI may then be stated as

$$\vartheta_{i,j}^n = \mathcal{G}[\vartheta^{n+1}] \equiv \max\{\mathcal{G}_{\Delta t}[\vartheta^{n+1}] ; \mathcal{G}_0[\vartheta^{n+1}]\}, \quad (18)$$

for $K \leq i \leq I - K$ and $1 \leq j \leq J - 1$ where $\vartheta_{i,j}^n \equiv \theta_{\Delta t}(n\Delta t, i\sigma_m, j)$ is a finite difference approximation to the dimensionally reduced optimal value function $\theta(t, s, q)$ at state $(n\Delta t, i\sigma_m, j)$, $\vartheta^{n+1} \equiv (\vartheta_{i,j}^{n+1}, \vartheta_{i,j+1}^{n+1}, \vartheta_{i,j-1}^{n+1}, \{\vartheta_{i\pm s,j}^{n+1}\}_{s=1}^K)^\top \in \mathbb{R}^{2K+3}$, and we have defined the finite difference operators

$$\begin{aligned} \mathcal{G}_{\Delta t}[\vartheta^{n+1}] &\equiv \vartheta_{i,j}^{n+1} + \Delta t \left\{ \gamma_{n+1} \sum_{k=1}^K \mathbb{P}_{n+1}\{Z = k\} [\vartheta_{i+k,j}^{n+1} - 2\vartheta_{i,j}^{n+1} + \vartheta_{i-k,j}^{n+1}] \right. \\ &\quad \left. + \sup_{\delta_{n+1}^- \in \mathcal{S}_+} \lambda_{n+1} e^{-\kappa_{n+1} \delta_{n+1}^-} [i\sigma_m + \delta_{n+1}^- + \vartheta_{i,j-1}^{n+1} - \vartheta_{i,j}^{n+1}] \mathbf{1}_{\{j > \Delta_{n+1}\}} \right\} \end{aligned}$$

<i>Quantity</i>	<i>Interpretation</i>	<i>Magnitude</i>	<i>Quantity</i>	<i>Interpretation</i>	<i>Magnitude</i>
K	strike	50.5	η	risk aversion	1
σ_m	tick-size	0.5	κ	limit decay	0.3
T	TTM	1	λ	limit intensity	100
\mathfrak{N}	no. of calls	100	γ	market intensity	200
Υ	half-spread	1	p	jump size prob.	0.9

TABLE 1. Parameter specifications used in simulation experiment. Some arbitrariness surrounds these numbers in the sense that no empirical calibration has been performed - nonetheless, they serve quite nicely for illustrative purposes.

$$\begin{aligned}
& + \sup_{\delta_{n+1}^+ \in \mathcal{S}_+} \lambda_{n+1} e^{-\kappa_{n+1} \delta_{n+1}^+} [-(i\sigma_m - \delta_{n+1}^+) + \vartheta_{i,j+1}^{n+1} - \vartheta_{i,j}^{n+1}] \mathbf{1}_{\{j < \Delta_{n+1}\}} \\
& - \eta(j - \Delta_{n+1})^2 \Big\}, \\
\mathcal{G}_0[\vartheta^{n+1}] & \equiv [i\sigma_m - \Upsilon_{n+1} + \vartheta_{i,j-1}^{n+1}] \mathbf{1}_{\{j > \Delta_{n+1}\}} - [i\sigma_m + \Upsilon_{n+1} - \vartheta_{i,j+1}^{n+1}] \mathbf{1}_{\{j < \Delta_{n+1}\}},
\end{aligned}$$

with analogous expressions near the boundaries. For completeness, note that the terminal condition of (18) takes on the form

$$\vartheta_{i,j}^N = \mathbf{1}_{\{0 < i\sigma_m < K\}} j(i\sigma_m - \Upsilon_N) + \mathbf{1}_{\{i\sigma_m \geq K\}} [\mathfrak{N}K - (\mathfrak{N} - j)(i\sigma_m + \Upsilon_N)], \quad (19)$$

whilst the first order conditions become

$$\delta_{n+1}^{*\pm} = \kappa_{n+1}^{-1} \pm i\sigma_m + \vartheta_{i,j}^{n+1} - \vartheta_{i,j\pm 1}^{n+1}.$$

Solving the control problem is thus algorithmically comparable to computing the no-arbitrage price of an American put option using finite difference methods: starting from the terminal condition and moving incrementally backwards in time, we must at each grid node decide whether a limit or a market order optimises our expected utility.

Proposition 3. *Assuming $2\gamma_n \Delta t \leq 1$ for $n = 1, 2, \dots, N$, the numerical scheme (18), (19) is (i) monotone, (ii) stable and (iii) consistent. Thus, if the scheme is also satisfying a comparison principle, then it converges locally uniformly to the unique viscosity solution of (13), (14).*

Proof. We refer the reader to appendix B for a proof of this result. \square

6. EXAMPLE

6.1. The Compound Poisson Model. With a converging numerical scheme at hand, we are in a position to compute the optimal limit order book hedge strategy called for in question 1. For simplicity, we will assume that all parameters are constant in time. In particular, the dynamics of the mid-price is assumed to be the difference between two compound Poisson processes:

$$dS_t = \sigma_m(dY_t^1 - dY_t^2),$$

with $S_0 = K$ and $Y_t^i = \sum_{j=1}^{N_t^i} Z_j^i$, for $i = 1, 2$, where $\{N_t^1, N_t^2\}_{t \in [0, T]}$ are independent Poisson processes of common intensity γt , whilst the jump sizes $\{Z_j^i\}_{j \in \mathbb{Z}^+}$ are i.i.d. random variables

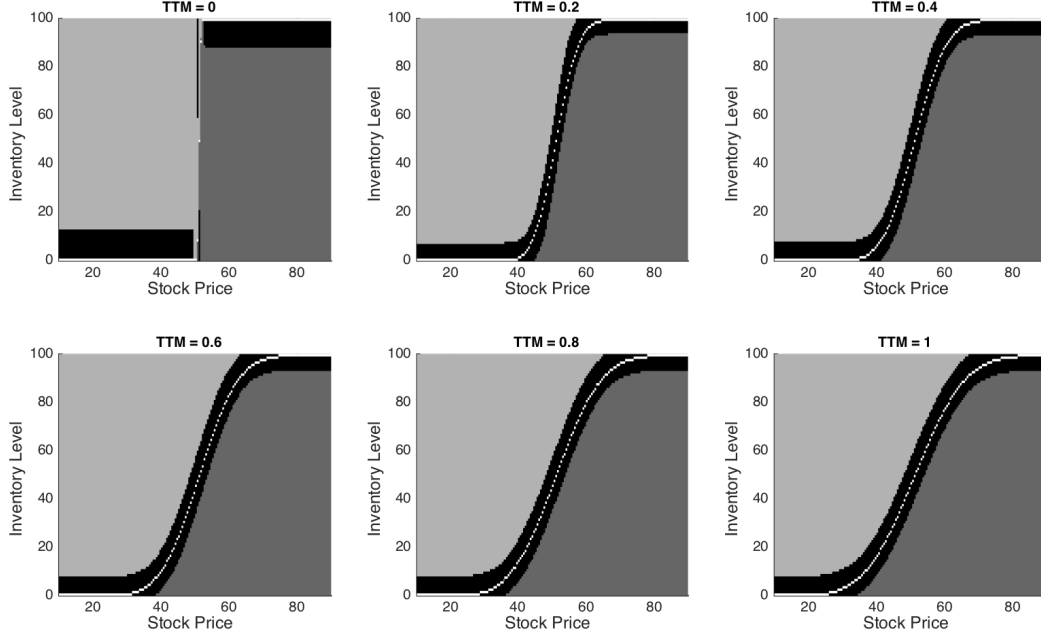


FIGURE 2. The optimal limit/market order regions for the portfolio manager at different times to maturity. The white line which cuts each figure across diagonally is the targeted inventory level Δ_t for different values of the underlying stock. The black region surrounding it, is stock-inventory levels for which the portfolio manager is contented with placing limit orders in an attempt to attain delta neutrality. Finally, the light grey region (top left) and the dark grey region (bottom right) correspond to market orders, i.e. the case where the portfolio manager deems it necessary to trade immediately to bounce back into the limit order region. For the corresponding surface plots see figure 3.

assumed to follow the geometric distribution⁷

$$\mathbb{P}\{Z_j^i = k\} = (1-p)^{k-1}p.$$

Here $k \in \mathbb{Z}_+$, $p \in (0, 1]$, and the first and second moments are of the form $\mathbb{E}[Z_j^i] = p^{-1}$, and $\mathbb{E}[(Z_j^i)^2] = (2-p)p^{-2}$. Jumps of unit tick size thus dominate the price process fluctuations, while larger jumps occur at a power decaying rate.⁸ Finally, as a direct consequence of (8) alongside the approximation (9), the targeted Bachelierian hedge strategy must be of the form

$$\Delta_t = \left\lceil \mathfrak{N}\Phi \left(\frac{S_t - K}{\sigma_m \sqrt{2\gamma p^{-2}(2-p)(T-t)}} \right) \right\rceil. \quad (20)$$

6.2. Simulation.

⁷Readers familiar with Lévy processes will notice that this essentially is a discrete symmetric version of the Kou model:

$$\nu(dz) = [p\lambda_+ e^{-\lambda_+ z} \mathbf{1}_{\{z>0\}} + (1-p)\lambda_- e^{-\lambda_- z} \mathbf{1}_{\{z<0\}}]dz,$$

where $\lambda_+, \lambda_- > 0$ and $p \in [0, 1]$.

⁸In practical applications, we cut off the mass function at a level k such that the probability of a jump of size k occurring is much less than one over the total number of incremental time steps (N).

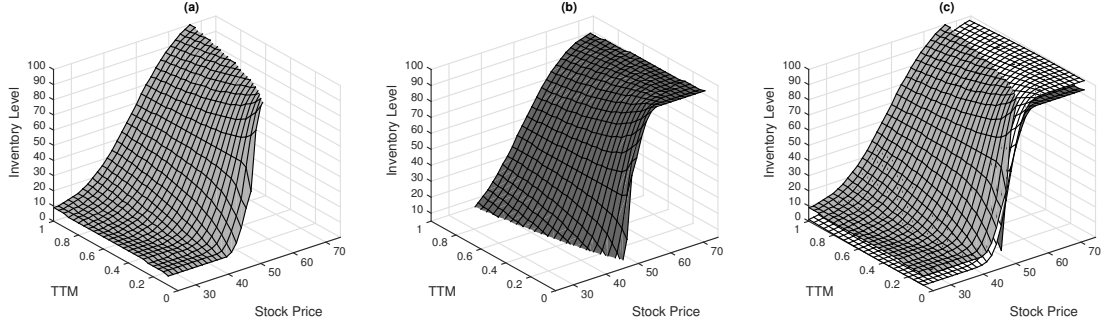


FIGURE 3. A three dimensional representation of figure 2. **(a)** The optimal market sell surface. States which lie above this surface call for the immediate liquidation of inventory through market orders. **(b)** The optimal market buy surface. States which lie below this surface call for the immediate acquisition of inventory through market orders. **(c)** The two market surfaces together with the targeted hedge strategy. The point here is that the market surfaces draw closer and closer to the targeted hedge surfaces reaching impact upon maturity. NB: for purely aesthetic reasons we are not exhibiting some of the shortest time to maturity data points (where the graphs converge toward (at least approximate) step functions).

Definition 1. We designate the joint process $\{S_t, Q_t\}_{t \in [0, T]}$ (i.e. the price path traced out by the underlying asset over $[0, T]$, together with the associated inventory level held by the portfolio manager) the **stock-inventory path**.

Remark 5. Clearly, while the stock path is exogenously determined by market forces, the portfolio manager is at liberty to choose his associated inventory level.

Consider the parametric specifications listed in table 1. Upon solving the HJB QVI numerically, we find the optimal trade regions exhibited cross-sectionally in figure 2 for various times to maturity. Jointly, these temporal hypersurfaces form the surface plots exhibited in figure 3. The interpretation is as follows: the portfolio manager aims to keep his portfolio approximately delta-neutral understood in the sense of keeping his stock-inventory path on a par with the hedge surface Δ_t (whatever the realised value of the underlying, S_t , may be). However, he is contended with merely posting limit sell or buy orders (depending on whether he is above or below Δ_t) in the immediately adjacent region marked by black in figure 2, thus allowing for minor deviations from the targeted hedge ratio. Should his stock-inventory path pierce either of the grey surface, his “risk aversion” kicks in, and he performs an immediate inventory alteration through unfavourable market orders (to the point where he re-enters the limit order region). Notice that as the market sell and the market buy surfaces both converge towards the strike K at maturity, it becomes increasingly difficult to avoid placing market orders with time if $S_t \approx K$.

The implications of this are forcefully demonstrated in figures 4 and 5 for the realisation of a single stock path.⁹ Here we investigate the performance of the portfolio manager’s combined limit-market order hedge strategy vis-à-vis the *naïve hedge strategy* which at all times utilises market orders to maintain delta neutrality. Unsurprisingly, since the former strategy permits all stock-inventory paths sandwiched between the optimal market surfaces, whilst the latter only allows those stock-inventory paths which are embedded¹⁰ in the optimal hedge surface, $\{Q_t\}_{t \in [0, T]}$ and $\{B_t\}_{t \in [0, T]}$ are considerably less erratic in case of the

⁹In simulating a Poisson path, it is helpful to recall that the time spent between consecutive jumps, τ , is a random variable which follows the exponential distribution $\mathbb{P}\{\tau \leq t\} = 1 - \exp\{-\gamma t\}$, see [52] [59].

¹⁰Since we assume that the portfolio manager only buys and sells integer quantities of the underlying, it is clear that the embedding should be read as “correct to the nearest integer”.

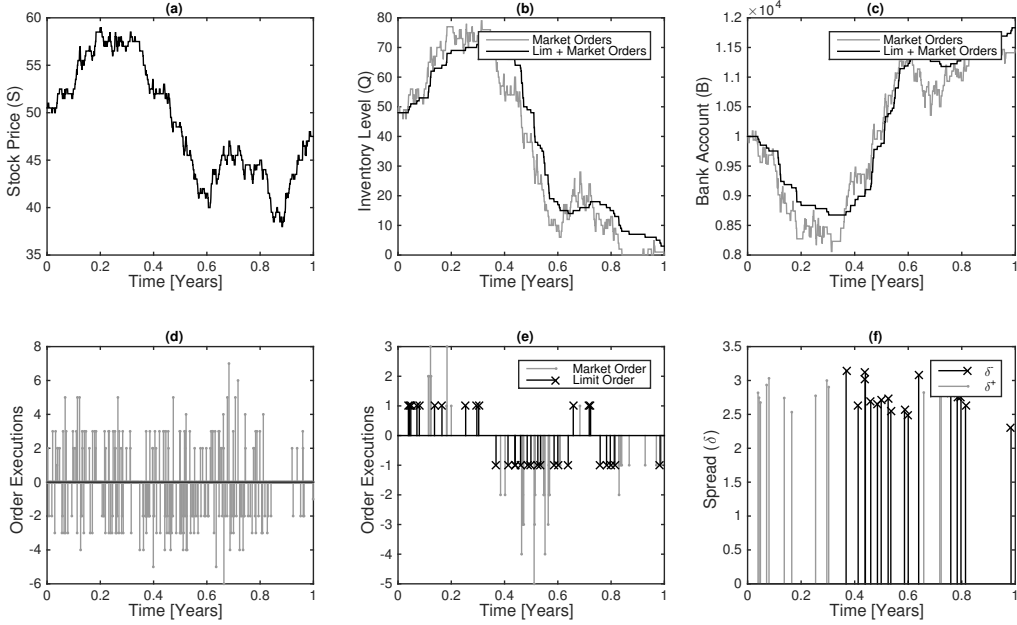


FIGURE 4. Instantiation of a stock-inventory path. **(a)** The evolution of the mid-price, seeded at $S_0 = K$. **(b)** The evolution of the portfolio manager's inventory when he hedges using market orders only (grey) or a combination of market and limit orders (black). **(c)** The evolution of the portfolio manager's bank account, again using market orders only (grey) or a combination of market and limit orders (black). Notice the latter outperforms the former. The bank wealth is (arbitrarily) seeded at $B_0 = 10^4$. **(d)** The magnitude of the orders placed by the portfolio manager when he hedges with market orders only. **(e)** The magnitude of the orders placed by the portfolio manager when he hedges with a combination of market (grey) and limit (black) orders. **(f)** The size of the spread posted by the portfolio manager in those cases where his limit orders are successful. Black stems are sells, while grey stems are buys.

former. Indeed, the lower frequency of trading, combined with the favourable prices associated with limit orders, yield a much more favourable cash position at the expiry of the options. Specifically, for the simulation at hand, the number of (not necessarily unit sized) market orders placed with the naïve strategy amounts to 216, while the number of market-limit orders placed with the control strategy come in at 37 & 30, with an average spread for the limit orders of 2.79. As for the mean quadratic variation of the inventory path, $\overline{(\Delta Q)^2} \equiv \frac{1}{N} \sum_{i=1}^N (\Delta Q_i)^2$, the naïve strategy amounts to 1.768, while the control strategy comes in at 0.205. The analogous quantity, computed for the bank path yields 4424 vs. 472.

To develop an understanding of the distributive properties of the hedge strategies above, we repeat the experiment 10^4 times. The resulting histograms are exhibited in figure 6, while the estimators are given in see table 2. As expected, the distribution of the terminal stock value (figure 6(a)) falls symmetrically around the strike in a normally distributed manner, with the best fit being obtained around the tails.¹¹ This highlights that the employ of the Bachelier dynamics in lieu of the jump processes is not altogether innocuous. As for the rate of return of the bank account, B_T/B_0 , we refer to figure 6(b). Whilst it is

¹¹This is hardly surprising: e.g. it is a well-known fact that the Skellam distribution (the law of the difference between two Poisson random variables, $N_1 - N_2$) tends to the normal distribution if (i) N_1 and N_2 have common intensity and, importantly, (ii) $N_1 - N_2 = k$ is large. See Abramowitz and Stegun [1].

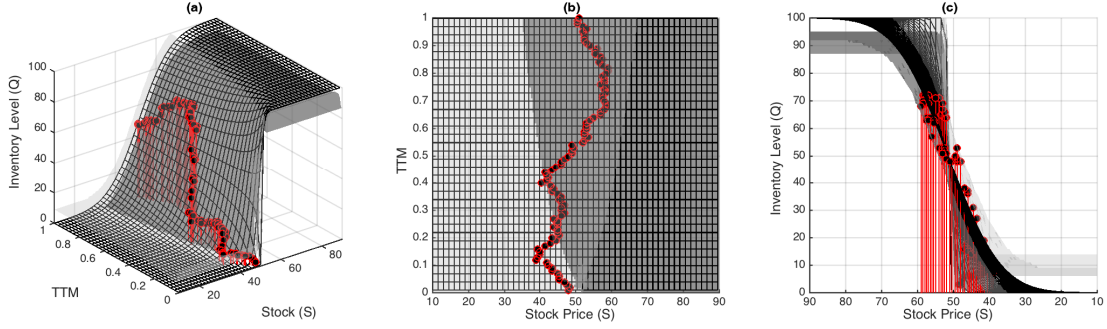


FIGURE 5. **(a)** A three dimensional view of the stock-inventory path described in figure 4 when the portfolio manager uses a combination of limit and market orders. Importantly, the path does not remain embedded in the gridded hedge surface, but is free to bounce back and forth between the encompassing market surfaces discretely highlighted in transparent grey. **(b)** The graph viewed from $Q = \mathfrak{N}$. Notice the symmetric nature of the market surfaces viewed from this angle. **(c)** The graph viewed from $t = 0$. Here it is quite apparent that the path moves within the pocket space defined by the surrounding market surfaces.

clear that the combined limit-market order strategy is superior to the naïve market order strategy, the “camel hump” nature of the histogram might prima facie strike one as puzzling. This mystery is quickly deflated, however, upon realising that the terminal value of the targeted hedge strategy - the step function $\mathfrak{N}\mathbf{1}\{S_T \geq K\}$ - invariably will drive the portfolio manager to either liquidate or acquire ~ 50 shares relative to his initial inventory. The asymmetry between the negative and positive return humps is likely to be symptomatic of the asymmetric boundary condition (i.e. the operations executed given that the calls are in or out of the money). Finally, figure 6(c) exhibits the return of the hedge portfolio, Π_T/Π_0 where

$$\Pi_t = B_t + \Delta_t S_t - \mathfrak{N} \cdot C_t.$$

Blatantly, $\Pi_{t < T}$ is a model dependent quantity which prompts us to suggest the at-the-money Bachelier price of the call option,

$$C_{t,\text{atm}}^{K,T} = \sigma_m \sqrt{\frac{\gamma(2-p)(T-t)}{\pi p^2}},$$

in accordance with equation (7). Again, while this is dynamically consistent with the choice of the Δ_t , (20), it is also in flagrant disregard for the jump nature of the underlying and the friction prevailing on the market. Hence, the Bachelier specification Π_0^B should not be seen as an absolute standard. Nevertheless, it is intriguing to note that it on average yields a positive return for the portfolio manager who uses a combination of limit and market orders to hedge his position, whilst the naïve hedge engenders a mean negative return. Indeed, a Welchian t -test shows that we can comfortably dismiss the null hypothesis of equal means at the 99% level. For more general values of Π_0 we simply observe that the mean of the control strategy return is higher than that of the naïve strategy, while the opposite is the case for the standard deviation.

7. CONCLUSION

The emergence of certain stylised properties of the otherwise hyper-complex limit order book beckons us to attempt mathematical modelling thereof. In this paper we proposed a simple jump model of the mid-price, alongside a probability of having a limit order met which is an

	B_T/B_0 LIMs	B_T/B_0 MOs	Π_T/Π_0^B LIMs	Π_T/Π_0^B MOs
Mean	0.9554	0.8868	1.0048	0.9445
Standard dev.	0.2345	0.2521	0.0077	0.0199
1% Quantile	0.6856	0.5828	0.9848	0.9044
25% Quantile	0.7183	0.6256	1.0000	0.9292
50% Quantile	0.9549	1.0574	1.0052	0.9438
75% Quantile	1.1926	1.1327	1.0102	0.9598
99% Quantile	1.2119	1.1763	1.0210	0.9843

TABLE 2. Estimators for figure see table 6. LIMs refers to the combined limit-market order strategy, while MOs refers to the naïve market order strategy.

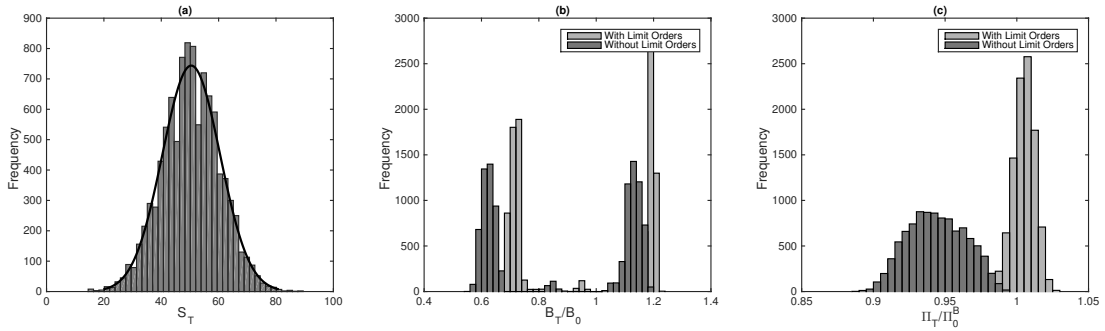


FIGURE 6. (a) Histogram of the terminal stock price with Bachelierian normal fit. (b) Histogram of the rate of return of the bank account. The “camel hump” shape is symptomatic of the binary “all or nothing” inventory position required for delta neutrality at maturity. (c) Histogram of the rate of return of the portfolio (bank + stock + options), where Π_0^B is the Bachelier value of the portfolio, defined below. Evidently (and unsurprisingly) the combined limit/market order strategy greatly outperforms the strategy which trades in market orders only.

exponentially decaying function of the distance to the mid-price. It was shown that the jump model converges weakly to Bachelier’s arithmetic Brownian motion for large values of the intensity. Furthermore, under the assumption that a portfolio manager wishing to Δ -hedge his short call position derives linear utility from his terminal wealth, but incurs quadratic lifetime penalisation from deviating from a targeted hedge strategy, we derived numerical values for his *optimal limit order quotes* and *stopping times* at which he should switch to pure market orders. Following the standard literature on convergence in the viscosity sense we proved that our scheme is monotone, stable, and consistent. We also exemplified the utility of our algorithm by comparing it to a naïve strategy which deploys nothing but market orders: a clear augmentation of the mean return and a clear reduction in the associated variance were established here. Finally, given the generic nature of our model, we reiterate that it is adaptable to more sophisticated expressions for the targeted hedge ratio in the limit order book.

7.1. Acknowledgements. The authors are grateful to Mark H. A. Davis (Imperial College London), Nicole Branger (University of Münster), Charles-Albert Lehalle (Capital Fund Management), and two anonymous referees who made detailed and helpful comments on various iterations of this manuscript.

APPENDIX A. PROOF OF PROPOSITION 1

To establish the convergence of the jump dynamics to the Bachelier dynamics for large jump intensities we invoke the following lemma:

Lemma 1. *Let $L_t \equiv L_t^1 - L_t^2$, where*

$$L_t^i \equiv L^i((0, t] \times \mathbb{R}) = \int_0^t \int_{\mathbb{R}} z J_i(dt \times dz)$$

for $i = 1, 2$. Furthermore, let us introduce the function $\zeta : [0, T] \mapsto \mathbb{R}_+$ by

$$\zeta_t \equiv \left(2 \int_0^t \mathbb{E}[Z_s^2] \gamma_s ds \right)^{-1/2}, \quad (21)$$

where $\mathbb{E}[Z_s^2] = \int_{\mathbb{R}} z^2 F_s(dz)$ and assume that $\sup_{s \in [0, t]} \int_{\mathbb{R}} z^4 F_s(dz) < \infty$. Then we have the following convergence in distribution:

$$\zeta_t L_t \xrightarrow{d} N(0, 1) \quad \text{as} \quad \int_0^t \gamma_s ds \rightarrow \infty, \quad (22)$$

where $N(0, 1)$ is the standard normal distribution.

Proof. By Levy's Continuity Theorem, it suffices to show that the characteristic function $\zeta_t L_t$ converges pointwise to the characteristic function of $N(0, 1)$.¹² To this end, suppose we partition the time interval into $n + 1$ segments: $\mathcal{T}_n = \{t_i | t_0 = 0, t_1 = t_0 + \Delta t_0, t_2 = t_1 + \Delta t_1, \dots, t_{n+1} = t = t_n + \Delta t_n\}$, where $\max_i \Delta t_i \rightarrow 0$ as $n \rightarrow \infty$. Similarly, let $\mathcal{A}_m = \{\Delta \mathbb{A}_1, \Delta \mathbb{A}_2, \dots, \Delta \mathbb{A}_m | \bigcup_{j=1}^m \Delta \mathbb{A}_j = \mathbb{A}\}$ be a partition of the mark space over disjoint subsets.¹³ From the elementary properties of Poisson random measures it follows that $J_i^{sq} \equiv J((t_s, t_s + \Delta t_s] \times \Delta \mathbb{A}_q)$ and $J_i^{ur} \equiv J((t_u, t_u + \Delta t_u] \times \Delta \mathbb{A}_r)$ are independent provided that $q \neq r$. Furthermore,

$$\mathbb{P}\{J_i^{sq} = k\} = e^{-\mu_i^{sq}} \frac{(\mu_i^{sq})^k}{k!},$$

where we have defined the discrete intensity measure $\mu_i^{sq} \equiv \mu_i((t_s, t_s + \Delta t_s], \Delta \mathbb{A}_q) = \gamma_s F_s(\Delta \mathbb{A}_q) \Delta t_s$. Thus, the characteristic function of $\zeta_t L_t$, i.e. $\varphi_{\zeta L}(a) \equiv \mathbb{E}[\exp\{ia\zeta_t L_t\}]$ where $i = \sqrt{-1}$ and $a \in \mathbb{R}$, can be decomposed as $\varphi_{\zeta L^1}(a) \varphi_{\zeta L^2}(-a)$ (by the independence of L^1 and L^2) whence

$$\begin{aligned} \varphi_{\zeta L}(a) &= \mathbb{E} \left[\exp \left\{ ia\zeta_t \int_0^t \int_{\mathbb{A}} z J_1(ds \times dz) \right\} \right] \mathbb{E} \left[\exp \left\{ -ia\zeta_t \int_0^t \int_{\mathbb{A}} z J_2(ds \times dz) \right\} \right] \\ &= \lim_{m, n \rightarrow \infty} \mathbb{E} \left[\exp \left\{ ia\zeta_{n+1} \sum_{s=0}^n \sum_{q=1}^m z_q J_1^{sq} \right\} \right] \mathbb{E} \left[\exp \left\{ -ia\zeta_{n+1} \sum_{u=0}^n \sum_{r=1}^m z_r J_2^{ur} \right\} \right] \\ &= \lim_{m, n \rightarrow \infty} \prod_{s, u=0}^n \prod_{q, r=1}^m \mathbb{E} [\exp\{ia\zeta_{n+1} z_q J_1^{sq}\}] \mathbb{E} [\exp\{-ia\zeta_{n+1} z_r J_2^{ur}\}] \\ &= \lim_{m, n \rightarrow \infty} \prod_{s, u=0}^n \prod_{q, r=1}^m \sum_{k_1=0}^{\infty} \mathbb{P}\{J_i^{sq} = k_1\} \mathbb{E} [\exp\{ia\zeta_{n+1} z_q J_1^{sq}\} | J_1^{sq} = k_1]. \end{aligned}$$

¹²Recall: if $\{X_n\}_{n \in \mathbb{N}^+}$ and X are random vectors in \mathbb{R}^k , and $\{\varphi_n\}_{n \in \mathbb{N}^+}$ and φ_X are the associated characteristic functions, then (I) X_n converges in distribution to X iff $\varphi_{X_n}(s) \rightarrow \varphi_X(s)$ for all $s \in \mathbb{R}^k$, and (II) if $\varphi_{X_n}(s) \rightarrow \varphi(s)$ pointwise for all $s \in \mathbb{R}^k$, and if φ is continuous on 0, then φ is the characteristic function of X and X_n converges in distribution to X .

¹³This discretisation idea is inspired by Hanson [39].

$$\begin{aligned}
& \sum_{k_2=0}^{\infty} \mathbb{P}\{J_i^{ur} = k_2\} \mathbb{E}[\exp\{-ia\zeta_{n+1}z_r J_2^{up}\} | J_2^{up} = k_2] \\
&= \lim_{m,n \rightarrow \infty} \prod_{s,u=0}^n \prod_{q,r=1}^m \sum_{k_1=0}^{\infty} e^{-\mu_1^{sq}} \frac{(\mu_1^{sq})^{k_1}}{k_1!} e^{ia\zeta_{n+1}z_q k_1} \sum_{k_2=0}^{\infty} e^{-\mu_2^{ur}} \frac{(\mu_2^{ur})^{k_2}}{k_2!} e^{-ia\zeta_{n+1}z_r k_2} \\
&= \lim_{m,n \rightarrow \infty} \prod_{s,u=0}^n \prod_{q,r=1}^m \exp\{\mu_1^{sq}(e^{ia\zeta_{n+1}z_q} - 1)\} \exp\{\mu_2^{ur}(e^{-ia\zeta_{n+1}z_r} - 1)\} \\
&= \lim_{m,n \rightarrow \infty} \exp\left\{\sum_{s=0}^n \sum_{q=1}^m \mu_1^{sq}(e^{ia\zeta_{n+1}z_q} - 1)\right\} \exp\left\{\sum_{u=0}^n \sum_{r=1}^m \mu_2^{ur}(e^{-ia\zeta_{n+1}z_r} - 1)\right\}, \\
&= \exp\left\{\int_0^t \int_{\mathbb{A}} (e^{ia\zeta_t z} - 1) \gamma_s F_s(dz) ds\right\} \exp\left\{\int_0^t \int_{\mathbb{A}} (e^{-ia\zeta_t z} - 1) \gamma_s F_s(dz) ds\right\}, \\
&= \exp\left\{\int_0^t (\mathbb{E}[e^{ia\zeta_t Z_s}] + \mathbb{E}[e^{ia\zeta_t Z_s}]^* - 2) \gamma_s ds\right\},
\end{aligned}$$

where $*$ designates the complex conjugate. Here, the fourth equality uses the *law of total expectation*, while the sixth equality uses the Taylor expansion of the exponential function. Finally, from the expansion

$$\mathbb{E}[e^{ia\zeta_t Z_s}] = 1 + ia\zeta_t \mathbb{E}[Z_s] - \frac{1}{2}a^2 \zeta_t^2 \mathbb{E}[Z_s^2] - \frac{1}{6}ia^3 \zeta_t^3 \mathbb{E}[Z_s^3] + \mathcal{O}(a^4 \zeta_t^4 \mathbb{E}[Z_s^4]),$$

we find after internal cancellation that

$$\varphi_{\zeta L}(a) = \exp\left\{-a^2 \zeta_t^2 \int_0^t \mathbb{E}[Z_s^2] \gamma_s ds + \mathcal{O}\left(a^4 \zeta_t^4 \int_0^t \mathbb{E}[Z_s^4] \gamma_s ds\right)\right\}.$$

Upon specifying ζ_t as in (21) and by noting that $\int_0^t \gamma_s ds \leq \int_0^t \mathbb{E}[Z_s^2] \gamma_s ds$ (the jump sizes Z_s takes values in \mathbb{N}^+) and $\int_0^t \mathbb{E}[Z_s^4] \gamma_s ds \leq \sup_{s \in [0,t]} \mathbb{E}[Z_s^4] \int_0^t \gamma_s ds$ (the fourth moment is bounded by assumption), letting $\int_0^t \gamma_s ds \rightarrow \infty$ it follows that $\Phi_{\zeta L}(a) \rightarrow \exp\{-\frac{1}{2}a^2\}$, which is the characteristic function of the standard normal distribution. \square

Proposition 1 may now immediately be established upon noting that for large $\bar{\gamma}$ the approximation $\zeta_t L_t \sim N(0, 1)$. Hence, $\sigma_m \zeta_t L_t \sim (\sigma_m/\sqrt{t})N(0, t)$. Rearranging this, the result follows.

APPENDIX B. PROOF OF PROPOSITION 3

Proof. Following [11] and [30] we first recall the following definitions, suitably adapted to the problem at hand

- (1) **(Monotonicity):** For all $\varepsilon \in \mathbb{R}_+$, and for all unit vectors $\{\hat{\mathbf{e}}_i\}_{i=1}^{2K+3}$ we require that

$$\mathcal{G}[\boldsymbol{\vartheta}^{n+1}] \leq \mathcal{G}[\boldsymbol{\vartheta}^{n+1} + \varepsilon \hat{\mathbf{e}}_i], \quad (23)$$

where $[\hat{\mathbf{e}}_i]_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta.

- (2) **(Stability):**

$$\|\boldsymbol{\vartheta}^n\|_{\infty} \leq C, \quad (24)$$

for some constant C independent of Δt , as $\Delta t \rightarrow 0$, where $\|\mathbf{x}\|_{\infty} \equiv \max_i |x_i|$.

- (3) **(Consistency):** Finally, we require that the system satisfies the basic consistency condition

$$\lim_{\Delta t \rightarrow 0} \left| \frac{1}{\Delta t} (\mathcal{G}[\boldsymbol{\vartheta}^{n+1}] - \vartheta_{i,j}^n) - \mathcal{H}[\boldsymbol{\theta}((n+1)\Delta t, i\sigma_m, q_{\min} + j)] \right| = 0, \quad (25)$$

where $\mathcal{H}[\cdot]$ is the HJB QVI, (13), centered at the coordinate

$$(t, s, q) = ((n+1)\Delta t, i\sigma_m, q_{\min} + j).$$

We will work through these items systematically, focussing on the case where the limit order equation applies (the market order equation trivially satisfies 1, 2, and 3):

- (1) **(Monotonicity)**: This is trivially true for all elements of \mathfrak{V}^{n+1} which only have a positive coefficient instantiation in $\mathcal{G}_{\Delta t}[\mathfrak{V}^{n+1}]$. In fact, the only non-trivial case is $\theta_{i,j}^{n+1}$. Specifically, upon eliminating all superfluous terms, the monotonicity condition requires:

$$\begin{aligned} 0 &\leq \varepsilon + \Delta t \left\{ -2\gamma_{n+1}\varepsilon + \sup_{\delta_{n+1}^- \in \mathbb{R}_+} [-\varepsilon\lambda_{n+1}e^{-\kappa_{n+1}\delta_{n+1}^-}] \mathbf{1}_{\{q_{\min}+j \geq \Delta_{n+1}\}} \right. \\ &\quad \left. + \sup_{\delta_{n+1}^+ \in \mathbb{R}_+} [-\varepsilon\lambda_{n+1}e^{-\kappa_{n+1}\delta_{n+1}^+}] \mathbf{1}_{\{q_{\min}+j < \Delta_{n+1}\}} \right\} \\ &= \varepsilon - \Delta t \left\{ 2\gamma_{n+1}\varepsilon + \varepsilon\lambda_{n+1} \inf_{\delta_{n+1}^- \in \mathbb{R}_+} [e^{-\kappa_{n+1}\delta_{n+1}^-}] \mathbf{1}_{\{q_{\min}+j \geq \Delta_{n+1}\}} \right. \\ &\quad \left. + \varepsilon\lambda_{n+1} \inf_{\delta_{n+1}^+ \in \mathbb{R}_+} [e^{-\kappa_{n+1}\delta_{n+1}^+}] \mathbf{1}_{\{q_{\min}+j < \Delta_{n+1}\}} \right\}. \end{aligned}$$

Since

$$\inf_{\delta_{n+1}^\pm} e^{-\kappa_{n+1}\delta_{n+1}^\pm} = 0,$$

we're left with the constraint

$$2\gamma_{n+1}\Delta t \leq 1, \quad \text{for } n = N-1, N-2, \dots, 0. \quad (26)$$

Indeed, numerical experiments corroborate that violating (26) leads to a failure in convergence of the algorithm. Notice that the inverse proportionality between γ and Δt in practice can lead to very fine grid spacings for realistic values of the jump rate intensity: hence, (26) significantly impedes the rapidity with which the algorithm can be executed.

- (2) **(Stability)**: Taking the L^∞ -norm of the HJB equation and using the triangle inequality we obtain

$$\|\vartheta_{i,j}^n\|_\infty \leq \|\vartheta_{i,j}^{n+1}\|_\infty + \Delta t \|\{\dots\}\|_\infty,$$

where $\{\dots\}$ signifies the content of the curly brackets in the definition of $\mathcal{G}_{\Delta t}$. Setting $\Delta t \rightarrow 0$ and using the fact that the terminal condition $\vartheta_{i,j}^N$ by assumption is bounded from above, we obtain the desired result.

- (3) **(Consistency)**: This follows immediately upon substituting in the definitions of $\mathcal{G}_{\Delta t}$ and \mathcal{H} in equation (25) and using the Taylor expansion

$$\vartheta_{i,j}^n = \vartheta_{i,j}^{n+1} - \Delta t \partial_t \vartheta_{i,j}^{n+1} + \mathcal{O}((\Delta t)^2).$$

To complete the proof we deploy similar arguments for the grid points near the boundaries (recall the rescaling of probabilities such as to avoid the price process leaving the grid). Since this effectively is a recapitulation of what has already been established we omit the details here. Finally, regarding the comparison principle, loosely put, what is required is that if θ_1 and θ_2 are two solutions of the HJB QVI with $\theta_1(T, s, q) \geq \theta_2(T, s, q)$ then we require $\theta_1(t, s, q) \geq \theta_2(t, s, q)$ for all t . For a more rigorous definition, see the viscosity references. \square

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