ON THE NUMERICAL SOLUTION OF MERTONIAN CONTROL PROBLEMS - A SURVEY OF THE MARKOV CHAIN APPROXIMATION METHOD FOR THE WORKING ECONOMIST

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Abstract. Analytic solutions to HJB equation in mathematical finance are relatively hard to come by, which stresses the need for numerical procedures. In this paper we provide a self-contained exposition of the finite-horizon Markov chain approximation method as championed by Kushner and Dupuis. Furthermore, we provide full details as to how well the algorithm fares when we deploy it in the context of Merton type optimisation problems. Assorted issues relating to implementation and numerical accuracy are thoroughly reviewed, including multidimensionality and the positive probability requirement, the question of boundary conditions, and the choice of parametric values.

Keywords. HJB equation, Finite Difference Approximation, Merton problem.

1. Introduction

It is well known to anyone who has dabbled in continuous time continuous state stochastic control problems *à la* Merton [25] that the search for closed form solutions is an onerous and all-too-often fruitless endeavour. This is unfortunate given the far-reaching scope of the field (see e.g. Cartea et al. [3], Munk [26], and Rogers [32] and the references therein), but hardly surprising given the highly non-linear nature of the governing PDEs. What is at least somewhat puzzling is the scarcity of pedagogical resources that "take the bull by the horns", as it were, and tackle these problems from a numerical angle. More concretely, while a plurality of such methods exist, there has been a widespread failure to disseminate their core principles to students of quantitative finance. Said principles include hands-on discretisations of the Hamilton-Jacobi-Bellman equation (see e.g. Forsyth and Labahn [13], [34]), Monte Carlo methods (see e.g. Cvitanić et al. [4] and Detemple et al. [6]), and forward-backward stochastic differential equations (see e.g. Ludwig et al. [22]). Most prominently, perhaps, stands Kushner and Dupuis' monograph on the so-called *Markov chain approximation method* [18], which although predominantly concerned with control-independent diffusions, straightforwardly can be generalised in this direction (Kushner [20]). This paper pertains to this latter method, the central premise of which is to substitute the continuous time continuous state controlled state process by a discrete time discrete state Markov chain. As shown by Kushner and Dupuis, by choosing the transition probabilities of the Markov chain based on standard finite difference discretisations of the governing Bellman equation, convergence can be established.

Date: July 17, 2018

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The purpose of this paper is two-fold: first, it serves as an exegesis of the Markov chain approximation method for anyone who seeks a swift and comparatively unconvoluted theoretical overview of the area. To this end, even numbered sections aim to equip the reader with the fundamental theoretical tools needed in order to implement numerical control problems in simple environments where uncertainty is driven by a Brownian motion. The second, and arguably more intriguing part of this paper is the odd numbered sections, which provide a detailed account of just how well the Markov chain approximation method fares when we deploy it in a Merton type portfolio optimisation context. As it will soon become apparent, actual implementations require a non-negligible amount of *Fingerspitzengefühl*, the key lessons of which are not communicated by Kushner and Dupuis. For example, inexact boundary conditions turn out to have a corrosive effect on the accuracy of the numerical controls for a rather large region into the finite difference grid. However, we show that if an exact relationship between adjacent grid nodes can be established, this problem altogether dissipates. Other matters of interest include the effect of the granularity of the mesh, the magnitude of the discount factor and the agent's risk aversion, and the sanctity of maintaining positive probabilities.

Admittedly, this is not the first paper to deal the numerical implementation of the Merton problem. Concrete examples include Fitzpatrick and Fleming [11] and Munk [27], [29]. Nevertheless, these papers deal with the infinite horizon single state process case, and only focus on the implicit implementation procedure. We, on the other hand, focus on finite horizon investment problems both from an explicit and implicit perspective, with generalisations to higher spatial dimensions. For readers interested in exploring the employ of the Markov chain approximation framework in less run-of-the-mill control problems in finance, we refer to Hindy et al. [16] in which a free boundary consumption problem is considered, Munk [30] in which rational reservation prices for European options are established, and Munk [28] in which optimal consumption-investment policies are determined when the agent receives stochastic undiversifiable labour income.

2. The Finite Horizon Stochastic Control Problem

We set out by restricting our attention to mono-dimensional controlled diffusion processes over finite temporal horizons. Let $\mathbb{T} = [0, T]$ where $T < \infty$ be the time interval of interest, and let $X_t : \Omega \times \mathbb{T} \mapsto \mathbb{R}$ be the stochastic process (the state variable) we are trying to control. As convention would have it, the latter is assumed to inhabit the stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathscr{F}_t^X\}_{t\in\mathbb{T}}$ is the canonical filtration of X. Defining the Brownian motion $\{W_t\}_{t\in\mathbb{T}}$ we stipulate the governing dynamics of X as

$$dX_s^\alpha = b(s, X_s^\alpha, \alpha_s)ds + \sigma(s, X_s^\alpha, \alpha_s)dW_s, \tag{1}$$

where $X_t^\alpha = x$, and $\alpha_s = \alpha(s, X_s)$ is an $\mathscr{F}_s^X$-adapted Markovian control, which takes values in the control space $\mathbb{A} \subset \mathbb{R}^m$, and b and σ are continuous functions $b : \mathbb{T} \times \mathbb{R} \times \mathbb{A} \mapsto \mathbb{R}$ and $\sigma : \mathbb{T} \times \mathbb{R} \times \mathbb{A} \mapsto \mathbb{R}$ chosen such as to guarantee the existence of a unique strong solution.[1]

The fundamental control problem we are trying solve is that of maximising the expectation

[1]Here, standard assumptions are that the SDE satisfies (I) the uniform Lipschitz condition $\exists K \in (0, \infty)$ s.t. $\forall t \in \mathbb{T}$, $\forall x, y \in \mathbb{R}$ and $\forall \alpha \in \mathbb{A}$: $|b(t, x, \alpha) - b(t, y, \alpha)| + |\sigma(t, x, \alpha) - \sigma(t, y, \alpha)| \leq K|x - y|$, and (II) $\mathbb{E}[\int_0^T |b(t, 0, \alpha)|^2 + |\sigma(t, 0, \alpha)|^2 dt] \leq \infty$.

$$W(t,x,\alpha) = \mathbb{E}_{t,x}\left[\int_t^T e^{-\int_t^s \beta_u du} f(s, X_s^\alpha, \alpha_s)ds + e^{-\int_t^T \beta_u du} g(X_T^\alpha)\right], \tag{2}$$

for all $(t,x) \in \mathbb{T} \times \mathbb{R}$ and $\alpha \in \mathscr{A}(t,x)$. Here, $f : \mathbb{T} \times \mathbb{R} \times \mathbb{A} \mapsto \mathbb{R}$ is a running reward function, whilst $g : \mathbb{R} \mapsto \mathbb{R}$ is a terminal reward function, both of which typically are taken to satisfy quadratic growth conditions. $\beta_u = \beta(u, X_u^\alpha, \alpha_u)$ is a function $\beta : \mathbb{T} \times \mathbb{R} \times \mathbb{A} \mapsto \mathbb{R}$ which present-values the reward functions as appropriate. Finally, we denote by $\mathscr{A}(t,x)$ the subset of controls which are *admissible*, i.e. the $\mathscr{F}_t^X$-adapted $\mathbb{A}$-valued controls which minimally satisfy $\mathbb{E}[\int_t^T |e^{-\int_t^s \beta_u du} f(s, X_s^\alpha, \alpha_s)|ds] < \infty$.

The crux of the matter is that the maximisation problem (2) generally is non-trivial. Defining the (optimal) *value function*

$$V(t,x) = \sup_{\alpha \in \mathscr{A}(t,x)} W(t,x,\alpha), \tag{3}$$

one may invoke the *dynamic programming principle* (DPP)

$$V(t,x) = \sup_{\alpha \in \mathscr{A}(t,x)} \mathbb{E}_{t,x}\left[\int_t^\tau e^{-\int_t^s \beta_u du} f(s, X_s^\alpha, \alpha_s)ds + e^{-\int_t^\tau \beta_u du} V(\tau, X_\tau^\alpha)\right], \tag{4}$$

for any stopping time τ, in order to set up the *Hamilton-Jacobi-Bellman* (HJB) equation

$$0 = \partial_t V + \sup_{a \in \mathbb{A}} \left\{-\beta_t V + b(t,x,a)\partial_x V + \tfrac{1}{2}\sigma^2(t,x,a)\partial_{xx}^2 V + f(t,x,a)\right\}, \tag{5}$$

s.t. $V(T,x) = g(x)$. The most striking feature of the HJB equation is arguably the change in the supremum: while the original control problem asked us to optimise over the set of $\mathbb{A}$-valued control processes $\{\alpha_s, s \geq 0\}$, we are now faced with "merely" having to optimise over the set $\mathbb{A}$. Insofar as a solution, ϕ, can be found to (5) we may invoke a *verification procedure* to check if ϕ coincides with the value function (i.e. to check if we have indeed solved the problem). Assuming the technicalities above, it suffices to check if (a) $\phi \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R}) \cap \mathcal{C}^0(\mathbb{T} \times \mathbb{R})$ and if (b) ϕ satisfies a quadratic growth condition. The reader is referred to Fleming & Soner [12], Pham [31], and Ross [33], for details.

3. The Merton Problem: An Analytic Reminder

The benchmark result against which we shall be comparing most of our numerical procedures stems right from the foundations of stochastic control theory in continuous time finance. Specifically, we are interested in Merton's quintessential problem of portfolio optimisation, [24], [25], over finite investment horizons $\mathbb{T} = [0,T]$. For the reader's convenience, we here provide a cursory overview of the problem.

Let X be the running wealth of an investor who trades continuously in a risk free asset and a stock, the respective price processes of which obey the usual dynamical equations

$$dB_s = rB_s ds, \quad \text{and} \quad dP_s = \mu P_s ds + \sigma P_s dW_s,$$

where r, μ and σ are constant parameters. If $\theta : \Omega \times \mathbb{T} \mapsto \mathbb{R}$ denotes the total [dollar] amount the investor has in stocks, and $c : \Omega \times \mathbb{T} \mapsto \mathbb{R}_+$ denotes the investor's rate of consumption, it follows from the *self-financing condition*[2] that

$$dX_s^{\theta,c} = [rX_s^{\theta,c} + \theta_s(\mu - r) - c_s]ds + \theta_s \sigma dW_s, \tag{6}$$

where $X_t^{\theta,c} = x$ is the initial endowment which we shall assume non-negative. Deriving utility from both his life-time consumption rate as well as his terminal bequest, the investor's problem is that of determining an optimal control pair $(\theta_t^*, c_t^*,) \in \mathscr{A}(t,x)$ s.t.

$$W(t,x,\theta,c) = \mathbb{E}_{t,x}\left[\int_t^T e^{-\beta(s-t)}u(c_s)ds + e^{-\beta(T-t)}u(X_T^{\theta,c})\right], \tag{7}$$

is maximal, where the controls are assumed to lie in the admissibility set $\mathscr{A}(t,x) := \{(\theta, c) : \int_t^T (c_s + \theta_s^2)ds < \infty \ \& \ X_t^{\theta,c} \geq 0 \text{ a.s. } \forall t \in \mathbb{T}\}$. The discount factor, β, is assumed constant, and the utility function $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is assumed to be isoelastic (CRRA), i.e. $u(x) = x^{1-\gamma}/(1-\gamma)$, where γ codifies the investor's level of risk aversion. In this paper we will restrict our attention to the root functions $\gamma \in (0,1)$.

As per the previous section, defining $V(t,x) = \sup_{(\theta,c)\in\mathscr{A}(t,x)} W(t,x,\theta,c)$, the following HJB equation will be satisfied

$$\beta V = \partial_t V + \sup_{(\theta,c)\in\mathbb{R}\times\mathbb{R}_+}\left\{[rx + \theta(\mu - r) - c]\partial_x V + \tfrac{1}{2}\theta^2\sigma^2\partial_{xx}^2 V + \frac{c^{1-\gamma}}{1-\gamma}\right\}, \tag{8}$$

where $V(T,x) = x^{1-\gamma}/(1-\gamma)$. Furthermore, straightforward differentiation gives the *first order conditions* i.e. the optimal policies expressed qua partial derivatives of the value function:

$$\theta^*(t,x) = -\frac{(\mu - r)}{\sigma^2}\frac{\partial_x V}{\partial_{xx}^2 V}, \quad \text{and} \quad c^*(t,x) = (\partial_x V)^{-1/\gamma}. \tag{9}$$

Based on the linearity of the wealth dynamics, we now make the important observation that if (θ^*, c^*) is the optimal policy for wealth level x and time t, then $(k\theta^*, kc^*)$ ought to be the optimal policy for wealth level kx and time t. As the reader can readily verify from (7) this entails that $V(t,kx) = k^{1-\gamma}V(t,x)$, in particular upon setting $k = x^{-1}$ and rearranging we conclude that $V(t,x) = x^{1-\gamma}V(t,1)$ which is to say that the optimal value function is separable in wealth and time. Rephrasing this slightly for our own mathematical convenience this boils down to the ansatz $V(t,x) = g(t)^\gamma x^{1-\gamma}/(1-\gamma)$, where g is some deterministic function of time. Combining this with (8) and (9) we reduce, after a little manipulation, the problem to a Bernoulli equation, the solution to which is $g(t) = A^{-1}(1 + [A-1]e^{-A(T-t)})$ where $A = [\beta - r(1-\gamma)]/\gamma - \frac{1}{2}(1-\gamma)[\mu - r]^2/(\gamma^2\sigma^2)$. Crucially, this allows us to calculate explicit expressions for the first order conditions of (8) viz.

$$\theta^*(t,x) = \frac{(\mu - r)x}{\gamma\sigma^2}, \quad \text{and} \quad c^*(t,x) = \frac{x}{g(t)}, \tag{10}$$

[2] See e.g. Björk [1] Lemma 6.4.

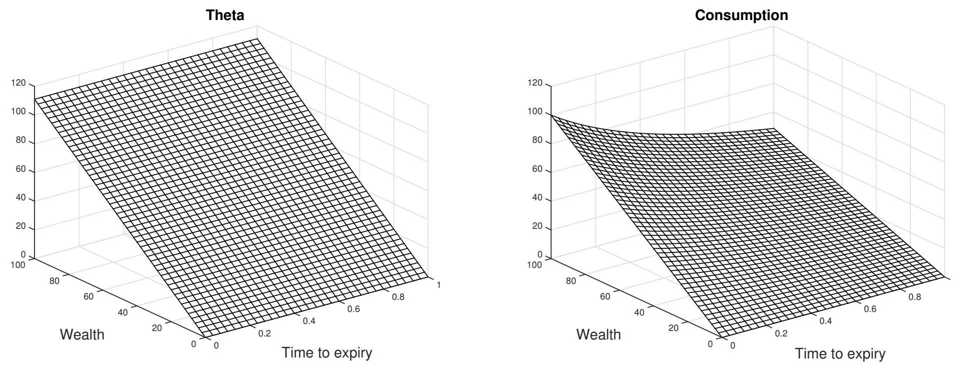

FIGURE 1. The optimal Merton policies plotted for various wealth levels at various holding periods ("time to expiry").

which, from verification, constitute the desired optimal controls. For illustrative purposes we exhibit these policies in figure 1 under the parametric specifications given in table 1. Note that the optimal investment policy is constant in time, but linear in wealth, implying that the investor has a constant fraction of his wealth invested in the stock. Meanwhile, his consumption again scales linearly with wealth but decays exponentially in time. At expiry he consumes everything, at bankruptcy he consumes nothing.

4. Towards a Trinomial/Explicit Markov Chain Approximation

4.1. **Establishing the Approximation.** Consider the discretisation $\mathscr{T}^\delta \times \mathscr{R}^h = \{0, \delta, 2\delta, ..., N\delta = T\} \times \{x_{\min}, x_{\min}+h, x_{\min}+2h, ..., x_{\min}+Ih =: x_{\max}\}$ of the full state space $\mathbb{T}\times\mathbb{R}$ of section 2, where, $x_{\min}$ and $x_{\max}$ are artificially imposed lower and upper boundaries, whilst h and δ are fixed spatial and temporal separations in the grid. Let $\{\xi_n^{h,\delta} | n \in \mathbb{N}_0\}$ be a controlled discrete parameter Markov chain on $\mathscr{R}^h$ that approximates (in a sense soon to be spelled out) the controlled process X_t^α. The stochastic evolution of said chain is determined by the set of probabilities $\{p^{h,\delta}(x, y|a, t) : x, y \in \mathscr{R}^h\}$ where $p^{h,\delta}(x, y|a, t)$ denotes the conditional probability that the Markov chain jumps from state x to state y given that the control $a \in \mathbb{A}$ is applied at time t. Needless to say, if these probabilities are to be construed as Markov chain transition probabilities, then they must satisfy the basic requirements of positivity and summing to unity. Let $a_n^{h,\delta} = a^{h,\delta}(n\delta, \xi_n^{h,\delta})$ denote the random variable, which is the actual control action for the chain, applied at the discrete time $n\delta$. We say the control policy $a^{h,\delta} = \{a_n^{h,\delta} | n \in \mathbb{N}_0\}$ for the chain is *admissible* provided that (i) $a_n^{h,\delta} \in \mathbb{A}$ and (ii) the chain has the Markov property under that policy, i.e. $\mathbb{P}[\xi_{n+1}^h = y | \xi_i^{h,\delta}, a_i^{h,\delta}, i \le n] = p^h(\xi_n^{h,\delta}, y | a_n^{h,\delta}, n\delta)$. Let $\mathscr{A}^{h,\delta}(n\delta, x)$ denote the set of admissible controls given that $\xi_n^{h,\delta} = x$ at time $n\delta$. We may then state the approximation to (2) as

$$
\begin{aligned}
W^{h,\delta}(n\delta, x, a^{h,\delta}) = \mathbb{E}\Bigg[\sum_{i=n}^{N-1} & e^{-\sum_{j=n}^{i-1} \beta(j\delta, \xi_j^{h,\delta}, a_j^{h,\delta})\delta} f(i\delta, \xi_i^{h,\delta}, a_i^{h,\delta})\delta \\
& + e^{-\sum_{j=n}^{N-1} \beta(j\delta, \xi_j^{h,\delta}, a_j^{h,\delta})\delta} g(\xi_N^{h,\delta}) \Bigg| \xi_n^{h,\delta} = x \Bigg],
\end{aligned}
$$

whilst the discretisation of the value function (3) is defined as

$$V^{h,\delta}(n\delta,x) = \sup_{a^{h,\delta}\in\mathscr{A}^{h,\delta}(n\delta,x)} W^{h,\delta}(n\delta,x,a^{h,\delta}). \tag{11}$$

Crucially, in order to secure convergence of $V^{h,\delta}(n\delta,x)$ to $V(n\delta,x)$ as $h\to 0$, it is incumbent that the Markov chain approximation is chosen in accordance with *local consistency conditions*. Specifically, defining $\Delta\xi_n^{h,\delta} := \xi_{n+1}^{h,\delta} - \xi_n^{h,\delta}$ we require that $\sup_{n,\omega}|\Delta\xi_n^{h,\delta}|\to 0$ as $h\to 0$ as well as

$$\mu_n^{h,\delta}(x,a) := \mathbb{E}[\Delta\xi_n^{h,\delta}|\xi_n^{h,\delta}=x, a_n^{h,\delta}=a] = b(n\delta,x,a)\delta + o(\delta), \tag{12a}$$

$$\Sigma_n^{h,\delta}(x,a) := \mathbb{E}[(\Delta\xi_n^{h,\delta} - \mu_n^{h,\delta}(x,a))^2|\xi_n^{h,\delta}=x, a_n^{h,\delta}=a] = \sigma^2(n\delta,x,a)\delta + o(\delta), \tag{12b}$$

$\forall x\in\mathscr{R}^h$ and $\forall a\in\mathbb{A}$. Here $o(y)$ is defined as a function which is small relative to y i.e: $\lim_{y\to 0} o(y)/y = 0$.

4.2. **Extracting the Solution.** Insofar as we have a locally consistent Markov chain approximation to X_t^α, we may solve for the discretised value function (11) through repeated application of the explicit DPP:[3]

$$V^{h,\delta}(n\delta,x) = \sup_{a\in\mathbb{A}}\left[f(n\delta,x,a)\delta + e^{-\beta(n\delta,x,a)\delta}\sum_{y\in\mathscr{R}^h} p^{h,\delta}(x,y|a,n\delta)V^{h,\delta}((n+1)\delta,y)\right] \tag{13}$$

starting from the terminal condition $V^{h,\delta}(N\delta,x)=g(x)$ and working our way incrementally backwards in time. As in a garden variety (linear) explicit procedure, the grid component $V^{h,\delta}(n\delta,x)$ is given entirely in terms of known quantities $\{V^{h,\delta}((n+1)\delta,y): y\in\mathscr{R}^h\}$, albeit with the added caveat that we must maximise the expression over all $a\in\mathbb{A}$ at every step in the process. Evidently, this aspect should (insofar as possible) be handled through the employ of the associated *first order conditions* i.e. the a^* which renders the partial derivative of the RHS of (13) with respect to a equal to zero. A considerably more time consuming procedure involves a search over a bounded mesh of possible controls at every point in the grid $\mathscr{T}^\delta\times\mathscr{R}^h$.

The final piece left of the puzzle is that of how we go about constructing a locally consistent Markov chain in the first place. A luminous beacon in night is in this context a fairly flexible programme involving finite difference approximations of the governing differential equation. Specifically, consider the PDE formally satisfied by (2). From Feynmac-Kac's theorem we have

$$\beta_t W = \partial_t W + b(t,x,a)\partial_x W + \tfrac{1}{2}\sigma^2(t,x,a)\partial_{xx}^2 W + f(t,x,a), \tag{14}$$

or in explicit discretised terms

[3]This equation is a one-step discrete approximation to (4) i.e.

$$V^{h,\delta}(n\delta,x) = \sup_{a\in\mathbb{A}}\left[f(n\delta,x,a)\delta + e^{-\beta(n\delta,x,a)\delta}\mathbb{E}\left[V^{h,\delta}((n+1)\delta,y)\middle|\xi_n^{h,\delta}=x, a_n^{h,\delta}=a\right]\right]$$

$$
\begin{aligned}
\beta_{n\delta} W^{h,\delta}((n+1)\delta, x, a) =& \delta^{-1}[W^{h,\delta}((n+1)\delta, x, a) - W^{h,\delta}(n\delta, x, a)] \\
&+ b(n\delta, x, a)\mathscr{D}_x W^{h,\delta}((n+1)\delta, x, a) \\
&+ \tfrac{1}{2}\sigma^2(n\delta, x, a)\mathscr{D}^2_{xx} W^{h,\delta}((n+1)\delta, x, a) + f(n\delta, x, a).
\end{aligned} \tag{15}
$$

The key step consists of defining the difference operators $\mathscr{D}_x$ and $\mathscr{D}^2_{xx}$ such that equation (15) yields transition probabilities which are uniformly non-negative. To this end we forgo central differencing and propose the following *up-wind* scheme:

- If $b(t,x,a) \geq 0$ let $\mathscr{D}_x W(t,x,a) = \mathscr{D}^+_x W(t,x,a) := h^{-1}[W(t,x+h,a) - W(t,x,a)]$.
- If $b(t,x,a) < 0$ let $\mathscr{D}_x W(t,x,a) = \mathscr{D}^-_x W(t,x,a) := h^{-1}[W(t,x,a) - W(t,x-h,a)]$.
- Let $\mathscr{D}^2_{xx} W(t,x,a) = h^{-2}[W(t,x+h,a) - 2W(t,x,a) + W(t,x-h,a)]$.

Defining $[y]^+ = \max\{y, 0\}$ and $[y]^- = \max\{-y, 0\}$, items 1 & 2 may be written as

- $b(t,x,a)\mathscr{D}_x W(t,x,a) = h^{-1}[W(t,x+h,a) - W(t,x,a)][b(t,x,a)]^+ - h^{-1}[W(t,x,a) - W(t,x-h,a)][b(t,x,a)]^-$.

Substituting these approximations into (15) and rearranging we obtain

$$
\begin{aligned}
&W^{h,\delta}(n\delta, x, a) \\
&\quad = \left(\delta h^{-1}[b(n\delta,x,a)]^+ + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)\right) W^{h,\delta}((n+1)\delta, x+h, a) \\
&\quad + \left(1 - \delta\beta(n\delta,x,a) - \delta h^{-1}|b(n\delta,x,a)| - \delta h^{-2}\sigma^2(n\delta,x,a)\right) W^{h,\delta}((n+1)\delta, x, a) \\
&\quad + \left(\delta h^{-1}[b(n\delta,x,a)]^- + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)\right) W^{h,\delta}((n+1)\delta, x-h, a) \\
&\quad + f(n\delta,x,a)\delta,
\end{aligned}
$$

where we have made use of the fact that $|y| = [y]^+ + [y]^-$. More succinctly, we can write this as

$$
W^{h,\delta}(n\delta, x, a) = \sum_{y \in \mathscr{R}^h(x)} p^{h,\delta}(x, y | a, n\delta) W^{h,\delta}((n+1)\delta, y, a) + f(n\delta, x, a)\delta,
$$

where we have defined the set $\mathscr{R}^h(x) := \{x+h, x, x-h\}$, alongside the pseudo-transition probabilities:

$$
p^{h,\delta}(x, x+h | a, n\delta) = \left(\delta h^{-1}[b(n\delta,x,a)]^+ + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)\right), \tag{16a}
$$

$$
\begin{aligned}
p^{h,\delta}(x, x | a, n\delta) = \Big(&1 - \delta\beta(n\delta,x,a) \\
&- \delta h^{-1}|b(n\delta,x,a)| - \delta h^{-2}\sigma^2(n\delta,x,a)\Big),
\end{aligned} \tag{16b}
$$

$$
p^{h,\delta}(x, x-h | a, n\delta) = \left(\delta h^{-1}[b(n\delta,x,a)]^- + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)\right), \tag{16c}
$$

$$
p^{h,\delta}(x, y | a, n\delta) = 0, \quad \forall y \notin \mathscr{R}^h(x). \tag{16d}
$$

The pseudo property arises from the fact that (16a)-(16d) clearly do not sum to unity. This prompts us to invoke a *renormalisation*: specifically, defining $\mathcal{N}^{h,\delta}_a := 1/(1 - \delta\beta(n\delta,x,a))$ we propose the following

$$p^{h,\delta}(x, x+h|a, n\delta) = \mathcal{N}_a^{h,\delta}\left(\delta h^{-1}[b(n\delta,x,a)]^+ + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)\right), \tag{17a}$$

$$\begin{aligned} p^{h,\delta}(x, x|a, n\delta) = \mathcal{N}_a^{h,\delta}\Big(&1 - \delta\beta(n\delta,x,a) \\ &- \delta h^{-1}|b(n\delta,x,a)| - \delta h^{-2}\sigma^2(n\delta,x,a)\Big), \end{aligned} \tag{17b}$$

$$p^{h,\delta}(x, x-h|a, n\delta) = \mathcal{N}_a^{h,\delta}\left(\delta h^{-1}[b(n\delta,x,a)]^- + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)\right), \tag{17c}$$

$$p^{h,\delta}(x, y|a, n\delta) = 0, \qquad \forall y \notin \mathscr{R}^h(x). \tag{17d}$$

which clearly satisfy $\sum_{y\in\mathscr{R}^h(x)} p(x,y|n\delta,a) = 1$. Furthermore, insofar as $1 > \delta\beta(n\delta,x,a) + \delta h^{-1}|b(n\delta,x,a)| + \delta h^{-2}\sigma^2(n\delta,x,a)$ $\forall x\forall n\forall a$ - or, equivalently,

$$\delta < \left[\beta(n\delta,x,a) + h^{-1}|b(n\delta,x,a)| + h^{-2}\sigma^2(n\delta,x,a)\right]^{-1}, \tag{18}$$

we can also guarantee the $p^{h,\delta}(x,x|a,n\delta)$ stays uniformly non-negative (the requirement of non-negativity is obviously satisfied by $p^{h,\delta}(x,x+h|a,n\delta)$ and $p^{h,\delta}(x,x-h|a,n\delta)$). Finally, it is easy to show that the probabilities satisfy the local consistency conditions (12). Thus, for (12a) we find that

$$\begin{aligned} \mu_n^{h,\delta}(x,a) &= h\cdot p^{h,\delta}(x,x+h|a,n\delta) + 0\cdot p^{h,\delta}(x,x|a,n\delta) - h\cdot p^{h,\delta}(x,x-h|a,n\delta) \\ &= h\mathcal{N}_a^{h,\delta}\left(\delta h^{-1}[b(n\delta,x,a)]^+ - \delta h^{-1}[b(n\delta,x,a)]^-\right) \\ &= (1-\delta\beta(n\delta,x,a))^{-1} b(n\delta,x,a)\delta \\ &= b(n\delta,x,a)\delta + o(\delta), \end{aligned}$$

as desired. Similarly, for (12b)

$$\begin{aligned} \Sigma_n^{h,\delta}(x,a) &= (h - b(n\delta,x,a)\delta)^2\cdot p^{h,\delta}(x,x+h|a,n\delta) + (b(n\delta,x,a)\delta)^2\cdot p^{h,\delta}(x,x|a,n\delta) \\ &\quad + (-h - b(n\delta,x,a)\delta)^2\cdot p^{h,\delta}(x,x-h|a,n\delta) + o(\delta) \\ &= \mathcal{N}_a^{h,\delta}\Big(h^2[\delta h^{-1}[b(n\delta,x,a)]^+ + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)] \\ &\quad + h^2[\delta h^{-1}[b(n\delta,x,a)]^- + \tfrac{1}{2}\delta h^{-2}\sigma^2(n\delta,x,a)]\Big) + o(\delta) \\ &= (1-\delta\beta(n\delta,x,a))^{-1}\Big(\delta\sigma^2(n\delta,x,a)\Big) + o(\delta) \\ &= \sigma^2(n\delta,x,a)\delta + o(\delta). \end{aligned}$$

Hence, the proposed transition probabilities are locally consistent.

Remark 4.1. In practice, the drift term $b(n\delta,x,a)$ is often found to be a linear sum of several different components. This allows us to deploy a method known as *splitting the operator*, which can reduce the computational complexity of (17) somewhat. Suppose $b(n\delta,x,a) = \sum_{i=1}^k b_i(n\delta,x,a)$ and that we apply the upwind criterion to each component of b individually rather than b as a whole. Specifically, writing $b(n\delta,x,a)\mathscr{D}_x W$ as $\sum_{i=1}^k b_i(n\delta,x,a)\mathscr{D}_{x,i}W$, suppose we choose a forward or backward differencing of $\mathscr{D}_{x,i}W$ based on the sign of $b_i(n\delta,x,a)$ $\forall i$, thus replacing $[b(n\delta,x,a)]^+$ in (17a) with $\sum_{i=1}^k[b_i(n\delta,x,a)]^+$, and $[b(n\delta,x,a)]^-$ in (17c) with $\sum_{i=1}^k[b_i(n\delta,x,a)]^-$. The advantage of this is apparent if the signs of the individual components are a priori given!

Remark 4.2. The skeptical reader might wonder whether our commitment to positive probabilities is mandatory. After all, there seems to be a growing community of iconoclasts who argue for the cogency of negative probabilities in finance, cf. Haug [14], Meissner and Burgin [23] and Zvan et el. [36]. Unfortunately, this heterodoxy is a dangerous game for our present purposes: specifically, upon proving convergence of the Markov chain approximation from viscosity principles, Kusher and Dupuis assume the *monotonicity property* (assumption A2.1. p. 449 [18]), which manifestly requires all probabilities to be non-negative.[4] Thus, we shall continue to abide by the positivity criterion (here, equation (18)), even though this inexorably will force us to adopt extremely small time steps in a Mertonian context. Of course, insofar as one is willing to face the possibility that one's algorithm might not converge, a Markov chain approximation with negative probabilities can still be invoked. Indeed, this author has had some success in this regard, although the line between stable and unstable input parameters is blurry at best: therefore, do tread carefully.

5. The Trinomial/Explicit Method and Merton's Problem

The DPP (13) and its tripartite probability structure (17), naturally allows for two different algorithmic interpretations with clear analogies in numerical option pricing. On the one hand, we can view it as recipe for a *trinomial tree*; on the other, as a full-fledged *finite difference grid.* Whilst this prima facie might appear like a minor detail, the difference of which boils down to algorithmic run time (the number of grid points evaluated), we shall argue that the rabbit hole goes deeper. Specifically, if we opt for the finite difference interpretation, then inevitably we will have to make specifications for the boundary points which can be a nebulous endeavour. This issue is side-stepped with a trinomial model, albeit at the cost of interlocking the number time steps with respect to the number of space steps, thereby complicating the positivity criterion (18) further. We provide a full exposition of these issues below: first, however, it is worthwhile phrasing the DPP and the associated transition probabilities for the Merton problem.

In line with non-negativity assumption on financial wealth we restrict x to the state-space $\mathscr{R}_1^h = \{0, h, 2h, ..., Ih = x_{\max}\}$. Let $\theta(n\delta, x)$ and $c(n\delta, x)$ be the discrete controls of the problem, both of which are supposed to be bounded by the interval $[0, Kx]$, for some constant K.[5] Then $\forall x \in \{h, 2h, ..., (I-1)h\}$ the DPP may be stated as

$$
\begin{aligned}
&V^{h,\delta}(n\delta, x) = \\
&\sup_{(\theta,c)\in\mathbb{R}\times\mathbb{R}_+}\left[\frac{c(n\delta,x)^{1-\gamma}}{1-\gamma}\delta + e^{-\beta\delta}\sum_{y\in\mathscr{R}^h(x)} p^{h,\delta}(x,y|\theta,c,n\delta)V^{h,\delta}((n+1)\delta,y)\right],
\end{aligned}
\tag{19}
$$

where $p^{h,\delta}(x,y|\theta,c,n\delta)$ is shorthand notation for $p^{h,\delta}(x,y|\theta(n\delta,x),c(n\delta,x),n\delta)$ and $\mathscr{R}^h(x) = \{x+h, x, x-h\}$. Specifically, we define the transition probabilities

[4] For an illuminating account on convergence of numerical HJB schemes to viscosity solutions the reader is referred to Forsyth and Labahn [13] particularly lemma 5.3 and theorem 5.1.

[5] Bounding different controls by different constants is obviously quite feasible: the reader should make a personal judgement call as to what makes sense in a given context.

$$
\begin{aligned}
p^{h,\delta}(x,x+h|\theta,c,n\delta) &= \frac{1}{1-\delta\beta}\Big(\delta h^{-1}\Big(rx+\theta(n\delta,x)(\mu-r)\Big)+\tfrac{1}{2}\delta h^{-2}\theta^2(n\delta,x)\sigma^2\Big),\\
p^{h,\delta}(x,x|\theta,c,n\delta) &= \frac{1}{1-\delta\beta}\Big(1-\delta\beta-\delta h^{-1}\Big(rx+\theta(n\delta,x)(\mu-r)+c(n\delta,x)\Big)\\
&\quad -\delta h^{-2}\theta^2(n\delta,x)\sigma^2\Big),\\
p^{h,\delta}(x,x-h|\theta,c,n\delta) &= \frac{1}{1-\delta\beta}\left(\delta h^{-1}c(n\delta,x)+\tfrac{1}{2}\delta h^{-2}\theta^2(n\delta,x)\sigma^2\right),\\
p^{h,\delta}(x,y|\theta,c,n\delta) &= 0, \qquad \forall y\notin\mathscr{R}^h(x),
\end{aligned}
$$

where we have made use of the splitting of the operator technique.[6] Finally, two straightforward differentiations of (19) yield the first order conditions (FOCs)

$$
\begin{aligned}
\theta^*(n\delta,x) &= -\frac{(\mu-r)}{\sigma^2}\frac{\mathscr{D}_x^+V^{h,\delta}((n+1)\delta,x)}{\mathscr{D}_{xx}^2V^{h,\delta}((n+1)\delta,x)},\\
c^*(n\delta,x) &= \left(\frac{e^{-\beta\delta}}{1-\beta\delta}\mathscr{D}_x^-V^{h,\delta}((n+1)\delta,x)\right)^{-1/\gamma}
\end{aligned}
$$

where the differencing operators $\mathscr{D}_x^+,\mathscr{D}_x^-$ and $\mathscr{D}_{xx}^2$ are as defined above, and we enforce the restriction $\theta^*(n\delta,x),c^*(n\delta,x)\in[0,KIh]$ $\forall n\forall x$. Given these analytic expressions for the optimal controls, the problem we are trying to solve is as simple as computing

$$
V^{h,\delta}(n\delta,x)=\left[\frac{c^*(n\delta,x)^\gamma}{1-\gamma}\delta+e^{-\beta\delta}\sum_{y\in\mathscr{R}^h(x)}p^{h,\delta}(x,y|\theta^*,c^*,n\delta)V^{h,\delta}((n+1)\delta,y)\right],
$$

incrementally backwards in time: $n=N-1,N-2,...,1,0$.

γ	β	r	μ	σ	$x_{\max}$	K
0.5	0.02	0.05	0.1	0.3	100	1.5

TABLE 1. The parametric specifications for the risk aversion γ, the subjective discounting β, the risk free rate r, the stock drift and volatility μ and σ, the upper bound on the state space $x_{\max}$, and the constant curbing the controls from above K.

5.1. **The Trinomial Method.** Let I be an even number ($\in 2\mathbb{N}$). The trinomial method is exactly what the name suggests: a recombining tree diagram where every node has exactly three child nodes (or, equivalently, a grid which cuts off both end nodes of its state space whenever we move backwards in time). Thus, at expiry, we set out by evaluating $V^{h,\delta}$ for all $x\in\{0,h,...,(I-1)h,Ih\}$ using the terminal condition. With these points at hand we proceed to compute $V^{h,\delta}$ for all $x\in\{h,...,(I-1)h\}$ at the previous time step using the DPP and continue thusly until $\frac{1}{2}I$ time steps into the past we compute $V^{h,\delta}$ for the singleton $x\in\{\frac{1}{2}Ih\}$. The advantages of this procedure over a full finite difference grid are clear: first, it saves us

[6] Specifically, since $x\geq 0$ $[rx]^+=rx$ ($\Rightarrow [rx]^-=0$); since $\mu>r$ the position in the risky asset must be positive whence $[\theta(\mu-r)]^+=\theta(\mu-r)$ ($\Rightarrow [\theta(\mu-r)]^-=0$). Lastly, since consumption is non-negative $[-c]^-=c$ ($\Rightarrow [-c]^+=0$).

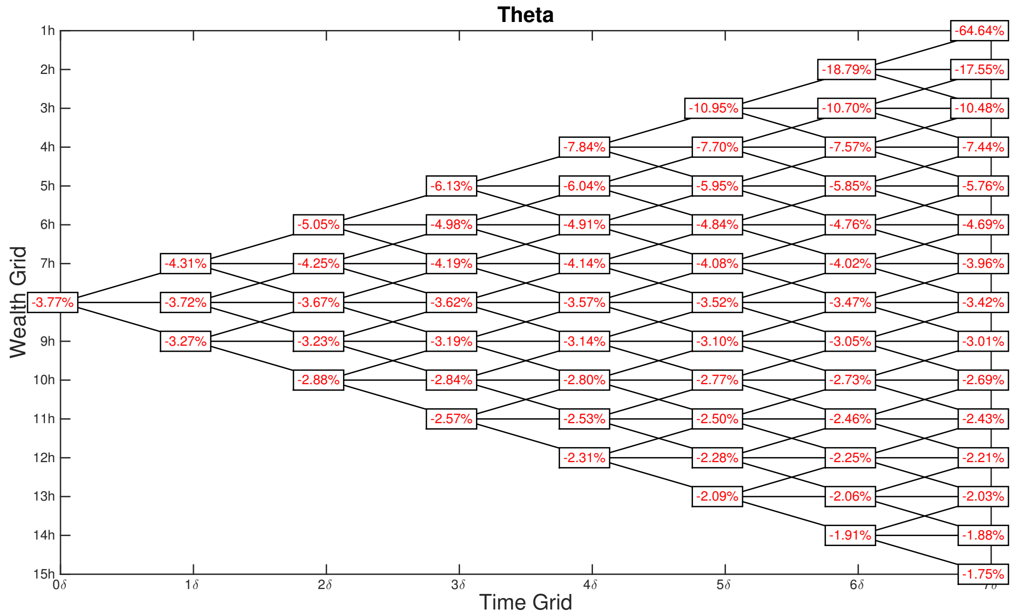

FIGURE 2. The percentage errors for the optimal stock investment obtained using the trinomial method.

the trouble of having to specify boundary conditions for $V^{h,\delta}(\cdot,0)$ and $V^{h,\delta}(\cdot,Ih)$. Secondly, assuming we are indeed only interested in the centre grid controls, there is a clear reduction in the number of nodes we need to evaluate ($(\frac{1}{2}I+1)^2$ for the tree versus $(\frac{1}{2}I+1)(I+1)$ for an analogously sized grid). However, the fact that we interlock the number of spatial separations in the grid $\#(h)=I$ with the number of temporal separations $\#(\delta)=N=\frac{1}{2}I$ does not bode well for the positivity criterion

$$\forall x\forall\theta\forall c:\delta<\left[\beta+h^{-1}\left(rx+\theta(n\delta,x)(\mu-r)+c(n\delta,x)\right)+h^{-2}\theta^2(n\delta,x,a)\sigma^2\right]^{-1}, \tag{20}$$

cf. equation (18). Specifically, upon setting $\delta=T/(\frac{1}{2}I)$, $h=x_{\max}/I$, $x=x_{\max}$ and $\theta=c=Kx_{\max}$ we get the *worst case scenario* inequality, which (if satisfied) surely will guarantee the positivity of all transition probabilities (and thence the convergence of the algorithm):

$$T<\tfrac{1}{2}I\left[\beta+(r+(\mu-r)K+K)I+\sigma^2K^2I^2\right]^{-1}. \tag{21}$$

This is a serious constraint. Suppose momentarily $I\in\mathbb{R}_+$: upon viewing the RHS as a function $f:\mathbb{R}_+\mapsto\mathbb{R}$ of I, we find that f assumes a maximum at $I^*=\sqrt{\beta/(\sigma^2K^2)}$. For realistic parametric values, $f(I^*)$ might prove considerably less than the desired level of T. Indeed, the true solution space (I should be a positive even number) will curb the allowed T values even further. Thus, the trinomial model might not be able to solve the Merton problem for the set of parameters we desire. Or at least not without some modification: e.g.

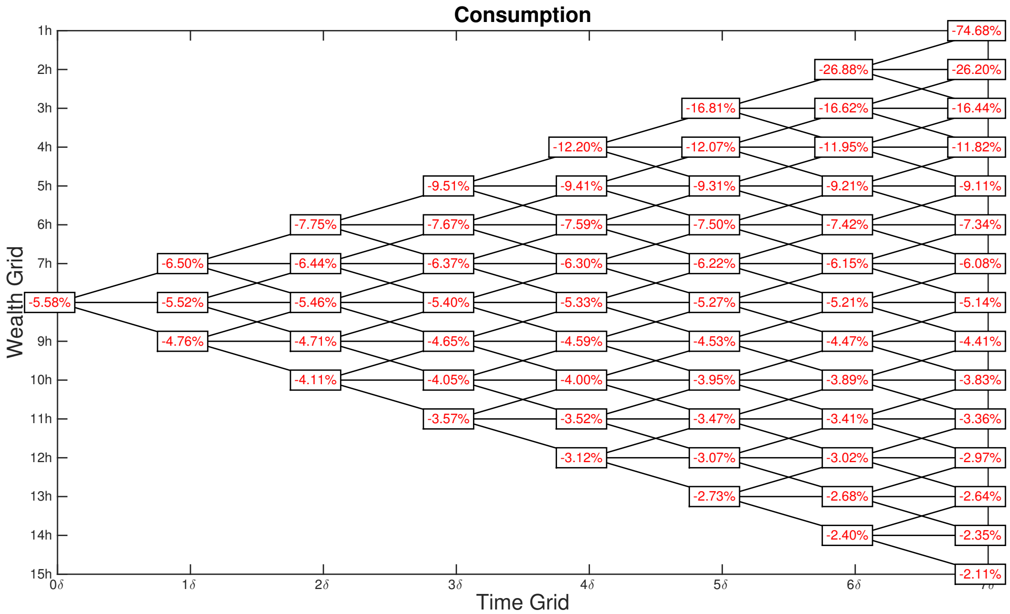

FIGURE 3. The percentage errors for the optimal consumption rate obtained using the trinomial method.

by introducing varying time steps in line with the fact that the state space decrease as we move backwards in time.

To get a feel for the gravity of this, consider a set-up with the parametric choices given in table 1. We find $I^* = 0.943$ and $f(I^*) = 0.300$: i.e. (21) makes us consider $T \leq 0.300$. Suppose we set $T = 0.1$: since $f(16) = 0.103$ an acceptable grid specification is $I = 16$ ($dt = 0.0125$, $h = 6.25$). Figures 2 and 3 exhibit the percentage errors (defined as $100 \cdot (a_{\text{approx.}} - a_{\text{true}})/a_{\text{true}}$) of the numerical controls incurred from the trinomial method. Unsurprisingly, the large grid spacing ($h = 6.25$) gives rise to non-negligible errors when evaluating the differencing operators $\mathscr{D}_x^{\pm} V$, $\mathscr{D}_{xx}^2 V$ *in lieu* of the derivatives $\partial_x V$ and $\partial_{xx}^2 V$. Indeed, these errors are most pronounced around $x = 0$ where the square root utility function ($\gamma = 0.5$) experiences the sharpest rise. For the mother of all nodes (the trunk of the tree, $(t, x) = (0, 50)$) we find the optimal numerical controls $(\theta^*, c^*) = (53.46, 42.76)$ vs. their analytic counterparts $(\theta^*, c^*) = (55.56, 45.29)$ and thence percentage errors of $(-3.77, -5.58)\%$.

5.2. **The Explicit Method.** The easiest way to circumvent the problems of the previous subsection is obviously to solve (19) as a full ("rectangular") finite difference grid, thereby decoupling $\#(\delta)$ from $\#(h)$. Obviously, we will still need to satisfy inequality (20) - however, this is now a matter of evaluating the RHS and specifying the δ accordingly. The main obstacle is undoubtedly the requirement that we must now make specifications for the boundary conditions alongside the grid. Compared to numerical problems in option pricing, this is considerably more obscure. E.g. whilst it is plausible that an option deep in or out of the

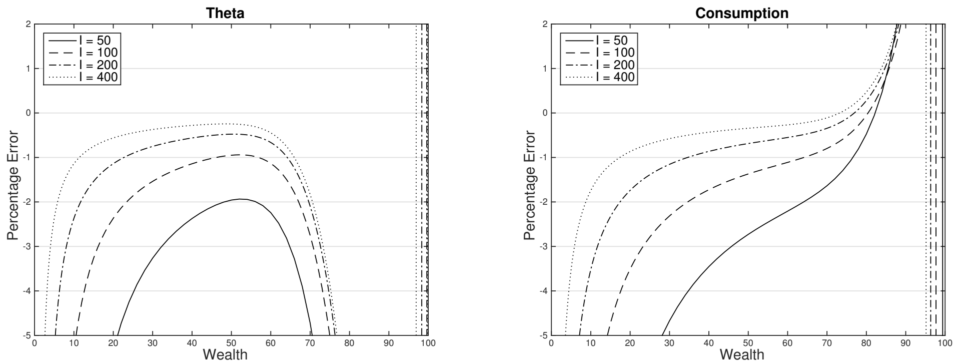

FIGURE 4. The percentage errors of the numerically computed optimal stock investment (**left**) and the optimal consumption (**right**) computed for various values of I using the explicit method. We assume $T = 1$ and $N = \lceil \beta + (r + (\mu - r)K + K)I + \sigma^2 K^2 I^2 \rceil$.

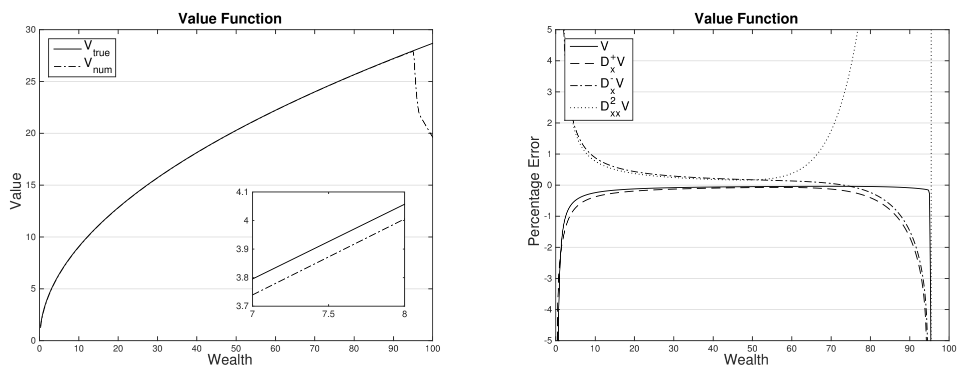

FIGURE 5. The analytic and numerical value functions (with miniature zoom) (**left**) and the percentage errors of the numerically computed value function including various derivatives (**right**) found using the explicit method. We assume $I = 100$, $T = 1$ and $N = \lceil \beta + (r + (\mu - r)K + K)I + \sigma^2 K^2 I^2 \rceil$.

money has vanishing gamma, an analogous argument does not carry over to a Mertonian value function problem.

A standing point of this paper is that the opacity of the boundary conditions is detrimental to the accuracy of our numerical controls for a *seizable* chunk of the state space. To this end, consider the *Dirichlet* conditions $\forall n \in \{0, 1, ..., N-1\}$:

$$V^{h,\delta}(n\delta, 0) = 0, \quad \text{and} \quad V^{h,\delta}(n\delta, Ih) = V^{h,\delta}(T, Ih).$$

The philosophy here is simple: a bankrupt investor ($x = 0$) can neither consume, nor build up a bequest. Thus, assuming $\gamma \in (0, 1)$ his value function is nil. At the other extreme ($x = Ih$), for relatively short investment horizons we do not expect drastic changes in the value function:

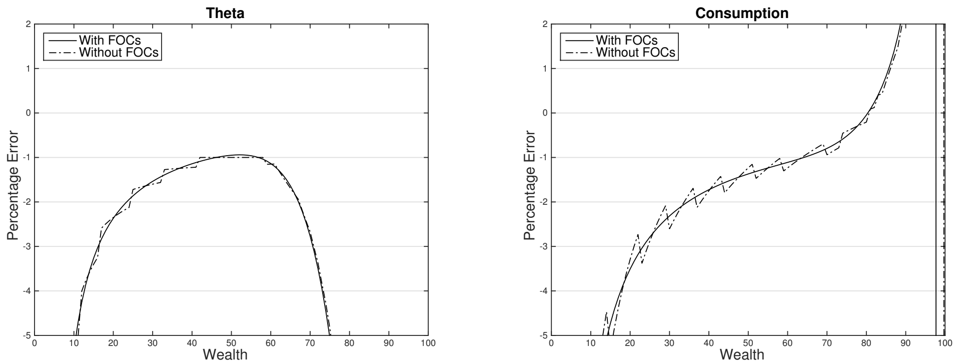

FIGURE 6. The percentage errors of the numerically computed optimal stock investment (**left**) and the optimal consumption (**right**) with and without the use of the first order conditions using the explicit method. We assume $I = 100$, $T = 1$ and $N = \lceil \beta + (r + (\mu - r)K + K)I + \sigma^2 K^2 I^2 \rceil$.

hence, we may approximate the upper boundary based on the terminal condition. To test the performance of the explicit method we plot the percentage errors of the numerically computed optimal controls in figure 4 for various values of I with an N just big enough to satisfy the worst case scenario inequality. These plots vividly demonstrate the grave shortcomings of our numerical method for large parts of the upper and lower state space. For low wealth levels this is hardly surprising: here the gradient of the utility function is at its steepest, and we wouldn't expect our relatively coarse grained difference operators to capture this adequately. The fact that the Dirichlet boundary happens to be analytically exact at $x = 0$ does to some degree seem to compensate for this as we increase I and N. On the other hand, the accuracy of θ^* near the upper boundary scarcely profits from increasing the fine-graining of the grid: seemingly, the mis-specification (however modest) of the upper Dirichlet boundary effectively kills our chances of reasonable convergence. Figure 5 offers a deeper investigation: the numerical value function is highly accurate for most wealth levels except at the upper boundary. The percentage error of the numerically computed second derivative is catastrophic in this region, which in turn propagates to the optimal stock investment (recall θ^* supervenes upon the ratio $\mathscr{D}_x^+ V / \mathscr{D}_{xx}^2 V$).

Remark 5.1. We previously stressed the importance of utilising the first order conditions when coding up the stochastic control problem. Nonetheless, there may be situations in which this is not feasible; i.e. where one has to resort to numerical optimisation of the DPP in stead. To test the feasibility of this method for the problem at hand, we discretised the space of possible controls $\mathbb{A} = [0, 150] \times [0, 150]$ into a mesh $\mathbb{A}_m$ of equidistant separation 0.01. At $t = T - \delta$ we performed a maximisation of the DPP over the full control mesh for every node in the optimal value grid. However, based on the principle that controls do not tend to vary drastically across incrementally close periods of time, all other maximisations were done over a subspace of $\mathbb{A}_m$ tailor-made to the optimal control pair of the subsequent time step. Specifically, if $(\theta^*(n\delta, x), c^*(n\delta, x)) = (\bar{\theta}, \bar{c})$ was found to be optimal for $V^{h,\delta}(n\delta, x)$, then the optimal solution for $V^{h,\delta}((n-1)\delta, x)$ was assumed to lie no further than ± 10 (in any direction) of $(\bar{\theta}, \bar{c})$. The numerical results of this method are illustrated in figure 6. While the algorithmic run time *with* the FOCs was found to be only 2.45 seconds on a 2.5 GHz Intel Core i5 processor, the corresponding run time *without* the FOCs was a staggering 1590.36 seconds.

6. Towards an Implicit Method

6.1. **A New Type of Markov Chain.** In subsection 4.1 we introduced a controlled discrete parameter Markov chain $\{\xi_n^{h,\delta} | n \in \mathbb{N}_0\}$ on the state space $\mathscr{R}^h = \{x_{\min}, x_{\min}+h, x_{\min}+2h, ..., x_{\min}+Ih =: x_{\max}\}$ in approximation of the controlled process X_t^α. Clearly, space and time play fundamentally different roles in this picture: while $\xi_n^{h,\delta}$ moves dynamically through space in accordance with locally consistent transition probabilities, time is a passive index ordered in terms of multiples of δ. The fundamental difference between the *explicit* and *implicit* method is precisely that the latter promotes time to a full state variable. Specifically, we now consider a controlled discrete parameter Markov chain $\{\zeta_n^{h,\delta} | n \in \mathbb{N}_0\}$ on the full time-space grid $\mathscr{T}^\delta \times \mathscr{R}^h = \{0, \delta, 2\delta, ..., N\delta = T\} \times \{x_{\min}, x_{\min}+h, x_{\min}+2h, ..., x_{\min}+Ih =: x_{\max}\}$, such that transition probabilities no longer carry the chain solely through space, but also through time.

Denote by $p^{h,\delta}(s,x;t,y|a)$ the conditional probability that the Markov chain jumps from state $(s,x) \in \mathscr{T}^\delta \times \mathscr{R}^h$ to state $(t,y) \in \mathscr{T}^\delta \times \mathscr{R}^h$ given that the control $a \in \mathbb{A}$ is applied. Furthermore, let $\Delta t_n^{h,\delta} := \Delta t^{h,\delta}(n\delta, \zeta_n^{h,\delta}, a_n^{h,\delta})$ denote a positive *interpolation interval*, such that $t_n^{h,\delta} = \sum_{j=0}^{n-1} \Delta t_j^{h,\delta}$. Then the discrete approximations to (2) and (3) may be stated as

$$
\begin{aligned}
W^{h,\delta}(n\delta, x, a^{h,\delta}) = \mathbb{E}\Bigg[&\sum_{i=n}^{N-1} e^{-\sum_{j=n}^{i-1} \beta(j\delta, \zeta_j^{h,\delta}, a_j^{h,\delta})\Delta t_j^{h,\delta}} f(i\delta, \zeta_i^{h,\delta}, a_i^{h,\delta}) \Delta t_i^{h,\delta} \\
&+ e^{-\sum_{j=n}^{N-1} \beta(j\delta, \zeta_j^{h,\delta}, a_j^{h,\delta})\Delta t_j^{h,\delta}} g(\zeta_N^{h,\delta}) \Bigg| \zeta_n^{h,\delta} = (n\delta, x) \Bigg],
\end{aligned}
$$

and

$$
V^{h,\delta}(n\delta, x) = \sup_{a^{h,\delta} \in \mathscr{A}^{h,\delta}(n\delta, x)} W^{h,\delta}(n\delta, x, a^{h,\delta}).
$$

Again, *local consistency* of some sort is the basic requirement for the convergence of the Markov chain. Let $\zeta_{n,0}^{h,\delta}$ and $\zeta_{n,1}^{h,\delta}$ denote the temporal and spatial parts of $\zeta_n^{h,\delta}$ respectively. We then require that $\sup_{n,\omega} |\Delta\zeta_{n,1}^{h,\delta}| \to 0$ as $h \to 0$ as well as

$$
\mu_n^{h,\delta}(x,a) := \mathbb{E}[\Delta\zeta_{n,1}^{h,\delta} | \zeta_{n,1}^{h,\delta} = x, a_n^{h,\delta} = a] = b(n\delta, x, a)\Delta t_n^{h,\delta} + o(\Delta t_n^{h,\delta}), \tag{22a}
$$

$$
\mathbb{E}[(\Delta\zeta_{n,1}^{h,\delta} - \mu_n^{h,\delta}(x,a))^2 | \zeta_{n,1}^{h,\delta} = x, a_n^{h,\delta} = a] = \sigma^2(n\delta, x, a)\Delta t_n^{h,\delta} + o(\Delta t_n^{h,\delta}), \tag{22b}
$$

$\forall x \in \mathscr{R}^h$ and $\forall a \in \mathbb{A}$, i.e. the spatial component of the chain must be consistent.

6.2. **Extracting the Solution.** As the designator clearly insinuates, the basic point of the *implicit method* is to set up a DPP, in which any given grid node at time $n+1$ is coupled to multiple grid nodes at the preceding time n (contrast this to the *explicit method* in which any given grid node is coupled to multiple grid nodes at the subsequent time step). Specifically, the implicit DPP we wish to solve is of the form

$$
\begin{aligned}
V^{h,\delta}(n\delta, x) = & \sup_{a\in\mathbb{A}} \Bigg[f(n\delta, x, a)\Delta t_n^{h,\delta} + e^{-\beta(n\delta,x,a)\Delta t_n^{h,\delta}} \sum_{y\in\mathscr{R}^h} p^{h,\delta}(n\delta, x; n\delta, y|a) V^{h,\delta}(n\delta, y) \\
& + e^{-\beta(n\delta,x,a)\Delta t_n^{h,\delta}} p(n\delta, x; (n+1)\delta, x|a) V^{h,\delta}((n+1)\delta, x) \Bigg],
\end{aligned} \tag{23}
$$

with the usual terminal condition $V^{h,\delta}(N\delta, x) = g(x)$. Notice that by virtue of implicitness the only non-zero cross temporal transition probability is the one that takes the spatial state into itself. Generalisations to this are quite feasible, although it would take us into the domain of so-called *θ-schemes*.

Before we derive locally consistent expressions for the transition probabilities and the interpolation interval, let us briefly consider how one should go about solving an expression like (23). Writing the implicit DPP on the general matrix form

$$
\sup_{a\in\mathbb{A}} \left[\mathbf{M}_n^{h,\delta}(a)\mathbf{V}_n^{h,\delta} + \mathbf{q}_n^{h,\delta}(a) \right] = \mathbf{V}_{n+1}^{h,\delta} \tag{24}
$$

where $\mathbf{V}_n^{h,\delta} := (V^{h,\delta}(n\delta, x_{\min}), V^{h,\delta}(n\delta, x_{\min}+h), ..., V^{h,\delta}(n\delta, x_{\min}+Ih))^\intercal$ and $\mathbf{q}_n^{h,\delta}(a)$ are vectors in $\mathbb{R}^{I+1}$, and $\mathbf{M}_n^{h,\delta}(a)$ is a matrix of transition probabilities in $\mathbb{R}^{(I+1)\times(I+1)}$, the challenges we face become quite apparent. Specifically, qua the supremum operator, (24) transcends the garden variety linear system of equations, which allows for immediate computation of $\mathbf{V}_n^{h,\delta}$ in terms of $\mathbf{V}_{n+1}^{h,\delta}$. Rather, the control dependence of $\mathbf{M}_n^{h,\delta}(a)$ and $\mathbf{q}_n^{h,\delta}(a)$ will generally render the LHS highly non-linear in $\mathbf{V}_n^{h,\delta}$ and thus expression (24) that much more difficult to solve.

6.2.1. *Approximations in Policy Space.* One of the easiest procedures to overcome the non-linearity issue of (24) is the employ of *policy space iterations*, based on a sequential computation of increasingly more accurate values for $\mathbf{V}_n^{h,\delta}$ and a. Specifically, let $a_0 \in \mathbb{A}$ be some initial admissible feedback policy (a reasonable starting point is to use the optimal control from the subsequent time step, $(n+1)\delta$). Then a first approximation $\mathbf{V}_{n,0}^{h,\delta}$ to $\mathbf{V}_n^{h,\delta}$ may be computed by solving the system

$$
\mathbf{M}_n^{h,\delta}(a_0)\mathbf{V}_{n,0}^{h,\delta} + \mathbf{q}_n^{h,\delta}(a_0) = \mathbf{V}_{n+1}^{h,\delta}.
$$

Indeed, knowledge of $\mathbf{V}_{n,0}^{h,\delta}$ allows us to compute an updated estimate, a_1, of a based on:[7]

$$
a_1 = \operatorname*{argmax}_{a\in\mathbb{A}} \left[\mathbf{M}_n^{h,\delta}(a)\mathbf{V}_{n,0}^{h,\delta} + \mathbf{q}_n^{h,\delta}(a) \right].
$$

Clearly, this procedure can now be repeated all over again. Indeed for a general $k = 0, 1, 2, ...$ we may perform the iterative steps

$$
\mathbf{M}_n^{h,\delta}(a_k)\mathbf{V}_{n,k}^{h,\delta} + \mathbf{q}_n^{h,\delta}(a_k) = \mathbf{V}_{n+1}^{h,\delta}, \tag{25a}
$$

$$
a_{k+1} = \operatorname*{argmax}_{a\in\mathbb{A}} \left[\mathbf{M}_n^{h,\delta}(a)\mathbf{V}_{n,k}^{h,\delta} + \mathbf{q}_n^{h,\delta}(a) \right]. \tag{25b}
$$

[7] Needless to say, this step is optimally handled through the FOCs.

Under mild conditions it can be shown that $\mathbf{V}^{h,\delta}_{n,k} \to \mathbf{V}^{h,\delta}_{n}$ as $k \to \infty$.[8] In practical terms, a reasonable place to stop the algorithm is when the value function stops changing notably in successive iterations: e.g. when $|\mathbf{V}^{h,\delta}_{n,k+1} - \mathbf{V}^{h,\delta}_{n,k}|_\infty < \varepsilon$, where ε is some small positive number.

6.2.2. *On ps and* Δts. Finally, let's expose the procedure by which we obtain the relevant transition probabilities and interpolation interval. Analogously to subsection 4.2 we consider an implicit upwind discretisation of (14)

$$\begin{aligned}\beta_{n\delta} W^{h,\delta}(n\delta, x, a) &= \delta^{-1}[W^{h,\delta}((n+1)\delta, x, a) - W^{h,\delta}(n\delta, x, a)] \\ &\quad + [b(n\delta, x, a)]^+ \mathscr{D}^+_x W^{h,\delta}(n\delta, x, a) - [b(n\delta, x, a)]^- \mathscr{D}^-_x W^{h,\delta}(n\delta, x, a) \\ &\quad + \tfrac{1}{2}\sigma^2(n\delta, x, a)\mathscr{D}^2_{xx} W^{h,\delta}(n\delta, x, a) + f(n\delta, x, a),\end{aligned} \tag{26}$$

where the differencing operators $\mathscr{D}^+_x, \mathscr{D}^-_x$ and $\mathscr{D}^2_{xx}$ are as defined above. This expression can be rearranged as

$$\begin{aligned}W^{h,\delta}(n\delta, x, a) &= \frac{\left(h^{-1}[b(n\delta, x, a)]^+ + \frac{1}{2}h^{-2}\sigma^2(n\delta, x, a)\right)}{Q^{h,\delta}(n\delta, x, a)} W^{h,\delta}(n\delta, x+h, a) \\ &\quad + \frac{\left(h^{-1}[b(n\delta, x, a)]^- + \frac{1}{2}h^{-2}\sigma^2(n\delta, x, a)\right)}{Q^{h,\delta}(n\delta, x, a)} W^{h,\delta}(n\delta, x-h, a) \\ &\quad + \frac{\delta^{-1}}{Q^{h,\delta}(n\delta, x, a)} W^{h,\delta}((n+1)\delta, x, a) + f(n\delta, x, a)\frac{1}{Q^{h,\delta}(n\delta, x, a)},\end{aligned} \tag{27}$$

where we have defined

$$Q^{h,\delta}(n\delta, x, a) := \beta(n\delta, x, a) + \delta^{-1} + h^{-1}|b(n\delta, x, a)| + h^{-2}\sigma^2(n\delta, x, a). \tag{28}$$

(27) is of the form

$$\begin{aligned}W^{h,\delta}(n\delta, x, a) &= \sum_{y \in \mathscr{R}^h_0(x)} p^{h,\delta}(n\delta, x; n\delta, y|a) W^{h,\delta}((n+1)\delta, y, a) \\ &\quad + p^{h,\delta}(n\delta, x; (n+1)\delta, x|a) W^{h,\delta}(n\delta, x, a) + f(n\delta, x, a)\Delta t^{h,\delta}_n,\end{aligned}$$

where $\mathscr{R}^h_0(x) := \{x+h, x-h\}$, which provides us with clear candidates for the transition probabilities and interpolation interval. However, again the associated $p^{h,\delta}(n\delta, x; n\delta, x+h|a)$, $p^{h,\delta}(n\delta, x; n\delta, x-h|a)$ and $p^{h,\delta}(n\delta, x; (n+1)\delta, x|a)$ fail to sum to unity unless $\beta = 0$. To compensate for this fact, we introduce a non-zero probability that the Markov chain stays the same

$$\begin{aligned}p(n\delta, x; n\delta, x|a) &= 1 - \sum_{y \in \mathscr{R}^h_0(x)} p^{h,\delta}(n\delta, x; n\delta, y|a) - p^{h,\delta}(n\delta, x; (n+1)\delta, x|a) \\ &= \frac{\beta(n\delta, x, a)}{Q^{h,\delta}(n\delta, x, a)}.\end{aligned}$$

All in all, we are therefore have

[8] See Kusher & Dupuis [18] theorem 6.2.1.

$$p^{h,\delta}(n\delta, x; n\delta, x+h|a) = \frac{h^{-1}[b(n\delta,x,a)]^+ + \frac{1}{2}h^{-2}\sigma^2(n\delta,x,a)}{Q^{h,\delta}(n\delta,x,a)}, \tag{29a}$$

$$p^{h,\delta}(n\delta, x; n\delta, x-h|a) = \frac{h^{-1}[b(n\delta,x,a)]^- + \frac{1}{2}h^{-2}\sigma^2(n\delta,x,a)}{Q^{h,\delta}(n\delta,x,a)}, \tag{29b}$$

$$p^{h,\delta}(n\delta, x; (n+1)\delta, x|a) = \frac{\delta^{-1}}{Q^{h,\delta}(n\delta,x,a)}, \tag{29c}$$

$$\begin{aligned} p^{h,\delta}(n\delta, x; n\delta, x|a) = 1 - &\sum_{y\in\mathscr{R}_0^h(x)} p^{h,\delta}(n\delta,x;n\delta,y|a) \\ &- p^{h,\delta}(n\delta,x;(n+1)\delta,x|a), \end{aligned} \tag{29d}$$

and

$$p^{h,\delta}(n\delta, x; n\delta, y|a, n\delta) = 0, \qquad \forall y \notin \mathscr{R}^h(x), \tag{29e}$$

along with the interpolation interval

$$\Delta t_n^{h,\delta} = \frac{1}{Q^{h,\delta}(n\delta,x,a)}. \tag{30}$$

It is readily seen that these probabilities are non-negative and comply with local consistency (22). Nevertheless, it remains computationally somewhat troubling (albeit mathematically correct) that the DPP (23) involves an optimisation over all $a \in \mathbb{A}$ of an expression which is heavily control dependent in its numerator and its denominator. To mitigate this tediousness, one may opt for redefining the denominator as the control independent quantity

$$\bar{Q}^{h,\delta}(n\delta,x) := \sup_{a\in\mathbb{A}} Q^{h,\delta}(n\delta,x,a), \tag{31}$$

assuming, of course, that $\bar{Q}^{h,\delta}(n\delta,x)$ is finite. If this fails to be the case, one can consider imposing artificial upper bounds on the controls (indeed, this will be the case for the Merton problem treated below - see also Fitzpatrick & Fleming [11]). The resulting transition probabilities are form-invariant expressions with respect to (29), albeit with the obvious proviso that (29d) will *not* equal $\beta(n\delta,x,a)/\bar{Q}^{h,\delta}(n\delta,x,a)$.

7. The Implicit Method and Merton's Problem

7.1. **Set-up.** Recall the definition of $Q^{h,\delta}$, (28), which in a Mertonian context this boils down to

$$Q^{h,\delta}(n\delta,x,\theta,c) = \beta + \delta^{-1} + h^{-1}\left(rx + \theta(n\delta,x)(\mu - r) + c(n\delta,x)\right) + h^{-2}\theta^2(n\delta,x,a)\sigma^2.$$

Clearly, the corresponding $\bar{Q}^{h,\delta}$, (31), is only finite insofar as we bound the controls from above (again, we will assume that $\forall n \forall x : \theta(n\delta,x), c(n\delta,x) \in [0, Kx]$). With this constraint

$$\bar{Q}^{h,\delta}(n\delta,x) = \beta + \delta^{-1} + h^{-1}\left(rx + Kx(\mu - r) + Kx\right) + h^{-2}K^2x^2\sigma^2.$$

Moreover, to save ourselves the trouble of writing multiple inverted hs in the transition probabilities, define

$$\tilde{Q}^{h,\delta}(n\delta,x) := h^2\bar{Q}^{h,\delta}(n\delta,x),$$

then $\forall x \in \{h, 2h, ..., (I-h)h\}$

$$\begin{aligned}
p^{h,\delta}(n\delta,x;n\delta,x+h|\theta,c) &= \frac{h\big(rx+\theta(n\delta,x)(\mu-r)\big)+\frac{1}{2}\theta(n\delta,x)^2\sigma^2}{\tilde{Q}^{h,\delta}(n\delta,x)},\\
p^{h,\delta}(n\delta,x;n\delta,x-h|\theta,c) &= \frac{hc(n\delta,x)+\frac{1}{2}\theta(n\delta,x)^2\sigma^2}{\tilde{Q}^{h,\delta}(n\delta,x)},\\
p^{h,\delta}(n\delta,x;(n+1)\delta,x|\theta,c) &= \frac{h^2\delta^{-1}}{\tilde{Q}^{h,\delta}(n\delta,x)},\\
p^{h,\delta}(n\delta,x;n\delta,x|\theta,c) &= 1-\sum_{y\in\mathscr{R}_0^h(x)} p^{h,\delta}(n\delta,x;n\delta,y|\theta,c)-p^{h,\delta}(n\delta,x;(n+1)\delta,x|a),\\
p^{h,\delta}(n\delta,x;n\delta,y|\theta,c) &= 0, \qquad \forall y\notin\mathscr{R}^h(x),
\end{aligned} \tag{32}$$

where $\Delta t_n^{h,\delta} = h^2/\tilde{Q}^{h,\delta}(n\delta,x)$. The main obstacle is again specifying plausible boundary conditions for the lower and upper boundaries. At $x=0$ we maintain that bankruptcy corresponds to a zero-consumption zero-investment strategy at all points in time, i.e. $V^{h,\delta}(0)=0$ with

$$\begin{aligned}
p(n\delta,0;n\delta,0|\theta,c) &= 1,\\
p(n\delta,0;n\delta,y|\theta,c) &= 0, \qquad \forall y\neq 0.
\end{aligned}$$

For $x = Ih$ the situation remains less transparent. Following [11] and [26] we make the assumption that there's a vanishing probability of leaving the grid, i.e.

$$\begin{aligned}
p^{h,\delta}(n\delta,Ih;n\delta,Ih-h|\theta,c) &= \frac{hc(n\delta,Ih)+\frac{1}{2}\theta(n\delta,Ih)^2\sigma^2}{\tilde{Q}^{h,\delta}(n\delta,Ih)},\\
p^{h,\delta}(n\delta,Ih;(n+1)\delta,Ih|\theta,c) &= \frac{h^2\delta^{-1}}{\tilde{Q}^{h,\delta}(n\delta,x)},\\
p^{h,\delta}(n\delta,Ih;n\delta,Ih|\theta,c) &= 1-p(n\delta,Ih;n\delta,Ih-h|\theta,c)-p(n\delta,Ih;(n+1)\delta,Ih|\theta,c),\\
p^{h,\delta}(n\delta,Ih;n\delta,y|\theta,c) &= 0, \qquad \forall y\notin\{Ih-h,Ih\}.
\end{aligned} \tag{33}$$

Thus, $\forall x \in \{h, 2h, ..., (I-1)h\}$ we have the implicit DPP

$$\begin{aligned}
V^{h,\delta}(n\delta,x) = \sup_{(\theta,c)\in\mathbb{R}\times\mathbb{R}_+}\Bigg[&\frac{c(n\delta,x)^{1-\gamma}}{1-\gamma}\frac{h^2}{\tilde{Q}^{h,\delta}(n\delta,x)} + e^{-\frac{\beta h^2}{\tilde{Q}^{h,\delta}(n\delta,x)}}\sum_{y\in\mathscr{R}^h(x)} p^{h,\delta}(n\delta,x;n\delta,y\\
&|\theta,c)V^{h,\delta}(n\delta,y) + e^{-\frac{\beta h^2}{\tilde{Q}^{h,\delta}(n\delta,x)}} p(n\delta,x;(n+1)\delta,x|\theta,c)V^{h,\delta}((n+1)\delta,x)\Bigg],
\end{aligned}$$

with terminal condition $V^{h,\delta}(N\delta,x) = x^{1-\gamma}/(1-\gamma)$. An analogous[9] expression holds for $V^{h,\delta}(n\delta,Ih)$ whilst $V^{h,\delta}(n\delta,0)=0$. We solve the DPP using iterations in policy space: specifically, at a given iterative step $k\in\mathbb{N}_0$ we solve the tridiagonal linear system

[9] I.e. with probabilities as in (33).

$$\mathbf{M}_n^{h,\delta}(\theta_k, c_k)\mathbf{V}_{n,k}^{h,\delta} + \mathbf{q}_n^{h,\delta}(c_k) = -h^2\delta^{-1}\mathbf{V}_{n+1}^{h,\delta}, \tag{34}$$

where $\mathbf{V}_{n,k}^{h,\delta} \in \mathbb{R}^{I+1}$ is a vector the $(i+1)^{\text{th}}$ component of which is $V_k^{h,\delta}(n\delta, ih)$, and $\mathbf{q}_n^{h,\delta}(c_k) \in \mathbb{R}^{I+1}$ is a vector the first component of which is 0, and more generally, the $(i+1)^{\text{th}} > 1$ component of which is $h^2 \exp\{\beta h^2/\tilde{Q}^{h,\delta}(n\delta, ih)\} c_k(n\delta, ih)^{1-\gamma}/(1-\gamma)$. Furthermore, $\mathbf{M}_n^{h,\delta}(\theta_k, c_k)$ is the $(I+1)\times(I+1)$ tridiagonal matrix

$$\mathbf{M}_n^{h,\delta}(\theta_k, c_k) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ M_{1,0} & M_{1,1} & M_{1,2} & 0 & \cdots & 0 \\ 0 & M_{2,1} & M_{2,2} & M_{2,3} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & M_{I,I-1} & M_{I,I} \end{pmatrix},$$

with components

$$\begin{aligned} M_{i,i-1} &= hc_k(n\delta, ih) + \tfrac{1}{2}\theta_k(n\delta, ih)^2\sigma^2, && i \in \mathbb{N}^I \\ M_{i,i+1} &= h\big(rih + \theta_k(n\delta, ih)(\mu - r)\big) + \tfrac{1}{2}\theta_k(n\delta, ih)^2\sigma^2, && i \in \mathbb{N}^{I-1} \\ M_{i,i} &= \tilde{Q}^{h,\delta}(n\delta, ih)(1 - e^{\beta h^2/\tilde{Q}^{h,\delta}(n\delta, ih)}) - M_{i,i-1} - M_{i,i+1} - h^2\delta^{-1}, && i \in \mathbb{N}^{I-1} \\ M_{I,I} &= \tilde{Q}^{h,\delta}(n\delta, Ih)(1 - e^{\beta h^2/\tilde{Q}^{h,\delta}(n\delta, Ih)}) - M_{I,I-1} - h^2\delta^{-1}, \end{aligned}$$

where $\mathbb{N}^I = \{1, 2, ..., I\}$ and $\mathbb{N}^{I-1} = \{1, 2, ..., I-1\}$. The associated policy update (cf. equation (25b)) is performed through the FOCs:

$$\begin{aligned} \theta_{k+1}(n\delta, ih) &= -\frac{(\mu - r)}{\sigma^2}\frac{\mathscr{D}_x^+ V_k^{h,\delta}((n+1)\delta, ih)}{\mathscr{D}_{xx}^2 V_k^{h,\delta}((n+1)\delta, ih)}, && i \in \mathbb{N}^{I-1} \\ c_{k+1}(n\delta, ih) &= \left(e^{-\frac{\beta h^2}{\tilde{Q}^{h,\delta}(n\delta, ih)}}\mathscr{D}_x^- V_k^{h,\delta}((n+1)\delta, ih)\right)^{-1/\gamma}, && i \in \mathbb{N}^I \end{aligned}$$

with all other control values zero: $\theta_{k+1}(n\delta, 0) = \theta_{k+1}(n\delta, Ih) = c_{k+1}(n\delta, 0) = 0$. Again we enforce $\theta(n\delta, ih), c(n\delta, ih) \in [0, KIh]\ \forall n, \forall i$.

Remark 7.1. At a given time step $n\delta < T$, it is opportune to set the initial controls $\theta_0(n\delta, :)$ and $c_0(n\delta, :)$ equal to $\theta^*((n+1)\delta, :)$ and $c^*((n+1)\delta, :)$ respectively.

Remark 7.2. Equation (34) can aptly be solved with Thomas' algorithm. If we insist on inverting the matrix, a Gaussian elimination procedure would cost us $\mathcal{O}((I+1)^3)$ binary operations per inversion. On the other hand, if we simply aim to solve the problem (and we do), an $\mathcal{O}(I+1)$ tridiagonal matrix algorithm will do just fine.

7.2. **Results.** We continue to work under the parameter specifications in table 1 and set the policy convergence parameter ε to 0.0001. Figure 7 plots the percentage errors of the numerically computed optimal controls, for various levels of I, with $T = 1$ and N fixed at 10 ($dt = 0.1$). Despite the comparatively large time steps, we find that the picture is almost identical to the corresponding explicit case, cf. figure 4, with the following provisos: (a) There is a small loss in accuracy in the optimal investment strategy between $I = 200$ and $I = 400$. (b) The optimal consumption seems to overshoot the 0% level. Further grid refinements (up

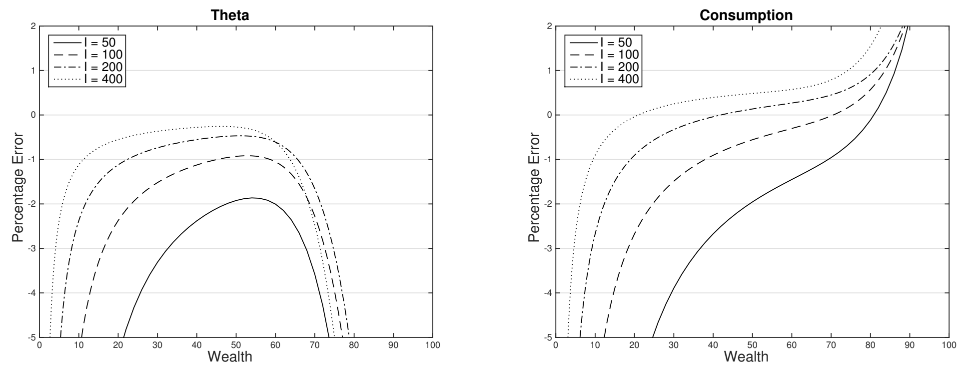

FIGURE 7. The percentage errors of the numerically computed optimal stock investment (**left**) and the optimal consumption (**right**) computed for various values of I using the implicit method. We assume $T = 1$ and $N = 10$. On average three policy iterations per time step are performed.

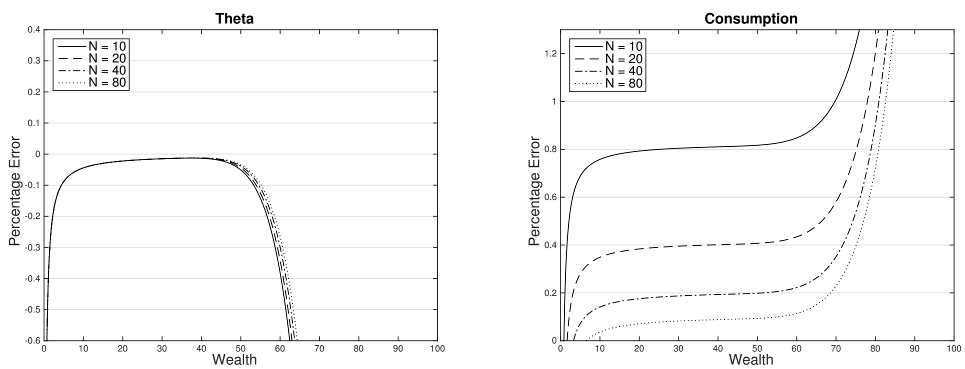

FIGURE 8. The percentage errors of the numerically computed optimal stock investment (**left**) and the optimal consumption (**right**) computed for various values of N using the implicit method. We assume $T = 1$ and $I = 10^4$.

to $I = 5 \cdot 10^4$) indicate that the upper boundary does not deteriorate much further. Indeed, if we simultaneously refine the grid along the temporal axis (recall, this is what was done in the explicit case in order to satisfy the worst case scenario inequality), the accuracy of the controls will improve. This is vividly illustrated in figure 8 for $I = 10^4$: notice in particular the improvements in the numerically computed optimal consumption, which quickly reverts back to the desired zeroth level of percentage error.

Ultimately, figures 7 and 8 are also a testimony to the fact that whilst increasing I quickly dampens numerical imprecision at low wealth levels, there is relatively little to be gained at the other end of the wealth spectrum. Only by augmenting N as well do we experience some rather modest improvements in the percentage error for the upper boundary. Thus, we are once more left with the impression that our inexact upper boundary effectively kills our chances of numerical accuracy in that region - at least for reasonable levels of computational

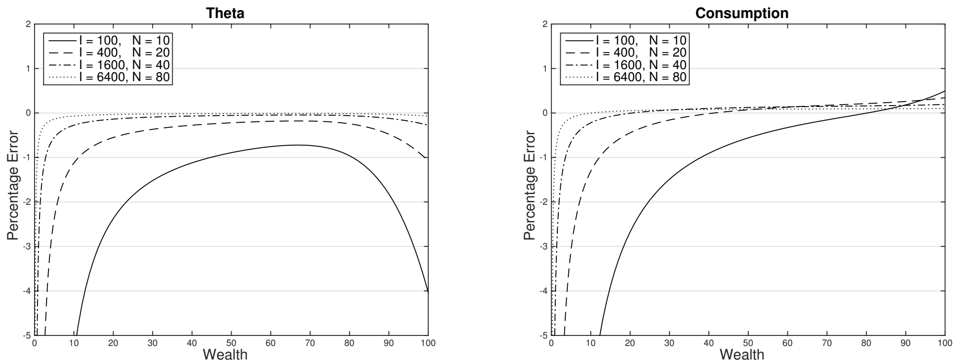

FIGURE 9. The percentage errors of the numerically computed optimal stock investment (**left**) and the optimal consumption (**right**) computed for various values of I and N, assuming a relational upper boundary, (35), using the implicit method.

time. Obviously, providing an analytically exact boundary defeats the very purpose of a numerical routine in the first place, so for most practical purposes it seems we have to *bite the bullet*. A somewhat milder strategy would be the deployment of an accurate ansatz for the value function at the upper boundary. Specifically, recall that the linearity of the wealth dynamics (6) previously lead us to conjecture that if (θ^*, c^*) is an optimal control pair at (t, x) then $(k\theta^*, kc^*)$ will be optimal for (t, kx), and thence that the value function is separable in time and wealth: $V(t,x) = g(t)^\gamma x^{1-\gamma}/(1-\gamma)$, where g is some function of time. Enforcing this equation at the upper boundary, it is readily shown that

$$V^{h,\delta}(n\delta, (I+1)h) = \left(1 + \frac{1}{I}\right)^{1-\gamma} V^{h,\delta}(n\delta, Ih). \tag{35}$$

We call this the *relational boundary*. Thus, at $x = Ih$, rather than using the inward probabilities (33) of Fitzpatrick & Fleming, we retain the expressions (32): a transition to $V^{h,\delta}(n\delta, (I+1)h)$ is simply handled through (35). Similarly, a FOC $(\theta(n\delta, Ih))$ which depends on $V^{h,\delta}(n\delta, (I+1)h)$ can be handled through (35). The results speak for themselves: in figure 9 we see that we effectively have eradicated numerical imprecision for high wealth levels for suitably fine grids. The price we had to pay was the correct assumption that the optimal value function is separable in space and time.

Finally, returning to the original boundary of Fitzpatrick & Fleming, let us say a few words about parametric choices and numerical reliability. In Munk [26] it is demonstrated that the subjective discount factor, β, is heavily correlated with the accuracy of the numerical procedure *in the infinite horizon case*, to the point that $\beta \approx 0.1$ yields deplorably inaccurate numerical controls, while $\beta \approx 0.8$ yields admirably accurate numerical controls. Hence, Munk argues that for economically plausible (i.e. low) values of β that grid must be so designed such that one can ignore "a rather wide neighbourhood of the imposed upper boundary". Having worked with $\beta = 0.02$ throughout this paper, the magnitude of the discounting factor is clearly less of an issue in the finite horizon case. However, as figure 10 clearly illustrates, this does not belie the fact that higher values of β generally lead to better numerical accuracy.

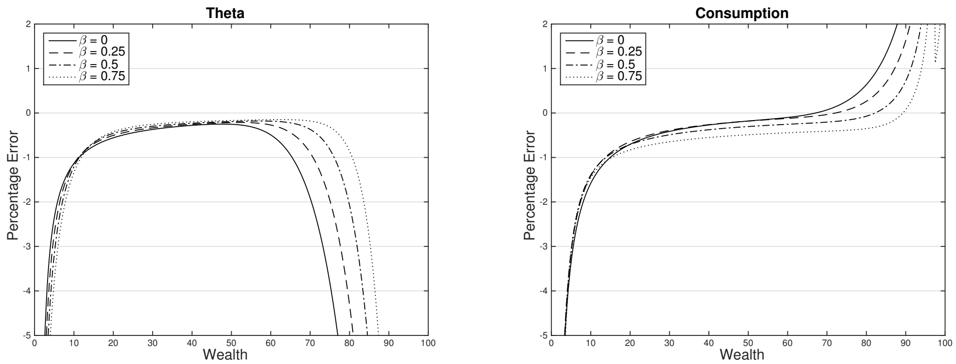

FIGURE 10. The percentage errors of the numerically computed optimal stock investment (**left**) and the optimal consumption (**right**) computed for various values of β using the implicit method. We assume $T = 1$, $N = 50$ and $I = 400$.

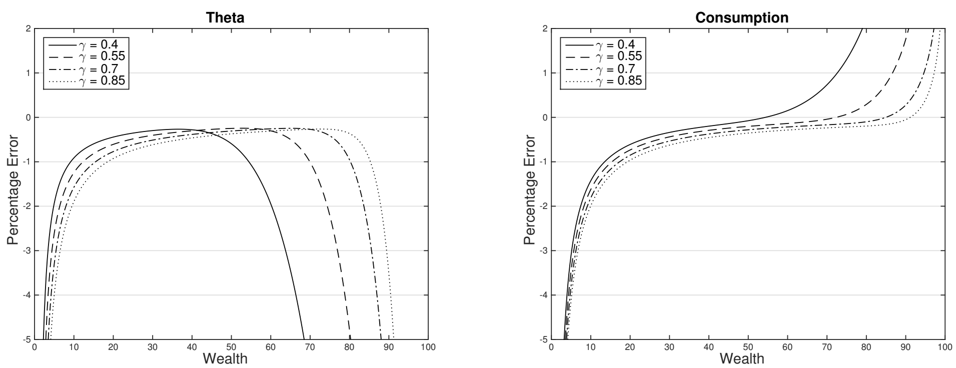

FIGURE 11. The percentage errors of the numerically computed optimal stock investment (**left**) and the optimal consumption (**right**) computed for various values of γ using the implicit method. We assume $T = 1$, $N = 50$ and and $I = 400$.

A similar conclusion extends to the level of risk aversion, γ, cf. figure 11. Low risk aversion ($\gamma \approx 0.35$) leads to numerical instability. On the other hand, a high risk aversion can significantly reduce the inaccuracy at the upper boundary. This is hardly surprising: at $x = Ih$, as γ increases, the *curvature*[10] of utility function (and thence also the optimal value function) decreases, which ultimately entails higher accuracy for the difference operators - particularly $\mathscr{D}^2_{xx}$.

[10]Per definitionem, if $f : \mathbb{R} \mapsto \mathbb{R}$ is a $\mathcal{C}^2$ function, then the curvature is given by $\kappa = |f''|/(1 + f'^2)^{3/2}$.

8. A Tale From Higher Dimensions

8.1. **The Multivariate Implicit Method.** Hitherto our concern has solely been with stochastic control problems with a singular spatial dimension. Prima facie, a generalisation to higher dimensions might seem like a conceptually trivial extension of what has already been covered (modulo the exponential increase in computational complexity - Bellman's so-called *curse of dimensionality*). Nonetheless, as we shall see, the verisimilitude of this claim rests heavily upon a rather severe constraint on the diffusion matrix, which invariably will bring us into trouble for Merton type problems. For our present purposes, we shall restrict our attention to an *implicit implementation*. The DPP we wish to solve is therefore still of the generic form (23), where y now runs over a multi-dimensional space grid.

We reinterpret equation (1) as an m-dimensional process, where $b:\mathbb{T}\times\mathbb{R}^m\times\mathbb{A}\mapsto\mathbb{R}^m$, $\sigma:\mathbb{T}\times\mathbb{R}^m\times\mathbb{A}\mapsto\mathbb{R}^{m\times r}$ and W is an $r-$dimensional standard Brownian motion. The PDE satisfied by (2) is the multi-dimensional extension of (14) viz.

$$\beta_t W = \partial_t W + \sum_{i=1}^{m} b_i(t,\boldsymbol{x},a)\partial_{x_i}W + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}(t,\boldsymbol{x},a)\partial^2_{x_i x_j}W + f(t,\boldsymbol{x},a), \tag{36}$$

where we have defined $\boldsymbol{x} := (x_1,x_2,...,x_m)^\intercal$ and $S(t,\boldsymbol{x},a) := \sigma(t,\boldsymbol{x},a)\sigma^\intercal(t,\boldsymbol{x},a)$. Again, to procure transition probabilities for the approximating Markov chain $\{\zeta_n^{h,\delta}|n\in\mathbb{N}_0\}$ on the multi-dimensional grid $\mathscr{T}^\delta\times\mathscr{R}^{h_1}\times...\times\mathscr{R}^{h_m} = \{0,\delta,...,N\delta\}\times\{x_{1,\min},x_{1,\min}+h_1,...,x_{1,\min}+h_1I_1 =: x_{1,\max}\}\times...\times\{x_{m,\min},x_{m,\min}+h_m,...,x_{m,\min}+h_mI_m =: x_{m,\max}\}$ we discretise (36) in the *upwind* sense in analogy with (26). Let $\hat{\mathbf{e}}_i = (0,0,...,1_i,0,...,0)^\intercal\in\mathbb{R}^m$ be a unit vector in the direction of the i^{th} spatial dimension, then the relevant difference operators may be stated as

- $\mathscr{D}^+_{x_i}W(t,\boldsymbol{x},a) := h_i^{-1}[W(t,\boldsymbol{x}+\hat{\mathbf{e}}_ih_i,a) - W(t,\boldsymbol{x},a)]$ (used if $b_i(t,\boldsymbol{x},a)\geq 0$).
- $\mathscr{D}^-_{x_i}W(t,\boldsymbol{x},a) := h_i^{-1}[W(t,\boldsymbol{x},a) - W(t,\boldsymbol{x}-\hat{\mathbf{e}}_ih_i,a)]$ (used if $b_i(t,\boldsymbol{x},a) < 0$).
- $\mathscr{D}^2_{x_ix_i}W(t,\boldsymbol{x},a) := h_i^{-2}[W(t,\boldsymbol{x}+\hat{\mathbf{e}}_ih_i,a) - 2W(t,\boldsymbol{x},a) + W(t,x-\hat{\mathbf{e}}_ih_i,a)]$.

Furthermore, we introduce the upwind cross derivatives:

- $\mathscr{D}^{+2}_{x_ix_j}W(t,\boldsymbol{x},a) := (2h_ih_j)^{-1}[2W(t,\boldsymbol{x},a) + W(t,\boldsymbol{x}+\hat{\mathbf{e}}_ih_i+\hat{\mathbf{e}}_jh_j,a) + W(t,\boldsymbol{x}-\hat{\mathbf{e}}_ih_i-\hat{\mathbf{e}}_jh_j,a) - W(t,\boldsymbol{x}+\hat{\mathbf{e}}_ih_i,a) - W(t,\boldsymbol{x}-\hat{\mathbf{e}}_ih_i,a) - W(t,\boldsymbol{x}+\hat{\mathbf{e}}_jh_j,a) - W(t,x-\hat{\mathbf{e}}_jh_j,a)]$ (used if $i\neq j$ and $S_{ij}(t,\boldsymbol{x},a)\geq 0$).
- $\mathscr{D}^{-2}_{x_ix_j}W(t,\boldsymbol{x},a) := -(2h_ih_j)^{-1}[2W(t,\boldsymbol{x},a) + W(t,\boldsymbol{x}+\hat{\mathbf{e}}_ih_i-\hat{\mathbf{e}}_jh_j,a) + W(t,\boldsymbol{x}-\hat{\mathbf{e}}_ih_i+\hat{\mathbf{e}}_jh_j,a) - W(t,\boldsymbol{x}+\hat{\mathbf{e}}_ih_i,a) - W(t,\boldsymbol{x}-\hat{\mathbf{e}}_ih_i,a) - W(t,\boldsymbol{x}+\hat{\mathbf{e}}_jh_j,a) - W(t,\boldsymbol{x}-\hat{\mathbf{e}}_jh_j,a)]$ (used if $i\neq j$ and $S_{ij}(t,\boldsymbol{x},a) < 0$).

Using these approximations in the discrete approximation of (36) we find (after the usual "unity" rescaling [29]) that the transition probabilities are of the form

$$\begin{aligned} p^{h,\delta}(n\delta,\boldsymbol{x};n\delta,\boldsymbol{x}\pm\hat{\mathbf{e}}_ih_i|a) = \frac{1}{Q^{h,\delta}(n\delta,\boldsymbol{x},a)}\Bigg(&\frac{[b_i(n\delta,\boldsymbol{x},a)]^\pm}{h_i} \\ &-\sum_{j\neq i}\frac{|S_{ij}(n\delta,\boldsymbol{x},a)|}{2h_ih_j} + \tfrac{1}{2}\frac{S_{ii}(n\delta,\boldsymbol{x},a)}{h_i^2}\Bigg), \end{aligned} \tag{37a}$$

$$p^{h,\delta}(n\delta,\boldsymbol{x};n\delta,\boldsymbol{x}\pm(\hat{\mathbf{e}}_i h_i+\hat{\mathbf{e}}_j h_j)|a)=\frac{S^{+}_{ij}(n\delta,\boldsymbol{x},a)}{2h_ih_jQ^{h,\delta}(n\delta,\boldsymbol{x},a)} \tag{37b}$$

$$p^{h,\delta}(n\delta,\boldsymbol{x};n\delta,\boldsymbol{x}\pm(\hat{\mathbf{e}}_i h_i-\hat{\mathbf{e}}_j h_j)|a)=\frac{S^{-}_{ij}(n\delta,\boldsymbol{x},a)}{2h_ih_jQ^{h,\delta}(n\delta,\boldsymbol{x},a)} \tag{37c}$$

$$p^{h,\delta}(n\delta,\boldsymbol{x};(n+1)\delta,\boldsymbol{x}|a)=\frac{\delta^{-1}}{Q^{h,\delta}(n\delta,\boldsymbol{x},a)}, \tag{37d}$$

$$\begin{aligned}p^{h,\delta}(n\delta,\boldsymbol{x};n\delta,\boldsymbol{x}|a)=1&-\sum_{y\in\mathscr{R}^h_0(\boldsymbol{x})}p^{h,\delta}(n\delta,\boldsymbol{x};n\delta,y|a)\\&-p^{h,\delta}(n\delta,\boldsymbol{x};(n+1)\delta,\boldsymbol{x}|a),\end{aligned} \tag{37e}$$

where, $p^{h,\delta}(n\delta,\boldsymbol{x};n\delta,\boldsymbol{y}|a)=0\ \forall\boldsymbol{y}\notin\mathscr{R}^h(\boldsymbol{x})$, and we have defined

$$\begin{aligned}Q^{h,\delta}(n\delta,\boldsymbol{x},a)&:=\beta(n\delta,\boldsymbol{x},a)+\delta^{-1}+\sum_{i=1}^{m}h_i^{-1}|b_i(n\delta,\boldsymbol{x},a)|\\&\quad+\sum_{i=1}^{m}h_i^{-2}S_{ii}(n\delta,\boldsymbol{x},a)-\sum_{i=1}^{m}\sum_{j\neq i}(2h_ih_j)^{-1}|S_{ij}(n\delta,\boldsymbol{x},a)|.\end{aligned} \tag{38}$$

Now, $\mathscr{R}^h(\boldsymbol{x}):=\{\forall i\forall j\text{ s.t. }i\neq j|\boldsymbol{x},\boldsymbol{x}\pm\hat{\mathbf{e}}_ih_i,\boldsymbol{x}\pm\hat{\mathbf{e}}_ih_i\pm\hat{\mathbf{e}}_jh_j\}$, while $\mathscr{R}^h_0(x):=\mathscr{R}^h(\boldsymbol{x})\backslash\{\boldsymbol{x}\}$. The interpolation interval remains $\Delta t^{h,\delta}_n=1/Q^{h,\delta}(n\delta,\boldsymbol{x},a)$.

Remark 8.1. Again, it is computationally advantageous to redefine $Q^{h,\delta}(n\delta,\boldsymbol{x},a)$ without control dependence in analogy with (31).

Provided that the formulae in (37) can be interpreted as probabilities, they satisfy the basic convergence requirement of being locally consistent in a sense analogous to (22).[11] Nonetheless, qua the negative coefficient on $|S_{ij}(n\delta,\boldsymbol{x},a)|$ there is potentially an issue with respect to keeping the probabilities uniformly non-negative. In fact, a closer look at (37a) and (38) reveals that a *sufficient condition* for non-negativity is the requirement that $\forall i\in\mathbb{N}$, $\forall\boldsymbol{x}\in\mathbb{R}^m$ and $\forall a\in\mathbb{A}$

$$\frac{1}{h_i}S_{ii}(n\delta,\boldsymbol{x},a)\geq\sum_{j\neq i}\frac{1}{h_j}|S_{ij}(n\delta,\boldsymbol{x},a)|. \tag{39}$$

Alas, this constraint is far from innocuous: indeed it fails to obtain for a rather large class of problems of interest. In some cases there may be easy fixes to this (e.g. if S is independent of $(\boldsymbol{x},a)$, which fails to satisfy (39), then the problem is as simple as rotating the coordinate system until the inequality *is* satisfied), yet, the general picture is less encouraging.

9. Multivariate Merton?

9.1. **The problem posed.** Consider the simple extension of Merton's problem in which we let a stochastic state process y drive variations in the parameters r,μ and σ: specifically, assume we are maximising the expectation

[11] σ^2 should be read as $\sigma\sigma^{\intercal}=S$.

$$W(t,x,y,\theta,c)=\mathbb{E}_{t,x,y}\left[\int_t^T e^{-\beta(s-t)}u(c_s)ds+e^{-\beta(T-t)}u(X_T^{\theta,c})\right], \tag{40}$$

over the space of admissible consumption-investment strategies, given that the wealth process evolves according to the coupled SDEs

$$dX_s^{\theta,c}=[r(y_s)X_s^{\theta,c}+\theta_s(\mu(y_s)-r(y_s))-c_s]ds+\theta_s\sigma(y_s)dW_{1s}, \tag{41a}$$

$$dy_s=m(y_s)ds+v(y_s)\{\rho(y_s)dW_{1s}+\sqrt{1-\rho(y_s)^2}dW_{2s}\}, \tag{41b}$$

where $W_1 \perp W_2$ are independent Brownian motions, while $m:\mathbb{R}\mapsto\mathbb{R}$, $v:\mathbb{R}\mapsto\mathbb{R}$, and $\rho:\mathbb{R}\mapsto[-1,1]$ are continuous functions codifying the drift, diffusion, and correlation respectively. The associated HJB is of the form

$$\begin{aligned}\beta V=\partial_t V+\sup_{(\theta,c)\in\mathbb{R}\times\mathbb{R}_+}\Big\{&[r(y)x+\theta(\mu(y)-r(y))-c]\partial_x V+m(y)\partial_y V\\&+\tfrac{1}{2}\theta^2\sigma(y)^2\partial_{xx}^2V+\tfrac{1}{2}v(y)^2\partial_{yy}^2V+\rho(y)\theta\sigma(y)v(y)\partial_{xy}^2V+\frac{c^{1-\gamma}}{1-\gamma}\Big\},\end{aligned} \tag{42}$$

where $V(T,x,y)=x^{1-\gamma}/(1-\gamma)$. Assuming uniformly equidistant grid spacing, condition (39) reduces to the diagonal dominance conditions

$$\theta_{n\delta}\sigma(y_{n\delta})\geq|\rho(y_{n\delta})v(y_{n\delta})|,\quad\text{and}\quad v(y_{n\delta})\geq|\rho(y_{n\delta})\theta_{n\delta}\sigma(y_{n\delta})|$$

which only are guaranteed to be simultaneously satisfied under the rather restrictive constraint $\rho=0$. To combat this, one could try an adaptive discretisation approach *a la* Wang et al. [34], where a suitable differencing scheme is custom picked for each individual grid node, with the aim of securing non-negative probabilities. However, such a search over uniformly positive coefficients will inevitably constitute a non-trivial exercise in coding for dimensionality $m\geq 1$, thus bringing the *practicality* of the approach into doubt. In [20] Kushner likewise describes a method which through the employ of non-local transitions aims to reduce the relative numerical noise of the off-diagonal elements in S - a method he stoically characterises as requiring some flexibility on the part of the programmer.

9.2. **The Joy of Downsizing.** Superficially the Markov chain approximation method thus appears to generalise rather poorly for Mertonian portfolio problems beyond the trivial case which has been the focus of this paper. Nonetheless, this notion completely ignores that some form of dimensionality reduction argument often can be invoked, leading to a considerable reduction in complexity. Consider e.g. the generic stochastic state extension of the Merton problem codified by equations (40) and (41). In direct analogy with the argument offered in section 3 it follows that the optimal value function is separable in time-state and wealth: in particular, we posit the form $V(t,x,y)=G(t,y)x^{1-\gamma}/(1-\gamma)$, where $G:\mathbb{T}\times\mathbb{R}\mapsto\mathbb{R}$. Substituting this expression into (42) alongside a change of control variables[12] $(\theta,c)=(x\pi,x\zeta)$ we find that the wealth variable, x, drops out, and the HJB equation collapses to

[12]Here, π represents the fraction of total wealth x invested in the stock, while ζ represents the rate of change of the fraction of total wealth being consumed.

$$0 = \partial_t G + \sup_{(\pi,\zeta)\in\mathbb{R}\times\mathbb{R}_+} \Big\{ - \tilde{\beta}(t,y,\pi,\zeta)G + \tilde{b}(t,y,\pi)\partial_y G + \tfrac{1}{2}v(y)^2\partial^2_{yy}G + \zeta^{1-\gamma} \Big\}, \tag{43}$$

s.t. $G(T,y) = 1$, where we have defined $\tilde{\beta} = \beta - (1-\gamma)[r(y) + \pi(\mu(y) - r(y)) - \zeta] + \frac{1}{2}(1-\gamma)\gamma\pi^2\sigma(y)^2$ and $\tilde{b} = m(y) + \rho(y)\pi\sigma(y)v(y)(1-\gamma)$. I.e. we are effectively solving the optimisation problem

$$G(t,y) = \sup_{(\pi_t,\zeta_t,)\in\mathscr{A}(t,y)} \mathbb{E}_{t,y}\left[\int_t^T e^{-\int_t^s \tilde{\beta}(u,y,\pi,\zeta)du}\zeta_s^{1-\gamma}ds + e^{-\int_t^T \tilde{\beta}(u,y,\pi,\zeta)du}\right], \tag{44}$$

where y is driven by the drift-adjusted dynamics $dy_s = \tilde{b}(s,y,\pi)dt + v(y)dW_s$. The point, of course, is that the control problem now falls inside the scope of something we know how to solve given the theory exposed in sections 2, 4, and 6. Whenever dimensional reduction permits itself, it should be enforced: not only does this serve to combat the Bellmanian curse, it can also greatly facilitate keeping the transitions probabilities non-negative and thence the numerical scheme convergent.

9.3. **An example with stochastic volatility:** As a simple extension to the Merton problem, consider the scenario in which the variance, $V := \sigma^2$, is driven by a CIR process: i.e. suppose the stock market is Hestonian, [15]. This problem was first tackled by Liu [21] and formally verified by Kraft [17][13] under the assumption that the investor solely cares about bequest maximisation. Here we shall maintain that the investor also cares about intertemporal consumption, which renders the problem somewhat less tractable; in fact, we are only able to extract closed-form expressions for the optimal controls in the event that the market is complete. To this end we posit that the variance process diffuses in perfect (anti)-correlation with the return on the stock. The maximisation problem we are trying to solve is thus of the form (40) where the wealth process evolves under the dynamical equations:

$$dX_s^{\theta,c} = [rX_s^{\theta,c} + \theta_s\lambda V_s - c_s]ds + \theta_s\sqrt{V_s}dW_s], \tag{45a}$$

$$dV_s = \kappa(\bar{V} - V_s)ds - \eta\sqrt{V_s}dW_s, \tag{45b}$$

where κ is the speed of mean reversion, $\bar{V}$ is the long-run equilibrium value, and η is a vol of vol parameter. Also note that we have made the standard assumption that the excess return is proportional (constant λ) to the instantaneous variance level.

Following general theory for affine models as documented in Munk [26][14] one may readily establish that the optimal controls are of the form

$$\theta^*(t,v,x) = \frac{\lambda x}{\gamma} - \eta x\frac{\partial_v g(t,v)}{g(t,v)}, \quad \text{and} \quad c^*(t,v,x) = \frac{x}{g(t,v)}. \tag{46}$$

where $g : \mathbb{T}\times\mathbb{R}_+ \mapsto \mathbb{R}_+$ is the function

$$g(t,v) = \int_t^T e^{-\frac{\gamma-1}{\gamma}(\mathscr{B}_0(s-t)+v\mathscr{B}_1(s-t))}ds + e^{-\frac{\gamma-1}{\gamma}(\mathscr{B}_0(T-t)+v\mathscr{B}_1(T-t))}, \tag{47}$$

[13]Note that square root processes do not formally satisfy the standard assumptions of the driving SDEs as exposed in section 2, hence the importance of Kraft's work.

[14]See theorem 7.8, chapter 7, "Stochastic investment opportunities: the general case".

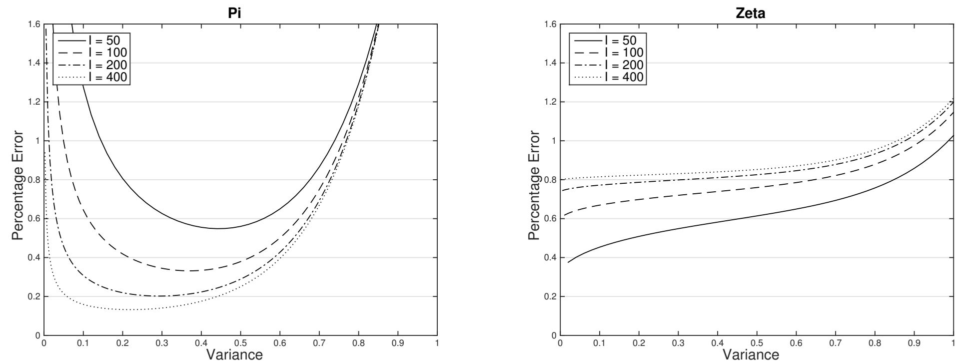

FIGURE 12. The percentage errors of the numerically computed optimal stock investment, π, (**left**) and the optimal consumption, ζ, (**right**) computed for various values of variance discretisation using the implicit method. We have assumed $\beta = 0.02$, $r = 0.05$, $\gamma = 0.5$, $\lambda = 0.3$, $\kappa = 3$, $\bar{V} = 0.3^2$, $\eta = 1$, $T = 1$, and $N = 10$.

and $\mathscr{B}_0, \mathscr{B}_1 : \mathbb{T} \mapsto \mathbb{R}$ are tedious expressions known in closed form.[15] Observe that while the integral in (47) must be solved numerically, this is computationally robust: in other words, we have a reliable way of extracting the optimal policies.

Against this benchmark, we wish to compare the performance of the Markov chain approximation method applied to the dimensionally reduced control problem codified by (43). The relevant parameters are here $\tilde{\beta}(t, v, \pi, \zeta) = \beta - (1-\gamma)(r + \pi\lambda v - \zeta) + \frac{1}{2}(1-\gamma)\gamma\pi^2 v$, $\tilde{b}(t, v, \pi) = \kappa(\bar{V} - v) - \pi\eta v(1-\gamma)$, and $v(v)^2 = \eta^2 v$. Our approach is that of the implicit method, so we can straightforwardly recycle the algorithm exposed in section 7 with minimal adjustments. With regards to the boundary conditions we assume that there is a vanishing probability of leaving the grid at either end of the variance grid. Figure 12 showcases the percentage error for the spatially reduced optimal policies (π^*, ζ^*) obtained this way. As it can be seen, they are in reasonable congruence.

10. CONCLUSION

It is well worth summarising some of our key findings: the main advantage of the *explicit* Markov chain approximation is the fact that it is straight-forward to implement. Alas, simplicity comes at the cost of satisfying the probabilistic positivity requirement (18), which for Merton type problem entails exceedingly small time steps. For a grid interpretation of the explicit method this is a nuisance, but does not cause difficulties in principle. As for the trinomial interpretation, a "worst case scenario" take on the inequality may altogether rule out the existence of a converging solution. To sidestep this issue, one may opt for the

[15] For completeness:

$$\mathscr{B}_0(\tau) = \left(\frac{\beta}{\gamma - 1} + r\right)\tau - \frac{\kappa\bar{V}}{\hat{\gamma}\eta^2}\left\{(\hat{\nu} + \hat{\kappa})\tau + 2\ln\left|\frac{2\hat{\nu}}{(\hat{\nu}+\hat{\kappa})(e^{\hat{\nu}\tau} - 1) + 2\hat{\nu}}\right|\right\}$$
$$\mathscr{B}_1(\tau) = \frac{\lambda^2}{\gamma}\frac{(e^{\hat{\nu}\tau} - 1)}{(\hat{\nu}+\hat{\kappa})(e^{\hat{\nu}\tau} - 1) + 2\hat{\nu}},$$

where we have defined the derived parameters $\hat{\gamma} = (\gamma - 1)/\gamma$, $\hat{\kappa} = \kappa - \hat{\gamma}\eta\lambda$, and $\hat{\nu} = \sqrt{\hat{\kappa}^2 + \hat{\gamma}\gamma^{-1}\lambda^2\eta^2}$.

implicit Markov chain approximation instead. Whilst this prima facie calls for a solution to a highly non-linear system of equations, one may invoke the *iterations in policy space algorithm* (25) in order to render the system (iteratively) linear and thence susceptible to a tridiagonal matrix algorithm. Irrespective of the procedure being implemented a further key insight pertains to the immense computational benefit of using discretised FOCs upon updating the controls. Searches over discrete control spaces are better avoided (or should at the very least be restricted to locally plausible regions based upon subsequent values of optimality).

In a Mertonian context, both procedures were found to be wildly inaccurate near the upper and lower boundaries. Considering the general opaqueness of what constitutes adequate boundaries this is hardly surprising: however, it is at least somewhat unsettling that the numerical accuracy near the upper boundary seemingly benefits so very little from increasing the grid refinement (and invariably, the computational run-time). Further numerical studies indicated that one may take some measures against this by choosing appropriate parameters (essentially, high values for β and γ), although this admittedly may fly in the face of empirical data. A better (but admittedly also bolder) move is that of introducing a relational upper boundary (35) based on the correct ansatz of time-space separability of the value function. This solved the problem satisfactorily even for relatively coarse grained grid specifications.

Finally, while a multi-dimensional extensions to the Markov chain approximation are theoretically possible, there are deep-seated issues pertaining to both computational speed (Bellman's curse of dimensionality) and convergence (keeping the transition probabilities non-negative for Merton type problems proves to be considerably more complex). Whilst one may be able to mitigate these problems using tricks such as a generalised Thomas algorithm and adaptive grids, it is clear that a fair share of ingenuity is called for. Fortunately, multi-variable HJB problems often have intuitive ways of being dimensionally reduced, which greatly facilitates their evaluation.

Acknowledgement. The author gratefully acknowledges support from the Danish Strategic Research Council, Program Committee for Strategic Growth Technologies, via the research center HIPERFIT: Functional High Performance Computing for Financial Information Technology (hiperfit.dk) under contract number 10-092299.

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