THE PDE METHOD DEMYSTIFIED

SIMON ELLERSGAARD NIELSEN

1. PDEs for Stochastic Processes

1.1. **The Feynman-Kac Formula.** Rather than solving the expected value problems inherent to much of mathematical finance, we may choose to reformulate them in deterministic terms qua partial differential equations. What bridges the gap between these fields is a fundamental result developed by Richard Feynman and Mark Kac:

Theorem 1. ***The Feynman-Kac Formula*** *Suppose $f : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ satisfies the PDE*

$$\frac{\partial f}{\partial t}(x,t) + \mu^{\mathbb{Q}}(x,t)\frac{\partial f}{\partial x}(x,t) + \tfrac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}(x,t) - r(x,t)f(x,t) + h(x,t) = 0,$$

$$\text{s.t.: } f(x,T) = \psi(x).$$

where $\mu^{\mathbb{Q}}, \sigma, r$ and h are known functions. Then the solution may be written as

$$f(x,t) = \mathbb{E}^{\mathbb{Q}}_{X_t=x}\left[\int_t^T D_t^u h(X_u,u)du + D_t^T\psi(x)\right], \tag{1}$$

where we have defined the discounting factor $D_t^{t'} := e^{-\int_t^{t'} r(X_s,s)ds}$ and X_t is a stochastic process defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{Q}, \mathbb{F})$ which follows the SDE

$$dX_t = \mu^{\mathbb{Q}}(X,t)dt + \sigma(X_t,t)dW_t^{\mathbb{Q}}, \quad X_0 = x. \tag{2}$$

Proof. We imagine that f solves the PDE listed above. Furthermore, define the process

$$Y_v := \int_t^v D_t^u h(X_u,u)du + D_t^v f(X_v,v).$$

From Itō's lemma

$$dY_v = d\int_t^v D_t^u h(X_u,u)du + dD_t^v f(X_v,v) + D_t^v df(X_v,v) + dD_t^v df(X_v,v).$$

Now we make the following observations:

- $dD_t^v = -r(X_v,v)D_t^v dv$. In particular, the fourth term is zero.
- The first term can be written as $d\int_t^v D_t^u h(X_u,u)du = D_t^v h(X_v,v)dv$.
- From Itō's lemma $df = f_v dv + \mu^{\mathbb{Q}} f_x dX + \frac{1}{2}\sigma^2 f_{xx} dX^2$.

Bringing these insights together we obtain

$$\begin{aligned} dY_v =& D_t^v\Big[h(X_v,v) - r(X_v,v)f(X_v,v) + \frac{\partial f}{\partial v}(X_v,v) + \mu^{\mathbb{Q}}(X_v,v)\frac{\partial f}{\partial x}(X_v,v) \\ &+ \tfrac{1}{2}\sigma^2(X_v,v)\frac{\partial^2 f}{\partial x^2}(X_v,v)\Big]dv + D_t^v\sigma(X_v,v)\frac{\partial f}{\partial x}(X_v,v)dW_v^{\mathbb{Q}} \end{aligned}$$

By definition of f, the square bracket is zero. Hence, after integrating we are left with

$$Y_T = Y_t + \int_t^T D_t^v\sigma(X_v,v)\frac{\partial f}{\partial x}(X_v,v)dW_v^{\mathbb{Q}}.$$

Taking $\mathbb{Q}$ expectations conditional on $X_t = x$ and using the fact that the Itō integral is a martingale we get

$$\mathbb{E}^{\mathbb{Q}}_{X_t=x}[Y_T] = Y_t = f(x,t),$$

which is exactly what needed to be shown. □

1.2. **The Kolmogorov Backward Equation.** In a certain sense, the PDE in Theorem B.1 *is* the backward equation: i.e. given the dynamics (2) we may infer that the quantity (1) satisfies the listed PDE. Nonetheless, when discussing the Kolmogorov backward equation, one usually thinks of a governing equation of *transition probabilities*.

Theorem 2. ***The Kolmogorov Backward Equation*** *Consider the filtered probability space $(\Omega, \mathscr{F}, \mathbb{Q}, \mathbb{F})$ and let X be a stochastic process on this space, which solves the SDE*

$$dX_t = \mu^{\mathbb{Q}}(X_t,t)dt + \sigma(X_t,t)dW_t^{\mathbb{Q}}.$$

Now consider the probability that X_s is within some set Y given that at time $t < s$ $X_t = x$, i.e.

$$\mathbb{Q}(X_s \in Y | X_t = x) = \int_{y\in Y} \rho(y,s|x,t)dy,$$

where ρ is the density function. Then ρ satisfies the PDE

$$\left\{\frac{\partial}{\partial t} + \mu^{\mathbb{Q}}(x,t)\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}\right\}\rho(y,s|x,t) = 0,$$

for $(x,t) \in \mathbb{R}\times(0,s)$ and where $\rho(y,s|x,t) \to \delta(y)$ as $t \nearrow s$.

Proof. We could consider the boundary value problem

$$\begin{aligned} &\frac{\partial f}{\partial t}(x,t) + \mu^{\mathbb{Q}}(x,t)\frac{\partial f}{\partial x}(x,t) + \tfrac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}(x,t) = 0, \qquad && (x,t)\in\mathbb{R}\times(0,s) \\ &f(x,s) = \mathbf{1}\{x \in Y\}, && x \in \mathbb{R} \end{aligned}$$

where $\mathbf{1}$ is the indicator function. From Feynman-Kac we deduce the solution

$$f(x,t) = \mathbb{E}^{\mathbb{Q}}_{X_t=x}[\mathbf{1}\{X_s \in Y\}].$$

But this is, of course, just equal to

$$f(x,t) = \mathbb{Q}(X_s \in Y|X_t = x) = \int_{y\in Y}\rho(y,s|x,t)dy$$

Since this argument works in both directions, we have effectively already verified the Backward theorem. In particular, substituting the transition density form of the solution into the PDE we get

$$\int_{y\in Y} \left\{ \frac{\partial}{\partial t} + \mu^{\mathbb{Q}}(x,t)\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2} \right\} p(y,s|x,t)dy = 0, \qquad (x,t)\in\mathbb{R}\times(0,s)$$

$$\int_{y\in Y} p(y,s|x,s)dy = \mathbf{1}\{x\in Y\}, \quad x\in\mathbb{R}.$$

But this holds for any set Y so we can drop the integrals (whence the indicator function becomes a Dirac delta function). □

1.3. **The Kolmogorov Forward Equation.** In the backward equation we observe that the differential operator operates on the variables upon which we condition (the "backward variables" if you will), i.e. (x,t). A corresponding result exists for the "forward variables" viz. (y,s):

Theorem 3. ***The Kolmogorov Forward Equation (a.k.a.The Fokker-Planck Equation)*** *Consider the filtered probability space $(\Omega,\mathscr{F},\mathbb{Q},\mathbb{F})$ and let X be a stochastic process on this space, which solves the SDE*

$$dX_t = \mu^{\mathbb{Q}}(X_t,t) + \sigma(X_t,t)dW_t^{\mathbb{Q}}.$$

Now consider the probability that X_s is within some set Y given that at time $t<s$ $X_t = x$, i.e.

$$\mathbb{Q}(X_s\in Y|X_t = x) = \int_{y\in Y} \rho(y,s|x,t)dy$$

where ρ is the density function. Then ρ satisfies the PDE

$$\frac{\partial \rho}{\partial s}(y,s|x,t) + \frac{\partial}{\partial y}(\mu^{\mathbb{Q}}(y,s)\rho(y,s|x,t)) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y,s)\rho(y,s|x,t)) = 0$$

for $(y,s)\in\mathbb{R}\times(0,T)$ and where $\rho(y,s|x,t)\to\delta(x)$ as $s\searrow t$.

Proof. Consider a test function $h(y,s)\in\mathscr{C}^{2,1}$ with compact support[1] in the set $\mathbb{R}\times(t,T)$ where $t<T$ are two fixed coordinates in time. From Itō's lemma,

$$\begin{aligned} h(X_T,T) =& h(X_t,t) + \int_t^T \left\{ \frac{\partial}{\partial s} + \mu^{\mathbb{Q}}(y,s)\frac{\partial}{\partial y} + \tfrac{1}{2}\sigma^2(y,s)\frac{\partial^2}{\partial y^2} \right\} h(X_s,s)ds \\ &+ \int_t^T \frac{\partial h}{\partial y}(X_s,s)dW_s^{\mathbb{Q}} \end{aligned}$$

Applying the expectation operator $\mathbb{E}^{\mathbb{Q}}_{X_t=x}[\cdots]$ and using the compact support condition ($h(x,T) = h(x,t) = 0$) we obtain

$$\int_{\mathbb{R}}\int_t^T \rho(y,s|x,t)\left\{ \frac{\partial}{\partial s} + \mu^{\mathbb{Q}}(y,s)\frac{\partial}{\partial y} + \tfrac{1}{2}\sigma^2(y,s)\frac{\partial^2}{\partial y^2} \right\} h(y,s)dsdy = 0$$

Integrating by parts "in time" we find that

[1] I.e. the function is non-zero only in a closed, bounded region.

$$\begin{aligned}\int_{\mathbb{R}}\int_t^T \rho(y,s|x,t)\frac{\partial h}{\partial s}(y,s)dsdy &= \int_{\mathbb{R}}\left\{[\rho h]_t^T - \int_t^T \frac{\partial \rho}{\partial s}hds\right\}dy \\ &= -\int_{\mathbb{R}}\int_t^T h(y,s)\frac{\partial \rho}{\partial s}(y,s|x,t)dsdy\end{aligned}$$

Likewise, using Fubini's rule and integrating by parts in "state space"

$$\begin{aligned}&\int_{\mathbb{R}}\int_t^T \rho(y,s|x,t)\left\{\mu^{\mathbb{Q}}(y,s)\frac{\partial}{\partial y} + \tfrac{1}{2}\sigma^2(y,s)\frac{\partial^2}{\partial y^2}\right\}h(y,s)dsdy \\ &= \int_t^T\int_{\mathbb{R}}\rho(y,s|x,t)\mu^{\mathbb{Q}}(y,s)\frac{\partial h}{\partial y}(y,s)dyds + \tfrac{1}{2}\int_t^T\int_{\mathbb{R}}\rho(y,s|x,t)\sigma^2(y,s)\frac{\partial^2 h}{\partial y^2}(y,s)dyds \\ &= \int_t^T\left\{[\rho\mu^{\mathbb{Q}}h]_{-\infty}^{\infty} - \int_{\mathbb{R}}\frac{\partial(\rho\mu^{\mathbb{Q}})}{\partial y}hdy\right\}ds + \tfrac{1}{2}\int_t^T\left\{\left[\rho\sigma^2\frac{\partial h}{\partial y}\right]_{-\infty}^{\infty} - \int_{\mathbb{R}}\frac{\partial\rho\sigma^2}{\partial y}\frac{\partial h}{\partial y}dy\right\}ds \\ &= -\int_t^T\int_{\mathbb{R}}\frac{\partial(\rho\mu^{\mathbb{Q}})}{\partial y}hdyds - \tfrac{1}{2}\int_t^T\left\{\left[\frac{\partial\rho\sigma^2}{\partial y}h\right]_{-\infty}^{\infty} - \int_{\mathbb{R}}\frac{\partial^2\rho\sigma^2}{\partial y^2}hdy\right\}ds \\ &= -\int_{\mathbb{R}}\int_t^T h(y,s)\left\{\frac{\partial(\rho(y,s|x,t)\mu^{\mathbb{Q}}(y,s))}{\partial y}h - \tfrac{1}{2}\frac{\partial^2(\rho(y,s|x,t)\sigma^2(y,s))}{\partial y^2}\right\}dyds\end{aligned}$$

Combining the last three results, we have

$$\int_{\mathbb{R}}\int_t^T h(y,s)\left\{\frac{\partial\rho}{\partial s}(y,s|x,t) + \frac{\partial(\rho(y,s|x,t)\mu^{\mathbb{Q}}(y,s))}{\partial y}h - \tfrac{1}{2}\frac{\partial^2(\rho(y,s|x,t)\sigma^2(y,s))}{\partial y^2}\right\}dyds = 0.$$

Indeed, since h was chosen as an arbitrary test function, the result now follows. $\square$