THE MARTINGALE METHOD DEMYSTIFIED

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Abstract. We consider the nitty gritty of the martingale approach to option pricing. These notes are largely based upon Björk's *Arbitrage Theory in Continuous Time* and Munk's *Fixed Income Securities*.

1. Changing the Measure

Consider the probability space $(\Omega, \mathscr{F})$ then we may think of how different allocations of probabilities to events in this space are interconnected. We say that two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent** (labelled $\mathbb{P} \sim \mathbb{Q}$) on $\mathscr{F}$ just in case

$$\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0, \ \ \forall A \in \mathscr{F}.$$

In particular, the **Radon-Nikodym theorem** instructs us that $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0 \, \forall A \in \mathscr{F}$ (i.e. $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$ on $\mathscr{F}$: $\mathbb{Q} \ll \mathbb{P}$) if and only if there exists an $\mathscr{F}$-measurable mapping $\xi : \Omega \mapsto \mathbb{R}_+$ such that

$$\int_A d\mathbb{Q}(\omega) = \int_A \xi(\omega) d\mathbb{P}(\omega), \ \ \forall A \in \mathscr{F}. \tag{1}$$

In the event that $A = \Omega$ the left-hand-side in this expression is unity (per definition of a probability measure). Likewise, the right-hand-side is defined as $\int_\Omega \xi d\mathbb{P} \equiv \mathbb{E}^{\mathbb{P}}[\xi]$. All in all, the quantity ξ is therefore a non-negative random variable with $\mathbb{E}^{\mathbb{P}}[\xi] = 1$. Since (1) infinitesimally can be written $\xi = d\mathbb{Q}/d\mathbb{P}$, ξ is commonly referred to as the **likelihood ratio** between $\mathbb{Q}$ and $\mathbb{P}$ or the **Radon-Nikodym derivative**. Three standard results surrounding ξ deserve mentioning:

(1) For any random variable X on $L^1(\Omega, \mathscr{F}, \mathbb{Q})$: $\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[\xi X]$ and $\mathbb{E}^{\mathbb{Q}}[\xi^{-1} X] = \mathbb{E}^{\mathbb{P}}[X]$. Proof: obvious using definitions.
(2) Assume $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$ on $\mathscr{F}$ and that $\mathscr{G} \subseteq \mathscr{F}$, then the likelihood ratios $\xi^{\mathscr{F}}$ and $\xi^{\mathscr{G}}$ are related by $\xi^{\mathscr{G}} = \mathbb{E}^{\mathbb{P}}[\xi^{\mathscr{F}} | \mathscr{G}]$.
(3) Finally, assume X is a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathbb{Q}$ be another measure on $(\Omega, \mathscr{F})$ with Radon-Nikodym derivative $\xi = d\mathbb{Q}/d\mathbb{P}$ on $\mathscr{F}$. Assume $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G} \subseteq \mathscr{F}$ then

$$\mathbb{E}^{\mathbb{Q}}[X|\mathscr{G}] = \frac{\mathbb{E}^{\mathbb{P}}[\xi X|\mathscr{G}]}{\mathbb{E}^{\mathbb{P}}[\xi|\mathscr{G}]}, \ \ \mathbb{Q} - \text{a.s.}$$

This result is sometimes referred to as the **Abstract Bayes' Theorem**.

Example: To get a fell for how these results are used in mathematical finance we consider the classical set-up: a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}\}_{t \in [0,T]})$ on a compact interval $[0, T]$. Typically, we are interested in some stochastic process $\{X_t\}_{t \in [0,T]}$ (e.g. a stock price) such that

Ω is the set of all possible paths of the process over $[0, T]$. Since all relevant uncertainty has been resolved at time T all (relevant) random variables will be known at time T. If we now consider the non-negative random variable ξ_T in $\mathscr{F}_T$, then provided $\mathbb{E}^{\mathbb{P}}[\xi_T] = 1$ we may define a new probability measure $\mathbb{Q}$ on $\mathscr{F}_T$ by setting $d\mathbb{Q} = \xi_T d\mathbb{P}$. Per definition, ξ_T is a Radon-Nikodym derivative of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on $\mathscr{F}_T$ so $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_T$. Thus, we will also have $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_t$ $\forall t \leq T$ so by the Radon-Nikodym Theorem there exists a random process $\{\xi_t\}_{t\in[0,T]}$ defined by $\xi_t = d\mathbb{Q}/d\mathbb{P}$ on $\mathscr{F}_t$, which we call the **likelihood process**. Item (2) above now immediately implies that the ξ-process is a $\mathbb{P}$-**martingale**:

$$\mathbb{E}^{\mathbb{P}}[\xi_{t'}|\mathscr{F}_t] = \xi_t, \quad t' > t.$$

Using this fact alongside item (3) also gives us the result that:

$$\mathbb{E}^{\mathbb{Q}}[X_{t'}|\mathscr{F}_t] = \mathbb{E}^{\mathbb{P}}\left[\frac{\xi_{t'}}{\xi_t}X_{t'}\middle|\mathscr{F}_t\right] \tag{2}$$

which turns out to be extremely useful in option pricing upon jumping between different numeraires.

2. The First and Second Fundamental Theorems

We consider a market model consisting of the non-dividend paying asset price processes $S_0, S_1, ...,$ S_N on the time interval $[0, T]$.

Theorem 1. ***The First Fundamental Theorem*** *The market model is free of arbitrage if and only there exists a martingale measure, i.e. a measure $\mathbb{Q} \sim \mathbb{P}$ such that the processes*

$$\frac{S_{0t}}{S_{0t}}, \frac{S_{1t}}{S_{0t}}, \dots, \frac{S_{Nt}}{S_{0t}},$$

are (local) martingales under $\mathbb{Q}$.

Notice that we don't commit ourselves to the interpretation that the numeraire, S_0, is the risk free asset. However, if indeed $S_{0t} = B_t \equiv \exp(\int_0^t r_s ds)$ where r is a possibly stochastic short rate, and we assume all processes are Wiener driven, meaning that $dS_{it} = S_{it}\mu_{it}dt + S_{it}\boldsymbol{\sigma}_{it}^{\intercal}d\boldsymbol{W}_t^{\mathbb{P}}$, then a measure $\mathbb{Q} \sim \mathbb{P}$ (the so **risk-neutral measure** associated with the risk free numeraire) is a martingale measure if and only if

$$dS_{it} = S_{it}r_t dt + S_{it}\boldsymbol{\sigma}_{it}^{\intercal}d\boldsymbol{W}_t^{\mathbb{Q}} \tag{3}$$

$\forall i \in \{0, 1, ..., N\}$, where $\boldsymbol{W}^{\mathbb{Q}}$ is a d-dimensional $\mathbb{Q}$-Wiener process. I.e. all assets have r as the short rate as their local rates of return. Proof: apply Itō's lemma to S_{it}/S_{0t}. Just in case $\mu_{it} = r_t$ do we obtain a local martingale (i.e. vanishing drift).

Next, we consider what it takes for us to be able to replicate (synthesise) assets on the market using existing products:

Theorem 2. ***The Second Fundamental Theorem*** *Assuming absence of arbitrage, the market model is complete if and only if the martingale measure $\mathbb{Q}$ is unique.*

Remark 1. This does clearly not say that there is only one martingale measure in existence. It only says that for this particular choice of numeraire (S_0) the measure is uniquely determined.

Theorem 3. ***Pricing Contingent Claims*** *Consider a contingent claim, X, that expires at time T. In order to avoid arbitrage we must price the claim according to*

$$X_t = S_{0t}\mathbb{E}^{\mathbb{Q}}\left[\frac{X_T}{S_{0T}}\middle|\mathscr{F}_t\right] \tag{4}$$

where $\mathbb{Q}$ is a martingale measure for $\{S_0, S_1, ..., S_N\}$ with S_0 as the numeraire. In particular, insofar as S_{0t} is the risk free asset $S_{0t} = \exp(\int_0^t r_s ds)$, then we obtain the classical pricing formula

$$X_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}X_T\middle|\mathscr{F}_t\right] \tag{5}$$

3. Lévy's Characterisation of Wiener Processes

A Wiener process (Brownian motion) W_t is a martingale with continuous paths and quadratic variation $[W,W](t) = t$. The properties actually suffice to characterise a Wiener process as demonstrated by Lévy:

Theorem 4. ***Lévy's Theorem*** *Let $\boldsymbol{M}_t = (M_{1t}, M_{2t}, ..., M_{dt})$ be a martingale with respect to the filtration $\{\mathscr{F}_t\}$, $t \geq 0$. Assume that (i) $\forall i : M_{i0} = 0$, (ii) $\forall i : M_{it}$ has continuous paths, (iii) $[M_i, M_j](t) = \delta_{ij}t$ $\forall t \geq 0$, then M_{1t}, M_{2t}, ..., M_{dt} are independent Wiener processes.*

Proof. We will prove this using characteristic functions. Consider the function $f(t, \boldsymbol{M}_t) = \exp\{\boldsymbol{h}^\intercal\boldsymbol{M} - \frac{1}{2}\boldsymbol{h}^\intercal\boldsymbol{h}t\}$. From Itō's lemma

$$\begin{aligned} df(t,\boldsymbol{M}_t) &= \left(\partial_t f(t,\boldsymbol{M}_t) + \tfrac{1}{2}\nabla_x^\intercal\nabla_x f(t,\boldsymbol{M}_t)\right)dt + \nabla_x f(t,\boldsymbol{M}_t)^\intercal d\boldsymbol{M}_t \\ &= \left(-\tfrac{1}{2}\boldsymbol{h}^\intercal\boldsymbol{h}f(t,\boldsymbol{M}_t) + \tfrac{1}{2}\boldsymbol{h}^\intercal\boldsymbol{h}f(t,\boldsymbol{M}_t)\right)dt + \nabla_x f(t,\boldsymbol{M}_t)^\intercal d\boldsymbol{M}_t \\ &= \nabla_x f(t,\boldsymbol{M}_t)^\intercal d\boldsymbol{M}_t. \end{aligned}$$

So f is clearly a martingale. It follows that

$$\mathbb{E}[f(t,\boldsymbol{M}_t)] = 1 \Leftrightarrow \mathbb{E}[e^{\boldsymbol{h}^\intercal\boldsymbol{M}_t}] = e^{\frac{1}{2}\boldsymbol{h}^\intercal\boldsymbol{h}t}.$$

The right-hand side is the moment generating function for independent normal random variables (mean 0 and variance t). The result follows. □

4. The Martingale Theorem and Girsanov's Theorem

Let $\boldsymbol{W}$ be a d-dimensional Wiener process and let X be a stochastic variable which is both $\mathscr{F}_T^{\boldsymbol{W}}$ measurable and L^1. Then there exists a uniquely determined $\mathscr{F}_T^{\boldsymbol{W}}$-adapted process $\boldsymbol{h} = (h_1, h_2, ..., h_d)$ such that X has the representation

$$X = \mathbb{E}[X] + \int_0^t \boldsymbol{h}_s^\intercal d\boldsymbol{W}_s. \tag{6}$$

Under the additional assumption that $\mathbb{E}[X^2] < \infty$ then $h_1, h_2, ..., h_d$ are in $\mathcal{L}^2$.

We can use this lemma to prove the following

Theorem 5. ***The Martingale Representation Theorem*** *Let $\boldsymbol{W}$ be a d-dimensional Wiener process, and assume that the filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ is defined as $\mathscr{F}_t = \mathscr{F}_t^{\boldsymbol{W}}$ for $t \in [0,T]$. Now let M be any $\mathscr{F}_t$ martingale. Then there exists a uniquely determined $\mathscr{F}_t$ adapted process $\boldsymbol{h} = (h_1, h_2, ..., h_d)$ such that M has the representation*

$$M_t = M_0 + \int_0^T \boldsymbol{h}_s^\intercal d\boldsymbol{W}_s, \quad t \in [0,T].$$

If the martingale M is square integrable, then $h_1, h_2, ..., h_d$ are in $\mathcal{L}^2$.

Recall from section 1 that the measure transformation $d\mathbb{Q} = \xi_T d\mathbb{P}$ on $\mathscr{F}_T$ (where ξ_T is a nonnegative random variable with $\mathbb{E}^{\mathbb{P}}[\xi_T] = 1$) generates a likelihood process $\{\xi_t\}_{t\in[0,T]}$ defined by $\xi_t \equiv d\mathbb{Q}/d\mathbb{P}$ on $\mathscr{F}_t$ which is a $\mathbb{P}$-martingale. It thus seems natural to define ξ_t as the solution to the SDE $d\xi_t = \phi_t \xi_t dW_t^{\mathbb{P}}$ with initial condition $\xi_0 = 1$ for some choice of the process ϕ (the initial condition guarantees unitary expectation under $\mathbb{P}$). In fact, using this SDE we should be able to generate a host of natural measure transformations from $\mathbb{P}$ to the new measure $\mathbb{Q}$, which indeed also is the upshot of Girsanov's theorem:

Theorem 6. ***Girsanov's Theorem*** *Let $\boldsymbol{W}^{\mathbb{P}}$ be a d-dimensional standard $\mathbb{P}$-Wiener process on $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\in[0,T]})$, and let $\boldsymbol{\phi}$ be any d-dimensional adapted column vector process (referred to as the Girsanov kernel). Now define the process ξ on $[0,T]$ by*

$$d\xi_t = \xi_t \boldsymbol{\phi}_t^\intercal d\boldsymbol{W}_t^{\mathbb{P}}, \quad \xi_0 = 1$$

or identically

$$\xi_t = \exp\left\{ \int_0^t \boldsymbol{\phi}_s^\intercal d\boldsymbol{W}_s^{\mathbb{P}} - \tfrac{1}{2} \int_0^t ||\boldsymbol{\phi}_s||^2 ds \right\}.$$

*Now **assume** that $\mathbb{E}^{\mathbb{P}}[\xi_T] = 1$ (see the Novikov condition) and **define** the new probability measure $\mathbb{Q}$ on $\mathscr{F}_T$ by $d\mathbb{Q} = \xi_T d\mathbb{P}$ on $\mathscr{F}_T$ then*

$$d\boldsymbol{W}_t^{\mathbb{P}} = \boldsymbol{\phi}_t dt + d\boldsymbol{W}_t^{\mathbb{Q}} \tag{7}$$

where $\boldsymbol{W}_t^{\mathbb{Q}}$ is a $\mathbb{Q}$ Wiener process.

Proof. Consider the uncorrelated case, where $\boldsymbol{\Sigma}$ is the identity matrix. We will use Lévy's theorem to verify that $\boldsymbol{W}_t^{\mathbb{Q}}$ is indeed a $\mathbb{Q}$ Wiener process. (i) $\boldsymbol{W}_0^{\mathbb{Q}} = 0$, (ii) $\boldsymbol{W}_t^{\mathbb{Q}}$ is continuous. (iii) Furthermore, it is cleat that $[W_i^{\mathbb{Q}}, W_j^{\mathbb{Q}}](t) = \delta_{ij} t$. The only thing left to show is that $\boldsymbol{W}_t^{\mathbb{Q}}$ is a $\mathbb{Q}$ martingale. To this end, it is already given that ξ_t is a $\mathbb{P}$ martingale with $\mathbb{E}^{\mathbb{P}}[\xi_T] = 1$. Applying Itō to the process $\xi_t \boldsymbol{W}_t^{\mathbb{Q}}$ we find that

$$\begin{aligned} d(\xi_t \boldsymbol{W}_t^{\mathbb{Q}}) &= \boldsymbol{W}_t^{\mathbb{Q}} d\xi_t + \xi_t d\boldsymbol{W}_t^{\mathbb{Q}} + d\xi_t d\boldsymbol{W}_t^{\mathbb{Q}} \\ &= \boldsymbol{W}_t^{\mathbb{Q}} \xi_t \boldsymbol{\phi}_t^\intercal d\boldsymbol{W}_t^{\mathbb{P}} + \xi_t (d\boldsymbol{W}_t^{\mathbb{P}} - \boldsymbol{\phi}_t dt) + (d\boldsymbol{W}_t^{\mathbb{P}} - \boldsymbol{\phi}_t dt) \xi_t \boldsymbol{\phi}_t^\intercal d\boldsymbol{W}_t^{\mathbb{P}} \\ &= \xi_t (\boldsymbol{W}_t^{\mathbb{Q}} \boldsymbol{\phi}_t^\intercal + 1) d\boldsymbol{W}_t^{\mathbb{P}} \end{aligned}$$

which proves that $\xi_t \boldsymbol{W}_t^{\mathbb{Q}}$ is a $\mathbb{P}$ martingale. Now, applying the abstract Bayes' theorem we quickly deduce that

$$\mathbb{E}^{\mathbb{Q}}[\boldsymbol{W}_{t'}^{\mathbb{Q}} | \mathscr{F}_t] = \xi_t^{-1} \mathbb{E}^{\mathbb{P}}[\xi_{t'} \boldsymbol{W}_{t'}^{\mathbb{Q}} | \mathscr{F}_t] = \xi_t^{-1} \xi_t \boldsymbol{W}_t^{\mathbb{Q}} = \boldsymbol{W}_t^{\mathbb{Q}},$$

which was to be proven. $\square$

- Assume that the Girsanov kernel $\boldsymbol{\phi}$ is such that
$$\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T ||\boldsymbol{\phi}_t||^2 dt}\right] < \infty$$
then ξ is a martingale and in particular $\mathbb{E}^{\mathbb{P}}[\xi_T] = 1$. This useful result is known as the **Novikov condition**.
- Girsonov's theorem holds in reverse. In particular, assume $\boldsymbol{W}^{\mathbb{P}}$ is a d-dimensional standard $\mathbb{P}$-Wiener process on $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\in[0,T]})$ and assume that $\mathscr{F}_t = \mathscr{F}_t^{\boldsymbol{W}}$ $\forall t$. Furthermore, assume there exists a measure $\mathbb{Q}$ such that $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_T$ then there exists an adapted process $\boldsymbol{\phi}$ such that the likelihood process ξ has the dynamics
$$d\xi_t = \xi_t \boldsymbol{\phi}_t^{\intercal} d\boldsymbol{W}_t^{\mathbb{P}}, \quad \xi_0 = 1.$$
- SDEs of the form $dX_t = \mu_t dt + \boldsymbol{\sigma}_t d\boldsymbol{W}_t^{\mathbb{P}}$ transform as $dX_t = (\mu_t + \boldsymbol{\sigma}_t\boldsymbol{\phi})dt + \boldsymbol{\sigma}_t d\boldsymbol{W}_t^{\mathbb{Q}}$ under $\mathbb{Q}$, which means that the drift changes $\mu_t \mapsto \mu_t + \boldsymbol{\sigma}_t\boldsymbol{\phi}$, but the diffusion remains unchanged.

Corollary 1. The Correlated Girsanov Theorem Let $\boldsymbol{W}^{\mathbb{P}}$ be a d-dimensional $\mathbb{P}$-Wiener process on $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\in[0,T]})$ with correlation matrix $\boldsymbol{\Sigma}$ (i.e. $d\boldsymbol{W}^{\mathbb{P}} \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}dt)$), and let $\boldsymbol{\phi}$ be any d-dimensional adapted column vector process. Now define the process ξ on $[0,T]$ by $d\xi_t = \xi_t(\boldsymbol{\Sigma}^{-1}\boldsymbol{\phi}_t)^{\intercal} d\boldsymbol{W}_t^{\mathbb{P}}$, where $\xi_0 = 1$, or identically
$$\xi_t = \exp\left\{\int_0^t (\boldsymbol{\Sigma}^{-1}\boldsymbol{\phi}_s)^{\intercal} d\boldsymbol{W}_s^{\mathbb{P}} - \tfrac{1}{2}\int_0^t \boldsymbol{\phi}_s^{\intercal}\boldsymbol{\Sigma}^{-1}\boldsymbol{\phi}_s ds\right\}.$$
Again, assume that $\mathbb{E}^{\mathbb{P}}[\xi_T] = 1$ and **define** the new probability measure $\mathbb{Q}$ on $\mathscr{F}_T$ by $d\mathbb{Q} = \xi_T d\mathbb{P}$ on $\mathscr{F}_T$ then
$$d\boldsymbol{W}_t^{\mathbb{P}} = \boldsymbol{\phi}_t dt + d\boldsymbol{W}_t^{\mathbb{Q}} \tag{8}$$
where $\boldsymbol{W}_t^{\mathbb{Q}}$ is a $\mathbb{Q}$ Wiener process with correlation matrix $\boldsymbol{\Sigma}$.

Proof. The proof follows immediately by Cholesky decomposing the correlated Wiener vector $\boldsymbol{W}_t^{\mathbb{P}} = \boldsymbol{L}\bar{\boldsymbol{W}}_{t,}^{\mathbb{P}}$ where $\bar{\boldsymbol{W}}_{t,}^{\mathbb{P}}$ is a standard Wiener vector, and $\boldsymbol{L}\boldsymbol{L}^{\intercal} = \boldsymbol{\Sigma}$. $\square$

5. The Market Price of Risk

Consider the case where we have N risky assets governed by the vector SDE system
$$d\boldsymbol{S}_t = \mathrm{diag}(\boldsymbol{S}_t)[\boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\boldsymbol{W}_t^{\mathbb{P}}]$$
where $\boldsymbol{W}$ is a d-dimensional Wiener process with *independent components* and μ and $\boldsymbol{\sigma}$ respectively are N and $N \times d$ dimensional tensors adapted to the Wiener filtration. From equation (3) we know that under the risk free numeraire, S_0, $\mathbb{Q}$ is a martingale measure just if all tradable assets $\{S_0, S_1, ..., S_N\}$ have the short rate as their local rate of return:
$$d\boldsymbol{S}_t = \mathrm{diag}(\boldsymbol{S}_t)[r_t\boldsymbol{1}dt + \boldsymbol{\sigma}_t d\boldsymbol{W}_t^{\mathbb{Q}}].$$
Girsanov's theorem informs us that the Wiener correlations are related by (8) so the question is, what is the kernel $\boldsymbol{\lambda}_t = -\boldsymbol{\phi}_t$ such that the drift changes as $\boldsymbol{\mu}_t \mapsto r_t\boldsymbol{1}$? From the last bullet point in the previous section, it is clear that $\boldsymbol{\lambda}_t$ must satisfy
$$\boldsymbol{\sigma}_t\boldsymbol{\lambda}_t = \boldsymbol{\mu}_t - r_t\boldsymbol{1}. \tag{9}$$

Clearly, the very existence of a risk neutral measure $\mathbb{Q}$ therefore necessitates that we can find a solution $\boldsymbol{\lambda}_t$ to this system. E.g, if $N < d$ then there are many solutions, one of which can be written as $\boldsymbol{\lambda}_t^* = \boldsymbol{\sigma}_t^\intercal(\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t^\intercal)^{-1}(\boldsymbol{\mu}_t - r_t\mathbf{1})$. On the other hand, if $N = d$ and $\boldsymbol{\sigma}$ is invertible then $\boldsymbol{\lambda}_t^* = \boldsymbol{\sigma}_t^{-1}(\boldsymbol{\mu}_t - r_t\mathbf{1})$ which is tantamount to the Sharpe ratio insofar as $\boldsymbol{\sigma}$ is the diagonal matrix $\text{diag}(\sigma_1, ..., \sigma_N)$. In any case, we refer to $\boldsymbol{\lambda}$ as the **market price of risk** vector, which makes sense insofar that each λ_{jt} codifies the factor loading for the individual risk factor W_{jt}.

Theorem 7. ***The Market Price of Risk***

- *Under absence of arbitrage, there will exist a market price of risk vector process $\boldsymbol{\lambda}_t$ satisfying $r_t\mathbf{1} = \boldsymbol{\mu}_t - \boldsymbol{\sigma}_t\boldsymbol{\lambda}_t$.*
- *The market price of risk $\boldsymbol{\lambda}_t$ is related to the Girsanov kernel through $\boldsymbol{\lambda}_t = -\boldsymbol{\phi}_t$ and thus to the risk neutral measure $\mathbb{Q}$ through*
$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\int_0^t \boldsymbol{\lambda}_s^\intercal d\boldsymbol{W}_s^{\mathbb{P}} - \tfrac{1}{2}\int_0^t ||\boldsymbol{\lambda}_s||^2 ds\right\}.$$
- *In a complete market, the market price of risk (or, alternatively, the martingale measure $\mathbb{Q}$) is uniquely determined and there is a unique price for every derivative.*
- *In an incomplete market there are several possible market prices of risk processes and several possible martingale measures which are consistent with no arbitrage.*
- *Thus, in an incomplete market $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, \mathbb{Q}\}$ are not determined by absence of arbitrage alone. Instead they will be determined by supply and demand on the market i.e. by the agents.*

Remark 2. Take care to notice the condition that the components in $d\boldsymbol{W}^{\mathbb{P}}$ are independent. If this is not the case, i.e. if $d\boldsymbol{W}^{\mathbb{P}} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}dt)$ for some $d \times d$ matrix $\boldsymbol{\Sigma}$, rewrite it as $d\boldsymbol{W}^{\mathbb{P}} = \boldsymbol{L}d\bar{\boldsymbol{W}}^{\mathbb{P}}$ where $d\bar{\boldsymbol{W}}^{\mathbb{P}}$ is a vector of i.i.d. Wiener increments and $\boldsymbol{L}$ is the lower triangular matrix arising from the Cholesky decomposetion $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^\intercal$. This has the effect that the market price of risk is defined through the equation $\boldsymbol{\sigma}_t\boldsymbol{L}_t\boldsymbol{\lambda}_t = \boldsymbol{\mu}_t - r_t\mathbf{1}$. In a complete market $N = d$ where $\boldsymbol{\sigma} = \text{diag}(\sigma_1, ..., \sigma_N)$ this means that $\boldsymbol{\lambda}_t = \boldsymbol{L}^{-1}\boldsymbol{R}$ where $\boldsymbol{R}$ is the vector of Sharpe ratios: $([\mu_1 - r]/\sigma_1, ..., [\mu_N - r]/\sigma_N)$.

6. Changing the Numeraire

As it was strongly suggested in section 2, there is no a priori reason why we should restrict ourselves to interpreting S_0 as the risk free asset in the First Fundamental Theorem as well as in the pricing equation (4). In fact, any **non-dividend paying tradeable asset** will do, although the martingale measures associated with each different numeraire will generally be distinct. To highlight this fact, we will write $\mathbb{Q}^0$ for a martingale measure under the numeraire S_0, $\mathbb{Q}^1$ for a martingale measure under the numeraire S_1 and so forth. We then have the following relationship between the different martingale measures

Theorem 8. *Assume that $\mathbb{Q}^i$ is a martingale measure for the numeraire S_i on $\mathscr{F}_T$ and assume S_j is a positive asset price process such that S_{jt}/S_{it} is a true $\mathbb{Q}^i$ martingale (not just a local one). If we define $\mathbb{Q}^j$ on $\mathscr{F}_T$ by the likelihood process*
$$\xi_{it}^j = \frac{d\mathbb{Q}^j}{d\mathbb{Q}^i} = \frac{S_{i0}}{S_{j0}} \cdot \frac{S_{jt}}{S_{it}}, \quad 0 \leq t \leq T \tag{10}$$

then $\mathbb{Q}^j$ is a martingale measure for S_j.

Proof. The result follows by equation (2). Let X_t be an arbitrage free price process, then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^j}\left[\frac{X_{t'}}{S_{jt'}}\middle|\mathscr{F}_t\right] &= \mathbb{E}^{\mathbb{Q}^i}\left[\frac{\xi^j_{it'}}{\xi^j_{it}}\frac{X_{t'}}{S_{jt'}}\middle|\mathscr{F}_t\right] = \mathbb{E}^{\mathbb{Q}^i}\left[\frac{1}{\xi^j_{it}}\frac{S_{i0}}{S_{j0}}\frac{X_{t'}}{S_{it'}}\middle|\mathscr{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^i}\left[\frac{S_{j0}}{S_{i0}}\frac{S_{it}}{S_{jt}}\frac{S_{i0}}{S_{j0}}\frac{X_{t'}}{S_{it'}}\middle|\mathscr{F}_t\right] = \frac{S_{it}}{S_{jt}}\mathbb{E}^{\mathbb{Q}^i}\left[\frac{X_{t'}}{S_{it'}}\middle|\mathscr{F}_t\right] \\ &= \frac{S_{it}}{S_{jt}}\frac{X_t}{S_{it}} = \frac{X_t}{S_{jt}}.\end{aligned}$$

So if $\mathbb{Q}^i$ is a martingale measure and $\mathbb{Q}^j$ is defined through ξ^j_i, then $\mathbb{Q}^j$ is a martingale measure. □

Theorem 9. *Assume that the price processes obey the $\mathbb{Q}^i$ dynamics*

$$d\boldsymbol{S}_t = diag(\boldsymbol{S}_t)[\boldsymbol{\mu}^i_t dt + \boldsymbol{\sigma}_t d\boldsymbol{W}^{\mathbb{Q}^i}_t].$$

Then the $\mathbb{Q}^i$ dynamics of the likelihood process ξ^j_i is given by

$$d\xi^j_{it} = \xi^j_{it}(\boldsymbol{\sigma}^\intercal_{jt} - \boldsymbol{\sigma}^\intercal_{it})d\boldsymbol{W}_{it}.$$

In particular, the Girsanov kernel $\boldsymbol{\phi}^j_i$ for the transition π^i to π^j is given by the volatility difference $\boldsymbol{\phi}^j_{it} = \boldsymbol{\sigma}_{jt} - \boldsymbol{\sigma}_{it}$.

Proof. Apply Itō's lemma to (10) remembering that ξ^j_i is a $\mathbb{Q}^i$ martingale. □

Essentially, a numeraire change thus boils down to the following: we start out with the conventional pricing formula $X_t = \mathbb{E}^{\mathbb{Q}}_t[\frac{B_t}{B_T}X_T]$. Then we introduce the RN derivative $\xi_T = \frac{B_T}{B_t}\frac{S_t}{S_T}$, such that the pricing formula becomes (from the Abstract Bayes' Theorem):

$$X_t = \mathbb{E}^{\mathbb{Q}^S}_t[\xi_T \tfrac{B_t}{B_T}X_T] = \mathbb{E}^{\mathbb{Q}^S}_t[\tfrac{S_t}{S_T}X_T].$$

To get the dynamics of S_T under the $\mathbb{Q}^S$ measure we use Girsanov's Theorem. Specifically, since $\xi_T = d\mathbb{Q}/d\mathbb{Q}^S$ we know that $\eta_T := \xi_T^{-1} = d\mathbb{Q}^S/d\mathbb{Q}$ is such that $d\eta_t = \phi\eta_t dW^{\mathbb{Q}}_t$ (find this ϕ). Furthermore, $dW^{\mathbb{Q}}_t = \phi dt + dW^{\mathbb{Q}^S}_t$, which is what we need. For example, if we move from the bank ($B_t = e^{rt}$) to the stock numeraire ($S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W^{\mathbb{Q}}_t\}$), then we readily see $d\eta_t = \sigma\eta_t dW^{\mathbb{Q}}_t$, whence $dW^{\mathbb{Q}}_t = \sigma dt + dW^{\mathbb{Q}^S}_t$.

7. Dividend Paying Stocks

Consider the case where S_{nt} is the price process of a dividend paying asset, then we **cannot** use the First Fundamental Theorem to infer that S_{nt}/B_t is a martingale under the risk free measure $\mathbb{Q}$ (or more generally, that S_{nt}/S_{jt} is a martingale under the $\mathbb{Q}^j$ measure). It turns out that to generalise the martingale property, we must include the "sum" of all incremental changes in the deflated cumulative dividend, meaning:

Theorem 10. ***Risk Neutral Valuation of Dividend Paying Assets*** *Let D_t be the cumulative dividend paid out by the asset S_n during the interval $[0, t]$. Then, under the risk* *neutral martingale measure $\mathbb{Q}$, the* ***normalised gain process***

$$G_t = \frac{S_{nt}}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

is a $\mathbb{Q}$-martingale.

Proof. We consider the dynamics of a self-financing portfolio which is long one unit of S_{nt} and where all dividends immediately are invested into the risk free bank account. Such a portfolio has the value process $\Pi_t = S_{nt} + X_t B_t$ where X_t denotes the instantaneous number of units of B_t. The point is, of course, that the portfolio can be viewed as a non-dividend paying asset, meaning that Π_t / B_t will be a $\mathbb{Q}$-martingale. Now, from Itō's lemma $d\Pi_t = dS_{nt} + X_t dB_t + B_t dX_t$. Combining this with the self-financing condition $d\Pi_t = dS_{nt} + dD_t + X_t dB_t$ we find that $dX_t = B_t^{-1} dD_t$. I.e. $\Pi_t = S_{nt} + \int_0^t B_s^{-1} B_t dD_s$ which will be a $\mathbb{Q}$ martingale upon being deflated by B_t. □

Theorem 11. ***General Valuation of Dividend Paying Assets*** *Assume now S_{nt} is an asset associated with the cumulative dividend D_t, and let S_{jt} be the price process of a* ***non-dividend*** *paying asset. Assuming absence of arbitrage we denote the martingale measure for the numeraire S_j by $\mathbb{Q}^j$ then the following holds*

- *The normalised gain process G defined by*

$$G_t = \frac{S_{nt}}{S_{jt}} + \int_0^t \frac{1}{S_{js}} dD_s - \int_0^t \frac{1}{S_{js}^2} dD_s dS_{js}$$

  is a $\mathbb{Q}^j$ martingale.
- *If the dividend process D has no driving Wiener component (or more generally, if $dDdS_j = 0$) then the last term vanishes.*