Report:
DYNAMIC SYSTEM

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Chen attractor is a strange attractor obtained from the following system :

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

With the following parameters : $a = 35$, $b = 3$, $c = 28$, we can observe a strange attractor similar to the Lorenz one. This attractor was first observed in simulations but then built using an electronic chaotic circuit (Chua circuit).

I Analytic Analysis

By looking the equations (1), we can figure that A=(0,0,0) is a fixed point. Let us find the others :

$$\begin{cases} 0 = a(y - x) \\ 0 = (c - a)x - xz + cy \\ 0 = xy - bz \end{cases} \implies \begin{cases} x = y \\ 0 = (c - a)x - \dfrac{x^3}{b} + cx \\ z = \dfrac{x^2}{b} \end{cases}$$

$$(c-a)x - \frac{x^3}{b} + cx = 0$$
$$x.\left((2c-a) - \frac{x^2}{b}\right) = 0$$

As we already saw that (0,0,0) is a fixed point and $x = 0 \implies y = z = 0$, we now suppose that $x \neq 0$, this leads to :

$$(2c-a) - \frac{x^2}{b} = 0$$
$$x = \pm\sqrt{b(2c-a)} \quad \text{Only if } 2c - a > 0$$

So if $2c > a$, there is two more fixed point $B_\pm = (\pm\sqrt{b(2c-a)}, \pm\sqrt{b(2c-a)}, 2c-a)$.

Now that we have all the fixed points of the system, we can try to determine their stability.
As the system isn't linear, we have to use the Jacobian Matrix and evaluate it at the fixed points.

$$\mathbf{J}(x,y,z) = \begin{pmatrix} -a & a & 0 \\ (c-a)-z & c & -x \\ y & x & -b \end{pmatrix}$$
$$\mathbf{J}(0,0,0) = \begin{pmatrix} -a & a & 0 \\ (c-a) & c & 0 \\ 0 & 0 & -b \end{pmatrix}$$

The stability depends on the sign of the three eigenvalues. Two eigenvalues are given by the submatrix

$$J' = \begin{pmatrix} -a & a \\ (c-a) & c \end{pmatrix}$$

and the third one is $\lambda_3 = -b$.

$$\lambda_\pm = \frac{\text{Tr}\, J' \pm \sqrt{\text{Tr}\, J'^2 - 4\det J'}}{2} = \frac{c - a \pm \sqrt{(c-a)^2 - 4(a^2 - 2ac)}}{2}$$
$$c - a + \sqrt{(c-a)^2 - 4(a^2 - 2ac)} < 0$$
$$\implies 2c < a$$

Thus $A = (0,0,0)$ is stable only if 2c<a, otherwise it is a saddle node.
Note : We have determined earlier that if 2c<a, A is the only fixed point.

The stability of $B_\pm$ is quite difficult to determine analytically, so we will skip this step. We can still notice that they have the same stability. To prove it, let us determine the characteristic polynomial of $\mathbf{J}(B_+)$ and $\mathbf{J}(B_-)$. We define $\alpha \equiv \sqrt{b(2c-a)}$

$$\det\left(\mathbf{J}(B_+) - \lambda I\right) = (-a-\lambda)\begin{vmatrix} c-\lambda & -\alpha \\ \alpha & -b-\lambda \end{vmatrix} - a\begin{vmatrix} -c & -\alpha \\ \alpha & -b-\lambda \end{vmatrix}$$
$$\det\left(\mathbf{J}(B_-) - \lambda I\right) = (-a-\lambda)\begin{vmatrix} c-\lambda & \alpha \\ -\alpha & -b-\lambda \end{vmatrix} - a\begin{vmatrix} -c & \alpha \\ -\alpha & -b-\lambda \end{vmatrix}$$

We have $\det(\mathbf{J}(B_-) - \lambda I) = \det(\mathbf{J}(B_+) - \lambda I)$, it means that the two matrices have the same eigenvalues so the two fixed points have the same stability. With the given parameters, these two fixed points exist and the characteristic polynomial is

$$f(\lambda) = 2a^2b + \lambda^2(-a - b + c) - 4abc - bc\lambda - \lambda^3$$

With the given parameters ($a = 35, b = 3, c = 28$)

$$f(\lambda) = -\lambda^3 - 10\lambda^2 - 84\lambda - 4410$$

The three roots (approximated) of this polynomial are :

$$\lambda_1 = -18.428$$
$$\lambda_2 = 4.21398 \ - 14.8846i$$
$$\lambda_3 = 4.21398 \ + 14.8846i$$

So with the given parameters, Chen's system has three fixed points and by using the Jacobian, we found that :
- $A = (0, 0, 0)$: Saddle node (one direction unstable, two stable)
- $B_\pm = (\pm\sqrt{b(2c - a)}, \pm\sqrt{b(2c - a)}, 2c - a)$: One direction stable and the others give unstable spirals

II Numerical Analysis

Let's study Chen's system using numerical tools.
Note : Most of the graphics provided in the appendix are vector images, you might be able to zoom as you wish.
First of all, we can simply plot the attractor. As we see on Figure 1, it looks like there is a part of Lorenz attractor in the Chen attractor. One may also compare the attractor to Saturn's rings (which is kind of a Cantor set). On Figure 2, we can see the attractor seen from above with the three fixed points. However it is not "rigorously" symmetric, we may suppose that there is some sort of symmetry, probably central symmetry. This might be due to the fact that B_+ and B_- are symmetric with respect to the origin. To highlight the symmetry, we can plot different Poincaré sections. The four graphs of Figure 5 are obtained by cutting the attractor along four different planes. They show that the attractor is more or less symmetric as we supposed.

By looking time series ($x(t)$,$y(t)$,$z(t)$) (Figure 7), we can see that they look quite random but the x and y seem relatively similar. We can plot both x and y on the same graph and also $x - y$ (cf Figure 8). Here again, the two series look similar but in fact, it's only their shape, the difference $x - y$ (and also of their derivative $dx/dt - dy/dt$) is important. In fact, I think the two series are linked by some sort of time shift and they don't have the same amplitude. Something like :

$$x(t) \approx A.f(t)$$
$$y(t) \approx B.f(t + \delta t)$$

It would be interesting to write some algorithm to find the amplitude and the time shift which minimize $|x(t) - y(t)|$.

We can also compute the Fast Fourier Transform (Figure 6). As there is no dominating frequency, I suppose this should not be a periodic spectrum. We might notice that the FFT of the x data and the y data are almost identical. We could make a connection between similarity in FFT and the fact that x and y component rapidly converge to about the same behavior/shape Figure 8.

How to characterize the chaotic behavior of the system ?
We can plot the distance between two points initially separated by 10^{-7} (Figure 3). The two trajectories stay close for about 6 time units but it suddenly spreads on the entire attractor. Then, the distance between the two points is erratic.

III Conclusion

Chen attractor can be classified as a chaotic system because of its sensitivity to initial conditions. Whatever the initial conditions, it seems that the produced attractor is pretty symmetric and it has the characteristic that x and y have the same behavior after a short time without being equal. Actually, I found this characteristic very interesting. I tried to figure how to put some numbers one this similarity (looking at the difference of the derivative) but it wasn't a success.

Appendices

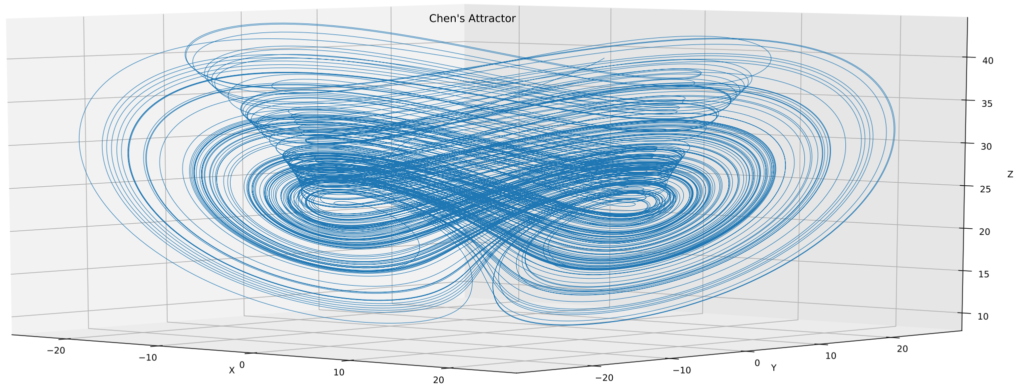

FIGURE 1 – Chen's Attractor

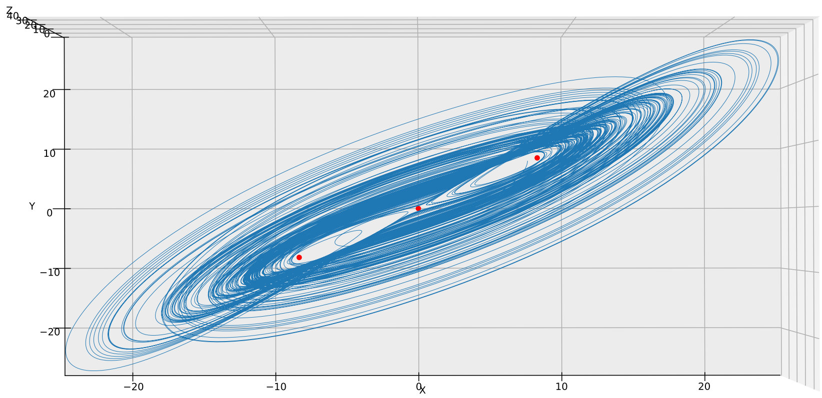

FIGURE 2 – Attractor seen from above with fixed points

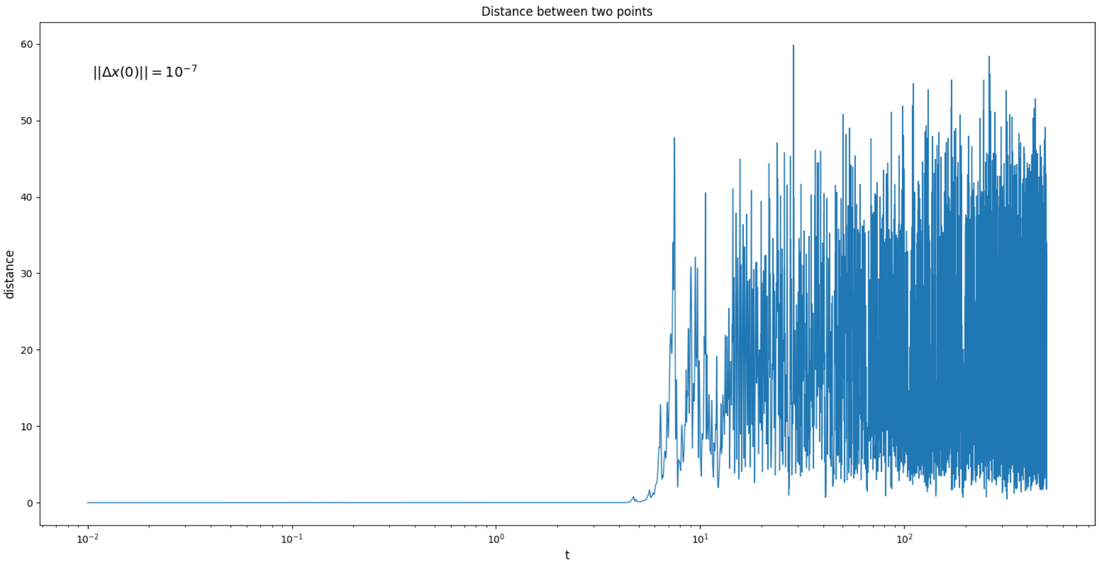

FIGURE 3 – Sensitivity to initial conditions

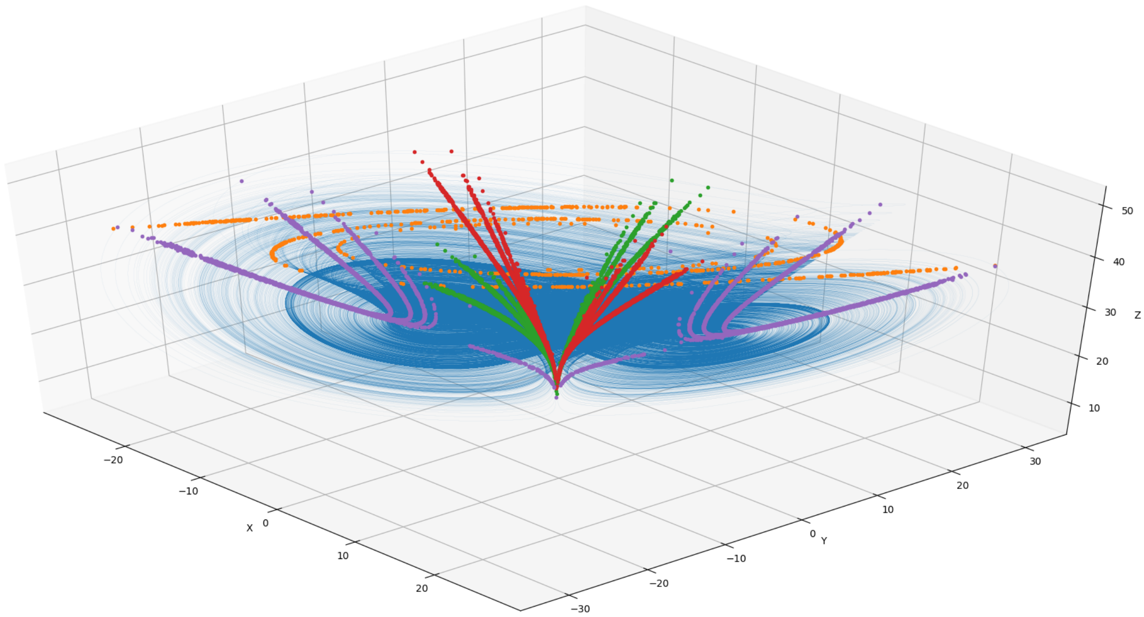

FIGURE 4 – Attractor with Poincaré Sections

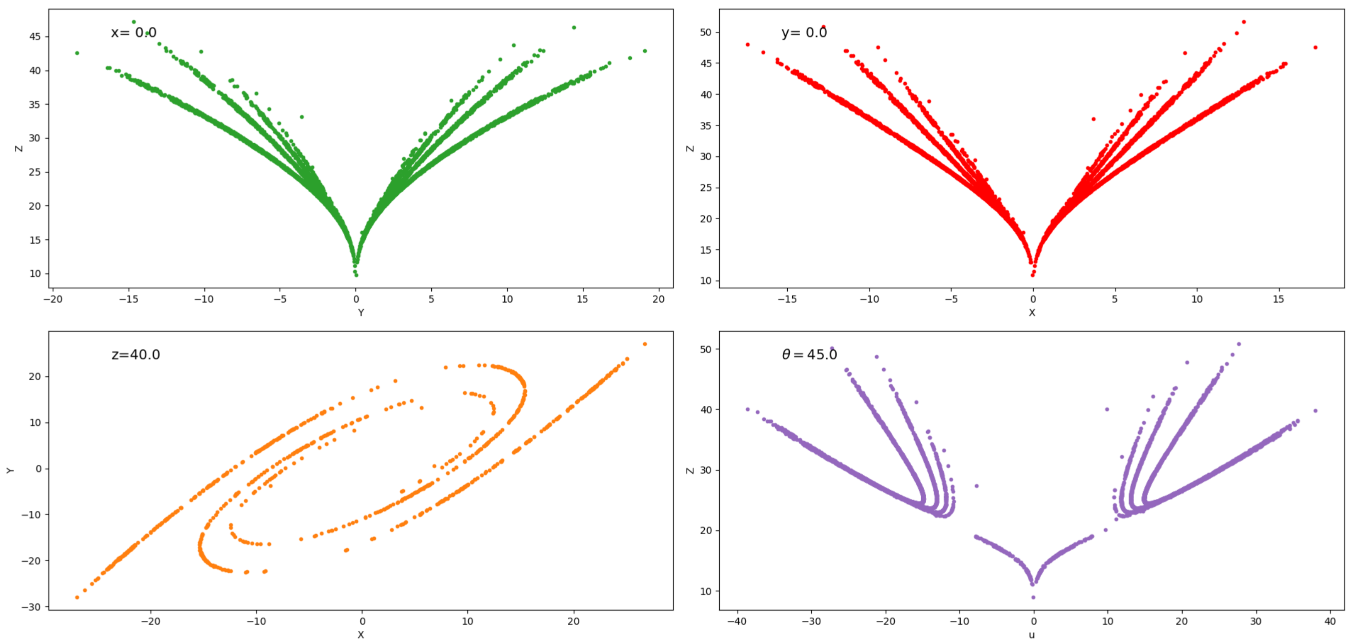

FIGURE 5 – Poincaré Sections. θ is the angle in cylindric coordinates

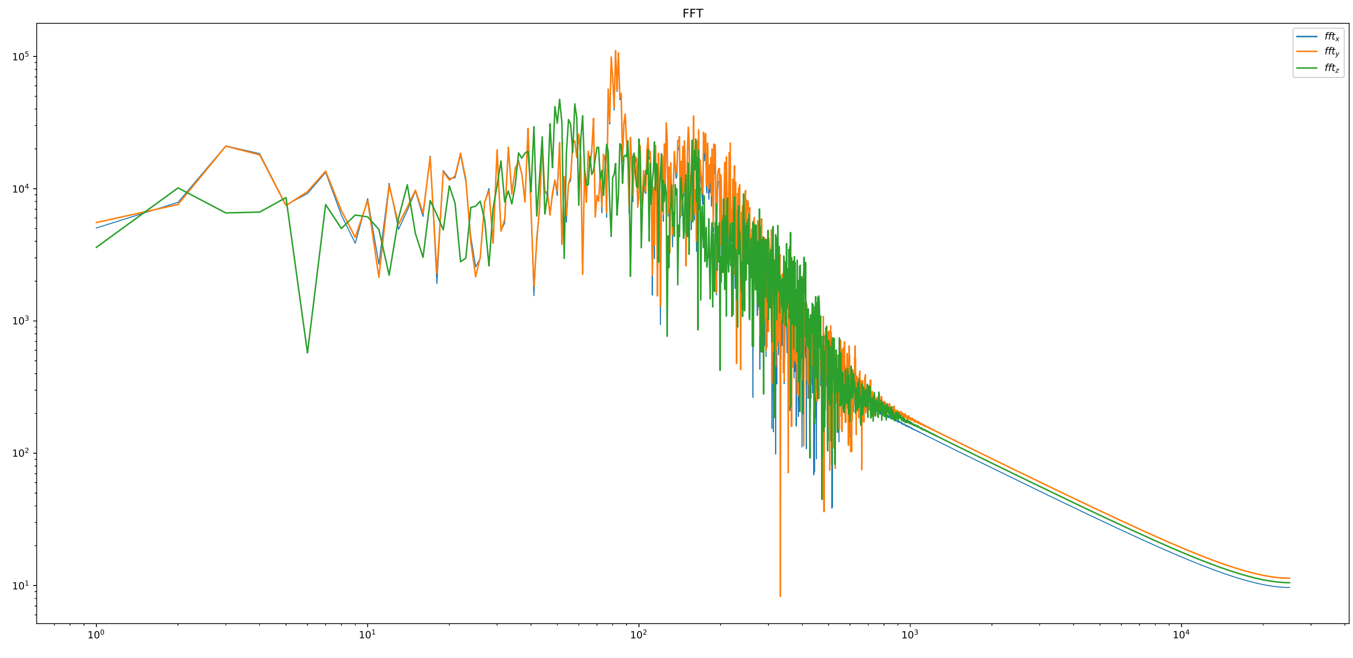

FIGURE 6 – FFT

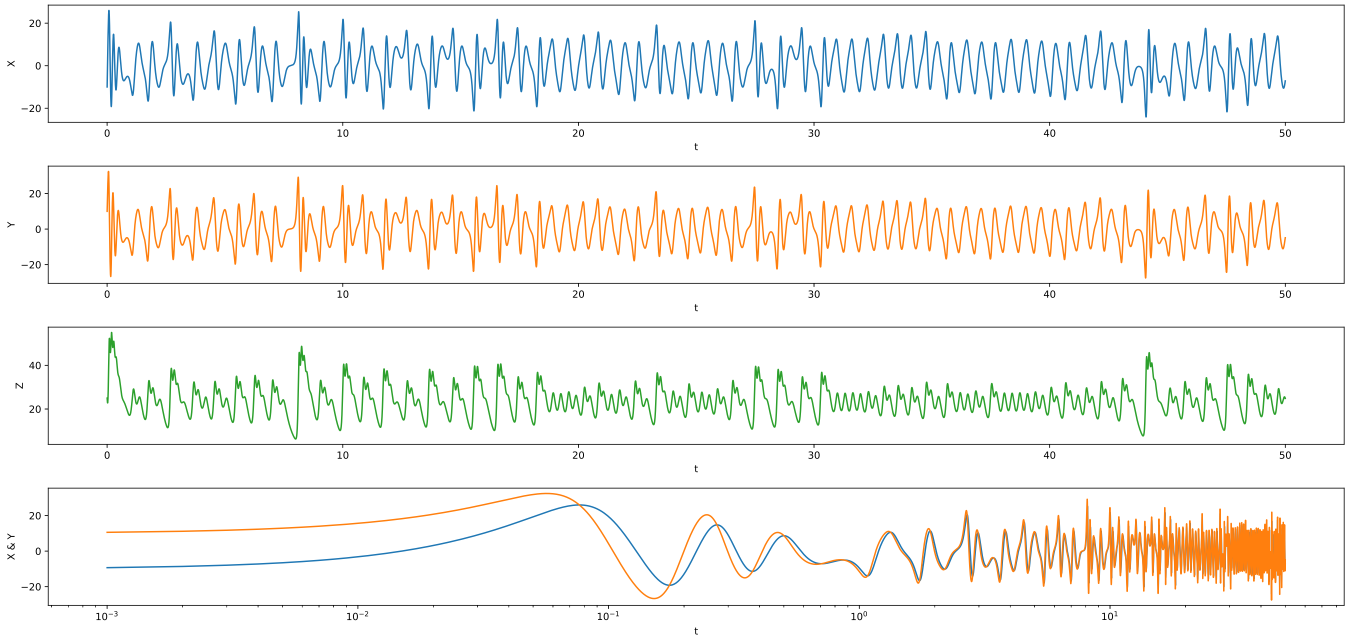

FIGURE 7 – Time Series

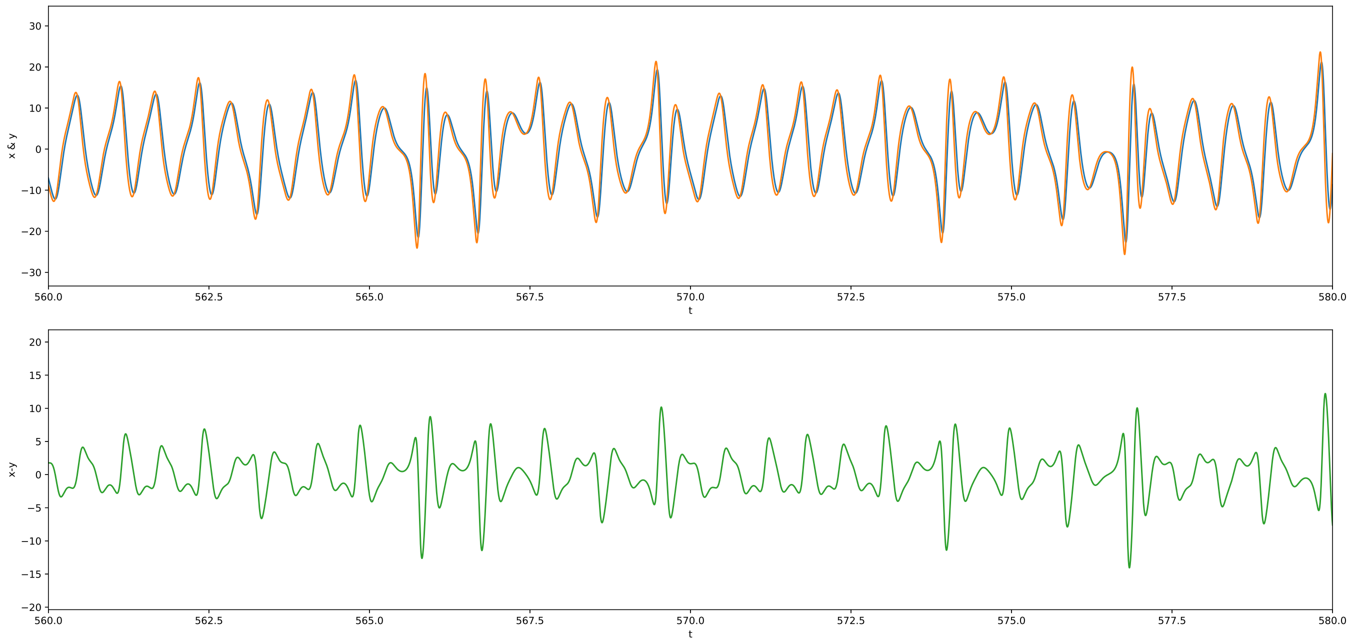

FIGURE 8 – Same shape for x and y data