Space Elevator Problem Set

Summary

ACTIVE TIME

45 minutes to 1 hour

TOTAL PROJECT TIME

45 minutes to 1 hour

KEY CONCEPTS

Forces, gravity, tension, circular motion, space exploration

CREDITS

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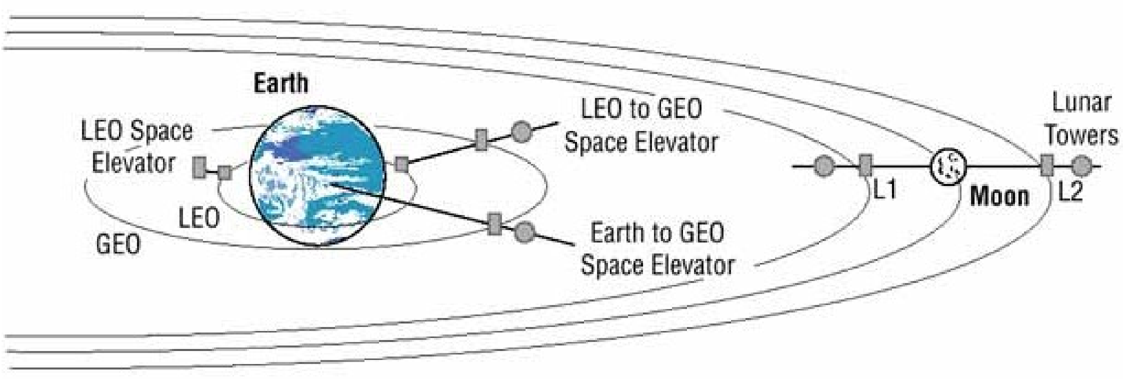
Introduction



Space elevators zipping people and materials up into space might seem like a very futuristic and improbable idea, but is it that difficult? This activity will guide you through the mathematics. Try it out and see what is possible with materials that can be produced with current technology.

This activity is not recommended for use as a science fair project. Good science fair projects have a stronger focus on controlling variables, taking accurate measurements, and analyzing data. To find a science fair project that is just right for you, browse our library of over 1,200 Science Fair Project Ideas or use the Topic Selection Wizard to get a personalized project recommendation.

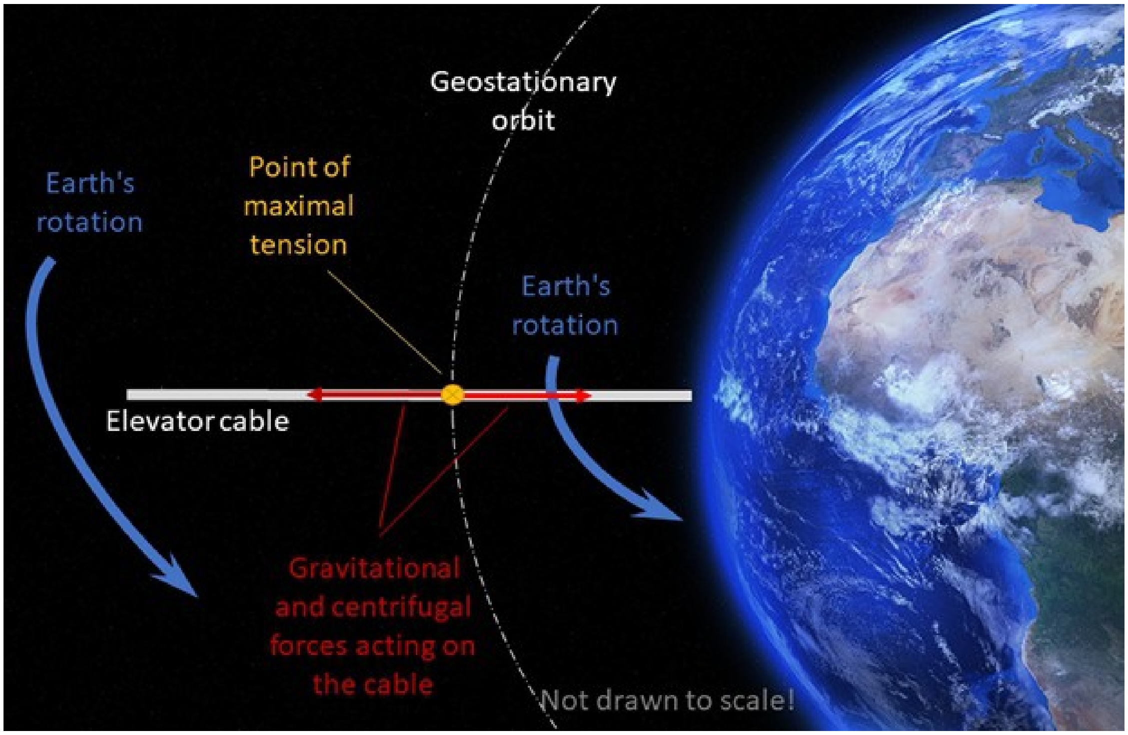
Materials

- Pen and paper
- Calculator
- Access to a computer program like Microsoft® Excel® that allows you to graph equations, or a graphing calculator

Background Information

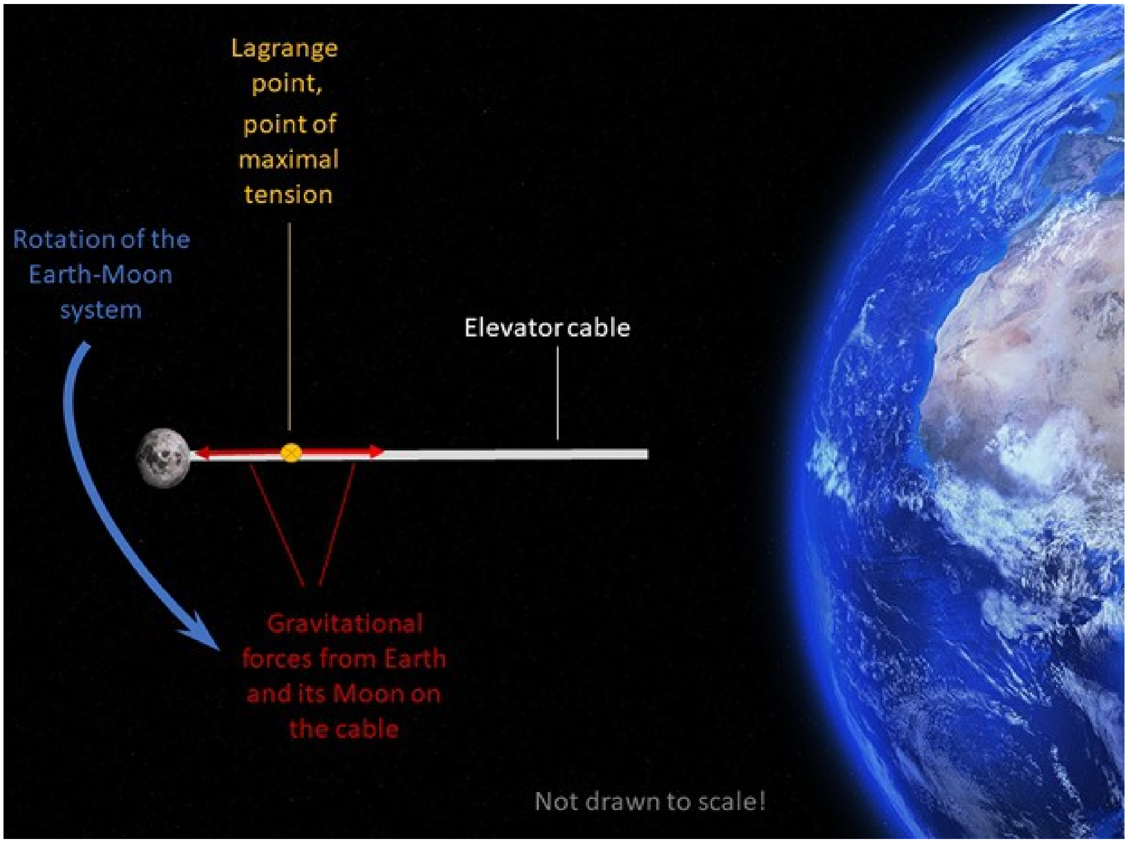
A space elevator is a transportation system that would be anchored to a celestial body and extend into space. Its purpose would be to move cargo through space without the use of spaceships. A space elevator would require much less fuel to transport cargo and might even be able to operate solely on solar power.

The most frequently proposed version suggests anchoring a cable to Earth at the equator. In this proposal, the elevator rotates with Earth, so it is stationary relative to Earth. The gravitational force generated by Earth's mass and the centrifugal force created by its rotation keep the elevator cable taut.



The gravitational force is strongest closer to Earth. As you move up along the cable, the gravitational force decreases and the centrifugal force increases. At geostationary orbit, these two forces are equal in strength. At this height, objects experience weightlessness and the cable experiences its maximal tension. The practical limitation on this idea is that the cable may break under this tension. Currently available mass-producible materials are not strong enough for this purpose.

But what if the elevator was anchored to a less-massive celestial body? With a reduced gravitational force, would a space elevator be attainable? This activity will guide you through the mathematics of a space elevator anchored to Earth's moon and extending toward Earth. In this proposal, the cable is kept taut by the competing forces of gravity created by the Moon and gravity created by Earth.



At a certain point along the cable (a *Lagrange point*), these two forces cancel each other, and one achieves weightlessness. This is the point along the cable where the tension is at its greatest. Would currently available materials be strong enough to sustain this tension? Try the activity to find out!

Prerequisites

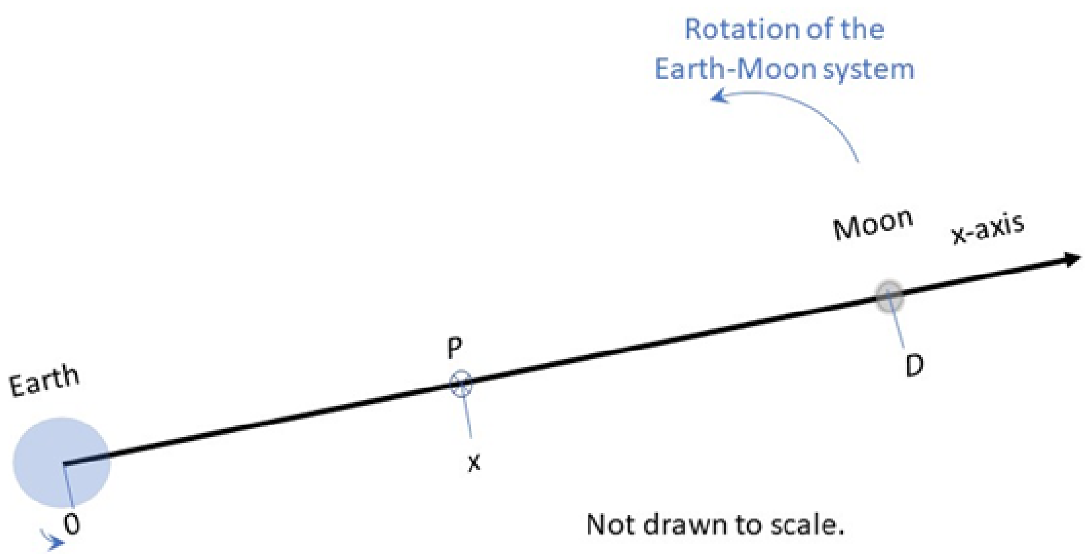
Knowledge of the following concepts is required:

- Newton's laws of motion
- Newton's law of universal gravitation
- Circular motion
- Algebra

Instructions

Teachers: A PDF of this activity and an answer key are available for classroom use.

1) In our calculations, we will use a coordinate system where the origin is the center of Earth, and the x-axis extends from the center of Earth to the center of the Moon. In this reference frame, the variable x represents the distance from the center of Earth to any point on the line extending to the center of the Moon. This reference frame will rotate with the Earth-Moon system.



We will study what happens at point P at a distance x from the center of Earth. We will use *D* to indicate the distance from the center of Earth to the center of the Moon, *M* for the mass of Earth, and *m* for the mass of the Moon.

Quantity	Variables	Value
Earth's Radius	R	6,371 km
The Moon's Radius	r	1,737 km
Distance Between the Center of Earth and the Center of the Moon	D	384,400 km
Earth's Mass	M	5.972×10^{24} kg
The Moon's Mass	m	7.348×10^{22} kg

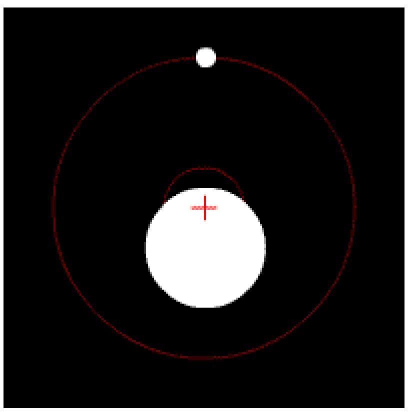
2)

A mass placed at a point P between Earth and the Moon will feel the gravitational force generated by both these masses. Use Newton's law of universal gravitation to write a formula for the gravitational force felt by a point mass m_{test} placed at point P. Use the same sign convention for force as you did for the *x*-axis (the direction from the center of Earth toward the Moon is positive).

3)

Consider the Earth-Moon system as a system of two point masses. Where, along the *x*-axis, is the center of mass of the Earth-Moon system located?

4) Gravity makes the Earth-Moon frame rotate around its center of mass, as shown in the figure (the red cross in the figure indicates the location of the center of mass of the system). The angular speed is $\omega = \sqrt{\frac{G(M+m)}{D^3}}$ radians per second, or approximately one full rotation per 27.3 days. Write the formula for the centrifugal force created by this rotation and felt by a point mass m_{test} placed at point *P* in the rotating reference frame.

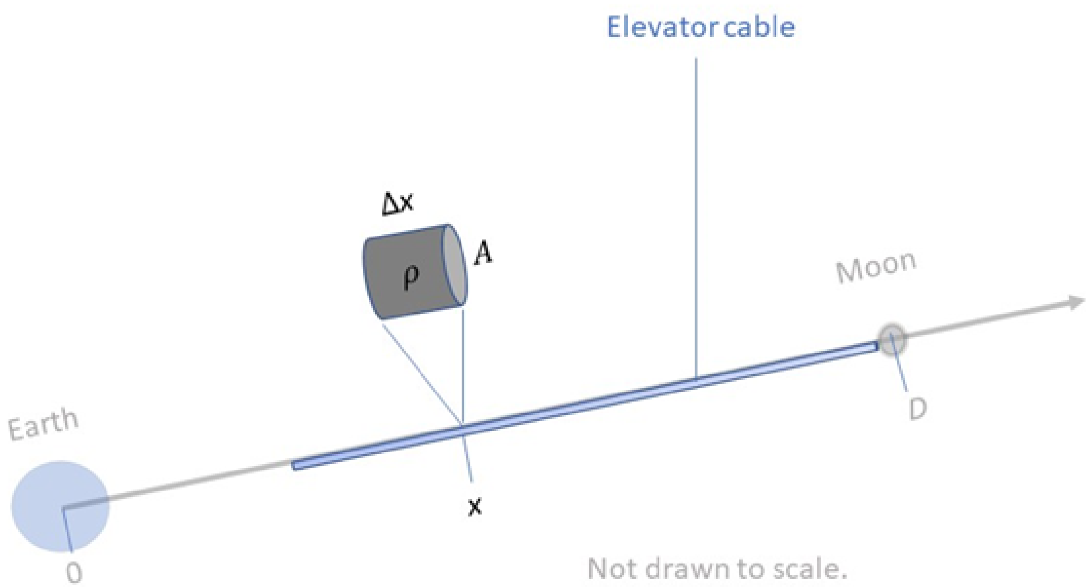


5)

Use your knowledge of how forces add up to find the formula for the gravitational and centrifugal forces combined.

6) Instead of a test mass m_{test} placed at point P, calculate this force (gravitational and centrifugal forces combined) on a tiny section of the cable with a length Δx placed at point P. Assume the cable has a fixed cross-sectional area A, and is made of a material with density ρ.

Hint: Find the formula for the mass of this tiny cylindrical piece of cable, and then substitute this mass in the formula for the force due to the gravitational and centrifugal forces combined.



7) Make a free-body diagram for this tiny section of cable. Make sure to include the gravitational forces of the Moon and Earth, the centrifugal force on the tiny section, and the tension force on each end of the tiny section of cable.

Hint: is the tension on each end of the cable the same or does the tension change along the cable?

8) Since in the rotating reference frame, the piece of cable does not move along the x-axis, there is no net force along the x-direction. In the formula, this becomes:

$$T\left(x+\Delta x\right)-T\left(x\right)-F\left(x\right)=0$$

Where:

$F\left(x\right)$ represents the gravitational and centrifugal forces combined.

$T\left(x\right)$ represents the tension in the cable at the point *P* with x-coordinate *x*.

This tells you how the tension changes along the length of the cable. The beginning of the section feels a tension $T\left(x\right)$, the end feels a tension $T\left(x+\Delta x\right)$. Because Δx is small, we can write this as $T\left(x\right)+\Delta T\left(x\right)$ where $\Delta T\left(x\right)$ represents the change in tension in the cable at point *P*.

Use this information to show that $\Delta T\left(x\right)$ must be due to the gravitational and centrifugal forces.

9)

Combine the formulas found in steps 6 and 8 to get a formula for the change in tension at each position along the cable (the change of tension as a function of x).

10) With calculus (using integration), this equation leads to an equation that shows how the tension in the cable varies as a function of *x*.

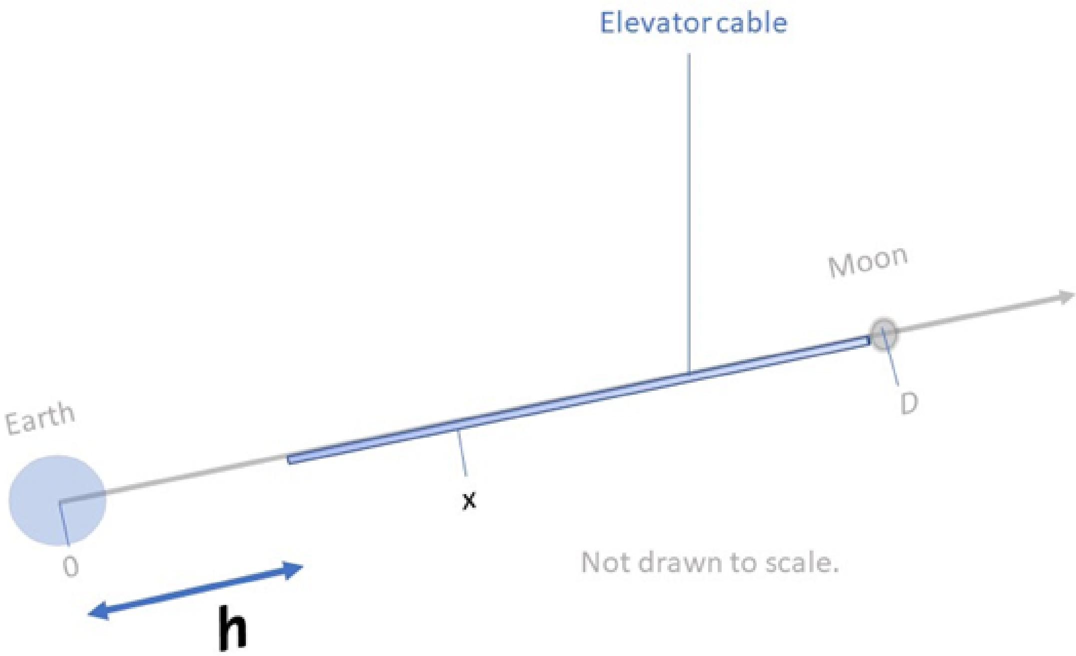
$$T\left(x\right)=\frac{GM}{D}A\,\rho\left\{\left(\frac{1+\mu}{2}\left(\frac{x}{D}\right)^2-\mu\frac{x}{D}+\frac{1}{\left(\frac{x}{D}\right)}+\frac{\mu}{\left(1-\frac{x}{D}\right)}\right)+constant\right\}$$

$$where\ \mu=\frac{m}{M}=\frac{mass\ Moon}{mass\ Earth}$$

11)

Since the formula you found in step 9 only provides the change in tension, integration gives an equation for the tension $T\left(x\right)$ including an unknown constant. In order to solve for that constant, we need to know the tension at one point along the cable. We can use the point at the free end of the cable (the point farthest from the Moon and closest to Earth) because the tension at that point is known. What is the tension at that point on the cable?

12) Define the endpoint of the cable as a distance h above the center of Earth.



The equation for $T\,(x)$ then becomes:

$$T\,(x) = \frac{GM}{D} A\,\rho \left\{ \left[\frac{1+\mu}{2} \left(\frac{h}{D}\right)^2 - \mu \frac{h}{D} + \frac{1}{\left(\frac{h}{D}\right)} + \frac{\mu}{\left(1-\frac{h}{D}\right)} \right] - \left[\frac{1+\mu}{2} \left(\frac{x}{D}\right)^2 - \mu \frac{x}{D} + \frac{1}{\left(\frac{x}{D}\right)} + \frac{\mu}{\left(1-\frac{x}{D}\right)} \right] \right\}$$

This seems like a huge equation. We can make it more readable by writing it in the following way:

$$T\,(x) = \frac{GM}{D} A\,\rho \left\{ \varepsilon\,(h) - \varepsilon\,(x) \right\}$$

where

$$\varepsilon\,(x) = \frac{1+\mu}{2} \left(\frac{x}{D}\right)^2 - \mu \frac{x}{D} + \frac{1}{\left(\frac{x}{D}\right)} + \frac{\mu}{\left(1-\frac{x}{D}\right)}$$

Use a computer program like Excel or a graphing calculator to plot how $\varepsilon\,(x)$ changes with x , with x between [0, *D*] or [0, 384,400 km].

13) In order for a cable to remain taut, the tension in the entire cable must always be positive. If the tension in the cable is negative at any point, the cable will collapse. In other words, think about how you can pull on a rope to make it taut, but you cannot push on a rope to make it taut—the rope must always be in positive tension when it is taut.

Use the formula $T\left(x\right)$ and the graph $\varepsilon\left(x\right)$ to estimate the limitations this restriction sets on the maximal value for h. In other words, how far must the elevator reach so that $\varepsilon\left(h\right)-\varepsilon\left(x\right)\geq 0$ for all x -values of points along the physical length of the cable.

Hint: The surface of the Moon is at $x=\left(D\text{–}r\right)\approx\left(384,400\ km\text{–}1,737\ km\right)$, *so the cable reaches from h to a height of 382,663 km. On your graph, you can find that* $\varepsilon\left(surface\ of\ the\ Moon\right)\approx 4.$

14) The elevator will also fail if the cable breaks, or if at any point along the line, the stress it experiences is larger than the breaking stress. $\left(Stress\ S\ =\frac{Tension\ T}{Cross\ section\ A}\right)$.
Use your graph of $\varepsilon\left(x\right)$ to find the point along the cable where the tension, and thus the stress, is the largest. (Assume a constant cross-section).

Hint: $T\left(x\right)$ *will be maximal where* $\varepsilon\left(x\right)$ *is minimal.*

Note: The Background section states that the tension is maximal at the point where the gravitational force of Earth cancels the gravitational force of the Moon (also called the Lagrange point). This is an alternative way to calculate the value for x where $T\left(x\right)$ is maximal.

15)

We can now concentrate on the point on the cable at the distance above Earth found in question 14 as that is the point along the cable where the tension is maximal.
Verify that at that point $\varepsilon\left(x\right)=1.6.$

16)

The cable will fail if the stress at that point is larger than the breaking stress of the cable material. Write the formula for the tension on the cable at that point and convert this formula to one for the stress on the cable at that point.

17)

Rearrange your answer from question 16 to solve for specific stress $\left(Specific\ stress=\frac{Stress}{Density}\right)$, then look at the right side of the equation. Does the specific stress of the cable at the point where it is at its most depend on the material of which it is made?

18)

Specific strength is defined as $\frac{Breaking\ Strength}{Density}$ or $\frac{S_{max}}{\rho}$. Materials with a larger specific strength are preferred as cable material.

Use what you found in step 17 to explain why scientists use a constraint on specific stress of the material and not a constraint on the stress or the density of the material. Then state what this constraint could be.

19) Table 1 lists a few materials, together with their breaking strength, density, and specific strength.

Table 1
Characteristics of Some Cable Materials

Material	Breaking Strength(*) [N/m²]	Density [kg/m³]	Specific Strength [Nm/kg]
Steel	5×10^8	8000	6.25×10^4
Titanium alloy	1.25×10^9	4800	2.6×10^5
Carbon fiber	4.3×10^9	1750	2.5×10^6
Carbon nanotube	6×10^{10}	~1000	6×10^7

* *Note:* Most metals—like steel, aluminum, and titanium alloy—are ductile, meaning they will stretch permanently before breaking (like bending a paper clip; it permanently deforms before it breaks). The yield strength is the maximum amount of stress that a material can bear without showing irreversible deformation. Typically, metal structures are designed to stay well below the yield strength, not the breaking strength. However, breaking strength and yield strength are usually the same order of magnitude, so this is a good approximation.

Calculate the specific stress experienced by the cable at the point where tension is maximal. Then, use the information from the table to evaluate if it exceeds the breaking strength of each material. Use the value $\varepsilon(h) = 4$, the smallest value for $\varepsilon(h)$ obtained in step 13, for your calculations.

20) Make a conclusion: What materials could be used to make a space elevator stretching from the Moon toward Earth?

What Happened?

The calculations explore a space elevator extending from the Moon. The calculations should have shown that the cable needs to extend to at least 76,880 km from the center of Earth for Earth's gravitational force to keep the cable taut. If you arrived at a different answer, check your calculations against our answer key.

The calculations show that for the cable to be able to support its own weight, it must be made from a material that is light and strong. Carbon fiber, as well as carbon nanotubes, are strong enough and, at the same time, light enough to support a cable with a constant cross-section.

Digging Deeper

In the calculations, we assumed a cable with a constant cross-section. More-advanced designs use a tapered cable, one that is thin at the ends where tension is low and becomes thicker as it nears the place of maximal tension. Other designs use a combination of these scenarios. Although these designs yield more-attainable results, the general conclusion holds: at the time this activity was written, mass-producible materials like carbon fiber can hold up a space elevator anchored on the Moon, but are not strong enough to hold up a space elevator anchored on Earth.

For Further Exploration

- Go through the calculations for a space elevator anchored on Earth. You can neglect any gravity generated by the Moon in this approach and set the tension at both ends of the cable to 0. What materials are strong enough to hold up this fixed cross-section cable?
- What would be the constraints of a space elevator anchored to Mars or another planet?
- What happens if you add a counterweight (a large point mass) at the end of an Earth-anchored or Mars-anchored cable?

Activities

Links

Careers