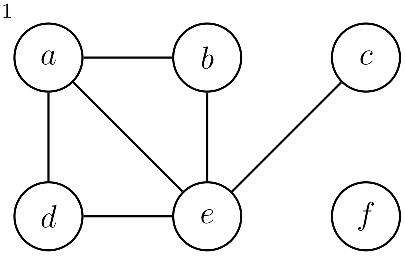
Lecture 10: Depth-First Search

Previously

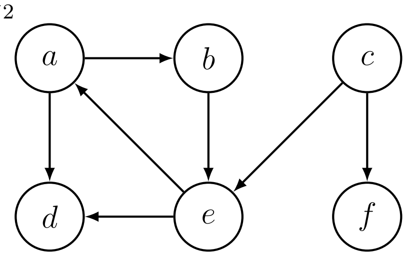
- Graph definitions (directed/undirected, simple, neighbors, degree)
- Graph representations (Set mapping vertices to adjacency lists)
- Paths and simple paths, path length, distance, shortest path
- Graph Path Problems
  - `Single_Pair_Reachability(`G`,`s`,`t`)`
  - `Single_Source_Reachability(`G`,`s`)`
  - `Single_Pair_Shortest_Path(`G`,`s`,`t`)`
  - `Single_Source_Shortest_Paths(`G`,`s`)` (SSSP)
- Breadth-First Search (BFS)
  - algorithm that solves Single Source Shortest Paths
  - with appropriate data structures, runs in $O(|V| + |E|)$ time (linear in input size)

Examples

G_1



G_2



Depth-First Search (DFS)

- Searches a graph from a vertex s, similar to BFS
- Solves Single Source Reachability, **not** SSSP. Useful for solving other problems (later!)
- Return (not necessarily shortest) parent tree of parent pointers back to s

---

- **Idea!** Visit outgoing adjacencies recursively, but never revisit a vertex
- i.e., follow any path until you get stuck, backtrack until finding an unexplored path to explore
- $P(s) = \texttt{None}$, then run $\text{visit}(s)$, where
- $\text{visit}(u)$:
  - for every $v \in \text{Adj}(u)$ that does not appear in P:
    * set $P(v) = u$ and recursively call $\text{visit}(v)$
  - (DFS finishes visiting vertex u, for use later!)

- **Example:** Run DFS on G_1 and/or G_2 from a

Correctness

- **Claim:** DFS visits v and correctly sets $P(v)$ for every vertex v reachable from s
- **Proof:** induct on k, for claim on only vertices within distance k from s
  - Base case ($k = 0$): $P(s)$ is set correctly for s and s is visited
  - Inductive step: Consider vertex v with $\delta(s, v) = k' + 1$
  - Consider vertex u, the second to last vertex on some shortest path from s to v
  - By induction, since $\delta(s, u) = k'$, DFS visits u and sets $P(u)$ correctly
  - While visiting u, DFS considers $v \in \text{Adj}(u)$
  - Either v is in P, so has already been visited, or v will be visited while visiting u
  - In either case, v will be visited by DFS and will be added correctly to P $\square$

Running Time

- Algorithm visits each vertex u at most once and spends $O(1)$ time for each $v \in \text{Adj}(u)$
- Work upper bounded by $O(1) \times \sum_{u \in V} \deg(u) = O(|E|)$
- Unlike BFS, not returning a distance for each vertex, so DFS runs in $O(|E|)$ time

Full-BFS and Full-DFS

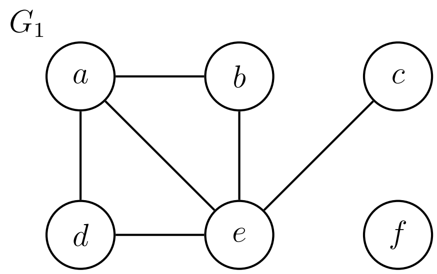
- Suppose want to explore entire graph, not just vertices reachable from one vertex
- **Idea!** Repeat a graph search algorithm A on any unvisited vertex

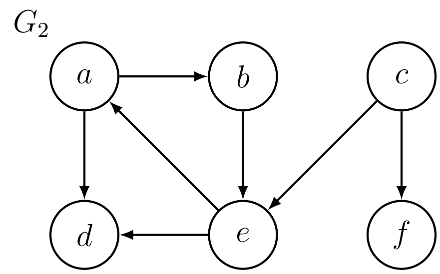
---

- Repeat the following until all vertices have been visited:
  - Choose an arbitrary unvisited vertex s, use A to explore all vertices reachable from s

---

- We call this algorithm **Full-*A***, specifically Full-BFS or Full-DFS if A is BFS or DFS
- Visits every vertex once, so both Full-BFS and Full-DFS run in $O(|V| + |E|)$ time
- **Example:** Run Full-DFS/Full-BFS on G_1 and/or G_2





Graph Connectivity

- An **undirected** graph is ***connected*** if there is a path connecting every pair of vertices
- In a directed graph, vertex u may be reachable from v, but v may not be reachable from u
- Connectivity is more complicated for directed graphs (we won't discuss in this class)
- `Connectivity(`G`)`: is undirected graph G connected?
- `Connected_Components(`G`)`: given undirected graph $G = (V, E)$, return partition of V into subsets $V_i \subseteq V$ (***connected components***) where each V_i is connected in G and there are no edges between vertices from different connected components
- Consider a graph algorithm A that solves Single Source Reachability
- **Claim:** A can be used to solve Connected Components
- **Proof:** Run Full-A. For each run of A, put visited vertices in a connected component $\square$

Topological Sort

- A ***Directed Acyclic Graph (DAG)*** is a directed graph that contains no directed cycle.
- A ***Topological Order*** of a graph $G = (V, E)$ is an ordering f on the vertices such that: every edge $(u, v) \in E$ satisfies $f(u) < f(v)$.
- **Exercise:** Prove that a directed graph admits a topological ordering if and only if it is a DAG.
- How to find a topological order?
- A ***Finishing Order*** is the order in which a Full-DFS **finishes visiting** each vertex in G
- **Claim:** If $G = (V, E)$ is a DAG, the reverse of a finishing order is a topological order
- **Proof:** Need to prove, for every edge $(u, v) \in E$ that u is ordered before v, i.e., the visit to v finishes before visiting u. Two cases:
  - If u visited before v:
    * Before visit to u finishes, will visit v (via (u, v) or otherwise)
    * Thus the visit to v finishes before visiting u
  - If v visited before u:
    * u can't be reached from v since graph is acyclic
    * Thus the visit to v finishes before visiting u $\square$

Cycle Detection

- Full-DFS will find a topological order if a graph $G = (V, E)$ is acyclic
- If reverse finishing order for Full-DFS is not a topological order, then G must contain a cycle
- Check if G is acyclic: for each edge (u, v), check if v is before u in reverse finishing order
- Can be done in $O(|E|)$ time via a hash table or direct access array
- To return such a cycle, maintain the set of **ancestors** along the path back to s in Full-DFS
- **Claim:** If G contains a cycle, Full-DFS will traverse an edge from v to an ancestor of v.
- **Proof:** Consider a cycle $(v_0, v_1, \ldots, v_k, v_0)$ in G
  - Without loss of generality, let v_0 be the first vertex visited by Full-DFS on the cycle
  - For each v_i, before visit to v_i finishes, will visit v_{i+1} and finish
  - Will consider edge (v_i, v_{i+1}), and if v_{i+1} has not been visited, it will be visited now
  - Thus, before visit to v_0 finishes, will visit v_k (for the first time, by v_0 assumption)
  - So, before visit to v_k finishes, will consider (v_k, v_0), where v_0 is an ancestor of v_k $\square$