Lecture 7: Binary Trees II: AVL

Last Time and Today's Goal

Sequence Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic		
	`build(X)`	`get_at(i)` `set_at(i,x)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Binary Tree	n	h	h	h	h
AVL Tree	n	$\log n$	$\log n$	$\log n$	$\log n$

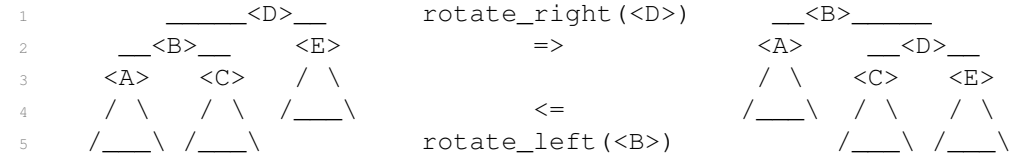
Set Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Binary Tree	$n \log n$	h	h	h	h
AVL Tree	$n \log n$	$\log n$	$\log n$	$\log n$	$\log n$

Height Balance

- How to maintain height $h = O(\log n)$ where n is number of nodes in tree?
- A binary tree that maintains $O(\log n)$ height under dynamic operations is called **balanced**
  - There are many balancing schemes (Red-Black Trees, Splay Trees, 2-3 Trees, . . .)
  - First proposed balancing scheme was the **AVL Tree** (Adelson-Velsky and Landis, 1962)

Rotations

- Need to reduce height of tree without changing its traversal order, so that we represent the same sequence of items
- How to change the structure of a tree, while preserving traversal order? **Rotations!**



- A rotation relinks $O(1)$ pointers to modify tree structure and maintains traversal order

Rotations Suffice

- **Claim:** $O(n)$ rotations can transform a binary tree to any other with same traversal order.
- **Proof:** Repeatedly perform last possible right rotation in traversal order; resulting tree is a canonical chain. Each rotation increases depth of the last node by 1. Depth of last node in final chain is $n-1$, so at most $n-1$ rotations are performed. Reverse canonical rotations to reach target tree. □
- Can maintain height-balance by using $O(n)$ rotations to fully balance the tree, but slow :(
- We will keep the tree balanced in $O(\log n)$ time per operation!

AVL Trees: Height Balance

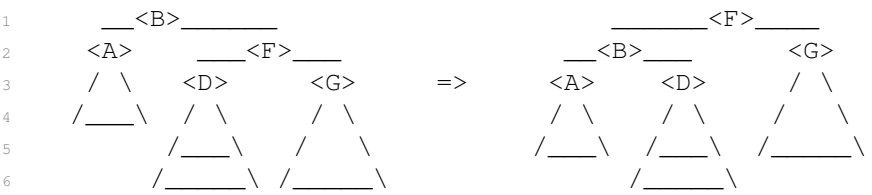
- AVL trees maintain **height-balance** (also called the **AVL Property**)
  - A node is **height-balanced** if heights of its left and right subtrees differ by at most 1
  - Let **skew** of a node be the height of its right subtree minus that of its left subtree
  - Then a node is height-balanced if its skew is $-1, 0,$ or 1

- **Claim:** A binary tree with height-balanced nodes has height $h = O(\log n)$ (i.e., $n = 2^{\Omega(h)}$)
- **Proof:** Suffices to show fewest nodes $F(h)$ in any height h tree is $F(h) = 2^{\Omega(h)}$

  $F(0) = 1,\ F(1) = 2,\ F(h) = 1 + F(h-1) + F(h-2) \geq 2F(h-2) \implies F(h) \geq 2^{h/2}$ □

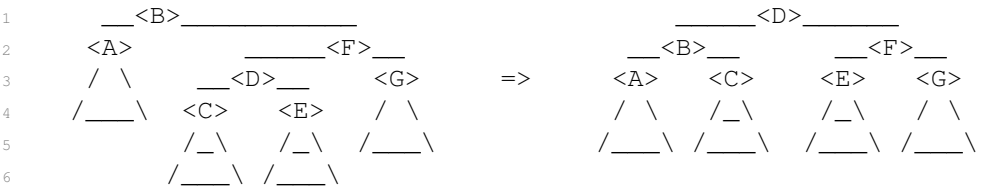
- Suppose adding or removing leaf from a height-balanced tree results in imbalance
  - Only subtrees of the leaf's ancestors have changed in height or skew
  - Heights changed by only ± 1, so skews still have magnitude ≤ 2
  - **Idea:** Fix height-balance of ancestors starting from leaf up to the root
  - Repeatedly rebalance lowest ancestor that is not height-balanced, wlog assume skew 2

- **Local Rebalance:** Given binary tree node <B>:
  - whose skew 2 and
  - every other node in <B>'s subtree is height-balanced,
  - then <B>'s subtree can be made height-balanced via one or two rotations
  - (after which <B>'s height is the same or one less than before)
- **Proof:**
  - Since skew of <B> is 2, <B>'s right child <F> exists
  - **Case 1:** skew of <F> is 0 or **Case 2:** skew of <F> is 1
    - Perform a left rotation on <B>



    - Let $h = \text{height}(\texttt{<A>})$. Then $\text{height}(\texttt{<G>}) = h + 1$ and $\text{height}(\texttt{<D>})$ is $h + 1$ in Case 1, h in Case 2
    - After rotation:
      - the skew of <B> is either 1 in Case 1 or 0 in Case 2, so <B> is height balanced
      - the skew of <F> is -1, so <F> is height balanced
      - the height of <B> before is $h + 3$, then after is $h + 3$ in Case 1, $h + 2$ in Case 2

  - **Case 3:** skew of <F> is -1, so the left child <D> of <F> exists
    - Perform a right rotation on <F>, then a left rotation on <B>



    - Let $h = \text{height}(\texttt{<A>})$. Then $\text{height}(\texttt{<G>}) = h$ while $\text{height}(\texttt{<C>})$ and $\text{height}(\texttt{<E>})$ are each either h or $h - 1$
    - After rotation:
      - the skew of <B> is either 0 or -1, so <B> is height balanced
      - the skew of <F> is either 0 or 1, so <F> is height balanced
      - the skew of <D> is 0, so D is height balanced
      - the height of <B> is $h + 3$ before, then after is $h + 2$

- **Global Rebalance:** Add or remove a leaf from height-balanced tree T to produce tree T'. Then T' can be transformed into a height-balanced tree T'' using at most $O(\log n)$ rotations.
- **Proof:**
  - Only ancestors of the affected leaf have different height in T' than in T
  - Affected leaf has at most $h = O(\log n)$ ancestors whose subtrees may have changed
  - Let <x> be lowest ancestor that is not height-balanced (with skew magnitude 2)
  - If a leaf was added into T:
    - Insertion increases height of <x>, so in Case 2 or 3 of Local Rebalancing
    - Rotation decreases subtree height: balanced after one rotation
  - If a leaf was removed from T:
    - Deletion decreased height of one child of <x>, not <x>, so only imbalance
    - Could decrease height of <x> by 1; parent of <x> may now be imbalanced
    - So may have to rebalance every ancestor of <x>, but at most $h = O(\log n)$ of them
- So can maintain height-balance using only $O(\log n)$ rotations after insertion/deletion!
- But requires us to evaluate whether possibly $O(\log n)$ nodes were height-balanced

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Computing Height

- How to tell whether node <x> is height-balanced? Compute heights of subtrees!
- How to compute the height of node <x>? Naive algorithm:
  - Recursively compute height of the left and right subtrees of <x>
  - Add 1 to the max of the two heights
  - Runs in $\Omega(n)$ time, since we recurse on every node :(
- **Idea:** Augment each node with the height of its subtree! (Save for later!)
- Height of <x> can be computed in $O(1)$ time from the heights of its children:
  - Look up the stored heights of left and right subtrees in $O(1)$ time
  - Add 1 to the max of the two heights
- During dynamic operations, we must **maintain** our augmentation as the tree changes shape
- Recompute subtree augmentations at every node whose subtree changes:
  - Update relinked nodes in a rotation operation in $O(1)$ time (ancestors don't change)
  - Update all ancestors of an inserted or deleted node in $O(h)$ time by walking up the tree

Steps to Augment a Binary Tree

- In general, to augment a binary tree with a **subtree property** `P`, you must:
  - State the subtree property `P(<X>)` you want to store at each node `<X>`
  - Show how to compute `P(<X>)` from the augmentations of `<X>`'s children in $O(1)$ time
- Then stored property `P(<X>)` can be maintained without changing dynamic operation costs

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Application: Sequence

- For sequence binary tree, we needed to know subtree **sizes**
- For just inserting/deleting a leaf, this was easy, but now need to handle rotations
- Subtree size is a subtree property, so can maintain via augmentation
  - Can compute size from sizes of children by summing them and adding 1

Conclusion

- Set AVL trees achieve $O(\lg n)$ time for all set operations,
  except $O(n \log n)$ time for build and $O(n)$ time for iter
- Sequence AVL trees achieve $O(\lg n)$ time for all sequence operations,
  except $O(n)$ time for build and iter

Application: Sorting

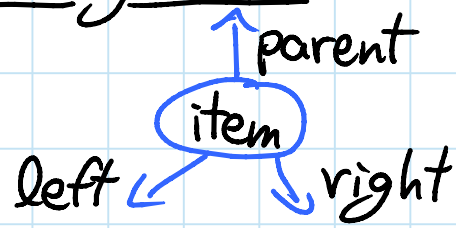
- Any Set data structure defines a sorting algorithm: build (or repeatedly insert) then iter
- For example, Direct Access Array Sort from Lecture 5
- AVL Sort is a new $O(n \lg n)$-time sorting algorithm

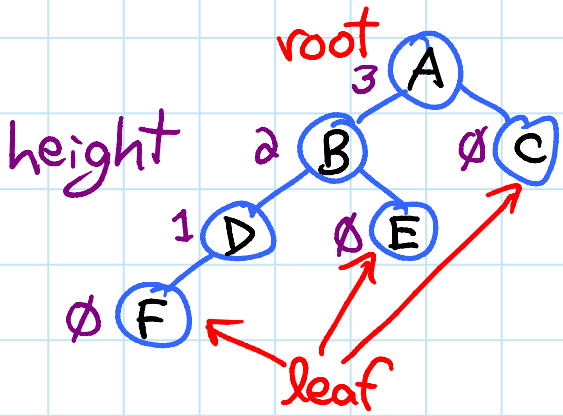
6.006	Lecture 7	Feb. 27, 2020

TODAY: Binary Trees II: AVL
- subtree augmentation
- sequence binary tree via subtree sizes
- height balance (AVL)
  - rotations
  - rebalancing
  - height augmentation

Recall: binary tree

- node:

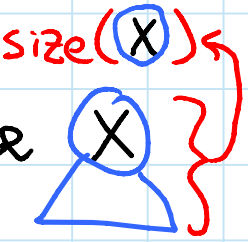




- height(X) = max. depth of node within subtree(X) = length of longest downward path
  - height ø ⇔ leaf
- h = height(root) = max. depth of node (leaf)
- traversal order = traverse left subtree, root, traverse right subtree
- subtree insert, delete, first/last, pred./successor in O(h) time/op.

Application: set binary tree (Binary Search Tree)
- traversal order = increasing key order
⇒ insert, delete, find/find-prev/next in O(h)/op.

Application: sequence binary tree
- suppose we knew size of every subtree
  - ↳ # nodes



subtree_at(node, i): ith node in traversal order of subtree(node)
- n_L = size(node.left)
- if $i < n_L$: subtree_at(node.left, i)
- if $i = n_L$: return node
- if $i > n_L$: subtree_at(node.right, $i - n_L - 1$)

get_at(i): subtree_at(root, i).item
set_at(i, x): subtree_at(root, i).item = x

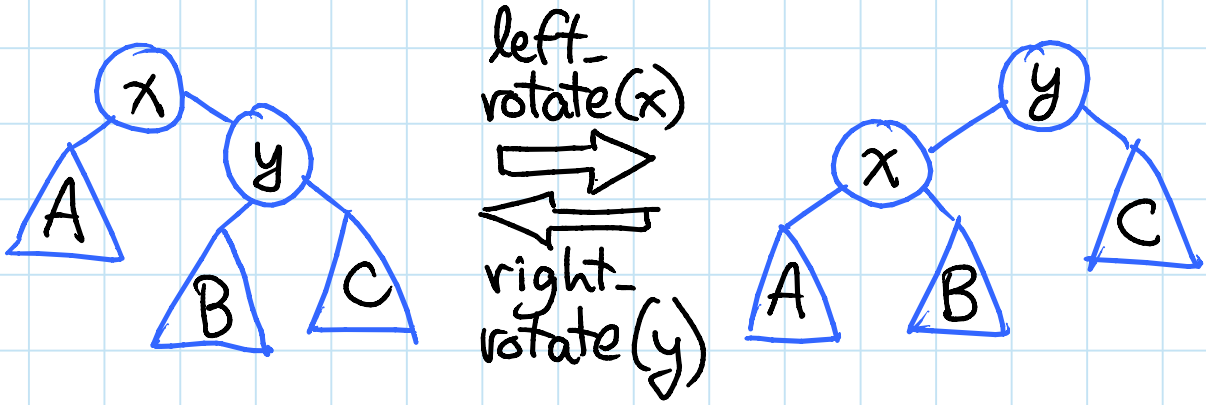
↓ for node

Subtree augmentation:
- maintain extra properties of nodes (subtrees)
- subtree property of a node can be computed in O(1) time given property values of node's children (& node)
  - e.g. sum/product/min/max of keys or other variable of items in subtree(node)
  - node.size = 1 + node.left.size + node.right.size
    where None.size = 0
  - NOT e.g. node's seq. index, or node's depth
- whenever insert/delete a leaf (e.g.), trigger augmentation update for all O(h) ancestors in order up the tree
  = subtrees changed by that update
  ⇒ get/set/insert/delete_at in O(h)

Balanced binary tree maintains $h = O(\lg n)$
⇒ all set & sequence ops. run in $O(\lg n)$ except build & iter

- need to re-arrange tree while preserving traversal order (to represent the same set/sequence)
- how?

Rotations:



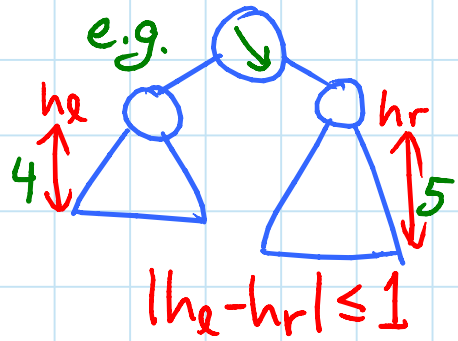
- preserves traversal order:
  A, x, B, y, C

- $O(1)$ pointer changes ⇒ $O(1)$ time
- need to update subtree augmentations of x & y (& their ancestors) too

First (& still one of simplest) balanced binary tree:

AVL trees: [Adel'son-Vel'skii & Landis 1962]

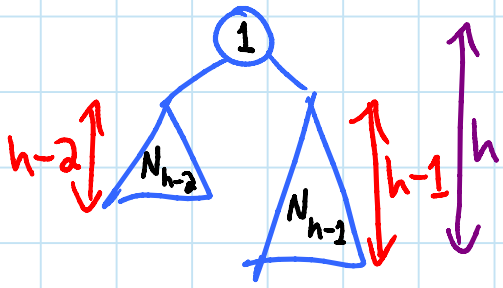
- require every node to be height-balanced: heights of children differ by at most ±1
- define height(None) = -1
- skew(node) = height(node.right) − height(node.left) $\Rightarrow \in \{-1, 0, 1\}$
  - notation: ↙ — ↘ ⇙ ⇘
    skew: -1 ∅ +1 -2 +2
    (⇙ ⇘ = temporary imbalance)
- each node stores node.height via subtree augmentation: node.height = 1 + max{node.left.height, node.right.height}
  (alternatively, can just store skews)



Height-balanced ⇒ Balanced

- worst when every node has skew ±1
- let N_h = min. # nodes in height-h AVL tree

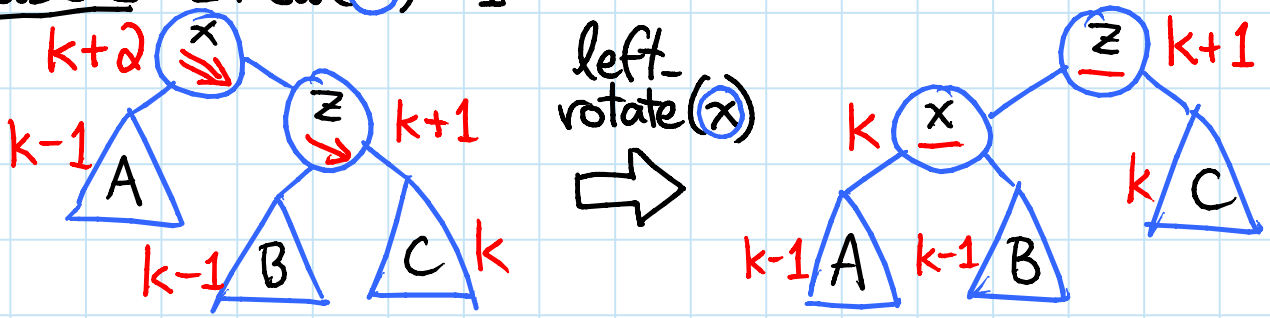
$\Rightarrow N_h = N_{h-1} + N_{h-2} + 1$
$> 2 N_{h-2}$
$\Rightarrow N_h > 2^{h/2}$
$\Rightarrow h < 2 \lg N_h$



AVL rebalancing:

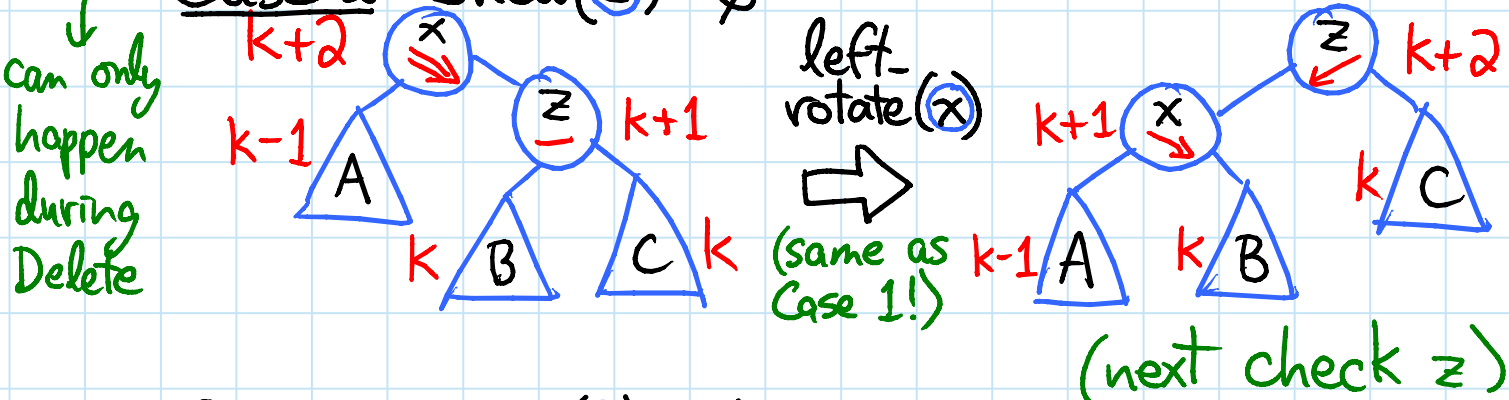
- binary tree insert/delete adds/removes a leaf, affecting ancestors' subtrees/heights by ± 1
  $\Rightarrow$ |skew| ≤ 2 still
- for each unbalanced ancestor x in order up tree:
  - assume skew(x) $= 2$ (if -2, flip left/right)
  $\Rightarrow$ x has right child z, balanced by induction

Case 1: skew(z) $= 1$



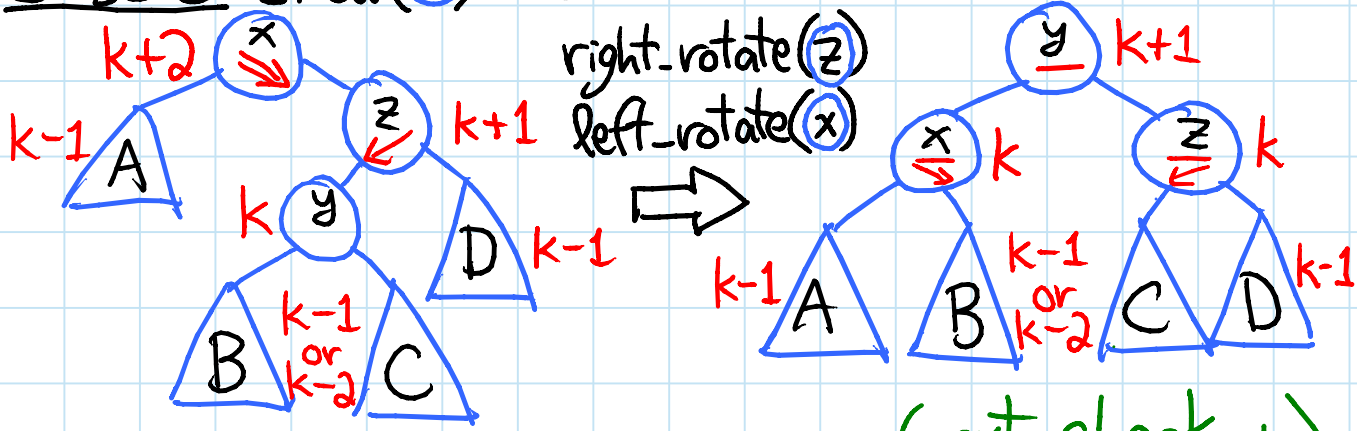
Case 2: skew(z) $= \emptyset$

can only happen during Delete



(next check z)

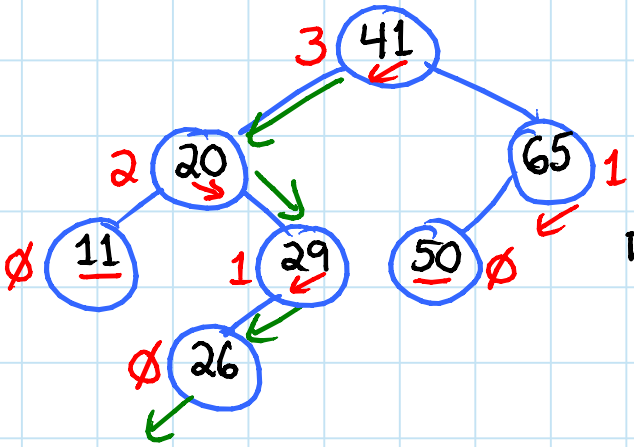
Case 3: skew(z) $= -1$



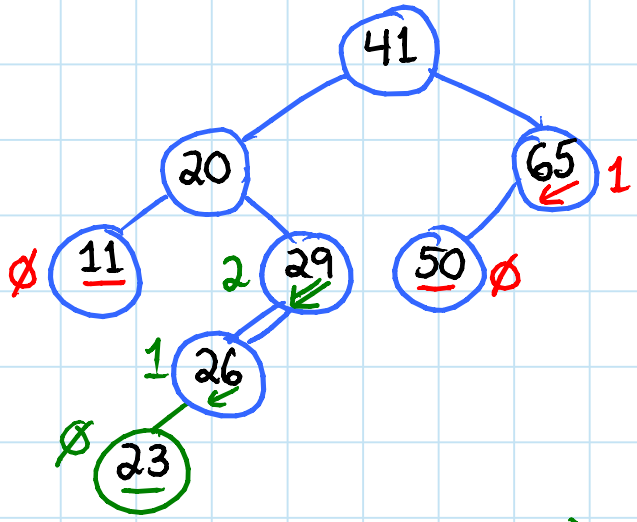
(next check y)

Example: set AVL

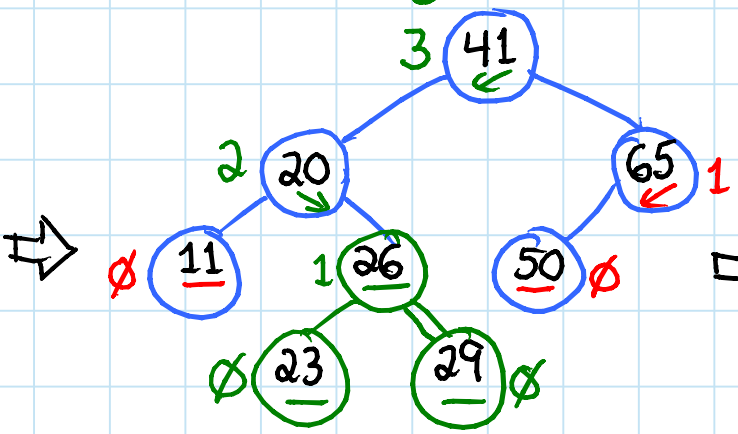
insert(23)



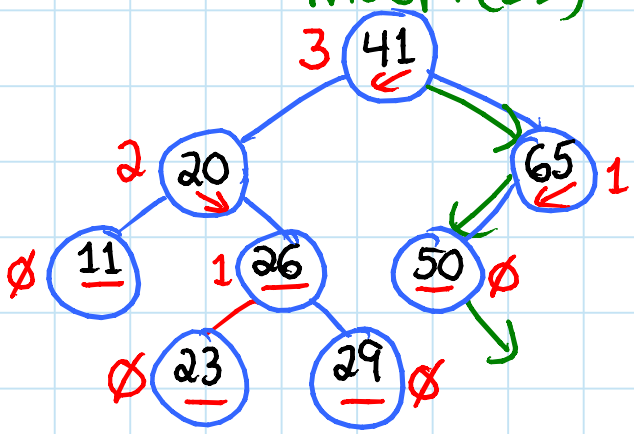
x=29: left-left case



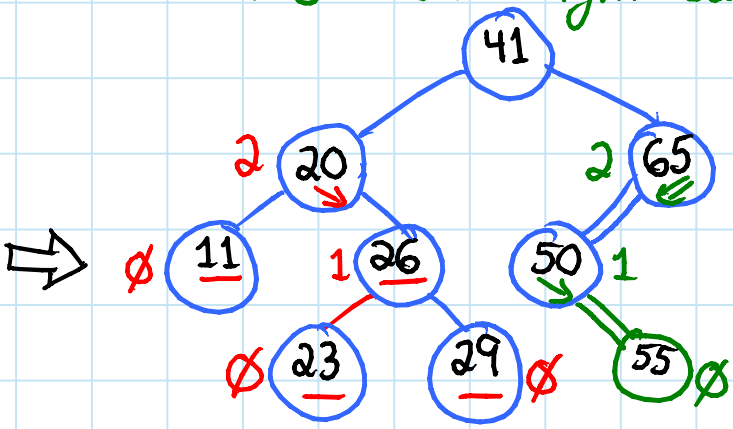
done.



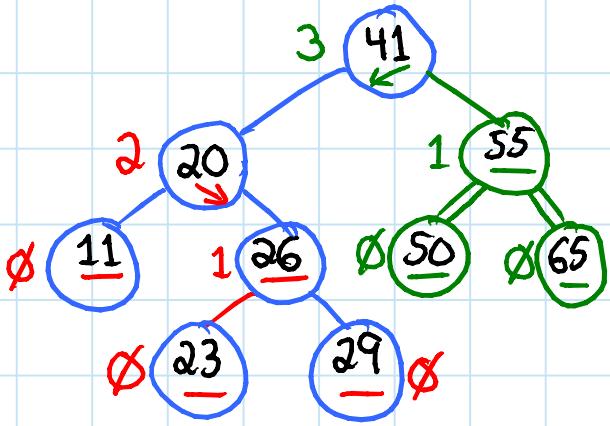
insert(55)



x=65: left-right case

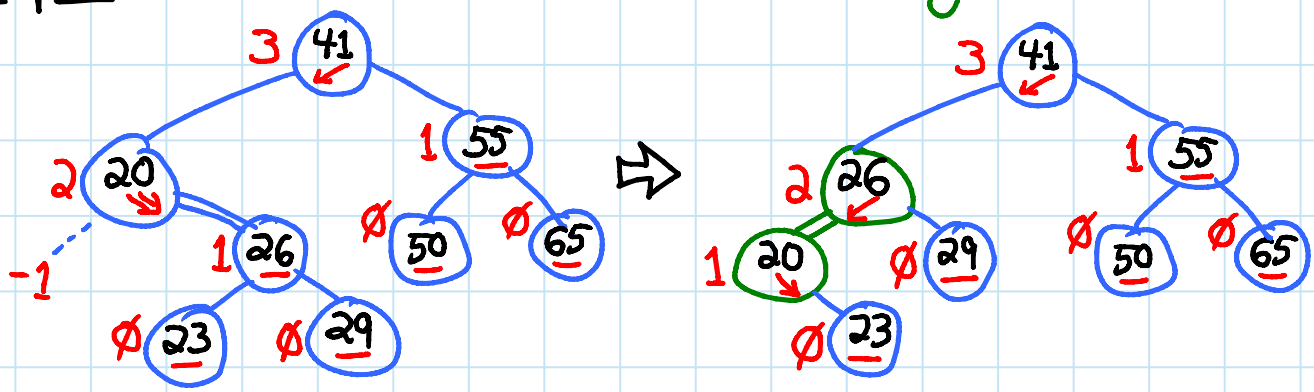


done.



- in fact never need >1 rotation pair for an Insert:
  height of subtree after rebalancing =
  height of subtree before insert

Example: delete(11) x=20: right-flat case



Wow! $O(\lg n)$ for everything!

Set data structure → sorting algorithm

① build(A) e.g. for x in A:
insert(x)

② iter_ord() → key order
– e.g. Direct Access Array Sort [L5]

AVL sort is $\Theta(n \lg n)$
– $\Theta(n \lg n)$ for ①
– $\Theta(n)$ for ②
↳ effectively maintaining the sorted order of a dynamic set!

Balanced trees:

there are many!

- AVL trees — [Adel'son-Velsii & Landis 1962]
- B-trees / 2-3-4 trees — [Bayer & McCreight 1972] [CLRS 18] [Nievergelt & Reingold 1973]
- BB[α] trees — [CLRS ch. 13]
- red-black trees — [Sleator & Tarjan 1985]
- (A) — splay trees — [Pugh 1989]
- (R) — skip lists — [Galperin & Rivest 1993]
- (A) — scapegoat trees — [Seidel & Aragon 1996]
- (R) — treaps

(R) = use random numbers to make decisions fast with high probability

(A) = "amortized": adding up costs for several operations ⇒ fast on average

e.g. splay trees are a current research topic
- see 6.854 (Advanced Algorithms) & 6.851 (Advanced Data Structures)