厦门大学 Xiamen University
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Statistics of Financial Markets
Project 4

Group No. 8
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CIR Model

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Definition

In Cox, Ingersoll, Roll (CIR) Model, the short-rate satisfies the SDE

$$dr(t) = a\{b - r(t)\}dt + \sigma\sqrt{r(t)}dW_t \tag{1}$$

where a, b, σ are constants and W_t is a Wiener process.

- proposed by Cox, Ingersoll and Ross (1985)
- equilibrium spot rate model
- mean reversion
- nonnegative r(t) if $2ab \geq \sigma^2$

Bond Pricing with CIR model

Under the corresponding numeraire:

$$V(t, T) = E_t[exp\{-\int_t^T r(s)ds\}V(T, T)]$$

with

$$dr(t) = \mu_r dt + \sigma_r dW_t$$

and

$$\mu_r = \mu\{r(t), t\}, \ \sigma_r = \sigma\{r(t), t\}$$

Bond Pricing with CIR model

By means of the condition that $V(T, T) = 1$, in combination with *Itô's Lemma* we get:

$$dV(T,T) = \{\frac{\partial V(t,T)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(t,T)}{\partial r^2} + \mu_r \frac{\partial V(t,T)}{\partial r}\}dt + \sigma \frac{\partial V(t,T)}{\partial r} dW_t \quad (2)$$

under the risk-neutral measure the PDE of the CIR model is:

$$r(t)V(t,T) = \frac{\partial V(t,T)}{\partial t} + \frac{1}{2} r(t)\sigma^2 \frac{\partial^2 V(t,T)}{\partial r^2} + a\{b - r(t)\}\frac{\partial V(t,T)}{\partial r} \quad (3)$$

Bond Pricing with CIR model

Assuming $V(t, T) = exp\{A(t) - r(t)B(t)\}$ and a nominal value 1, we can consider:

$$\begin{aligned}
\frac{\partial V(t,T)}{\partial t} =& \{A'(t) - r(t)B'(t)\}V(t) \\
\frac{\partial V(t,T)}{\partial r} =& - B(t)V(t) \\
\frac{\partial^2 V(t,T)}{\partial r^2} =& B^2(t)V(t)
\end{aligned}$$

Bond Pricing with CIR model

With the boundary conditions $V(T,T) = 1$ and $A(T,T) = B(T,T) = 0$:

$$V(t,T) = exp\{A(t) - r(t)B(t)\}$$

where

$$\begin{aligned}
A(t) =& \frac{2ab}{\sigma^2} log \frac{2\psi \, exp\{(a+\psi)(T-t)/2\}}{2\psi + (a+\psi)exp\{\psi(T-t)-1\}} \\
B(t) =& \frac{2exp\{\psi(T-t)-1\}}{2\psi + (a+\psi)exp\{\psi(T-t)-1\}} \\
\psi =& \sqrt{a^2 + 2\sigma^2}
\end{aligned}$$

Bond Pricing with CIR model

For increasing time perilds τ the term structure curve $Y_T(t)$ converges to the value:

$$Y_{lim} = \frac{2ab}{\psi + a}$$

and the term structure has the following properties:

- $r(t) > b$: decreasing term structure
- $r(t) < Y_{lim}$: increasing term structure
- $b > r(t) > Y_{lim}$: term structure first rises and then falls

Dataset

The data we use is consist of daily observations of the annualized yield on Chinese Treasury Bond with 1 year to maturity. The time span is from 01. July 2006 to 01. July 2016. The data is downloaded from Wind Database.

CIR Process Densities

For MLE of the parameter vector θ, transition are required. The transition density of CIR process has a closed form expression

The density of $r_{t+\Delta t}$ at time $t+\Delta t$ is :

$$p(r_{t+\Delta t}|r_t,\theta,\Delta t) = c\, exp(-u-v)(\frac{v}{u})^{\frac{q}{2}}\, I_q(2\sqrt{uv})$$

where

$$\begin{aligned} c =& \frac{2a}{\sigma^2\{1-exp(-a\Delta t)\}} \\ u =& c r_t\, exp(-a\Delta t) \\ v =& c r_{t+\Delta t} \\ q =& 2ab/\sigma^2 - 1 \end{aligned}$$

and $I_q(2\sqrt{uv})$ is the modified Bessel function of the first order q.

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The likelihood function for interest rate time series is:

$$L(\theta) = \prod_{t=1}^{n-1} p(r_{t+1}|r_t, \theta, \Delta t)$$

then the log-likelihood function of the CIR process is given by:

$$\begin{aligned} logL(\theta) =& \sum_{t=1}^{n-1} logp(r_{t+1}|r_t, \theta, \Delta t) \\ =& (n-1) log\, c + \sum_{t=1}^{n-1} [-u_t - v_{t+1} + 0.5q\, log \frac{v_{t+1}}{u_t} + log\{I_q(2\sqrt{u_t v_t} \end{aligned} \tag{4}$$

where $u_t = cr_t exp(-a\Delta t)$, and $v_{t+1} = cr_{t+1}$.

Since MLE needs good starting values, we can collect the starting values for parameter by OLS. The conditional mean function for CIR is:

$$m(r;\theta) = E_\theta(r_t | r_{t-1} = r) = \gamma_0 + \gamma_1 r$$

with

$$\gamma_0 = -b\{exp(-a\Delta t) - 1\}$$

and

$$\gamma_1 = exp(-a\Delta t)$$

Conditional LSE for a and b are:

$$\hat{a} = -\frac{1}{\Delta t}[\{n^{-1}\sum_{t=1}^{n}(r_t - \bar{r_n})(r_{t-1} - \bar{r_n}')\}/\{n^{-1}\sum_{t=1}^{n}(r_{t-1} - \bar{r_n}')^2\}]$$

$$\hat{b} = -\frac{\bar{r_n} - exp(-a\Delta t)\bar{r_n}'}{exp(-a\Delta t) - 1}$$

where $\bar{r_n} = n^{-1}\sum_{t=1}^{n} r_t$ and $\bar{r_n}' = n^{-1}\sum_{t=1}^{n} r_{t-1}$.

Calibrating the CIR Model

Initial Estimates

Conditional second moment is:

$$\begin{aligned} v(r;\theta) =& E_\theta[\{r_t - E_\theta(r_t|r_{t-1}=r)\}^2|r_{t-1}=r] = \sigma^2(\eta_0+\eta_1 r) \\ \eta_0 =& \frac{b}{2a}\{exp(-a\Delta t)-1\}^2 \\ \eta_1 =& -\frac{1}{a}exp(-a\Delta t)\{exp(-a\Delta t)-1\} \end{aligned}$$

Estimator for σ:

$$\hat{\sigma}^2 = n^{-1}\sum_{t=1}^{n}\frac{\{r_t - m(r_{t-1};\hat{a},\hat{b})\}^2}{\hat{\eta}_0+\hat{\eta}_1 r_{t-1}}$$

where $\hat{\eta}_0$ and $\hat{\eta}_1$ are evaluated at $(\hat{a},\hat{b})$.

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The starting values for parameter we get using the dataset are $\hat{\theta}_0 = (\hat{a}_0, \hat{b}_0, \hat{\sigma}_0) = (0.3161, 0.0275, 0.0372)$.

Then maximize the log-likelihood function in equation (2) over its parameter space we can get

$$\hat{\theta} = (\hat{a}, \hat{b}, \hat{\sigma}) = arg\ \max_{\theta}\ log\ L(\theta) = (0.2452, 0.0279, 0.0373)$$

Using the estimated parameters $\hat{\theta} = (0.2452, 0.0279, 0.0373)$, we simulate the spot rate of CIR Model, and it's as figure 1 shows:

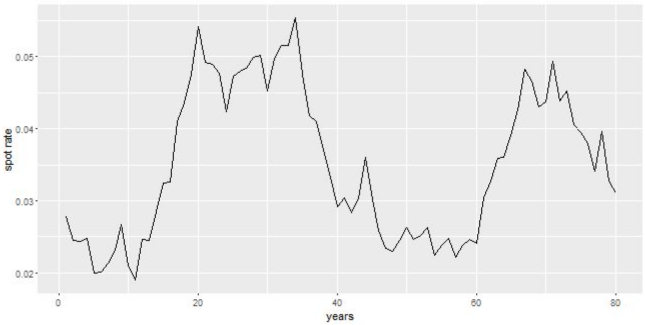

Figura 1: Simulated Spot Rate

Spot Rate Series and Term Structure

Term Structure implied by the estimated parameters

For the spot interest rate is greater than b, the term structure is as figure 2 shows, the yield reduces as the time to maturity increases.

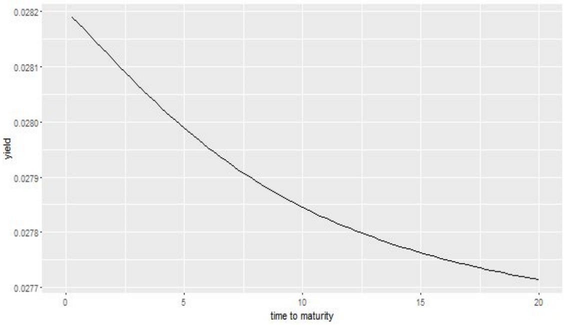

Figura 2: Term Structure with $r(t) > b$

Spot Rate Series and Term Structure

Term Structure implied by the estimated parameters

For the spot interest rate is smaller than Y_{lim}, the term structure is as figure 3 shows, the yield rises as the time to maturity increases.

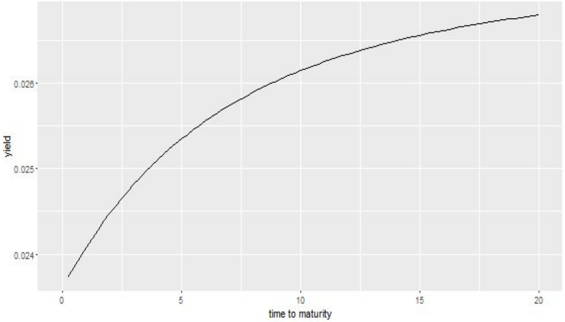

Figura 3: Term Structure with $r(t) < Y_{lim}$

Spot Rate Series and Term Structure

Term Structure implied by the estimated parameters

For the spot interest rate is between b and Y_{lim}, the term structure is as figure 4 shows, the term structure first rises and then falls.

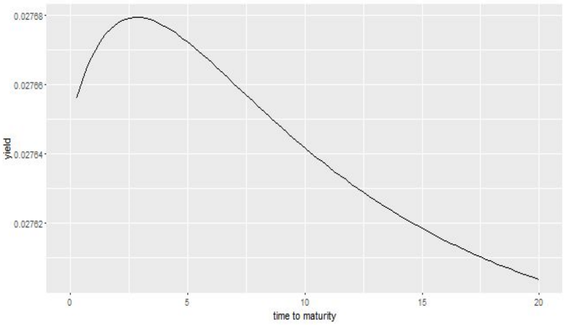

Figura 4: Term Structure with $b > r(t) > Y_{lim}$