Week 4: Hasse diagrams and logical statements

Click on a question number to see how your answers were marked and, where available, full solutions.

Question Number	Score			
1	4	/	4	Review
2	4	/	4	Review
3	4	/	4	Review
4	5	/	5	Review
5	4	/	4	Review
6	4	/	4	Review
Total	**25**	**/**	**25 (100%)**	

Performance Summary

Exam Name:	Week 4: Hasse diagrams and logical statements
Session ID:	1657818450
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Exam Start:	Sat Oct 10 2020 17:42:04
Exam Stop:	Sat Oct 10 2020 18:25:18
Time Spent:	0:43:14

Question 1

Another important relation is called a **partial order**. We say that a relation $\leq$ is a partial order when it has the three characteristic qualities:

- $a \leq a$ always
- if $a \leq b$ and $b \leq a$ then $b = a$
- if $a \leq b$ and $b \leq c$ then $a \leq c$.

Typically, we take a set together with partial ordering such as "less than or equal to" $\leq$, "subset" $\subseteq$ or "divides" $|$. Together, the partial ordering and set are called a **partially ordered set** or **poset**.

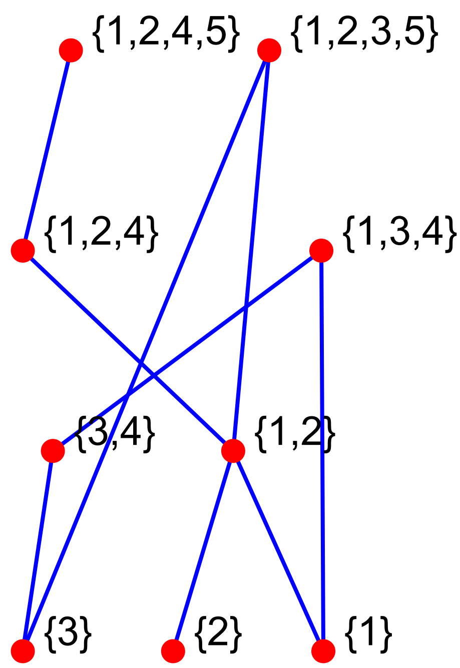
Instead of an arrow diagram, we use a Hasse Diagram to visualise a poset. This communicates the relation between elements of the set with as few edges as possible.

Consider the set

$$A = \{\{1\},\{2\},\{3\},\{1,2\},\{3,4\},\{1,2,4\},\{1,3,4\},\{1,2,4,5\},\{1,2,3,5\}\}$$

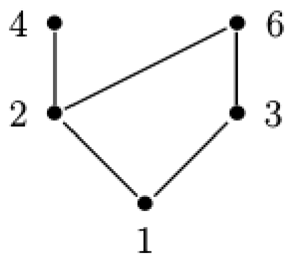
together with the partial ordering $\subseteq$.

Make a Hasse Diagram for this poset. Click on two verticies to make or destroy the edge between them.



- If $A \subsetneq B$ and there is no C such that $A \subsetneq C \subsetneq B$, draw a line between A and B, with A positioned lower than B.
- Do not draw any lines that can be deduced by the transitive property.
- Do not draw any loops to indicate the reflexive property.

For example, the poset $(\{1, 2, 3, 4, 5, 6\}, |)$ is a poset. Its corresponding Hasse diagram is



(Your score will not be affected.)

You have created all of the correct edges. You were awarded **4** marks.

You scored **4** marks for this part.

Score: 4/4

Advice

See the steps section above for a general description for how to design Hasse diagrams.

A good rule of thumb is to work from the bottom up. Join all 1-element sets to those 2-element sets that contain them as a subset. Then do the same for 2-element sets to 3-element sets, and 3-element sets to 4-element sets.

Next check if any of the 1-element sets are contained in any of the 3-element sets that they are not already connected to by a path of edges. If they are, join these vertices. Do the same for 2-element sets contained in 4-element sets.

Finally, check if any of the 1-element sets are contained in any of the 4-element sets that they are not already connected to by a path of edges.

Question 2

In 1742, Christian Goldbach wrote in a letter to Leonhard Euler claiming that

Every even integer greater than 2 can be written as the sum of two primes.

This is now called Goldbach's conjecture, and is the oldest and best-known unsolved problem in number theory. While the proof has remained elusive, the conjecture has been computer verified for numbers up to 4×10^{18}.

Prove the following weakened version of the Goldbach conjecture:

Every even integer between 16 and 23 is the sum of two primes.

Prove that it is possible to write these numbers as a sum of two primes, by actually writing them as sum of two primes:

- $16 =$ 13+3 $13 + 3$ ✔ Expected answer: "11+5" $11+5$
- $18 =$ 11+7 $11 + 7$ ✔ Expected answer: "11+7" $11+7$
- $20 =$ 17+3 $17 + 3$ ✔ Expected answer: "13+7" $13+7$
- $22 =$ 19+3 $19 + 3$ ✔ Expected answer: "5+17" $5+17$.

Gap 0

✔ Good, $16 = 13 + 3$. You were awarded **1** mark.

Gap 1

✔ Good, $18 = 11 + 7$. You were awarded **1** mark.

Gap 2

✔ Good, $20 = 17 + 3$. You were awarded **1** mark.

Gap 3

✔ Good, $22 = 19 + 3$. You were awarded **1** mark.

You scored **4** marks for this part.

Score: 4/4 ✔

Advice

This is a question about proof. You must show that each of the given numbers can be written as the sum of two prime numbers. The best way to do this is to exhibit the numbers as a sum of primes:

- $16 =$ 11+5
- $18 =$ 11+7
- $20 =$ 13+7
- $22 =$ 5+17.

Question 3

In this question we introduce the notion of **existence** (denoted $\exists$) through the familiar concept of odd and even numbers. If $n \in \mathbb{Z}$ then we say

n is **even** when there exists some integer k such that $n = 2k$,

n is **odd** when there exists some integer k such that $n = 2k + 1$.

Prove the following statements.

a)

Prove 6 is even.

By definition 6 is even when there exists some integer k such that $6 = 2k$. It is easy to demonstrate that there is some integer k with this property, because we can show exactly what it is: $k =$ 3 ✔ Expected answer: 3 .

✔ Your answer is correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

b)

Prove 17 is odd.

By definition 17 is odd when there exists some integer k such that $17 = 2k + 1$. It is easy to demonstrate that there is some integer k with this property, because we can show exactly what it is: $k =$ 8 ✔ Expected answer: 8 .

✔ Your answer is correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

c)

There exists some numbers which are even, and some numbers which are odd. But *all* numbers are even or odd. Formally we write this as

$$\forall n \in \mathbb{Z}, n \text{ is even or } n \text{ is odd.}$$

This is a statement about every possible integer. So we can not consider each integer individually and instead we consider an unspecified $n \in \mathbb{Z}$. There are an infinite number of individual existence proofs to do here, but we can reduce the workload to two possibilities by considering n in modulo 2.

If $n =$ 0 ✔ Expected answer: 0 $\pmod 2$ then $\exists k \in \mathbb{Z}$ such that $n = 2k$, hence n is even.

If $n =$ 1 ✔ Expected answer: 1 $\pmod 2$ then $\exists k \in \mathbb{Z}$ such that $n = 1 + 2k$, hence n is odd.

Since these are the only possible values for $n \pmod 2$, every integer n is either even or odd.

Gap 0

✔ Your answer is correct. You were awarded **1** mark.

Gap 1

✔ Your answer is correct. You were awarded **1** mark.

You scored **2** marks for this part.

Score: 2/2 ✔

Advice

The purpose of these questions is to use the definition of even and odd.

- 6 is even because there exists some number ($k = 3$) such that $6 = 2k$. We prove the existence of k by writing it down.
- 17 is odd because there exists some number ($k = 8$) such that $17 = 2k + 1$. Again, we prove the existence of k by writing it down.

These are examples of *constructive* proofs, where we prove existence by constructing number with the required property.

Not all proofs are constructive. For example, to prove that every integer is even or odd we rely upon the properties of arithmetic in modulo 2 to show that there must exist some number k such that $n = 2k$ or $n = 2k + 1$. But, in contrast with the above examples, we do not actually say what that value is.

Question 4

A function $f(x)$ is said to be **unbounded** if

$$\forall M,\ \exists x,\ |f(x)| > M.$$

Consider the function $f(x) = \sqrt{x}$.

a)

Before we tackle the general statement, let's consider some examples.

For $M = 8$ there exists a number x such that $|f(x)| > M$. Prove this by finding such an x.

81 81 ✔ Expected answer: 65 65

✔ $|f(81)| > 8$. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

b)

For $M = 14$, there exists a number x such that $|f(x)| > M$. Prove this by finding such an x.

225 225 ✔ Expected answer: 197 197

You scored **1** mark for this part.

Score: 1/1 ✔

c)

Now prove that $f(x)$ is unbounded, which is to say that for any M we must show that there exists a number x such that $|f(x)| > M$. Select all expressions for x that make $|f(x)| > M$.

- [x] $x = M^2 + 2$
- [x] $x = M^2 + 1$
- [ ] $x = M^2 - 1$
- [ ] $x = M^2$

✔

Expected answer:

- [x] $x = M^2 + 2$
- [x] $x = M^2 + 1$
- [ ] $x = M^2 - 1$
- [ ] $x = M^2$

✔ You chose a correct answer. You were awarded **1** mark.

✔ You chose a correct answer. You were awarded **1** mark.

You scored **2** marks for this part.

Score: 2/2 ✔

d)

A function is **bounded** if it is not unbounded. What logical expression corresponds to being bounded?

- ( ) $\forall M, \exists x, |f(x)| \geq M.$
- ( ) $\exists x, \forall M, |f(x)| \leq M.$
- (•) $\exists M, \forall x, |f(x)| \leq M.$

✔

Expected answer:

- ( ) $\forall M, \exists x, |f(x)| \geq M.$
- ( ) $\exists x, \forall M, |f(x)| \leq M.$
- (•) $\exists M, \forall x, |f(x)| \leq M.$

✔ You chose a correct answer. You were awarded **1** mark.

You scored **1** mark for this part.

You scored 1 mark for this part.

Score: 1/1 ✔

Advice

First we want to find some x such that $|f(x)| > 8$. Since $f(x) = \sqrt{x}$ and $\sqrt{x}$ is never negative for any real x, we are really just solving $\sqrt{x} > 8$. So any x for which $x > 64$ will work.

In the general case, any x for which $x > M^2$ will work. Note that $x = M^2$ will not satisfy the strict inequality.

For the last part, we wish to construct the negation of the unbounded statement. This is done by replacing all $\forall$ symbols with $\exists$ symbols and vice versa, as well as writing the opposite of any mathematical statements.

Question 5

A consequence of the Mean Value Theorem applied to the parabola $f(x) = x^2$ is that for every non-empty interval $[a, b]$ there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

a)

Before we tackle the general statement. Let's consider some examples.

For the interval $[3, 5]$, find the value of c whose existence was foretold by the Mean Value Theorem.

The Mean Value Theorem states that there is some $c \in (3, 5)$ such that

$$f'(c) = \frac{f(5)-f(3)}{5-3}.$$

Since $f'(c) = 2c$ we are able to solve this expression and find c.

(Your score will not be affected.)

Answer: 4 ✔

Expected answer: 4

You revealed the steps.

You revealed the steps.

Your answer is correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1

b)

For the interval $[2, 6]$, find the value of c whose existence was foretold by the Mean Value Theorem.

4

Expected answer: 4

Your answer is correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1

c)

Now prove that for every non-empty interval $[a, b]$ there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

where $f(x) = x^2$.

Suppose $[a, b]$ is some unknown interval with $a < b$.

There exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, because the value of c is

(b+a)/2 $\frac{b+a}{2}$

Expected answer: (a + b)/2 $\frac{a+b}{2}$

Your answer is numerically correct. You were awarded **2** marks.

You scored **2** marks for this part.

Score: 2/2

Advice

Since $f(x) = x^2$, on the interval $[3, 5]$ we have that the right-hand side is $\frac{f(b)-f(a)}{b-a} = \frac{5^2-3^2}{5-3} = 8$.

Since $f'(x) = 2x$, the left-hand side is $f'(c) = 2c$. So solving $2c = 8$ gives $c = 4$.

In general, the right-hand side is $\frac{f(b)-f(a)}{b-a} = \frac{b^2-a^2}{b-a} = \frac{(b-a)(b+a)}{b-a} = b + a$ (assuming $b - a \neq 0$).

So if $f'(c) = 2c = b + a$, we must have $c = \frac{b+a}{2}$. As it is the arithmetic mean of a and b, clearly $c \in (a, b)$ as expected.

Question 6

The statement 'all integers are prime' can be expressed formally as

$$\forall n \in \mathbb{Z},\ n \text{ is prime.}$$

Of course, this is false. We can prove this by contradiction or by proving that the **negation** is true. The negation of 'all integers are prime' is 'some integers are not prime', which can be expressed formally as

$$\exists n \in \mathbb{Z},\ n \text{ is not prime.}$$

a)

Enter a positive integer (less than 100) which is not prime, and threby prove with counterexample that not all integers are prime.

4 4 ✔ Expected answer: 4 4

✔ The number 4 is not prime because it is divisible by 2. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

b)

Which expression is the negation of

$$\forall m \in \mathbb{Q},\ \exists n \in \mathbb{Q}, m + n = m \times n.$$

Negate a statement by negating each part separately. Negation will turn each $\forall$ into an $\exists$ and each $\exists$ into a $\forall$. This expression has three parts, whose negations are:

- $\neg(\forall m \in \mathbb{Q}) \leftrightarrow \exists m \in \mathbb{Q}$,
- $\neg(\exists n \in \mathbb{Q}) \leftrightarrow \forall n \in \mathbb{Q}$,
- $\neg(m + n = m \times n) \leftrightarrow (m + n \neq m \times n)$,

(Your score will not be affected.)

- ○ $\forall m \in \mathbb{Q},\ \exists n \in \mathbb{Q}, m + n \neq m \times n.$
- ◉ $\exists m \in \mathbb{Q},\ \forall n \in \mathbb{Q}, m + n \neq m \times n.$
- ○ $\forall m \in \mathbb{Q},\ \exists n \in \mathbb{Q}, m + n = m \times n.$
- ○ $\exists m \in \mathbb{Q},\ \forall n \in \mathbb{Q}, m + n = m \times n.$

✔

Expected answer:

- ○ $\forall m \in \mathbb{Q},\ \exists n \in \mathbb{Q}, m + n \neq m \times n.$
- ◉ $\exists m \in \mathbb{Q},\ \forall n \in \mathbb{Q}, m + n \neq m \times n.$
- ○ $\forall m \in \mathbb{Q},\ \exists n \in \mathbb{Q}, m + n = m \times n.$
- ○ $\exists m \in \mathbb{Q},\ \forall n \in \mathbb{Q}, m + n = m \times n.$

You revealed the steps.

✔ You chose a correct answer. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

c)

The statement

$$\forall m \in \mathbb{Q}, \exists n \in \mathbb{Q}, m + n = m \times n$$

is false, and we can prove this by contradiction.

Suppose the statement is true. Then it is true for $m = 1$, which means there exists an $n \in \mathbb{Q}$ such that

$$1 + n = 1 \times n.$$

But if we subtract n from both sides this means that the left-hand side becomes

1 ✔ Expected answer: 1 while the right-hand side becomes

0 ✔ Expected answer: 0 , which is the contradiction we were expecting.

Gap 0

✔ Your answer is correct. You were awarded **0.5** marks.

Gap 1

✔ Your answer is correct. You were awarded **0.5** marks.

You scored **1** mark for this part.

Score: 1/1 ✔

d)

Prove that the expression

$$\forall m \in \mathbb{Q} - \{1\}, \exists n \in \mathbb{Q}, m + n = m \times n$$

is true.

Consider some fixed $m \in \mathbb{Q}$ where $m \neq 1$. Then choose

$$n =$$

(-m)/(1-m) $-\dfrac{m}{1-m}$ ✔ Expected answer: m/(m - 1) $\dfrac{m}{m-1}$

This choice of n has the desired property that $m + n = m \times n$, because

$$m + n(1 - m) = 0$$

$$m + n - mn = 0$$

$$m + n = m \times n.$$

When you are doing this yourself it is helpful to start with $m + n = m \times n$ and then deduce n. But when presenting your proof it is important that you preserve the order of the expression you are proving.

✔ Your answer is numerically correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

Advice

For part (a), any composite number between 1 and 99 will suffice. Notice that 1 is also a correct answer, since 1 is not prime by definition.

For part (b), see the steps provided for a detailed explanation. In general, the negation of $\forall$ is $\exists$ and vice-versa. The negation of $A = B$ is simply $A \neq B$.

Notice that in parts (c) and (d), the proof follows the structure of the expression.

In part (d), you can deduce the required value for n by rearranging the expression $m + n = mn$ to make n the subject.