习题 8.3

$$\int \frac{t}{t^2+1}dt = \frac{1}{2}\ln(t^2+1)+C, \quad \int \frac{1}{(t^2+1)^2}dt = \frac{1}{2}\left(\arctan t + \frac{t}{t^2+1}\right)+C$$

$$\Downarrow$$

$$\int \frac{t}{t^2+a^2}dt = \frac{1}{2}\ln(t^2+a^2)+C,$$

$$\int \frac{1}{(t^2+a^2)^2}dt = \frac{1}{a^4}\int \frac{1}{[(\frac{t}{a})^2+1]^2}dt = \frac{1}{a^3}\int \frac{1}{[(\frac{t}{a})^2+1]^2}d\left(\frac{t}{a}\right)$$

$$= \frac{1}{a^3}\cdot\frac{1}{2}\left[\arctan\frac{t}{a} + \frac{\frac{t}{a}}{(\frac{t}{a})^2+1}\right]+C$$

$$= \frac{1}{2a^3}\left(\arctan\frac{t}{a} + \frac{at}{t^2+a^2}\right)+C.$$

Ex1. (1) $\int \frac{x^3}{x-1}dx$ $\qquad x^3-1=(x-1)(x^2+x+1)$

$$= \int \frac{(x^3-1)+1}{x-1}dx = \int \frac{(x-1)(x^2+x+1)+1}{x-1}dx = \int \left(x^2+x+1+\frac{1}{x-1}\right)dx$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+C$$

(2) $\int \frac{x-2}{x^2-7x+12}dx$ $\qquad x^2-7x+12=(x-3)(x-4)$

$$= \int \frac{(x-3)+1}{(x-3)(x-4)}dx = \int \left[\frac{1}{x-4}+\frac{1}{(x-3)(x-4)}\right]dx$$

$$= \int \left(\frac{1}{x-4}+\frac{1}{x-4}-\frac{1}{x-3}\right)dx = \int \left(\frac{2}{x-4}-\frac{1}{x-3}\right)dx$$

$$= 2\ln|x-4| - \ln|x-3| + C$$

$$= \ln\frac{(x-4)^2}{|x-3|}+C.$$

(3) $\int \frac{dx}{1+x^3}$ $\qquad x^3+1=(x+1)(x^2-x+1)$

$$= \int\left(\frac{1}{3}\cdot\frac{1}{x+1}+\frac{1}{3}\cdot\frac{-x+2}{x^2-x+1}\right)dx \qquad \frac{1}{1+x^3}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

$$A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3}.$$

$$= \frac{1}{3}\int\frac{1}{x+1}dx - \frac{1}{3}\int\frac{x-2}{x^2-x+1}dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{(x-\frac{1}{2})-\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx$$

令 $t=x-\frac{1}{2}$，则

$$\int\frac{(x-\frac{1}{2})-\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx = \int\frac{t-\frac{3}{2}}{t^2+\frac{3}{4}}dt = \int\frac{t}{t^2+\frac{3}{4}}dt - \frac{3}{2}\int\frac{1}{t^2+(\frac{\sqrt{3}}{2})^2}dt$$

$$= \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\cdot\arctan\frac{2t}{\sqrt{3}} + C$$

$$= \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) - \sqrt{3}\arctan\frac{2t}{\sqrt{3}} + C$$

$$= \frac{1}{2}\ln(x^2-x+1) - \sqrt{3}\arctan\frac{2x-1}{\sqrt{3}} + C$$

综上. $\int \frac{1}{1+x^3}dx = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} + C.$

(4) $\int \frac{dx}{1+x^4}$

$x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2$

$= (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

$$= \int \frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}dx$$

$\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$

$A = \frac{1}{2\sqrt{2}},\ B = \frac{1}{2},\ C = -\frac{1}{2\sqrt{2}},\ D = \frac{1}{2}$

$$= \frac{1}{2\sqrt{2}}\int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx - \frac{1}{2\sqrt{2}}\int\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx$$

$$= \frac{1}{2\sqrt{2}}\int \frac{\left(x+\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx - \frac{1}{2\sqrt{2}}\int \frac{\left(x-\frac{\sqrt{2}}{2}\right)-\frac{\sqrt{2}}{2}}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx$$

$$= \frac{1}{2\sqrt{2}}\left[\int \frac{x+\frac{\sqrt{2}}{2}}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx + \frac{\sqrt{2}}{2}\int\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx\right]$$

$\frac{1}{2} = \left(\frac{\sqrt{2}}{2}\right)^2$

$$- \frac{1}{2\sqrt{2}}\left[\int \frac{x-\frac{\sqrt{2}}{2}}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx - \frac{\sqrt{2}}{2}\int\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx\right]$$

$$= \frac{1}{2\sqrt{2}}\left[\frac{1}{2}\ln(x^2+\sqrt{2}x+1) + \underbrace{\left(\frac{\sqrt{2}}{2}\cdot\frac{2}{\sqrt{2}}\right)}_{=1}\arctan\frac{2}{\sqrt{2}}\left(x+\frac{\sqrt{2}}{2}\right)\right]$$

$$- \frac{1}{2\sqrt{2}}\left[\frac{1}{2}\ln(x^2-\sqrt{2}x+1) - \frac{\sqrt{2}}{2}\cdot\frac{2}{\sqrt{2}}\arctan\frac{2}{\sqrt{2}}\left(x-\frac{\sqrt{2}}{2}\right)\right] + C$$

$$= \frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\left[\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)\right] + C$$

(5) $\int \frac{dx}{(x-1)(x^2+1)^2}$

$\frac{1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$A = \frac{1}{4},\ B = C = -\frac{1}{4},\ D = E = -\frac{1}{2}$

$$= \int\left(\frac{1}{4}\cdot\frac{1}{x-1} + \frac{1}{4}\cdot\frac{-x-1}{x^2+1} + \frac{1}{2}\cdot\frac{-x-1}{(x^2+1)^2}\right)dx$$

$$= \underbrace{\frac{1}{4}\int\frac{1}{x-1}dx}_{①} - \underbrace{\frac{1}{4}\int\frac{x+1}{x^2+1}dx}_{②} - \underbrace{\frac{1}{2}\int\frac{x+1}{(x^2+1)^2}dx}_{③}$$

① $= \ln|x-1| + C.$

② $= \int\frac{x}{x^2+1}dx + \int\frac{1}{x^2+1}dx = \frac{1}{2}\ln(x^2+1) + \arctan x + C.$

③ $= \int\frac{x}{(x^2+1)^2}dx + \int\frac{1}{(x^2+1)^2}dx = \frac{1}{2}\int\frac{1}{(x^2+1)^2}d(x^2) + \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)$

$$= -\frac{1}{2(x^2+1)} + \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C.$$

综上, $\int \frac{dx}{(x-1)(x^2+1)^2}$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\left[\frac{1}{2}\ln(x^2+1) + \arctan x\right]$$

$$-\frac{1}{2}\left[-\underline{\frac{1}{2(x^2+1)}} + \frac{1}{2}\arctan x + \underline{\frac{x}{2(x^2+1)}}\right]$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{8}\ln(x^2+1) - \frac{1}{2}\arctan x - \frac{x-1}{4(x^2+1)} + C$$

(6) $\int \frac{x-2}{(2x^2+2x+1)^2}dx$

$$= \frac{1}{4}\int \frac{x-2}{(x^2+x+\frac{1}{2})^2}dx \qquad x^2+x+\frac{1}{2} = (x+\frac{1}{2})^2+\frac{1}{4}.$$

$$= \frac{1}{4}\int \frac{(x+\frac{1}{2})-\frac{5}{2}}{[(x+\frac{1}{2})^2+\frac{1}{4}]^2}dx$$

$$= \frac{1}{4}\int \frac{t-\frac{5}{2}}{(t^2+\frac{1}{4})^2}dt \qquad (t = x+\frac{1}{2})$$

$$= \frac{1}{4}\int \frac{t}{(t^2+\frac{1}{4})^2}dt - \frac{5}{8}\int \frac{1}{(t^2+\frac{1}{4})^2}dt$$

$$= \frac{1}{4}\cdot\frac{1}{2}\cdot\left(-\frac{1}{t^2+\frac{1}{4}}\right) - \frac{5}{8}\cdot 16\cdot\int \frac{1}{[(2t)^2+1]^2}dt$$

$$= -\frac{1}{8}\cdot\frac{1}{t^2+\frac{1}{4}} - 5\int \frac{1}{[(2t)^2+1]^2}d(2t)$$

$$= -\frac{1}{8}\cdot\frac{1}{t^2+\frac{1}{4}} - 5\cdot\frac{1}{2}\left[\arctan(2t) + \frac{2t}{1+4t^2}\right] + C$$

$$= -\frac{1}{8t^2+2} - \frac{5}{2}\arctan(2t) - \frac{10t}{8t^2+2} + C$$

$$= -\frac{10t+1}{8t^2+2} - \frac{5}{2}\arctan(2t) + C$$

$$= -\frac{10(x+\frac{1}{2})+1}{8(x+\frac{1}{2})^2+2} - \frac{5}{2}\arctan(2x+1) + C$$

$$= -\frac{5x+1}{4x^2+4x+2} - \frac{5}{2}\arctan(2x+1) + C$$

Ex2. (1) $\int \frac{1}{5-3\cos x}dx \qquad t = \tan\frac{x}{2},\quad \cos x = \frac{1-t^2}{1+t^2},\quad dx = \frac{2}{1+t^2}dt$

$$= \int \frac{1}{5-3\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt \qquad (t = \tan\frac{x}{2})$$

$$= \int \frac{2}{4t^2+1}dt = \frac{1}{2}\arctan(2t) + C$$

$$= \frac{1}{2}\arctan(2\tan\frac{x}{2}) + C.$$

(2) $\int \frac{1}{2+\sin^2 x}dx$ $\qquad t=\tan x=\frac{\sin x}{\cos x}$, $\quad t^2=\frac{\sin^2 x}{\cos^2 x}=\frac{\sin^2 x}{1-\sin^2 x}$.

$\sin^2 x=\frac{t^2}{t^2+1}$, $\quad dx=\frac{1}{t^2+1}dt$

$$=\int \frac{1}{2+\frac{t^2}{t^2+1}}\cdot\frac{1}{t^2+1}dt \qquad (t=\tan x)$$

$$=\int \frac{1}{3t^2+2}dt$$

$$=\frac{1}{3}\cdot\sqrt{\frac{3}{2}}\arctan\left(\sqrt{\frac{3}{2}}t\right)+C$$

$$=\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{3}{2}}t\right)+C$$

(3) $\int \frac{1}{1+\tan x}dx$

$$=\int \frac{1}{1+t}\cdot\frac{1}{1+t^2}dt \qquad (t=\tan x,\ x=\arctan t)$$

$$=\frac{1}{2}\int\left(\frac{1}{1+t}-\frac{t-1}{t^2+1}\right)dt$$

$$=\frac{1}{2}\int\frac{1}{t+1}dt-\frac{1}{2}\int\frac{t}{t^2+1}dt+\frac{1}{2}\int\frac{1}{t^2+1}dt$$

$$=\frac{1}{2}\ln|t+1|-\frac{1}{4}\ln(t^2+1)+\frac{1}{2}\arctan t+C$$

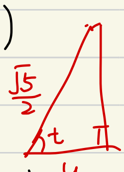
$$=\underline{\frac{1}{2}\ln|\tan x+1|}-\underline{\frac{1}{4}\ln(\tan^2 x+1)}+\frac{1}{2}x+C$$

$$=\frac{1}{2}\ln|\sin x+\cos x|+\frac{1}{2}x+C$$

(4) $\int \frac{x^2}{\sqrt{1+x-x^2}}dx$

方法1. 直角三角形换元. $1+x-x^2=\frac{5}{4}-(x-\frac{1}{2})^2$.

$$\int\frac{x^2}{\sqrt{1+x-x^2}}dx=\int\frac{(u+\frac{1}{2})^2}{\sqrt{\frac{5}{4}-u^2}}du \qquad (u=x-\frac{1}{2})$$



$u=\frac{\sqrt{5}}{2}\cos t$

$\sqrt{\frac{5}{4}-u^2}=\frac{\sqrt{5}}{2}\sin t$

$du=-\frac{\sqrt{5}}{2}\sin t\,dt$.

$$=\int\frac{(\frac{\sqrt{5}}{2}\cos t+\frac{1}{2})^2}{\frac{\sqrt{5}}{2}\sin t}\cdot\left(-\frac{\sqrt{5}}{2}\right)\sin t\,dt \qquad (u=\frac{\sqrt{5}}{2}\cos t)$$

$$=-\int\left(\frac{\sqrt{5}}{2}\cos t+\frac{1}{2}\right)^2dt$$

$$=-\int\left(\frac{5}{4}\cos^2 t+\frac{\sqrt{5}}{2}\cos t+\frac{1}{4}\right)dt$$

$$=-\frac{5}{4}\cdot\frac{1}{2}\int(\cos 2t+1)dt-\frac{\sqrt{5}}{2}\int\cos t\,dt-\frac{1}{4}t$$

$$=-\frac{5}{16}\sin 2t-\frac{5}{8}t-\frac{\sqrt{5}}{2}\sin t-\frac{1}{4}t+C$$

$$=-\frac{5}{8}\sin t\cos t-\frac{7}{8}t-\frac{\sqrt{5}}{2}\sin t+C$$

$\sin t=\frac{2}{\sqrt{5}}\sqrt{\frac{5}{4}-u^2}$

$\cos t=\frac{2}{\sqrt{5}}u$

$t=\arccos(\frac{2}{\sqrt{5}}u)$

$$= -\frac{5}{8}\cdot\frac{2}{\sqrt5}\sqrt{\frac54-u^2}\cdot\frac{2}{\sqrt5}u - \frac78\arccos\left(\frac{2}{\sqrt5}u\right) - \frac{\sqrt5}{2}\cdot\frac{2}{\sqrt5}\sqrt{\frac54-u^2}+C$$

$$= -\frac12 u\sqrt{\frac54-u^2} - \frac78\arccos\left(\frac{2}{\sqrt5}u\right) - \sqrt{\frac54-u^2}+C \qquad u=x-\frac12$$

$$= -\frac12\left(x-\frac12\right)\sqrt{1+x-x^2} - \frac78\arccos\frac{2x-1}{\sqrt5} - \sqrt{1+x-x^2}+C$$

$$= -\frac{2x+3}{4}\sqrt{1+x-x^2} - \frac78\arccos\frac{2x-1}{\sqrt5}+C.$$

方法2. Euler 变换. 令 $\sqrt{1+x-x^2}=xt+1$, 则 $1+x-x^2=x^2t^2+2xt+1$,

$$x=\frac{1-2t}{t^2+1},\quad \sqrt{1+x-x^2}=\frac{1-2t}{t^2+1}t+1=\frac{1+t-t^2}{t^2+1}.$$

$$dx=\frac{-2(t^2+1)-2t(1-2t)}{(t^2+1)^2}dt=\frac{2(t^2-t-1)}{(t^2+1)^2}dt$$

$$\int\frac{x^2}{\sqrt{1+x-x^2}}dx=\int\frac{(1-2t)^2}{(t^2+1)^2}\cdot\frac{t^2+1}{1+t-t^2}\cdot\frac{2(t^2-t-1)}{(t^2+1)^2}dt$$

$$=2\int\frac{(1-2t)^2}{(t^2+1)^3}dt$$

$$=2\int\frac{4t^2-4t+1}{(t^2+1)^3}dt$$

$$=2\int\frac{4(t^2+1)-4t-3}{(t^2+1)^3}dt$$

$$=2\int\left[\frac{4}{(t^2+1)^2}-\frac{4t+3}{(t^2+1)^3}\right]dt$$

$$\int\frac{1}{(t^2+1)^2}dt=\frac12\left(\arctan t+\frac{t}{t^2+1}\right)+C$$

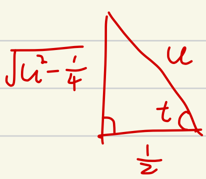
$$\int\frac{t}{(t^2+1)^3}dt=\frac12\int\frac{1}{(t^2+1)^3}d(t^2)=-\frac14\cdot\frac{1}{(t^2+1)^2}+C.$$

$$\int\frac{1}{(t^2+1)^3}dt$$

(5) $\displaystyle\int\frac{1}{\sqrt{x^2+x}}dx$

方法1. $x^2+x=\left(x^2+x+\frac14\right)-\frac14=\left(x+\frac12\right)^2-\frac14$. 令 $u=x+\frac12$, 则

$$\int\frac{1}{\sqrt{x^2+x}}dx=\int\frac{1}{\sqrt{u^2-\frac14}}du$$



$u=\frac12\sec t$

$\sqrt{u^2-\frac14}=\frac12\tan t$

$du=\frac12\sec t\tan t\,dt$

$$=\int\frac{1}{\frac12\tan t}\cdot\frac12\sec t\tan t\,dt$$

$$=\int\sec t\,dt$$

$$=\ln|\sec t+\tan t|+C=\ln\left|2u+2\sqrt{u^2-\frac14}\right|+C$$

$$=\ln\left|2x+1+2\sqrt{x^2+x}\right|+C$$

方法2. Euler变换　令 $\sqrt{x^2+x} = t - x$，则 $x^2+x = x^2-2tx+t^2$，

$$x = \frac{t^2}{2t+1}, \quad \sqrt{x^2+x} = t - \frac{t^2}{2t+1} = \frac{t^2+t}{2t+1}$$

$$dx = \frac{2t(2t+1)-2t^2}{(2t+1)^2}dt = \frac{2(t^2+t)}{(2t+1)^2}dt.$$

$$\int \frac{1}{\sqrt{x^2+x}}dx = \int \frac{2t+1}{t^2+t}\cdot\frac{2(t^2+t)}{(2t+1)^2}dt$$

$$= 2\int \frac{1}{2t+1}dt$$

$$= \ln|2t+1| + C \qquad t = x+\sqrt{x^2+x}$$

$$= \ln|2x+1+2\sqrt{x^2+x}| + C$$

(6) $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx$

令 $t = \sqrt{\frac{1-x}{1+x}}$，则 $t^2 = \frac{1-x}{1+x}$，$x = \frac{1-t^2}{t^2+1}$.

$$dx = \frac{-2t(t^2+1)-2t(1-t^2)}{(t^2+1)^2}dt = \frac{-4t}{(t^2+1)^2}dt$$

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx = \int \frac{(t^2+1)^2}{(1-t^2)^2}\cdot t\cdot\frac{-4t}{(t^2+1)^2}dt$$

$$= 4\int \frac{-t^2}{(t^2-1)^2}dt$$

$$= 4\int \frac{(1-t^2)-1}{(t^2-1)^2}dt$$

$$= 4\Big(\underbrace{\int \frac{1}{t^2-1}dt}_{①} - \underbrace{\int \frac{1}{(t^2-1)^2}dt}_{②}\Big)$$

$$① = \int \frac{1}{t^2-1}dt = \frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t-1}\right)dt = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C.$$

$$② = \int \frac{1}{(t^2-1)^2}dt = \int \frac{1}{(t-1)^2(t+1)^2}dt \qquad \frac{1}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

$$= \frac{1}{4}\int\left[-\frac{1}{t-1} + \frac{1}{(t-1)^2} + \frac{1}{t+1} + \frac{1}{(t+1)^2}\right]dt \qquad A = -\frac{1}{4},\ B = C = D = \frac{1}{4}.$$

$$= \frac{1}{4}\left[-\ln|t-1| - \frac{1}{t-1} + \ln|t+1| - \frac{1}{t+1}\right] + C$$

$$= \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - \frac{t}{2(t^2-1)} + C$$

所以 $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx = 4\left[\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| - \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| + \frac{t}{2(t^2-1)}\right] + C$

$$= \ln\left|\frac{t+1}{t-1}\right| + \frac{2t}{t^2-1} + C \qquad t=\sqrt{\frac{1-x}{1+x}}$$

$$= \ln\left|\frac{\sqrt{\frac{1-x}{1+x}}+1}{\sqrt{\frac{1-x}{1+x}}-1}\right| + \frac{2\cdot\sqrt{\frac{1-x}{1+x}}}{\frac{1-x}{1+x}-1} + C$$

$$= \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| - \frac{\sqrt{1-x^2}}{x} + C.$$

第8章 总练习题

Ex1. (1) $\int \frac{\sqrt{x}-2\sqrt[3]{x}-1}{\sqrt[4]{x}}\,dx$

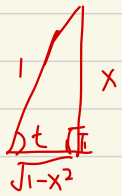
$= \int \left(x^{\frac{1}{4}} - 2\cdot x^{\frac{1}{12}} - x^{-\frac{1}{4}}\right) dx$

$= \frac{4}{5}x^{\frac{5}{4}} - \frac{24}{13}x^{\frac{13}{12}} - \frac{4}{3}x^{\frac{3}{4}} + C$

(2) $\int x \arcsin x \, dx$

$= \frac{1}{2}\int \arcsin x \, d(x^2) = \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int x^2 \, d(\arcsin x)$

$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$



$x = \sin t$

$\sqrt{1-x^2} = \cos t$

$dx = \cos t \, dt$

$\int \frac{x^2}{\sqrt{1-x^2}}dx$

$= \int \frac{\sin^2 t}{\cos t}\cdot \cos t\, dt = \int \sin^2 t \, dt = \frac{1}{2}\int (1-\cos 2t)\,dt$

$= \frac{1}{2}t - \frac{1}{4}\sin 2t + C = \frac{1}{2}t - \frac{1}{2}\sin t \cos t + C$

$= \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C.$

综上, $\int x \arcsin x \, dx = \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C$

$= \frac{2x^2-1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C.$

(3) $\int \frac{1}{1+\sqrt{x}}dx$

$= \int \frac{1}{1+t}\cdot 2t\,dt \qquad (t=\sqrt{x},\ x=t^2,\ dx = 2t\,dt)$

$= 2\int \frac{t}{t+1}dt = 2\int \frac{(t+1)-1}{t+1}dt = 2\int \left(1-\frac{1}{t+1}\right)dt$

$= 2t - 2\ln|t+1| + C$

$= 2\sqrt{x} - 2\ln|\sqrt{x}+1| + C$

$= 2\sqrt{x} - \ln(\sqrt{x}+1)^2 + C.$

(4) $\int e^{\sin x}\sin 2x \, dx$

$= 2\int e^{\sin x}\sin \underline{\cos x\, dx}$

$= 2\int e^{\sin x}\sin x \, d(\sin x)$

$$= 2\int t e^t \, dt \qquad (t = \sin x)$$

$$= 2\int t \, de^t = 2\left(te^t - \int e^t dt\right)$$

$$= 2te^t - 2e^t + C = 2(t-1)e^t + C$$

$$= 2(\sin x - 1)e^{\sin x} + C.$$

(5) $\int e^{\sqrt{x}} \, dx$

$$= \int e^t \cdot 2t \, dt \qquad (t = \sqrt{x},\ x = t^2,\ dx = 2t\, dt)$$

$$= 2(t-1)e^t + C$$

$$= 2(\sqrt{x}-1)e^{\sqrt{x}} + C$$

(6) $\int \frac{dx}{x\sqrt{x^2-1}}$ (§8.2. 例10. $\int \frac{1}{x^2\sqrt{x^2-1}} dx$)

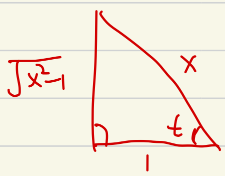
方法1. $\frac{1}{x\sqrt{x^2-1}}$ 存在域为 $(-\infty,-1)\cup(1,+\infty)$.

当 $x \in (1,+\infty)$,

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \int \frac{1}{x^2} \cdot \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} dx = -\int \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} d\left(\frac{1}{x}\right)$$

$$= \underbrace{-\arcsin\frac{1}{x}}_{\text{奇函数}} + C.$$

方法2. $\int \frac{1}{x\sqrt{x^2-1}} dx$



$x = \sec t$, $t = \arccos\frac{1}{x}$

$\sqrt{x^2-1} = \tan t$

$dx = \sec t \tan t \, dt$

$$= \int \cos t \cdot \cot t \cdot \sec t \tan t \, dt$$

$$= \int 1 \, dt = t + C = \arccos\frac{1}{x} + C$$

$$\left(\arcsin\frac{1}{x} + \arccos\frac{1}{x} = \frac{\pi}{2} \right)$$

(7) $\int \frac{1-\tan x}{1+\tan x} dx$

方法1. 令 $t = \tan x$, 则 $x = \arctan t$, $dx = \frac{1}{t^2+1} dt$.

$$\int \frac{1-\tan x}{1+\tan x} dx = \int \frac{1-t}{1+t} \cdot \frac{1}{t^2+1} dt = \int \left(\frac{1}{t+1} - \frac{t}{t^2+1}\right) dt$$

$$= \ln|t+1| - \frac{1}{2}\ln(t^2+1) + C$$

$$= \ln\left|\frac{t+1}{\sqrt{t^2+1}}\right| + C = \ln\left|\frac{\tan x + 1}{\sqrt{\tan^2 x + 1}}\right| + C = \ln\left|\frac{\frac{\sin x}{\cos x} + 1}{\sqrt{\frac{\sin^2 x}{\cos^2 x} + 1}}\right| + C$$

$$= \ln\left|\frac{\sin x+\cos x}{1}\right| + C = \ln|\sin x+\cos x| + C.$$

方法2. $\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{\cos x-\sin x}{\cos x+\sin x}dx = \int \frac{1}{\cos x+\sin x}d(\cos x+\sin x)$

$$= \ln|\sin x+\cos x| + C$$

方法3. $\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}dx$

$\int \tan x\,dx = -\ln|\cos x| + C$

$$= \int \tan\left(\frac{\pi}{4}-x\right)dx$$

$$= \ln\left|\cos\left(\frac{\pi}{4}-x\right)\right| + C$$

$$= \ln\left|\frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x\right| + C$$

$$= \ln|\sin x+\cos x| + C.$$

(8) $\int \frac{x^2-x}{(x-2)^3}dx$

$$= \int \frac{(x-2)^2+4x-4-x}{(x-2)^3}dx = \int \frac{(x-2)^2+3(x-2)+2}{(x-2)^3}dx$$

$$= \int \left[\frac{1}{x-2}+\frac{3}{(x-2)^2}+\frac{2}{(x-2)^3}\right]dx$$

$$= \ln|x-2| - \frac{3}{x-2} - \frac{1}{(x-2)^2} + C.$$

(9) $\int \frac{1}{\cos^4 x}dx$

$$= \int \sec^4 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx = \int (1+\tan^2 x)\,d(\tan x)$$

$$= \tan x + \frac{1}{3}\tan^3 x + C$$

(10) $\int \sin^4 x\,dx$

$\cos^{2k} x$, $\sin^{2k} x \longrightarrow \cos(2nx)$

$$= \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right)dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{4}\cdot\frac{1}{2}\int(\cos 4x+1)\,dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\left(\frac{1}{4}\sin 4x + x\right) + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$$

(11) $\displaystyle\int \frac{x-5}{x^3-3x^2+4}\,dx$

$x^3-3x^2+4 = (x^3-2x^2)-(x^2-4)$

$= x^2(x-2)-(x-2)(x+2)$

$= (x-2)(x^2-x-2)$

$= (x-2)(x-2)(x-1)$

$= (x-2)^2(x-1)$

$\dfrac{x-5}{x^3-3x^2+4} = \dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

$A=-\frac{2}{3}$, $B=\frac{2}{3}$, $C=-1$

$= -\frac{2}{3}\displaystyle\int\frac{1}{x-1}\,dx+\frac{2}{3}\int\frac{1}{x-2}\,dx$

$-\displaystyle\int\frac{1}{(x-2)^2}\,dx$

$= -\frac{2}{3}\ln|x-1|+\frac{2}{3}\ln|x-2|$

$+\dfrac{1}{x-2}+C$

$= \frac{2}{3}\ln\left|\dfrac{x-2}{x-1}\right|+\dfrac{1}{x-2}+C.$