习题 11.3

Ex4. 计算 $(n\in\mathbb{N}_+)$

(1) $\int_0^1 (\ln x)^n dx$ (2) $\int_0^1 \frac{x^n}{\sqrt{1-x}} dx$.

解：(1) 瑕点 $x=0$.

$$\left(x^p\cdot(\ln x)^n = \frac{(\ln x)^n}{x^{-p}},\quad \frac{[(\ln x)^n]'}{(x^{-p})'} = \frac{n\cdot(\ln x)^{n-1}\cdot\frac{1}{x}}{-p\cdot x^{-p-1}} = -\frac{n}{p}\cdot\frac{(\ln x)^{n-1}}{x^{-p}} \quad \cdots\quad (-1)^n\cdot\frac{n!}{p^n}\cdot x^p, \right)$$

令 $f(x) = (-\ln x)^n = (-1)^n(\ln x)^n$, $x\in(0,1]$, 则 $f(x)\geqslant 0$, 由于

$$\lim_{x\to 0^+} x^{\frac{1}{2}}\cdot f(x) = \lim_{x\to 0^+}\frac{f(x)}{x^{-\frac{1}{2}}} = \lim_{x\to 0^+}\frac{n!\cdot x^{\frac{1}{2}}}{(\frac{1}{2})^n} = 0,$$

由 Cauchy 判别法，瑕积分 $\int_0^1(\ln x)^n dx$ 收敛.

记 $I_n = \int_0^1(\ln x)^n dx$, 则

$$I_1 = \int_0^1 \ln x\, dx = \lim_{u\to 0^+}\int_u^1 \ln x\, dx = \lim_{u\to 0^+}(x\ln x - x)\Big|_u^1 = \lim_{u\to 0^+}[-1-(u\ln u - u)] = -1.$$

当 $n\geqslant 2$ 时,

$$\begin{aligned}
I_n = \int_0^1(\ln x)^n dx &= \lim_{u\to 0^+}\int_u^1(\ln x)^n dx \\
&= \lim_{u\to 0^+}\left[x(\ln x)^n\Big|_u^1 - n\int_u^1(\ln x)^{n-1}dx\right] \\
&= (0-0) - n\cdot\lim_{u\to 0^+}\int_u^1(\ln x)^{n-1}dx \\
&= -n\int_0^1(\ln x)^{n-1}dx = -nI_{n-1},
\end{aligned}$$

所以，$I_n = (-1)^n n!$

(2) 瑕点为 $x=1$. 由于 $\frac{x^n}{\sqrt{1-x}}\geq 0$, $\forall x\in[0,1)$, 并且

$$\lim_{x\to 1^-}(1-x)^{\frac{1}{2}}\cdot\frac{x^n}{\sqrt{1-x}} = \lim_{x\to 1^-}x^n = 1,$$ 由 Cauchy 判别法，瑕积分 $\int_0^1\frac{x^n}{\sqrt{1-x}}dx$ 收敛.

令 $t=\sqrt{1-x}$，$x\in[0,1)$，则 $x=1-t^2$，$t\in(0,1]$，对 $\forall u\in[0,1)$，有

$$\int_0^u \frac{x^n}{\sqrt{1-x}}dx \xlongequal{t=\sqrt{1-x}} \int_1^{\sqrt{1-u}} \frac{(1-t^2)^n}{t}\cdot(-2t)dt$$

$$= 2\int_{\sqrt{1-u}}^1 (1-t^2)^n dt$$

令 $t=\cos\theta$，$\theta\in[0,\arccos\sqrt{1-u}]$，则 $1-t^2=\sin^2\theta$，$dt=-\sin\theta d\theta$，

$$\int_{\sqrt{1-u}}^1 (1-t^2)^n dt = -\int_{\arccos\sqrt{1-u}}^0 \sin^{2n}\theta\cdot\sin\theta d\theta$$

$$= \int_0^{\arccos\sqrt{1-u}} \sin^{2n+1}\theta\, d\theta,$$

所以，$$\int_0^1 \frac{x^n}{\sqrt{1-x}}dx = \lim_{u\to1^-}\int_0^u \frac{x^n}{\sqrt{1-x}}dx$$

$$= \lim_{u\to1^-} 2\int_0^{\arccos\sqrt{1-u}} \sin^{2n+1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\, d\theta$$

$$= 2\cdot\frac{(2n)!!}{(2n+1)!!}$$

Ex5. 瑕积分 $J=\int_0^{\frac{\pi}{2}}\ln(\sin x)dx$ 收敛，并且 $J=-\frac{\pi}{2}\ln 2$.

证：$x=0$ 是瑕点。令 $f(x)=-\ln(\sin x)$，$x\in(0,\frac{\pi}{2}]$，则 $f(x)\geq 0$.

对 $\forall p>0$，有

$$\lim_{x\to0^+} x^p\cdot f(x) = \lim_{x\to0^+}\frac{-\ln(\sin x)}{x^{-p}} = \lim_{x\to0^+}\frac{\frac{-1}{\sin x}\cdot\cos x}{-p\cdot x^{-p-1}}$$

$$= \lim_{x\to0^+}\frac{1}{p}\cdot x^p\cdot\frac{x}{\sin x}\cdot\cos x = 0,$$

由 Cauchy 判别法，瑕积分 $\int_0^{\frac{\pi}{2}}\ln(\sin x)dx$ 收敛.

（$\int\ln(\sin x)dx$ 不能用初等函数表示）

令 $t=\frac{\pi}{2}-x$，则对 $\forall u\in(0,\frac{\pi}{2}]$，有

$$\int_u^{\frac{\pi}{2}}\ln(\sin x)dx = -\int_{\frac{\pi}{2}-u}^0 \ln[\sin(\tfrac{\pi}{2}-t)]dt = \int_0^{\frac{\pi}{2}-u}\ln(\cos t)dt = \int_0^{\frac{\pi}{2}-u}\ln(\cos x)dx.$$

由于 $\int_0^{\frac{\pi}{2}} \ln(\sin x)dx$ 收敛，则瑕积分 $\int_0^{\frac{\pi}{2}} \ln(\cos x)dx$ 也收敛，并且

$$J=\int_0^{\frac{\pi}{2}} \ln(\sin x)dx=\int_0^{\frac{\pi}{2}} \ln(\cos x)dx.$$

所以
$$\begin{aligned}2J&=\int_0^{\frac{\pi}{2}} \ln(\sin x)dx+\int_0^{\frac{\pi}{2}} \ln(\cos x)dx\\&=\int_0^{\frac{\pi}{2}}\left[\ln(\sin x)+\ln(\cos x)\right]dx\\&=\int_0^{\frac{\pi}{2}}\ln(\sin x\cos x)dx\\&=\int_0^{\frac{\pi}{2}}\ln\frac{\sin 2x}{2}dx\\&=\int_0^{\frac{\pi}{2}}\ln(\sin 2x)dx-\int_0^{\frac{\pi}{2}}\ln 2\,dx\end{aligned}$$

$=\frac{\pi}{2}\ln 2$

从而 $\int_0^{\frac{\pi}{2}}\ln(\sin 2x)dx$ 收敛于 $2J+\frac{\pi}{2}\ln 2$.

对 $\forall u\in(0,\frac{\pi}{4}]$，有

$$\int_u^{\frac{\pi}{4}}\ln(\sin 2x)dx\ \overset{t=2x}{=\!=}\ \frac{1}{2}\int_{2u}^{\frac{\pi}{2}}\ln(\sin t)dt,$$

所以 $\int_0^{\frac{\pi}{4}}\ln(\sin 2x)dx=\lim\limits_{u\to 0^+}\int_u^{\frac{\pi}{4}}\ln(\sin 2x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin t)dt=\frac{1}{2}J.$

对 $\forall v\in[\frac{\pi}{4},\frac{\pi}{2})$，有

$$\begin{aligned}\int_{\frac{\pi}{4}}^{v}\ln(\sin 2x)dx\ &\overset{t=2x-\frac{\pi}{2}}{=\!=}\ \frac{1}{2}\int_0^{2v-\frac{\pi}{2}}\ln\left[\sin\left(t+\frac{\pi}{2}\right)\right]dt\\&=\frac{1}{2}\int_0^{2v-\frac{\pi}{2}}\ln(\cos t)dt\end{aligned}$$

所以 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln(\sin 2x)dx=\lim\limits_{v\to(\frac{\pi}{2})^-}\int_{\frac{\pi}{4}}^{v}\ln(\sin 2x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\cos t)dt=\frac{1}{2}J.$

综上，$\int_0^{\frac{\pi}{2}}\ln(\sin 2x)dx=J$.

所以 $J=-\frac{\pi}{2}\ln 2$.

Ex6. 证明

(1) $\int_0^{\pi}\theta\ln(\sin\theta)d\theta=-\frac{\pi^2}{2}\ln 2$;　　(2) $\int_0^{\pi}\frac{\theta\sin\theta}{1-\cos\theta}d\theta=2\pi\ln 2$.

证：(1) 由于 $\lim\limits_{\theta\to 0^+}\theta\ln(\sin\theta)=0$，$\lim\limits_{\theta\to\pi^-}\theta\ln(\sin\theta)=-\infty$，所以

只有 $x=\pi$ 是瑕点.

由于对 $\forall p>0$，

$$\lim_{\theta\to\pi^-}(\pi-\theta)^p\cdot[-\theta\ln(\sin\theta)]=\pi\cdot\lim_{\theta\to\pi^-}(\pi-\theta)^p\cdot[-\ln(\sin\theta)]=\pi\cdot 0=0.$$

由 Cauchy 判别法，瑕积分 $\int_0^\pi\theta\ln(\sin\theta)\,d\theta$ 收敛.

令 $x=\pi-\theta$，则对 $\forall u\in[\frac{\pi}{2},\pi)$，有

$$\int_{\frac{\pi}{2}}^{u}\theta\ln(\sin\theta)\,d\theta=-\int_{\frac{\pi}{2}}^{\pi-u}(\pi-x)\ln[\sin(\pi-x)]\,dx$$

$$=-\pi\int_{\frac{\pi}{2}}^{\pi-u}\ln(\sin x)\,dx+\int_{\frac{\pi}{2}}^{\pi-u}x\ln(\sin x)\,dx$$

$$=\pi\int_{\pi-u}^{\frac{\pi}{2}}\ln(\sin x)\,dx-\int_{\pi-u}^{\frac{\pi}{2}}\theta\ln(\sin\theta)\,d\theta. \quad (1)$$

由于 $\int_0^{\frac{\pi}{2}}\ln(\sin\theta)d\theta$ 收敛于 $-\frac{\pi}{2}\ln 2$，$\int_{\frac{\pi}{2}}^{\pi}\theta\ln(\sin\theta)\,d\theta$ 收敛，则

$$\int_0^\pi\theta\ln(\sin\theta)\,d\theta=\int_0^{\frac{\pi}{2}}\theta\ln(\sin\theta)\,d\theta+\int_{\frac{\pi}{2}}^{\pi}\theta\ln(\sin\theta)\,d\theta$$

$$\overset{(1)}{=}\pi\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx$$

$$=-\frac{\pi^2}{2}\ln 2.$$

(2) 由于 $\lim\limits_{\theta\to 0^+}\dfrac{\theta\sin\theta}{1-\cos\theta}=\lim\limits_{\theta\to 0^+}\dfrac{\theta\cdot\theta}{\frac{1}{2}\theta^2}=2$，所以 $\int_0^\pi\frac{\theta\sin\theta}{1-\cos\theta}d\theta$ 为定积分.

$$\int\frac{\theta\sin\theta}{1-\cos\theta}\,d\theta$$

$$=\int\frac{\theta\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}\,d\theta$$

$$=\int\theta\cot\frac{\theta}{2}\,d\theta$$

$$\underline{\underline{t=\frac{1}{2}\theta}} \quad 2\int 2t\cot t\,dt$$

$$= 4\int t\cot t\,dt$$

$$= 4\int t\,d[\ln(\sin t)]$$

$$= 4t\ln(\sin t) - 4\int \ln(\sin t)\,dt.$$

对 $\forall u, v \in (0,\pi)$，且 $u<v$，有

$$\int_u^v \frac{\theta\sin\theta}{1-\cos\theta}\,d\theta = 4t\ln(\sin t)\Big|_{\frac{1}{2}u}^{\frac{1}{2}v} - 4\int_{\frac{1}{2}u}^{\frac{1}{2}v} \ln(\sin t)\,dt,$$

在上式中分别令 $u\to0^+$，$v\to\pi^-$，由于

$$\lim_{u\to0^+} 4\cdot\frac{1}{2}u\cdot\ln\left(\sin\frac{1}{2}u\right)=0,$$

$$\lim_{v\to\pi^-} 4\cdot\frac{1}{2}v\cdot\ln\left(\sin\frac{1}{2}v\right)=0$$

$$\int_0^{\frac{\pi}{2}} \ln(\sin t)\,dt = -\frac{\pi}{2}\ln2,$$

所以，$$\int_0^\pi \frac{\theta\sin\theta}{1-\cos\theta}\,d\theta = (0-0) - 4\cdot\left(-\frac{\pi}{2}\ln2\right) = 2\pi\ln2.$$

第十一章总练习题

Ex1. (1) $\int_0^1 \frac{x^{p-1}}{x+1}dx = \int_1^{+\infty}\frac{x^{-p}}{1+x}dx$, $p>0$.

证：对 $\forall u\in(0,1]$，有

$$\int_u^1 \frac{x^{p-1}}{x+1}dx \overset{t=\frac{1}{x}}{=\!=\!=} \int_{\frac{1}{u}}^1 \frac{(\frac{1}{t})^{p-1}}{\frac{1}{t}+1}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$= \int_1^{\frac{1}{u}} \frac{t^{-p}}{t+1}dt$$

$$= \int_1^{\frac{1}{u}} \frac{x^{-p}}{1+x}dx$$

由于 $\lim\limits_{u\to0^+}\frac{1}{u}=+\infty$，则

$$\int_0^1\frac{x^{p-1}}{x+1}dx = \lim_{u\to0^+}\int_u^1\frac{x^{p-1}}{x+1}dx = \lim_{u\to0^+}\int_1^{\frac{1}{u}}\frac{x^{-p}}{1+x}dx = \int_1^{+\infty}\frac{x^{-p}}{1+x}dx.$$

(2) $\int_0^{+\infty}\frac{x^{p-1}}{x+1}dx = \int_0^{+\infty}\frac{x^{-p}}{x+1}dx$，$0<p<1$

证：对 $\forall u\geqslant 1$，有

$$\int_1^u \frac{x^{p-1}}{x+1}dx \overset{t=\frac{1}{x}}{=\!=\!=} \int_1^{\frac{1}{u}} \frac{(\frac{1}{t})^{p-1}}{\frac{1}{t}+1}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$= \int_{\frac{1}{u}}^1 \frac{t^{-p}}{t+1}dt$$

$$= \int_{\frac{1}{u}}^1 \frac{x^{-p}}{x+1}dx.$$

所以 $\int_1^{+\infty}\frac{x^{p-1}}{x+1}dx = \lim\limits_{u\to+\infty}\int_1^u\frac{x^{p-1}}{x+1}dx = \lim\limits_{u\to+\infty}\int_{\frac{1}{u}}^1\frac{x^{-p}}{x+1}dx = \int_0^1\frac{x^{-p}}{x+1}dx$

对 $\forall v\geqslant 1$，有

$$\int_1^v \frac{x^{-p}}{x+1}dx \overset{t=\frac{1}{x}}{=\!=\!=} \int_1^{\frac{1}{v}} \frac{(\frac{1}{t})^{-p}}{\frac{1}{t}+1}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$= \int_{\frac{1}{v}}^1 \frac{t^{p-1}}{t+1}dt = \int_{\frac{1}{v}}^1\frac{x^{p-1}}{x+1}dx.$$

所以 $\int_1^{+\infty} \frac{x^{-p}}{x+1} dx = \lim_{v \to +\infty} \int_1^{v} \frac{x^{-p}}{x+1} dx = \lim_{v \to +\infty} \int_{\frac{1}{v}}^{1} \frac{x^{p-1}}{x+1} dx = \int_0^1 \frac{x^{p-1}}{x+1} dx.$

综上, $\int_0^{+\infty} \frac{x^{p-1}}{x+1} dx = \int_0^{+\infty} \frac{x^{-p}}{x+1} dx.$

Ex2. (1) $\frac{\pi}{2\sqrt{2}} < \int_0^1 \frac{dx}{\sqrt{1-x^4}} < \frac{\pi}{2}$

解: 对 $\forall x \in (0,1)$, 有

$$\frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{2(1-x^2)}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} = \frac{1}{\sqrt{1-x^4}} < \frac{1}{\sqrt{1-x^2}},$$

所以,

$$\frac{\sqrt{2}}{2} \arcsin\frac{1}{2} = \frac{1}{2\sqrt{2}} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx < \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^4}} dx < \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} = \arcsin\frac{1}{2} \qquad (1)$$

由比较原则可知, 瑕积分 $\int_0^1 \frac{1}{\sqrt{1-x^4}} dx$ 收敛, 并且

$$\frac{\sqrt{2}}{2}\left(\frac{\pi}{2} - \arcsin\frac{1}{2}\right) = \frac{\sqrt{2}}{2} \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-x^2}} dx \le \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-x^4}} dx \le \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2} - \arcsin\frac{1}{2} \qquad (2)$$

综合(1)(2)两式, 得

$$\frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}}{2} \cdot \frac{\pi}{2} < \int_0^1 \frac{1}{\sqrt{1-x^4}} dx < \frac{\pi}{2}.$$

(2) $\frac{1}{2}\left(1 - \frac{1}{e}\right) < \int_0^{+\infty} e^{-x^2} dx < 1 + \frac{1}{2e}$

证: 令 $f(x) = e^{-x^2}$, $x \in [0, +\infty)$, 则 $f(x) > 0$. 对 $\forall p > 0$, 有

$$\lim_{x \to +\infty} x^p \cdot f(x) = \lim_{x \to +\infty} \frac{x^p}{e^{x^2}} = 0.$$

由 Cauchy 判别法可知, $\int_0^{+\infty} e^{-x^2} dx$ 收敛, 并且

$$\int_0^{+\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{+\infty} e^{-x^2} dx$$

一方面, $\int_0^{+\infty} e^{-x^2} dx \geq \int_0^1 e^{-x^2} dx > \int_0^1 x e^{-x^2} dx = \frac{1}{2} \int_0^1 e^{-x^2} d(x^2) = \frac{1}{2}(1 - e^{-1})$

$$= \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

另一方面，$\int_0^{+\infty} e^{-x^2}dx < \int_0^1 1\,dx + \int_1^{+\infty} x e^{-x^2}dx$

$= 1 + \frac{1}{2e}$.

Ex3 计算反常积分的值

(1) $\int_0^{+\infty} e^{-ax}\cos bx\,dx \quad (a>0)$

解：$\int e^{-ax}\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx) + C$,

对 $\forall u\geq 0$,

$$\int_0^u e^{-ax}\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx)\Big|_0^u$$

$$= \frac{e^{-au}}{a^2+b^2}(-a\cos bu + b\sin bu) + \frac{a}{a^2+b^2}$$

从而 $\lim\limits_{u\to+\infty}\int_0^u e^{-ax}\cos bx\,dx = 0 + \frac{a}{a^2+b^2} = \frac{a}{a^2+b^2}$,

于是 $\int_0^{+\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$.

(2) $\int_0^{+\infty} e^{-ax}\sin bx\,dx \quad (a>0)$

解：$\int e^{-ax}\sin bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\sin bx - b\cos bx) + C$

对 $\forall u\geq 0$,

$$\int_0^u e^{-ax}\sin bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\sin bx - b\cos bx)\Big|_0^u$$

$$= \frac{e^{-au}}{a^2+b^2}(-a\sin bu - b\cos bu) + \frac{b}{a^2+b^2},$$

$\lim\limits_{u\to+\infty}\int_0^u e^{-ax}\sin bx\,dx = 0 + \frac{b}{a^2+b^2} = \frac{b}{a^2+b^2}$.

所以 $\int_0^{+\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

(3) $\int_0^{+\infty} \frac{\ln x}{1+x^2}dx$

解: Step 1. 验证 瑕积分 $\int_0^1 \frac{\ln x}{1+x^2}dx$ 与无穷积分 $\int_1^{+\infty}\frac{\ln x}{1+x^2}dx$ 都收敛.

对 $\forall p>0$, 有

$$\lim_{x\to 0^+} x^p\cdot\frac{-\ln x}{1+x^2} = \lim_{x\to 0^+}\frac{1}{1+x^2}\cdot\lim_{x\to 0^+}\frac{-\ln x}{x^{-p}}$$

$$= 1\cdot\lim_{x\to 0^+}\frac{-\frac{1}{x}}{-p\,x^{-p-1}}$$

$$= \frac{1}{p}\lim_{x\to 0^+} x^p = 0,$$

取 $p=\frac{1}{2}\in(0,1)$, 由 Cauchy 判别法, 瑕积分 $\int_0^1\frac{\ln x}{1+x^2}dx$ 收敛.

对 $\forall p>0$, 有

$$\frac{(x^p\ln x)'}{(1+x^2)'} = \frac{px^{p-1}\ln x + x^{p-1}}{2x} = \frac{1}{2}\left(x^{p-2}\ln x + x^{p-2}\right)$$

取 $p=\frac{3}{2}\in(1,2)$, 则

$$\lim_{x\to+\infty} x^{\frac{3}{2}}\cdot\frac{\ln x}{1+x^2} = \lim_{x\to+\infty}\frac{x^{\frac{3}{2}}\ln x}{1+x^2} = \lim_{x\to+\infty}\frac{1}{2}\left(x^{-\frac{1}{2}}\ln x + x^{-\frac{1}{2}}\right) = 0,$$

由 Cauchy 判别法可知, 无穷积分 $\int_1^{+\infty}\frac{\ln x}{1+x^2}dx$ 也收敛.

于是 $\int_0^{+\infty}\frac{\ln x}{1+x^2}dx$ 收敛.

Step 2. 对 $\forall u\in[1,+\infty)$, 令 $t=\frac{1}{x}$, $x\in[1,u]$, 则

$$\int_1^u\frac{\ln x}{1+x^2}dx = \int_1^{\frac{1}{u}}\frac{\ln\frac{1}{t}}{1+\frac{1}{t^2}}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$= \int_{\frac{1}{u}}^1\frac{\ln\frac{1}{t}}{t^2+1}dt$$

$$= -\int_{\frac{1}{u}}^1\frac{\ln t}{1+t^2}dt = -\int_{\frac{1}{u}}^1\frac{\ln x}{1+x^2}dx$$

于是，$\int_1^{+\infty}\frac{\ln x}{1+x^2}dx = \lim\limits_{u\to+\infty}\int_1^u \frac{\ln x}{1+x^2}dx$

$$= -\lim_{u\to+\infty}\int_{\frac{1}{u}}^1 \frac{\ln x}{1+x^2}dx$$

$$= -\int_0^1 \frac{\ln x}{1+x^2}dx,$$

$$\int_0^{+\infty}\frac{\ln x}{1+x^2}dx = \int_0^1\frac{\ln x}{1+x^2}dx + \int_1^{+\infty}\frac{\ln x}{1+x^2}dx = 0.$$

(4) $\int_0^{\frac{\pi}{2}} \ln(\tan\theta)\,d\theta$

解：(习题 11.3 Ex5) 由于瑕积分 $\int_0^{\frac{\pi}{2}}\ln(\sin\theta)d\theta$ 与 $\int_0^{\frac{\pi}{2}}\ln(\cos\theta)d\theta$

都收敛，并且

$$\int_0^{\frac{\pi}{2}}\ln(\sin\theta)d\theta = \int_0^{\frac{\pi}{2}}\ln(\cos\theta)d\theta = -\frac{\pi}{2}\ln 2,$$

则 $\int_0^{\frac{\pi}{2}}\ln(\tan\theta)d\theta = \int_0^{\frac{\pi}{2}}\ln\left(\frac{\sin\theta}{\cos\theta}\right)d\theta$

$$= \int_0^{\frac{\pi}{2}}\left[\ln(\sin\theta) - \ln(\cos\theta)\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}}\ln(\sin\theta)d\theta - \int_0^{\frac{\pi}{2}}\ln(\cos\theta)d\theta$$

$$= 0.$$