§ 10.4 旋转曲面的面积

一. 微元法

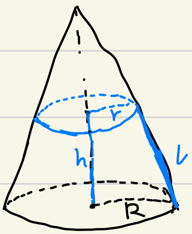
略

面积. 体积. 弧长　可求长曲线. 弧长

§ 21.1

二. 旋转曲面的面积　(§21.6)

圆台侧面面积的直观推导



圆台上底为 r, 下底半径为 R, $r<R$. 高为 h.

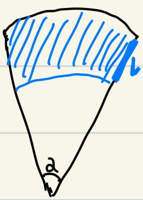
母线 $l=\sqrt{(R-r)^2+h^2}$. 该圆台可视为下底半径为 R,

高为 $H=\frac{R}{R-r}h$　$\frac{H-h}{H}=\frac{r}{R}$

的圆锥的底部, 该圆锥的母线

$$L=\sqrt{R^2+H^2}=\sqrt{R^2+\frac{R^2}{(R-r)^2}h^2}=\sqrt{(R-r)^2+h^2}\cdot\frac{R}{R-r}.$$

将该圆锥侧面展开, 得到半径为 L, 弧长为 $2\pi R$ 的扇形,



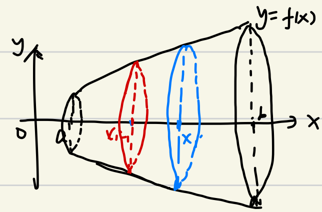
于是顶角 $\alpha=\frac{2\pi R}{L}=\frac{2\pi(R-r)}{\sqrt{(R-r)^2+h^2}}$.

所以圆台侧面面积为

$$S=\frac{1}{2}\alpha L^2-\frac{1}{2}\alpha(L-l)^2=\frac{1}{2}\alpha(2Ll-l^2)$$

$$=\pi(R+r)\sqrt{(R-r)^2+h^2}$$

曲线 $y=f(x)$, $x\in[a,b]$

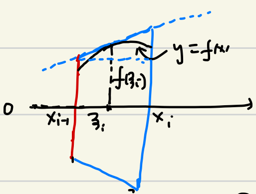


$f(x)\geq 0$. f 在 $[a,b]$ 上连续可导.

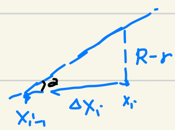
对 $[a,b]$ 作分割 $T=\{\Delta_i\}$, 令 $\xi_i=\frac{1}{2}(x_{i-1}+x_i)$.

将旋转体位于 Δ_i 的部分近似为如下圆台:

上、下底分别为 r, R, 母线经过点 $(\xi_i, f(\xi_i))$, 母线为曲线 $y=f(x)$ 在点 $(\xi_i, f(\xi_i))$ 处的切线的一部分, 则



$$\frac{1}{2}(R+r)=f(\xi_i), \quad \frac{R-r}{\Delta x_i}=|f'(\xi_i)|$$



从而该圆台侧面面积为

$$S_i=\pi\cdot 2f(\xi_i)\cdot\sqrt{|f'(\xi_i)|^2(\Delta x_i)^2+(\Delta x_i)^2}$$

$$=2\pi f(\xi_i)\sqrt{1+f'^2(\xi_i)}\cdot\Delta x_i$$

作 Riemann 和, 得 $2\pi\sum_{i=1}^{n} f(\xi_i)\sqrt{1+f'^2(\xi_i)}\cdot\Delta x_i$

由于 f 在 $[a,b]$ 上连续可导, 则 $f(x)\sqrt{1+f'^2(x)}$ 在 $[a,b]$ 上连续, 从而可积.

$$S=\lim_{\|T\|\to 0} 2\pi\sum_{i=1}^{n} f(\xi_i)\sqrt{1+f'^2(\xi_i)}\cdot\Delta x_i = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx.$$

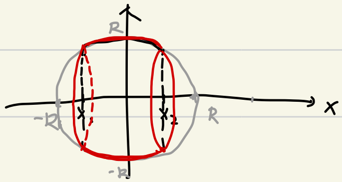
注: 若曲线方程为参数方程

$$x=x(t),\quad y=y(t),\quad \alpha\leq t\leq\beta.$$

$x(t)$, $y(t)$ 在 $[\alpha,\beta]$ 上连续可导, $y(t)\geq 0$, 则由该曲线所绕成的旋转曲面面积为

$$S=2\pi\int_\alpha^\beta y(t)\sqrt{x'^2(t)+y'^2(t)}\,dt.$$

例11. 计算 $x^2+y^2=R^2$ 在 $[x_1, x_2] \subset [-R,R]$ 上的弧段绕着 x 轴旋转所得球带的面积



解: $y=f(x)=\sqrt{R^2-x^2}$, $x\in[x_1, x_2]$

$$S=2\pi\int_{x_1}^{x_2} f(x)\sqrt{1+f'^2(x)}\,dx$$

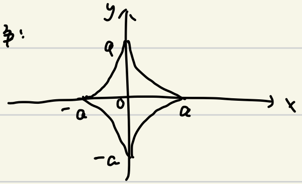
$$=2\pi\int_{x_1}^{x_2}\sqrt{R^2-x^2}\cdot\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx$$

$$=2\pi\int_{x_1}^{x_2} R\,dx = 2\pi R(x_2-x_1)$$

当 $x_1=-R$, $x_2=R$. 得到球的表面积 $S=4\pi R^2$.

例12. 内摆线 $x=a\cos^3 t$, $y=a\sin^3 t$ 绕 x 轴旋转所得的旋转曲面的面积.

解:



$$S=2\cdot 2\pi\int_0^{\frac{\pi}{2}} a\sin^3 t\sqrt{(-3a\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2}\,dt$$

$$=12\pi a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos t\,dt$$

$$=12\pi a^2\left(\int_0^{\frac{\pi}{2}}\sin^4 t\,d\sin t\right)$$

$$=\frac{12}{5}\pi a^2.$$

习题 10.4

Ex1. 求旋转曲面面积

(1) $y=\sin x,\ 0\le x\le \pi$. 绕 x 轴.

$\int \sqrt{x^2+a^2}\,dx \quad (a>0)$

$= \frac{1}{2}\left(x\sqrt{x^2+a^2}+a^2\ln\left|x+\sqrt{x^2+a^2}\right|\right)+C$

解: $S = 2\pi\int_0^\pi \sin x\sqrt{1+\cos^2 x}\,dx$

$= -2\pi\int_0^\pi \sqrt{1+\cos^2 x}\,d\cos x$

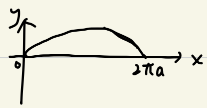
$= -2\pi\int_1^{-1}\sqrt{1+t^2}\,dt \qquad (t=\cos x)$

$= 2\pi\int_{-1}^{1}\sqrt{1+t^2}\,dt.$

$= 2\pi\cdot\frac{1}{2}\left(t\sqrt{1+t^2}+\ln\left|t+\sqrt{1+t^2}\right|\right)\Big|_{-1}^{1} = \left[2\sqrt{2}+2\ln(\sqrt{2}+1)\right]\pi$

(2) $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi\ (a>0)$. 绕 x 轴.

解:



$x'(t)=a(1-\cos t),\quad y'(t)=a\sin t.$

$x'^2(t)+y'^2(t)=a^2(1-\cos t)^2+a^2\sin^2 t = 2a^2(1-\cos t).$

$S = 2\pi\int_0^{2\pi} y(t)\sqrt{x'^2(t)+y'^2(t)}\,dt$

$= 2\pi\int_0^{2\pi} a(1-\cos t)\cdot\sqrt{2a^2(1-\cos t)}\,dt$

$= 2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos t)^{\frac{3}{2}}\,dt \qquad 1-\cos t = 2\sin^2\frac{t}{2}$

$= 2\sqrt{2}\pi a^2\cdot 2\sqrt{2}\int_0^{2\pi}\left(\sin\frac{t}{2}\right)^{\frac{3}{2}}dt$

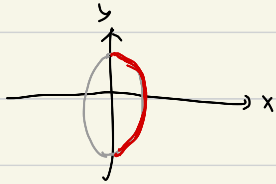
$= 8\pi a^2\cdot 2\int_0^{\pi}\sin^3 u\,du \qquad (u=\frac{t}{2})$

$= 32\pi a^2\int_0^{\frac{\pi}{2}}\sin^3 u\,du$

$= 32\pi a^2\cdot\frac{2}{3}$

$= \frac{64}{3}\pi a^2.$

(3) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 绕 y 轴



解: $x = x(t) = a\cos t$, $y = y(t) = b\sin t$,

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

$x'(t) = -a\sin t$, $y'(t) = b\cos t$.

所以, $\sqrt{x'^2(t)+y'^2(t)} = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$,

$$S = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x(t)\cdot\sqrt{x'^2(t)+y'^2(t)}\,dt$$

$$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos t\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,dt$$

$$= 2\pi a\int_{-1}^{1}\sqrt{(a^2-b^2)u^2+b^2}\,du \qquad (u=\sin t)$$

① 当 $a=b$, $S = 4\pi a^2$.

② 当 $a>b$.

$$S = 2\pi a\int_{-1}^{1}\sqrt{\underbrace{(a^2-b^2)}_{>0}u^2+b^2}\,du$$

$$= 2\pi a\sqrt{a^2-b^2}\cdot\int_{-1}^{1}\sqrt{u^2+\frac{b^2}{a^2-b^2}}\,du$$

$$= 2\pi a\sqrt{a^2-b^2}\cdot\frac{1}{2}\left(u\sqrt{u^2+\frac{b^2}{a^2-b^2}}+\frac{b^2}{a^2-b^2}\ln\left|u+\sqrt{u^2+\frac{b^2}{a^2-b^2}}\right|\right)\Bigg|_{-1}^{1}$$

$$= \pi a\sqrt{a^2-b^2}\cdot\left(2\sqrt{1+\frac{b^2}{a^2-b^2}}+\frac{b^2}{a^2-b^2}\ln\left|u+\sqrt{u^2+\frac{b^2}{a^2-b^2}}\right|\right)$$

$$= 2\pi a^2+\frac{\pi ab^2}{\sqrt{a^2-b^2}}\ln\frac{(a+\sqrt{a^2-b^2})^2}{b^2}$$

$$= 2\pi a^2+\frac{2\pi ab^2}{\sqrt{a^2-b^2}}\ln\frac{a+\sqrt{a^2-b^2}}{b}.$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right) + C$$

③ 当 $a<b$ 时，

$$S = 2\pi a\cdot\sqrt{b^2-a^2}\int_{-1}^{1}\sqrt{\frac{b^2}{b^2-a^2}-u^2}\,du$$

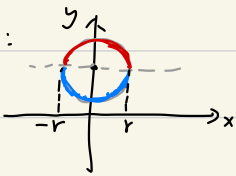
$$= 2\pi a\cdot\sqrt{b^2-a^2}\cdot\frac{1}{2}\left(u\sqrt{\frac{b^2}{b^2-a^2}-u^2} + \frac{b^2}{b^2-a^2}\arcsin\frac{\sqrt{a^2-b^2}\cdot x}{b}\right)\Bigg|_{-1}^{1}$$

$$= \pi a\sqrt{b^2-a^2}\cdot\left(2\sqrt{\frac{b^2}{b^2-a^2}-1} + \frac{2b^2}{b^2-a^2}\arcsin\frac{\sqrt{a^2-b^2}}{b}\right)$$

$$= 2\pi a^2 + \frac{2\pi a b^2}{\sqrt{b^2-a^2}}\arcsin\frac{\sqrt{a^2-b^2}}{b}.$$

(4) $x^2+(y-a)^2=r^2 \quad (r<a)$，绕 x 轴.

解：



$y=f_1(x)=a+\sqrt{r^2-x^2}, \quad x\in[-r,r]$,

$y=f_2(x)=a-\sqrt{r^2-x^2}, \quad x\in[-r,r]$.

$$f_1'(x) = -\frac{x}{\sqrt{r^2-x^2}}, \quad f_2'(x)=\frac{x}{\sqrt{r^2-x^2}},$$

所以 $\quad 1+f_1'^2(x)=\dfrac{r^2}{r^2-x^2}, \quad 1+f_2'^2(x)=\dfrac{r^2}{r^2-x^2}$.

$$S = 2\pi\int_{-r}^{r} f_1(x)\sqrt{1+f_1'^2(x)}\,dx + 2\pi\int_{-r}^{r} f_2(x)\sqrt{1+f_2'^2(x)}\,dx$$

$$= 2\pi\int_{-r}^{r}(a+\sqrt{r^2-x^2})\frac{r}{\sqrt{r^2-x^2}}\,dx + 2\pi\int_{-r}^{r}(a-\sqrt{r^2-x^2})\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$= 4\pi a\int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$= 4\pi a r\arcsin\frac{x}{r}\Bigg|_{-r}^{r} = 4\pi^2 a r.$$

Ex2. 平面光滑曲线由极坐标方程

$$r=r(\theta),\quad \alpha\le\theta\le\beta \qquad ([\alpha,\beta]\subset[0,\pi],\ r(\theta)\ge 0)$$

给出，求该曲线绕极轴旋转所得旋转曲面的面积.

解: $x=x(\theta)=r(\theta)\cos\theta$, $\quad y=y(\theta)=r(\theta)\sin\theta$, $\quad \alpha\le\theta\le\beta$,

$x'(\theta)=r'(\theta)\cos\theta-r(\theta)\sin\theta$,

$y'(\theta)=r'(\theta)\sin\theta+r(\theta)\cos\theta$.

则 $x'^2(\theta)+y'^2(\theta)=r^2(\theta)+r'^2(\theta)$.

于是，光滑曲线 $r=r(\theta)$, $\alpha\le\theta\le\beta$ 绕极轴旋转所得旋转曲面的面积

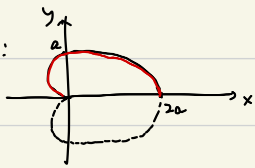
$$S=2\pi\int_\alpha^\beta y(\theta)\sqrt{x'^2(\theta)+y'^2(\theta)}\,d\theta$$

$$=2\pi\int_\alpha^\beta r(\theta)\sin\theta\cdot\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$$

Ex3. 求由极坐标曲线绕极轴旋转所得旋转曲面的面积.

(1) 心形线 $r=a(1+\cos\theta)$ $\quad(a>0)$

解:



$r'(\theta)=-a\sin\theta$,

$r^2(\theta)+r'^2(\theta)=a^2(1+\cos\theta)^2+a^2\sin^2\theta$

$=2a^2(1+\cos\theta)$.

$$S=2\pi\int_0^\pi r(\theta)\sin\theta\cdot\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$$

$$=2\pi\int_0^\pi a(1+\cos\theta)\sin\theta\cdot\sqrt{2a^2(1+\cos\theta)}\,d\theta$$

$$=2\sqrt{2}\pi a^2\int_0^\pi(1+\cos\theta)^{\frac{3}{2}}\underbrace{\sin\theta\,d\theta}_{=-d(1+\cos\theta)}$$

$$= -2\sqrt{2}\pi a^2 \int_0^{\pi} (1+\cos\theta)^{\frac{3}{2}} \, d(1+\cos\theta)$$

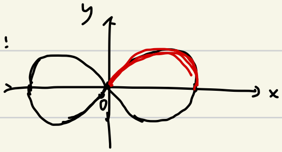
$$= -2\sqrt{2}\pi a^2 \int_2^0 t^{\frac{3}{2}} dt \qquad (t = 1+\cos\theta)$$

$$= 2\sqrt{2}\pi a^2 \int_0^2 t^{\frac{3}{2}} dt$$

$$= 2\sqrt{2}\pi a^2 \cdot \frac{2}{5} t^{\frac{5}{2}} \Big|_0^2 = \frac{32}{5}\pi a^2$$

(2) 双纽线 $r^2 = 2a^2\cos 2\theta \quad (a>0)$

解:



只需考虑 $\theta \in [0, \frac{\pi}{4}]$ 的部分:

$$r = \sqrt{2a^2\cos 2\theta}, \quad \theta \in [0, \frac{\pi}{4}]$$

$$r(\theta) = \sqrt{2a^2\cos 2\theta} = \sqrt{2}\, a \sqrt{\cos 2\theta}, \quad 0 \le \theta \le \frac{\pi}{4}$$

$$r'(\theta) = \sqrt{2}\, a \cdot \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}, \quad 0 \le \theta \le \frac{\pi}{4},$$

则 $r^2(\theta) + r'^2(\theta) = 2a^2\cos 2\theta + 2a^2 \cdot \frac{\sin^2 2\theta}{\cos 2\theta} = 2a^2 \cdot \frac{1}{\cos 2\theta}$,

所以由曲线 $r = \sqrt{2a^2\cos 2\theta}$, $\theta \in [0, \frac{\pi}{4}]$ 绕极轴旋转所得的旋转曲面面积为

$$S_0 = 2\pi \int_0^{\frac{\pi}{4}} r(\theta)\sin\theta \cdot \sqrt{r^2(\theta) + r'^2(\theta)} \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sqrt{2}\, a \sqrt{\cos 2\theta} \sin\theta \cdot \sqrt{\frac{2a^2}{\cos 2\theta}} \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} 2a^2 \sin\theta \, d\theta$$

$$= 4\pi a^2 \int_0^{\frac{\pi}{4}} \sin\theta \, d\theta$$

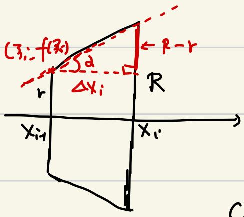
$$= 2(2-\sqrt{2})\pi a^2.$$

所以整个双纽线绕极轴旋转所得旋转曲面面积 $S = 2S_0 = 4(2-\sqrt{2})\pi a^2$.

$(\xi_i = \frac{1}{2}(x_{i-1}+x_i))$

Ex4. ξ_i 取 x_{i-1}，是否可以推导出同样的旋转曲面面积公式？

解: $\xi_i = x_{i-1}$, $\quad r = f(\xi_i) = f(x_{i-1})$



$$\frac{R-r}{\Delta x_i} = f'(\xi_i)\text{，所以}$$

$$R = f'(\xi_i)\Delta x_i + r = f'(\xi_i)\Delta x_i + f(\xi_i).$$

$$S_i = \pi(R+r)\sqrt{(R-r)^2+h^2}$$

$$= \pi\left[f'(\xi_i)\Delta x_i + 2f(\xi_i)\right]\sqrt{(f'(\xi_i)\Delta x_i)^2+(\Delta x_i)^2}$$

$$= \pi\left[f'(\xi_i)\Delta x_i + 2f(\xi_i)\right]\cdot\sqrt{1+f'^2(\xi_i)}\cdot\Delta x_i$$

$$= 2\pi f(\xi_i)\sqrt{1+f'^2(\xi_i)}\,\Delta x_i + \pi f'(\xi_i)\Delta x_i\cdot\Delta x_i$$

$$\sum_{i=1}^n S_i = 2\pi\sum_{i=1}^n f(\xi_i)\sqrt{1+f'^2(\xi_i)}\,\Delta x_i + \pi\sum_{i=1}^n f'(\xi_i)\Delta x_i\cdot\Delta x_i$$

$\downarrow \|T\|\to 0$

$$2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$$

$\|T\| = \max\limits_{1\le i\le n}\{\Delta x_i\}.$

由于 $\left|\pi\sum_{i=1}^n f'(\xi_i)\Delta x_i\cdot\Delta x_i\right| \le \left|\pi\sum_{i=1}^n f'(\xi_i)\Delta x_i\right|\cdot\|T\|.$

$$\lim_{\|T\|\to 0}\pi\sum_{i=1}^n f'(\xi_i)\Delta x_i = \pi\int_a^b f'(x)dx = \pi(f(b)-f(a)).$$

所以 $\lim\limits_{\|T\|\to 0}\pi\sum_{i=1}^n f'(\xi_i)\Delta x_i\cdot\Delta x_i = 0.$

$$\lim_{\|T\|\to 0}\sum_{i=1}^n S_i = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx.$$