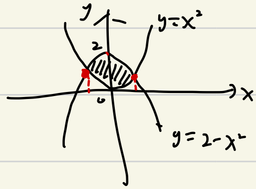
习题10.1

Ex1. $y=x^2$, $y=2-x^2$ 所围成平面图形的面积.

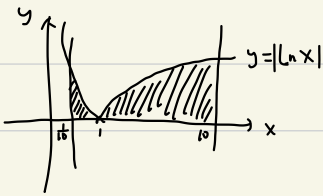


解: 解 $\begin{cases} y=x^2 \\ y=2-x^2 \end{cases}$ 得 $\begin{cases} x_1=-1 \\ y_1=1 \end{cases}$ 或 $\begin{cases} x_2=1 \\ y_2=1 \end{cases}$

所围成平面图形的面积

$$A=\int_{-1}^{1}\left[(2-x^2)-x^2\right]dx=\int_{-1}^{1}(2-2x^2)\,dx=\frac{8}{3}.$$

Ex2. 由曲线 $y=|\ln x|$ 与直线 $x=\frac{1}{10}$, $x=10$, $y=0$ 所围平面图形的面积.



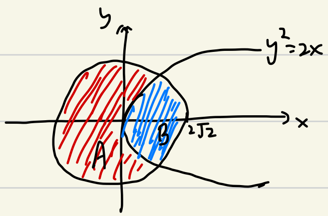
解: $A=\int_{\frac{1}{10}}^{1}(-\ln x)\,dx+\int_{1}^{10}\ln x\,dx$

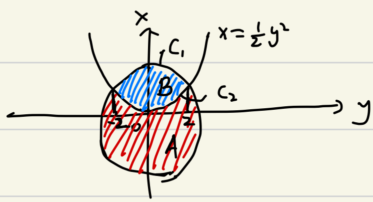
$=-(x\ln x-x)\Big|_{\frac{1}{10}}^{1}+(x\ln x-x)\Big|_{1}^{10}$

$=\frac{9}{10}-\frac{1}{10}\ln 10+10\ln 10-9$

$=-\frac{81}{10}+\frac{99}{10}\ln 10.$

Ex3. 抛物线 $y^2=2x$ 把圆 $x^2+y^2\le 8$ 分成两部分. 求两部分面积之比.





C_1: $x=\sqrt{8-y^2}$, $y\in[-2,2]$

C_2: $x=\frac{1}{2}y^2$, $y\in[-2,2]$

解: $\begin{cases} y^2=2x \\ x^2+y^2=8 \end{cases}$ 解得 $\begin{cases} x_1=-2 \\ y_1=2 \end{cases}$ 或 $\begin{cases} x_2=2 \\ y_2=2 \end{cases}$

$$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right)+C$$

$$B = \int_{-2}^{2}\left(\sqrt{8-y^2}-\frac{1}{2}y^2\right)dy$$

$$= \left[\frac{1}{2}\left(y\sqrt{8-y^2}+8\arcsin\frac{y}{2\sqrt{2}}\right)-\frac{1}{6}y^3\right]\Bigg|_{-2}^{2}$$

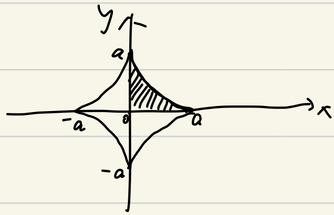
$$= \left[\frac{1}{2}\left(2\cdot\sqrt{8-4}+8\arcsin\frac{\sqrt{2}}{2}\right)-\frac{1}{6}\cdot 8\right]$$

$$-\left[\frac{1}{2}\left(-2\sqrt{8-4}-8\arcsin\frac{\sqrt{2}}{2}\right)+\frac{1}{6}\cdot 8\right]$$

$$= \frac{4}{3}+2\pi.$$

于是 $A = 8\pi - B = 6\pi - \frac{4}{3}$,

所以 $\frac{A}{B} = \frac{9\pi-2}{3\pi+2}$.



Ex4. 内摆线 $x = a\cos^3 t$, $y = a\sin^3 t$ $(a>0)$

所围成的平面图形的面积、

解: $\frac{dx}{dt} = -3a\cos^2 t\sin t$, 则

$$A = 4\int_0^{\frac{\pi}{2}} |x'(t)\,y(t)|\,dt$$

$$= 4\int_0^{\frac{\pi}{2}} 3a\cos^2 t\sin t\cdot a\sin^3 t\,dt$$

$$= 12a^2\int_0^{\frac{\pi}{2}}(1-\sin^2 t)\sin^4 t\,dt$$

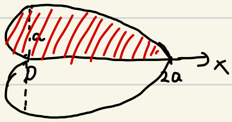
$$= 12a^2\left(\int_0^{\frac{\pi}{2}}\sin^4 t\,dt-\int_0^{\frac{\pi}{2}}\sin^6 t\,dt\right)$$

$$= 12a^2\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{8}a^2.$$

$$\int_0^{\frac{\pi}{2}}\sin^n t\,dt = \int_0^{\frac{\pi}{2}}\cos^n t\,dt = \begin{cases}\frac{(2m-1)!!}{(2m)!!}\cdot\frac{\pi}{2}, & n=2m\\ \frac{(2m)!!}{(2m+1)!!}, & n=2m+1\end{cases}$$

Ex5. 心形线 $r=a(1+\cos\theta)$ $(a>0)$. 所围成的平面图形的面积.



解: $A = 2\cdot\frac{1}{2}\int_0^{\pi} r^2(\theta)\,d\theta$

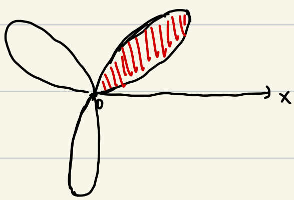
$$= a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \pi a^2 + a^2\cdot 2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$= \frac{3}{2}\pi a^2.$$

Ex6. 三叶曲线 $r=a\sin 3\theta$ 所围平面图形的面积.

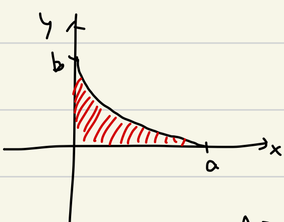


解: $A = 3\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} r^2(\theta)\,d\theta$

$$= \frac{3}{2}a^2\int_0^{\frac{\pi}{3}}\sin^2 3\theta\,d\theta$$

$$= \frac{\pi}{4}a^2.$$

Ex7. 曲线 $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ $(a,b>0)$ 与坐标轴所围平面图形的面积.



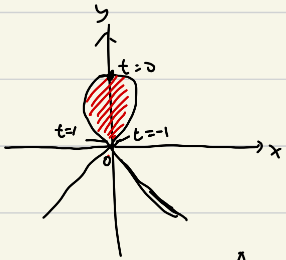
解: 由 $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ 得

$$y = b\left(1-\sqrt{\frac{x}{a}}\right)^2 = \frac{b}{a}x-\frac{2b}{\sqrt{a}}\sqrt{x}+b,\quad x\in[0,a].$$

所以 所围图形的面积

$$A=\int_0^a|y|\,dx=\int_0^a y\,dx=\int_0^a\left(\frac{b}{a}x-\frac{2b}{\sqrt{a}}\sqrt{x}+b\right)dx=\frac{1}{6}ab.$$

Ex8. 曲线 $\begin{cases} x = t-t^3 \\ y = 1-t^4 \end{cases}$ 所围平面图形的面积.



解：解 $t-t^3=0$，得

$t_1=0,\ t_2=1,\ t_3=-1.$

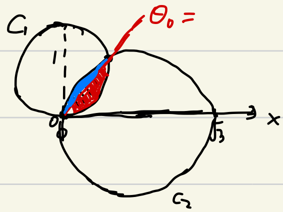
所围平面图形的面积.

$$A=2\int_{-1}^{0}|x(t)\,y'(t)|\,dx$$

$$=2\int_{-1}^{0}|(t-t^3)\cdot(-4)t^3|\,dt$$

$$=8\int_{-1}^{0}t^3(t-t^3)\,dt$$

$$=\frac{16}{35}$$



Ex9. $r=\sin\theta$、$r=\sqrt{3}\cos\theta$ 所围图形的面积.

解：解 $\begin{cases} r=\sin\theta \\ r=\sqrt{3}\cos\theta \end{cases}$ 得 $\begin{cases} \theta=\frac{\pi}{3} \\ r=\frac{\sqrt{2}}{2} \end{cases}$

所以 $$A=\frac{1}{2}\int_{0}^{\frac{\pi}{3}}(\sin\theta)^2\,d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}3\cos^2\theta\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{3}}(1-\cos2\theta)\,d\theta+\frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1+\cos2\theta)\,d\theta$$

$$=\frac{1}{4}\left(\theta-\frac{1}{2}\sin2\theta\right)\Big|_{0}^{\frac{\pi}{3}}+\frac{3}{4}\left(\theta+\frac{1}{2}\sin2\theta\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

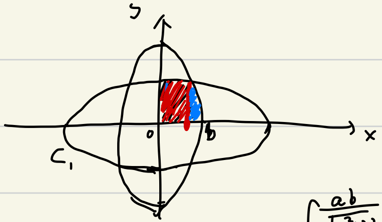
$$=\frac{1}{4}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)+\frac{3}{4}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$$

$$=\frac{5}{24}\pi-\frac{\sqrt{3}}{4}$$

Ex10 两椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 与 $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ $(a>0, b>0)$

公共部分的面积.

解: 不妨设 $a>b$. 解 $\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\\ \frac{x^2}{b^2}+\frac{y^2}{a^2}=1\\ x, y\geq 0\end{cases}$ 得 $x=y=\frac{ab}{\sqrt{a^2+b^2}}$



$$A_0=\int_0^{\frac{ab}{\sqrt{a^2+b^2}}}\sqrt{b^2-\frac{b^2}{a^2}x^2}\,dx+\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{b}\sqrt{a^2-\frac{a^2}{b^2}x^2}\,dx$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right)+C$$

$$=\frac{b}{a}\int_0^{\frac{ab}{\sqrt{a^2+b^2}}}\sqrt{a^2-x^2}\,dx+\frac{a}{b}\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{b}\sqrt{b^2-x^2}\,dx$$

$$=\frac{b}{2a}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right)\Bigg|_0^{\frac{ab}{\sqrt{a^2+b^2}}}+\frac{a}{2b}\left(x\sqrt{b^2-x^2}+b^2\arcsin\frac{x}{b}\right)\Bigg|_{\frac{ab}{\sqrt{a^2+b^2}}}^{b}$$

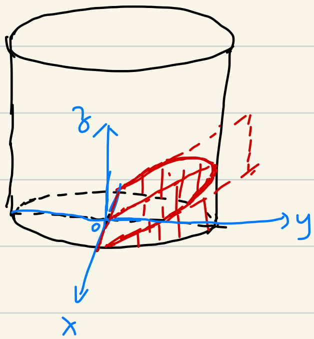
$$\frac{\pi}{2}-\arcsin\frac{a}{\sqrt{a^2+b^2}}=\arcsin\frac{b}{\sqrt{a^2+b^2}}.$$

$$=ab\arcsin\frac{b}{\sqrt{a^2+b^2}},$$

从而整个公共部分面积为

$$A=4A_0=4ab\arcsin\frac{b}{\sqrt{a^2+b^2}}$$

习题 10.2

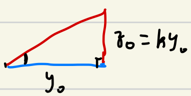


Ex1 直椭圆柱被通过底面短轴的斜平面所截，求所截得楔形体的体积。

解：建立立体直角坐标系 xyz，底面椭圆为 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (0<a<b)$， $a=4,\ b=10$

斜平面为 $z=ky \quad (k>0)$ $k=\frac{1}{2}$

任取 $x_0\in(-a,a)$，平面 $x=x_0$ 与楔形体截面为直角三角形。



其中位于 xoy 平面中的直角边长为

$$y_0=\sqrt{b^2-\frac{b^2}{a^2}x_0^2}.$$

另一条直角边长为

$$z_0=ky_0=k\sqrt{b^2-\frac{b^2}{a^2}x_0^2}.$$

所以截面面积函数为

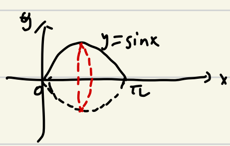
$$A(x)=\frac{1}{2}\sqrt{b^2-\frac{b^2}{a^2}x^2}\cdot k\sqrt{b^2-\frac{b^2}{a^2}x^2}=\frac{kb^2}{2a^2}(a^2-x^2),\quad x\in[-a,a].$$

楔形体体积为

$$V=\int_{-a}^{a}A(x)\,dx=\frac{kb^2}{2a^2}\int_{-a}^{a}(a^2-x^2)\,dx=\frac{2kab^2}{3}.\qquad \left(\frac{2}{3}\cdot\frac{1}{2}\cdot4\cdot100=\frac{400}{3}\right)$$

Ex2. 求旋转体体积。

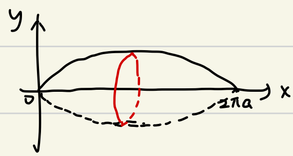
(1) $y=\sin x,\ 0\le x\le\pi$，绕 x 轴



解： $V=\pi\int_0^{\pi}y^2\,dx$

$$=\pi\int_0^{\pi}\sin^2x\,dx=\frac{\pi}{2}\int_0^{\pi}(1-\cos2x)\,dx=\frac{\pi}{2}\left(x-\frac{1}{2}\sin2x\right)\Big|_0^{\pi}=\frac{\pi^2}{2}.$$

(2) $x=a(t-\sin t)$, $y=a(1-\cos t)$ $(a>0)$, $0\le t\le 2\pi$. 绕x轴.



解: $V=\int_0^{2\pi a}\pi y^2 dx$

$$=\pi\int_0^{2\pi} a^2(1-\cos t)^2\cdot a(1-\cos t)dt$$

$$=\pi a^3\int_0^{2\pi}(1-\cos t)^3 dt$$

$$=\pi a^3\int_0^{2\pi}\left(2\sin^2\frac{t}{2}\right)^3 dt$$

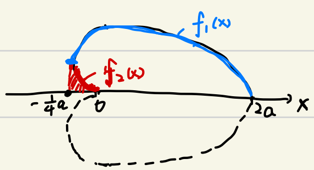
$$=8\pi a^3\int_0^{2\pi}\sin^6\frac{t}{2}dt$$

$$=8\pi a^3\cdot 2\int_0^{\pi}\sin^6 u\,du \qquad \left(u=\frac{t}{2},\ t\in[0,2\pi]\right)$$

$$=8\pi a^3\cdot 4\int_0^{\frac{\pi}{2}}\sin^6 u\,du$$

$$=32\pi a^3\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=5\pi^2a^3$$

(3) $r=a(1+\cos\theta)$ $(a>0)$ 绕极轴



解: $x=x(\theta)=r(\theta)\cos\theta=a(\cos\theta+\cos^2\theta)$,

$y=y(\theta)=r(\theta)\sin\theta=a(\sin\theta+\sin\theta\cos\theta)$.

$0\le\theta\le\pi$

$x_{max}=x(0)=2a$, $x_{min}=x\left(\frac{2}{3}\pi\right)=-\frac{1}{4}a$

$$V=\pi\int_{-\frac{1}{4}a}^{2a}f_1^2(x)dx-\pi\int_{-\frac{1}{4}a}^{0}f_2^2(x)dx$$

$$=\pi\int_{\frac{2}{3}\pi}^{0}y^2(\theta)\cdot x'(\theta)d\theta-\pi\int_{\frac{2}{3}\pi}^{\pi}y^2(\theta)\cdot x'(\theta)d\theta$$

$$=\pi\int_{\pi}^{0}y^2(\theta)\cdot x'(\theta)d\theta$$

$$=\pi\int_{\pi}^{0}r^2(\theta)\sin^2\theta\cdot(r(\theta)\cos\theta)'d\theta$$

$$= \pi \int_{\pi}^{0} \overbrace{a^2(1+\cos\theta)^2 \sin^2\theta}^{r^2(\theta)\sin^2\theta} \cdot \left[-a\sin\theta\cdot\cos\theta - a(1+\cos\theta)\sin\theta\right] d\theta$$

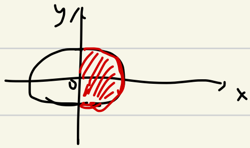
$$= \pi a^2 \int_{\pi}^{0} (1+\cos\theta)^2(1-\cos^2\theta)(-2\sin\theta\cos\theta - \sin\theta)\, d\theta$$

$$= \pi a^2 \int_0^{\pi} (1+\cos\theta)^2(1-\cos^2\theta)(2\cos\theta+1)\cdot \sin\theta\, d\theta$$

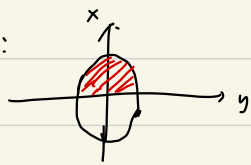
$$= \pi a^2 \int_{-1}^{1} (1-t)^2(1-t^2)(-2t+1)\, dt \qquad (t=-\cos\theta)$$

$$= \frac{8\pi}{3}a^2.$$

(4) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a,b>0)$ 绕y轴



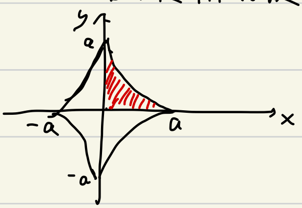
解:



$x=\sqrt{a^2-\frac{a^2}{b^2}y^2}$, $y\in[-b,b]$.

$$V=\pi\int_{-b}^{b} x^2 dy = \pi\int_{-b}^{b}\left(a^2-\frac{a^2}{b^2}y^2\right)dy = \frac{4\pi}{3}a^2 b.$$

Ex4. 由内摆线 $x=a\cos^3 t$, $y=a\sin^3 t$ 所围平面图形绕x轴旋转所得旋转体体积.



解: $V = 2\cdot\pi\int_0^a y^2 dx$

$$= 2\pi\int_{\frac{\pi}{2}}^{0} a^2\sin^6 t\, dt\cdot(-3a\cos^2 t\sin t)\, dt$$

$$= 6\pi a^3\int_0^{\frac{\pi}{2}} \sin^7 t\cos^2 t\, dt$$

$$= 6\pi a^3\int_0^{\frac{\pi}{2}} \sin^7 t(1-\sin^2 t)\, dt$$

$$= 6\pi a^3\left(\int_0^{\frac{\pi}{2}}\sin^7 t\, dt - \int_0^{\frac{\pi}{2}}\sin^9 t\, dt\right)$$

$$= 6\pi a^3\left(\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} - \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\right)$$

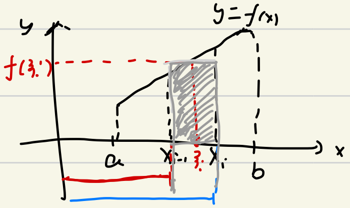
$$= \frac{32}{105}\pi a^3.$$

$0 \le a \le x \le b$

Ex5. 导出曲边梯形 $0 \le y \le f(x)$, $a \le x \le b$ 绕y轴旋转所得立体体积。

在 $[a,b]$ 上可积

$$V = 2\pi \int_a^b x f(x)\,dx$$



解: ① $a \cdot b \ge 0$. 不妨设 $0 \le a \le b$.

对 $[a,b]$ 作分割 $T = \{\Delta_i\}$.

令 $\xi_i = \frac{x_{i-1} + x_i}{2}$, 得到

以 x_{i-1} 为内径, 以 x_i 为外径, 以 $f(\xi_i)$ 为长度的管状立体,

体积为 $\pi x_i^2 f(\xi_i) - \pi x_{i-1}^2 f(\xi_i) = \pi (x_i^2 - x_{i-1}^2) f(\xi_i) = 2\pi \xi_i f(\xi_i) \Delta x_i$

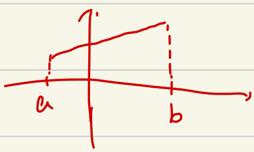
作 Riemann 和 $\sum_{i=1}^n 2\pi \xi_i f(\xi_i) \cdot \Delta x_i$, 由于 f 在 $[a,b]$ 可积, 则

$$2\pi \int_a^b x f(x)\,dx = \lim_{\|T\| \to 0} \sum_{i=1}^n 2\pi \xi_i f(\xi_i) \Delta x_i.$$

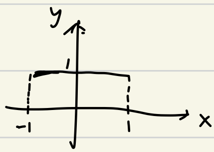
所以可以定义 $y = f(x)$, $a \le x \le b$ 绕 y 轴旋转所得立体体积为

$$V = 2\pi \int_a^b x f(x)\,dx$$

② $a \cdot b < 0$. $a < 0 < b$.



反例: $y = f(x) \equiv 1$, $x \in [-1, 1]$.



该曲边梯形绕y轴旋转所得立体为底面半径为1, 高为1的圆柱体, 体积为 π.

但是 $2\pi \int_{-1}^1 x\,dx = 0$.

Ex6. $0 \le y \le \sin x$, $0 \le x \le \pi$ 绕y轴旋转所得立体体积。

解： $V = 2\pi \int_0^{\pi} x \sin x \, dx$

$= 2\pi \left(- \int_0^{\pi} x \, d \cos x \right)$

$= -2\pi x \cos x \Big|_0^{\pi} + 2\pi \int_0^{\pi} \cos x \, dx$

$= -2\pi^2 \cdot (-1) + \underbrace{2\pi \sin x \Big|_0^{\pi}}_{=0}$

$= 2\pi^2$

习题 10.3

Ex1. 求弧长.

$y=f(x),\ x\in[a,b].\quad S=\int_a^b\sqrt{1+f'^2(x)}\,dx.$

$x=x(t),\ y=y(t),\ t\in[\alpha,\beta].\quad S=\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)}\,dt$

$r=r(\theta),\ \theta\in[\alpha,\beta],\quad S=\int_\alpha^\beta\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta.$

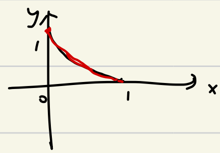
(1) $y=x^{\frac{3}{2}},\quad 0\le x\le 4$

解: $y'=\frac{3}{2}x^{\frac{1}{2}},\quad \sqrt{1+y'^2}=\sqrt{1+\frac{9}{4}x}.$

$$S=\int_0^4\sqrt{1+\frac{9}{4}x}\,dx=\frac{4}{9}\int_0^4\left(1+\frac{9}{4}x\right)^{\frac{1}{2}}d\left(1+\frac{9}{4}x\right)=\frac{4}{9}\cdot\frac{2}{3}\cdot\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\Big|_0^4$$

$$=\frac{8}{27}\left(10\sqrt{10}-1\right)$$

(2) $\sqrt{x}+\sqrt{y}=1.$



解: $y=(1-\sqrt{x})^2=x-2\sqrt{x}+1,\quad x\in[0,1],$

$y'=1-\frac{1}{\sqrt{x}},$

$t=\sqrt{x}$

$$S=\int_0^1\sqrt{1+y'^2}\,dx=\int_0^1\sqrt{2+\frac{1}{x}-\frac{2}{\sqrt{x}}}\,dx=\int_0^1\sqrt{2+\frac{1}{x}-\frac{2}{\sqrt{x}}}\cdot 2\sqrt{x}\,d\sqrt{x}$$

$$=2\int_0^1\sqrt{2x+1-2\sqrt{x}}\,d\sqrt{x}$$

$$\overset{t=\sqrt{x}}{=}2\int_0^1\sqrt{2t^2+1-2t}\,dt$$

$$\int\sqrt{x^2+a^2}\,dx=\frac{1}{2}\left(x\sqrt{x^2+a^2}+a^2\ln\left|x+\sqrt{x^2+a^2}\right|\right)+C$$

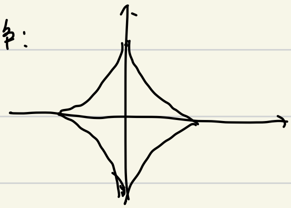
$$=2\sqrt{2}\int_0^1\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}\,dt$$

$$=2\sqrt{2}\cdot\frac{1}{2}\left[\left(t-\frac{1}{2}\right)\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}+\frac{1}{4}\ln\left|\left(t-\frac{1}{2}\right)+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}\right|\right]\Bigg|_0^1$$

$$=2\sqrt{2}\cdot\frac{1}{2}\left[\frac{\sqrt{2}}{2}+\frac{1}{2}\ln(\sqrt{2}+1)\right]=\sqrt{2}+\frac{\sqrt{2}}{2}\ln(\sqrt{2}+1).$$

(3) $x = a\cos^3 t,\ y = a\sin^3 t \quad (a>0).\quad 0 \le t \le 2\pi$

解:



$x'(t) = -3a\cos^2 t\sin t.$

$y'(t) = 3a\sin^2 t\cos t.$

$x'^2(t)+y'^2(t) = 9a^2\sin^2 t\cos^2 t$

$$S = \int_0^{2\pi}\sqrt{x'^2(t)+y'^2(t)}\,dt$$

$$= 3a\int_0^{2\pi}\sqrt{\sin^2 t\cos^2 t}\,dt$$

$$= 3a\cdot 4\int_0^{\frac{\pi}{2}} \underbrace{\sin t\cos t}_{\geq 0}\,dt$$

$$= 6a\int_0^{\frac{\pi}{2}}\sin 2t\,dt = -3a\cdot\cos 2t\Big|_0^{\frac{\pi}{2}} = 6a.$$

(4) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)\quad (a>0).\quad 0\le t\le 2\pi.$

解: $x'(t) = a(-\sin t + \sin t + t\cos t) = at\cos t,$

$y'(t) = a(\cos t - \cos t + t\sin t) = at\sin t$

$x'^2(t)+y'^2(t) = a^2t^2$

$$S = \int_0^{2\pi}\sqrt{a^2t^2}\,dt = a\int_0^{2\pi}t\,dt = 2\pi^2 a.$$

(5) $r = a\sin^3\frac{\theta}{3},\quad 0\le\theta\le 3\pi.$

解. $r'(\theta) = a\cdot 3\sin^2\frac{\theta}{3}\cdot\cos\frac{\theta}{3}\cdot\frac{1}{3} = a\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$

$r^2(\theta)+r'^2(\theta) = a^2\sin^6\frac{\theta}{3} + a^2\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3} = a^2\sin^4\frac{\theta}{3}.$

$$s = \int_0^{3\pi}\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$$

$$= \int_0^{3\pi}\sqrt{a^2\sin^4\frac{\theta}{3}}\,d\theta = a\int_0^{3\pi}\sin^2\frac{\theta}{3}\,d\theta \xlongequal{t=\frac{\theta}{3}} 3a\int_0^{\pi}\sin^2 t\,dt = \frac{3\pi a}{2}.$$

(6) $r = a\theta \quad (a>0), \quad 0 \le \theta \le 2\pi.$

解: $s = \int_0^{2\pi} \sqrt{r^2(\theta) + r'^2(\theta)}\, d\theta$

$= \int_0^{2\pi} \sqrt{a^2\theta^2 + a^2}\, d\theta$

$= a\int_0^{2\pi} \sqrt{\theta^2+1}\, d\theta$

$= a \cdot \frac{1}{2}\left(\theta\sqrt{\theta^2+1} + \ln\left|\theta+\sqrt{\theta^2+1}\right|\right)\Big|_0^{2\pi}$

$= \frac{a}{2}\left[2\pi\sqrt{4\pi^2+1} + \ln(2\pi+\sqrt{4\pi^2+1})\right].$

Ex2. 求曲线在指定点的曲率

$C: x=x(t), y=y(t).\qquad k = \frac{|x'y''-x''y'|}{(x'^2+y'^2)^{\frac{3}{2}}}$

$C: y=f(x).\qquad k = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}$

(1) $xy=4$. 点 $(2,2)$

解: $y=\frac{4}{x}$, $y' = -\frac{4}{x^2}$, $y'' = \frac{8}{x^3}$.

$y'|_{x=2} = -1$, $y''|_{x=2} = 1$,

所以 $k = \frac{1}{(1+1)^{\frac{3}{2}}} = \frac{1}{2\sqrt{2}}$.

(2) $y=\ln x$, 点 $(1,0)$

解: $y' = \frac{1}{x}$, $y'' = -\frac{1}{x^2}$,

$y'|_{x=1} = 1$, $y''|_{x=1} = -1$,

所以 $k = \frac{1}{(1+1)^{\frac{3}{2}}} = \frac{1}{2\sqrt{2}}$.

(3) $x=a(t-\sin t)$. $y=a(1-\cos t)$. $(a>0)$. 点 $t=\frac{\pi}{2}$.

解: $x'(t)=a(1-\cos t)$, $x''(t)=a\sin t$,

$y'(t)=a\sin t$, $y''(t)=a\cos t$,

$x'(\frac{\pi}{2})=a$, $x''(\frac{\pi}{2})=a$, $y'(\frac{\pi}{2})=a$, $y''(\frac{\pi}{2})=0$.

所以 $$K=\frac{|x'y''-x''y'|}{(x'^2+y'^2)^{\frac{3}{2}}}=\frac{|0-a^2|}{(a^2+a^2)^{\frac{3}{2}}}=\frac{1}{2\sqrt{2}a}.$$

(4) $x=a\cos^3 t$, $y=a\sin^3 t$, $(a>0)$. 点 $t=\frac{\pi}{4}$.

解: $x'(t)=-3a\cos^2 t\sin t$, $x''(t)=6a\cos t\sin^2 t-3a\cos^3 t$,

$y'(t)=3a\sin^2 t\cos t$, $y''(t)=6a\cos^2 t\sin t-3a\sin^3 t$.

$x'(\frac{\pi}{4})=-\frac{3\sqrt{2}}{4}a$. $x''(\frac{\pi}{4})=\frac{3\sqrt{2}}{4}a$, $y'(\frac{\pi}{4})=\frac{3\sqrt{2}a}{4}$, $y''(\frac{\pi}{4})=\frac{3\sqrt{2}}{4}a$.

所以 $$K=\frac{|x'y''-x''y'|}{(x'^2+y'^2)^{\frac{3}{2}}}=\frac{2}{3a}.$$

Ex4. 略

Ex5. 极坐标方程 $r=r(\theta)$ 表示的曲线的曲率公式.

$r(\theta)$ 二阶可导.

解: $x=x(\theta)=r(\theta)\cos\theta$, $y=y(\theta)=r(\theta)\sin\theta$.

$x'(\theta)=r'(\theta)\cos\theta-r(\theta)\sin\theta$, $y'(\theta)=r'(\theta)\sin\theta+r(\theta)\cos\theta$,

$$\begin{aligned}x''(\theta)&=r''(\theta)\cos\theta-r'(\theta)\sin\theta-r'(\theta)\sin\theta-r(\theta)\cos\theta\\&=r''(\theta)\cos\theta-2r'(\theta)\sin\theta-r(\theta)\cos\theta,\end{aligned}$$

$$y''(\theta) = r''(\theta)\sin\theta + r'(\theta)\cos\theta + r'(\theta)\cos\theta - r(\theta)\sin\theta$$

$$= r''(\theta)\sin\theta + 2r'(\theta)\cos\theta - r(\theta)\sin\theta.$$

$$x'(\theta)y''(\theta) - x''(\theta)y'(\theta) = \cdots = r^2(\theta) + 2r'^2(\theta) - r(\theta)r''(\theta)$$

$$x'^2(\theta) + y'^2(\theta) = \cdots = r^2(\theta) + r'^2(\theta).$$

所以曲率

$$K = \frac{|x'y'' - x''y'|}{(x'^2+y'^2)^{\frac{3}{2}}} = \frac{|r^2 + 2r'^2 - rr''|}{(r^2+r'^2)^{\frac{3}{2}}}.$$

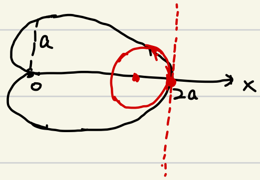
Ex6 心形线 $r = a(1+\cos\theta)$ $(a>0)$ 在 $\theta=0$ 处的曲率、曲率半径和曲率圆.

解: $r'(\theta) = -a\sin\theta$, $r''(\theta) = -a\cos\theta$,

$r(0) = 2a$, $r'(0) = 0$, $r''(0) = -a$,

曲率 $K = \dfrac{|r^2+2r'^2-rr''|}{(r^2+r'^2)^{\frac{3}{2}}} = \dfrac{3}{4a}$,

曲率半径 $R = \dfrac{1}{K} = \dfrac{4}{3}a$.



曲率圆的圆心为 $\left(\frac{2}{3}a, 0\right)$.

曲率圆的方程

$$\left(x - \frac{2}{3}a\right)^2 + y^2 = \frac{4}{9}a^2.$$

Ex7. 证明抛物线 $y=ax^2+bx+c$ 在顶点处曲率最大.

证: $y'=2ax+b$, $y''=2a$.

所以抛物线的曲率

$$K=\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}=\frac{|2a|}{[1+\underbrace{(2ax+b)^2}_{\geq 0}]^{\frac{3}{2}}}$$

当 $x=-\frac{b}{2a}$ 时, $2ax+b=0$, K 取最大值 $|2a|$.

也即顶点 $\left(-\frac{b}{2a}, \frac{b^2-4ac}{4a}\right)$ 处抛物线的曲率最大.

Ex8. 求曲线 $y=e^x$ 上曲率最大的点.

解: $y'=e^x$, $y''=e^x$, 从而曲线 $y=e^x$ 的曲率

$$K=\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}=\frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}.$$

$$\left[\frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}\right]'=\frac{e^x(1+e^{2x})^{\frac{3}{2}}-\frac{3}{2}\cdot(1+e^{2x})^{\frac{1}{2}}\cdot e^{2x}\cdot 2\cdot e^x}{(1+e^{2x})^3}$$

$$=\underbrace{\frac{(1+e^{2x})^{\frac{1}{2}}}{(1+e^{2x})^3}\cdot e^x}_{>0}\cdot(1-2e^{2x})$$

当 $x\in(-\infty, \ln\frac{\sqrt{2}}{2}]$, $\left[\frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}\right]'>0$, 即 $\frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$ 在 $(-\infty, \ln\frac{\sqrt{2}}{2}]$ 上严格增.

当 $x\in(\ln\frac{\sqrt{2}}{2}, +\infty)$, $\frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$ 在 $(\ln\frac{\sqrt{2}}{2}, +\infty)$ 上严格减.

所以当 $x=\ln\frac{\sqrt{2}}{2}$, K 取最大值.

也即曲线 $y=e^x$ 在点 $\left(\ln\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ 处曲率取最大值 $\frac{2\sqrt{3}}{9}$.