§8.3. 有理函数和可化为有理函数的不定积分

一. 有理函数的不定积分.

设 $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad (a_n \neq 0)$,

$Q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m \quad (b_m \neq 0)$

称 $R(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}$ 为有理函数.

若 $n < m$, 则称 $R(x)$ 为真分式; 若 $n \geq m$, 则称 $R(x)$ 为假分式.

Step1. 处理 $R(x)$.

代数学命题:

① 任何一个假分式都可分解为一个多项式和一个真分式之和 (多项式除法)

② 多项式 $Q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m \quad (b_m \neq 0)$ 都实数系内作标准分解:

$$Q(x) = b_m (x-a_1)^{\lambda_1} \cdot (x-a_2)^{\lambda_2} \cdots (x-a_s)^{\lambda_s} \cdot \underbrace{(x^2+p_1 x+q_1)^{u_1} \cdots (x^2+p_t x+q_t)^{u_t}}_{\text{不可约},\ \Delta<0} \qquad (1)$$

③ 设 $R(x) = \dfrac{P(x)}{Q(x)}$ 为真分式, 若 $Q(x)$ 可以标准分解为 (1) 式, 则

$R(x)$ 可解为

$$R(x) = \sum_{i=1}^{\lambda_1} \frac{A_i^1}{(x-a_1)^i} + \sum_{i=1}^{\lambda_2} \frac{A_i^2}{(x-a_2)^i} + \cdots + \sum_{i=1}^{\lambda_s} \frac{A_i^s}{(x-a_s)^i}$$

$$+ \sum_{i=1}^{u_1} \frac{B_i^1 x + C_i^1}{(x^2+p_1 x+q_1)^i} + \cdots + \sum_{i=1}^{u_t} \frac{B_i^t x + C_i^t}{(x^2+p_t x+q_t)^i}$$

由上述三命题, 可将任何一个有理函数 $R(x)$ 分解为多项式和部分分式的和.

Step2. 求 $\int R(x)dx$

部分分式可以分为4类：

(I) $\frac{A}{x-a}$　　(II) $\frac{A}{(x-a)^k}$ $(k\geq 2)$

(III) $\frac{Bx+C}{x^2+px+q}$　　(IV) $\frac{Bx+C}{(x^2+px+q)^k}$ $(k\geq 2)$　　$(\Delta = p^2-4q<0)$

(I): $\int \frac{A}{x-a}dx = A\ln|x-a| + C$

(II): $\int \frac{A}{(x-a)^k}dx = \frac{A}{(1-k)(x-a)^{k-1}} + C.$

(III): $x^2+px+q = \left(x+\frac{p}{2}\right)^2 + \underbrace{\left(q-\frac{p^2}{4}\right)}_{>0}.$

令 $t = x+\frac{p}{2}$，$r = \sqrt{q-\frac{p^2}{4}} > 0$，则

$$\frac{Bx+C}{x^2+px+q}dx = \frac{Bt+\tilde{C}}{t^2+r^2}dt,\ \text{其中}\ \tilde{C} = C-\frac{p}{2}B.$$

所以 $\int \frac{Bx+C}{x^2+px+q}dx = \int \frac{Bt+\tilde{C}}{t^2+r^2}dt = B\int\frac{t}{t^2+r^2}dt + \tilde{C}\int\frac{1}{t^2+r^2}dt$

$= B\cdot\frac{1}{2}\ln(t^2+r^2) + \tilde{C}\cdot\frac{1}{r}\arctan\frac{t}{r} + C$

$= \frac{B}{2}\ln(x^2+px+q) + \frac{C-\frac{p}{2}B}{r}\arctan\frac{x+\frac{p}{2}}{r} + C$

(IV). $\int\frac{Bx+C}{(x^2+px+q)^k}dx$ $(k\geq 2)$

令 $t = x+\frac{p}{2}$，$r = \sqrt{q-\frac{p^2}{4}} > 0$，则

$$\frac{Bx+C}{(x^2+px+q)^k}dx = \frac{Bt+\tilde{C}}{(t^2+r^2)^k}dt$$

$$\int\frac{Bt+\tilde{C}}{(t^2+r^2)^k}dt = B\underline{\int\frac{t}{(t^2+r^2)^k}dt} + \tilde{C}\int\frac{1}{(t^2+r^2)^k}dt$$

$$\int \frac{t}{(t^2+r^2)^k}dt = \frac{1}{2}\int \frac{1}{(t^2+r^2)^k}d(t^2) = \frac{1}{2(1-k)(t^2+r^2)^{k-1}} + C$$

$$= \frac{1}{2(1-k)(x^2+px+q)^{k-1}} + C$$

令 $I_k = \int \frac{1}{(t^2+r^2)^k}dt$.

(递推公式) $I_k = \int \frac{1}{(t^2+r^2)^k}dt$

$$= \frac{1}{r^2}\int \frac{r^2}{(t^2+r^2)^k}dt = \frac{1}{r^2}\int \frac{(t^2+r^2)-t^2}{(t^2+r^2)^k}dt$$

$$= \frac{1}{r^2}\int \frac{1}{(t^2+r^2)^{k-1}}dt - \frac{1}{r^2}\int \frac{t^2}{(t^2+r^2)^k}dt$$

$$\left[\frac{1}{(t^2+r^2)^{k-1}}\right]' = (1-k)\cdot\frac{1}{(t^2+r^2)^k}\cdot 2t$$

$$= \frac{1}{r^2}I_{k-1} - \frac{1}{r^2}\int t^2\cdot\frac{1}{2(1-k)t}d\left[\frac{1}{(t^2+r^2)^{k-1}}\right]$$

$$= \frac{1}{r^2}I_{k-1} - \frac{1}{2(1-k)r^2}\int t\, d\left[\frac{1}{(t^2+r^2)^{k-1}}\right]$$

$$= \frac{1}{r^2}I_{k-1} - \frac{1}{2(1-k)r^2}\left[\frac{t}{(t^2+r^2)^{k-1}} - \underbrace{\int \frac{1}{(t^2+r^2)^{k-1}}dt}_{I_{k-1}}\right]$$

$$= \frac{1}{r^2}I_{k-1} - \frac{1}{2(1-k)r^2}\cdot\frac{t}{(t^2+r^2)^{k-1}} + \frac{1}{2(1-k)r^2}I_{k-1}$$

$$= \frac{2k-3}{2(1-k)r^2}I_{k-1} - \frac{1}{2(1-k)r^2}\cdot\frac{t}{(t^2+r^2)^{k-1}}$$

(不要记结论！记住过程)

难点：1. 对 $Q(x)$ 作标准分解.

2. 真分式 $R(x)$ 分解成部分分式之和.

3. 第IV类部分分式的不定积分.

} 计算比较复杂

例1. $R(x)=\dfrac{2x^4-x^3+4x^2+9x-10}{x^5+x^4-5x^2-2x^2+4x-8}$

解: $Q(x)=x^5+x^4-5x^2-2x^2+4x-8=(x-2)(x+2)^2(x^2-x+1)$

设 $R(x)=\dfrac{A_1}{x-2}+\left[\dfrac{A_2}{x+2}+\dfrac{A_3}{(x+2)^2}\right]+\dfrac{Bx+C}{x^2-x+1}$, (1)

其中 A_1, A_2, A_3, B和C都是待定常数, 在(1)式两端同乘 $Q(x)$, 得

$$2x^4-x^3+4x^2+9x-10=A_1(x+2)^2(x^2-x+1)+A_2(x-2)(x+2)(x^2-x+1)$$
$$+A_3(x-2)(x^2-x+1)+(Bx+C)(x-2)(x+2)^2$$
$$=(\cdots)x^4+(\cdots)x^3+(\cdots)x^2+(\cdots)x+(\cdots)$$

得到关于 A_1, A_2, A_3, B和C的5元一次方程组, 解得

$A_1=1$, $A_2=2$, $A_3=-1$, $B=-1$, $C=1$.

于是, $R(x)=\dfrac{1}{x-2}+\dfrac{2}{x+2}-\dfrac{1}{(x+2)^2}-\dfrac{x-1}{x^2-x+1}$

① $\int\dfrac{1}{x-2}dx=\ln|x-2|+C$. ② $\int\dfrac{2}{x+2}dx=2\ln|x+2|+C$.

③ $\int-\dfrac{1}{(x+2)^2}dx=\dfrac{1}{x+2}+C$.

④ 由于 $x^2-x+1=(x-\frac12)^2+\frac34$. 令 $t=x-\frac12$, 则

$$-\frac{x-1}{x^2-x+1}dx=-\frac{t-\frac12}{t^2+\frac34}dt.$$

于是 $\int-\dfrac{x-1}{x^2-x+1}dx=-\int\dfrac{t-\frac12}{t^2+\frac34}dt=-\int\dfrac{t}{t^2+\frac34}dt+\dfrac12\int\dfrac{1}{t^2+\frac34}dt$ $\quad \frac34=(\frac{\sqrt3}{2})^2$

$$=-\frac12\int\frac{1}{t^2+\frac34}d(t^2)+\frac12\cdot\frac{2}{\sqrt3}\arctan\frac{2}{\sqrt3}t+C$$

$$=-\frac12\ln(t^2+\frac34)+\frac{\sqrt3}{3}\arctan\frac{2\sqrt3}{3}t+C$$

$$=\frac12\ln(x^2-x+1)+\frac{\sqrt3}{3}\arctan\frac{2\sqrt3x-\sqrt3}{3}+C$$

综上. $\int R(x)\,dx = \ln|x-2| + 2\ln|x+2| + \frac{1}{x+2} - \frac{1}{2}\ln(x^2-x+1)$
$+ \frac{\sqrt{3}}{3}\arctan\frac{2\sqrt{3}x-\sqrt{3}}{3} + C.$

例2. $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

解: $R(x) = \frac{x^2+1}{(x^2-2x+2)^2} = \frac{(x^2-2x+2)+(2x-2)+1}{(x^2-2x+2)^2} = \frac{1}{x^2-2x+2} + \frac{2x-1}{(x^2-2x+2)^2}$

由于 $x^2-2x+2 = (x-1)^2+1$, 令 $t = x-1$, 则

① $\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{t^2+1}\,dt = \arctan t + C = \arctan(x-1) + C.$

② $\int \frac{2x-1}{(x^2-2x+2)^2}\,dx = \int \frac{2t+1}{(t^2+1)^2}\,dt = \int \frac{2t}{(t^2+1)^2}\,dt + \int \frac{1}{(t^2+1)^2}\,dt$

$= -\frac{1}{t^2+1} + \int \frac{(t^2+1)-t^2}{(t^2+1)^2}\,dt$

$\left(\frac{1}{t^2+1}\right)' = (-1)\cdot\frac{1}{(t^2+1)^2}\cdot 2t.$

$= -\frac{1}{t^2+1} + \int \frac{1}{t^2+1}\,dt - \int \frac{t^2}{(t^2+1)^2}\,dt.$

$= -\frac{1}{t^2+1} + \arctan t - \int t^2 \cdot \frac{1}{-2t}\,d\left(\frac{1}{t^2+1}\right)$

$= -\frac{1}{t^2+1} + \arctan t + \frac{1}{2}\int t\,d\left(\frac{1}{t^2+1}\right)$

$= -\frac{1}{t^2+1} + \arctan t + \frac{1}{2}\cdot\frac{t}{t^2+1} - \frac{1}{2}\int \frac{1}{t^2+1}\,dt$

$= -\frac{1}{t^2+1} + \arctan t + \frac{t}{2(t^2+1)} - \frac{1}{2}\arctan t$

$= \frac{t-2}{2(t^2+1)} + \frac{1}{2}\arctan t + C$

$= \frac{x-3}{2(x^2-2x+2)} + \frac{1}{2}\arctan(x-1) + C$

综上. $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx = \frac{x-3}{2(x^2-2x+2)^2} + \frac{3}{2}\arctan(x-1) + C.$

二. 三角函数有理式的不定积分.

有理式：由函数 $u(x)$，$v(x)$ 和常数经过有限次四则运算所得的函数称为关于 $u(x)$, $v(x)$ 的有理式，记为 $R(u(x), v(x))$.

$R(\sin x, \cos x)$ —— 三角函数有理式.

注：带有 $\tan x$, $\cot x$, $\sin 2x$, $\cos 2x$，$\sec x$ 或 $\csc x$ 等的有理式在本质上都是 $R(\sin x, \cos x)$

求 $\int R(\sin x, \cos x)\,dx$

一般方法：万能变换 $t = \tan\frac{x}{2}$，则

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{1-t^2}{1+t^2}$$

$x = 2\arctan t$, $dx = \frac{2}{1+t^2}dt$

从而 $\int R(\sin x, \cos x)\,dx = \int R\left(\frac{2t}{t^2+1}, \frac{1-t^2}{1+t^2}\right)\cdot\frac{2}{1+t^2}\,dt$

$$= \int R(t)\,dt$$

例3. $\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx$

解：令 $t = \tan\frac{x}{2}$，则 $\sin x = \frac{2t}{1+t^2}$, $\cos t = \frac{1-t^2}{1+t^2}$,

$$\frac{1+\sin x}{\sin(1+\cos x)} = \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)} = \frac{(1+t^2)^2+2t(1+t^2)}{2t(1+t^2+1-t^2)} = \frac{(1+t^2)(t^2+2t+1)}{2t\cdot 2}$$

$$= \frac{(1+t^2)(t+1)^2}{4t}. \qquad dx = \frac{2}{1+t^2}dt$$

所以 $\displaystyle\int \frac{1+\sin x}{\sin x(1+\cos x)}dx = \int \frac{(t+1)^2}{2t}dt = \frac{1}{2}\int \left(t+2+\frac{1}{t}\right)dt$

$$= \frac{1}{2}\left(\frac{1}{2}t^2+2t+\ln|t|\right)+C$$

$$= \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C.$$

注：万能变换理论上可处理任何一个三角函数有理式的不定积分，

但是它并不是最简便的.

例14. $\displaystyle\int \frac{1}{a^2\sin^2 x + b^2\cos^2 x}dx \quad (ab\neq 0)$

解：(方法1). 令 $t=\tan\frac{x}{2}$，则

$$\int \frac{1}{a^2\sin^2 x + b^2\cos^2 x}dx = \int \frac{1}{a^2\cdot\frac{4t^2}{(1+t^2)^2} + b^2\cdot\frac{(1-t^2)^2}{(1+t^2)^2}}\cdot\frac{2}{1+t^2}dt$$

$$= \int \frac{2(1+t^2)}{4a^2t^2+b^2(1-t^2)^2}dt$$

$$= \int \frac{2(1+t^2)}{b^2t^4+(4a^2-2b^2)t^2+b^2}dt$$

$$= \frac{2}{b^2}\int \frac{t^2+1}{t^4+\left(4\cdot\frac{a^2}{b^2}-2\right)t^2+1}dt$$

$$= \cdots$$

(方法2). $\displaystyle\int \frac{1}{a^2\sin^2 x+b^2\cos^2 x}dx = \int \frac{\sec^2 x}{a^2\tan^2 x+b^2}dx = \int \frac{1}{a^2\tan^2 x+b^2}d\tan x$

$$\overset{t=\tan x}{=\!=\!=} \int \frac{1}{a^2t^2+b^2}\,dt = \frac{1}{ab}\arctan\frac{at}{b}+C$$

$$= \frac{1}{ab}\arctan\frac{a\tan x}{b}+C$$

特殊方法：当被积函数是 $\sin^2 x$，$\cos^2 x$ 以及 $\sin x\cos x$ 的有理式时，可采用变量代换 $t=\tan x$.

三．含有根式的有理式的不定积分.

1. $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$，其中 $ad-bc\neq 0$.

令 $t=\sqrt[n]{\frac{ax+b}{cx+d}}$，则 $t^n=\frac{ax+b}{cx+d}$，$x=\frac{b-dt^n}{ct^n-a}$.

$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$ 转化为关于 t 的有理函数的不定积分 $\int R(t)\,dt$.

例5. $\int \frac{1}{x}\sqrt{\frac{x+2}{x-2}}\,dx$

解：令 $t=\sqrt{\frac{x+2}{x-2}}$，则 $t^2=\frac{x+2}{x-2}$，$x=\frac{2+2t^2}{t^2-1}$.

$$\frac{1}{x}\sqrt{\frac{x+2}{x-2}} = \frac{t^2-1}{2(1+t^2)}\cdot t = \frac{t^3-t}{2(1+t^2)}, \quad dx = 2\cdot\frac{2t(t^2-1)-2t(t^2+1)}{(t^2-1)^2}\,dt$$

$$= \frac{-8t}{(t^2-1)^2}\,dt$$

于是 $$\int \frac{1}{x}\sqrt{\frac{x+2}{x-2}}\,dx = \int \frac{t^3-t}{2(1+t^2)}\cdot\frac{-8t}{(t^2-1)}\,dt$$

$$= \int \frac{-4t^2}{(t^2+1)(t^2-1)}\,dt = \int\left(\frac{2}{1-t^2}-\frac{2}{1+t^2}\right)dt$$

$$= 2\int \frac{1}{1-t^2}dt - 2\int \frac{1}{1+t^2}dt$$

$$= 2\int \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right)dt - 2\arctan t$$

$$= \ln|t+1| - \ln|t-1| - 2\arctan t + C$$

$$= \ln\left|\frac{t+1}{t-1}\right| - 2\arctan t + C \qquad \left(t = \sqrt{\frac{x+2}{x-2}}\right)$$

$$= \ln\left|\frac{1+\sqrt{\frac{x+2}{x-2}}}{\sqrt{\frac{x+2}{x-2}}-1}\right| - 2\arctan\sqrt{\frac{x+2}{x-2}} + C$$

$$= \ln\left|\frac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}\right| - 2\arctan\sqrt{\frac{x+2}{x-2}} + C.$$

例16 $\int \frac{1}{(1+x)\sqrt{2+x-x^2}}dx$ $\qquad R\left(x, \sqrt{\frac{ax+b}{cx+d}}\right)$

解: $2+x-x^2 = -(x^2-x-2) = (2-x)(x+1).$

$$\frac{1}{(1+x)\sqrt{2+x-x^2}} = \frac{1}{(1+x)\sqrt{(1+x)(2-x)}} = \frac{\sqrt{1+x}}{(1+x)^2\sqrt{2-x}} = \frac{1}{(1+x)^2}\sqrt{\frac{1+x}{2-x}}$$

令 $t = \sqrt{\frac{1+x}{2-x}}$, 则 $x = \frac{2t^2-1}{1+t^2}$,

$$\frac{1}{(1+x)^2}\sqrt{\frac{1+x}{2-x}} = \frac{(1+t^2)^2}{9t^4}\cdot t = \frac{(1+t^2)^2}{9t^3}, \quad dx = \frac{6t}{(1+t^2)^2}dt.$$

所以 $\int \frac{1}{(1+x)\sqrt{2+x-x^2}}dx = \int \frac{(1+t^2)^2}{9t^3}\cdot\frac{6t}{(1+t^2)^2}dt = \frac{2}{3}\int\frac{1}{t^2}dt$

$$= -\frac{2}{3t} + C = -\frac{2}{3}\sqrt{\frac{2-x}{1+x}} + C.$$

2. $\int R(x, \sqrt{ax^2+bx+c}\,)\,dx$ $\begin{cases} ① \ a>0, \ \Delta = b^2-4ac \neq 0 \\ ② \ a<0, \ \Delta = b^2-4ac > 0. \end{cases}$

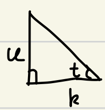
方法1. 直角三角形技巧.

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

令 $u = x+\frac{b}{2a}$, $k^2 = \frac{4ac-b^2}{4a^2}$ 且 $k>0$, 则 $\int R(x, \sqrt{ax^2+bx+c})\,dx$

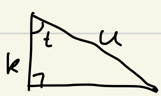
可化为以下三种不定积分:

① $\int R(u, \sqrt{u^2+k^2}\,)\,du$, 对应 $a>0$ 且 $\Delta = b^2-4ac<0$.



令 $u = k\tan t$, 则 $\sqrt{u^2+k^2} = k\sec t$, $du = k\sec^2 t\,dt$.

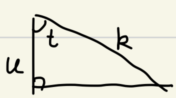
② $\int R(u, \sqrt{u^2-k^2}\,)\,du$, 对应 $a>0$ 且 $\Delta = b^2-4ac>0$



$\frac{k}{u} = \cos t$. 令 $u = k\sec t$, 则 $\sqrt{u^2-k^2} = k\tan t$,

$du = k\sec t\tan t\,dt$.

③ $\int R(u, \sqrt{k^2-u^2}\,)\,du$, 对应 $a<0$ 且 $\Delta = b^2-4ac>0$.



令 $u = k\cos t$, 则 $\sqrt{k^2-u^2} = k\sin t$,

$du = -k\sin t\,dt$

方法2. Euler 变换法.

① 若 $a>0$, 可令 $\sqrt{ax^2+bx+c} = t-\sqrt{a}\,x$, 则

$ax^2+bx+c = t^2-2\sqrt{a}\,t\,x+ax^2$.

整理后可得 $x = \frac{t^2-c}{2\sqrt{a}\,t+b}$, $\sqrt{ax^2+bx+c} = \frac{\sqrt{a}\,t^2+bt+c\sqrt{a}}{2\sqrt{a}\,t+b}$

$$dx = \frac{2(\sqrt{a}\,t^2+bt+c\sqrt{a})}{(2\sqrt{a}\,t+b)^2}\,dt.$$

② 若 $c>0$，可令 $\sqrt{ax^2+bx+c} = xt+\sqrt{c}$，则

$$x = \frac{2\sqrt{c}\,t-b}{a-t^2}, \quad \sqrt{ax^2+bx+c} = \frac{\sqrt{c}\,t^2-bt+\sqrt{c}\,a}{a-t^2},$$

$$dx = \frac{2(\sqrt{c}\,t^2-bt+\sqrt{c}\,a)}{(a-t^2)^2}\,dt.$$

③ 若 $\Delta = b^2-4ac>0$，则 ax^2+bx+c 可分解为

$$a(x-\lambda)(x-u) \qquad (\lambda\neq u)$$

可令 $\sqrt{ax^2+bx+c} = t(x-\lambda)$，则

$$a(x-\lambda)(x-u) = ax^2+bx+c = t^2(x-\lambda)^2,$$

$$a(x-u) = t^2(x-\lambda)$$

于是 $x = \dfrac{-au+\lambda t^2}{t^2-a}$，$\sqrt{ax^2+bx+c} = \dfrac{a(\lambda-u)t}{t^2-a}$，

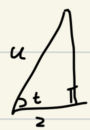
$$dx = \frac{2a(u-\lambda)}{(t^2-a)^2}\,dt.$$

例1. $\displaystyle\int \frac{1}{x\sqrt{x^2-2x-3}}\,dx$

解：(方法1). $x^2-2x-3 = (x-1)^2-4 = (x-1)^2-2^2$.

$$\int \frac{1}{x\sqrt{x^2-2x-3}}\,dx = \int \frac{1}{(u+1)\sqrt{u^2-2^2}}\,du \qquad (u=x-1)$$

$$= \int \frac{1}{(2\sec t+1)\cdot 2\tan t}\cdot 2\sec t\tan t\,dt \qquad (u=2\sec t)$$



$\sqrt{u^2-2^2} = 2\tan t$

$du = 2\sec t\tan t\,dt$

$$= \int \frac{1}{2+\cos t}\,dt$$

$$= \int \frac{1}{2+\frac{1-s^2}{1+s^2}} \cdot \frac{2}{1+s^2} ds$$

$$= 2\int \frac{1}{s^2+3} ds$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{s}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{\sqrt{x^2-2x-3}}{\sqrt{3}(x+1)} + C.$$

$(s = \tan\frac{t}{2}$

$$= \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} = \frac{\sin\frac{t}{2}\cos\frac{t}{2}}{\cos^2\frac{t}{2}}$$

$$= \frac{\frac{1}{2}\sin t}{\frac{1}{2}(1+\cos t)} = \frac{\sin t}{1+\cos t}$$

$$= \frac{\frac{\sqrt{u^2-4}}{u}}{1+\frac{2}{u}} = \frac{\sqrt{u^2-4}}{u+2}$$

$$= \frac{\sqrt{x^2-2x-3}}{x+1}$$

(方法2). Euler变换. 令 $\sqrt{x^2-2x-3} = t-x$.

则 $x = \frac{t^2-1}{2(t-1)}$, $\sqrt{x^2-2x-3} = t - \frac{t^2-1}{2(t-1)} = \frac{t^2-2t-3}{2(t-1)}$,

$$dx = \frac{t^2-2t-3}{2(t-1)^2} dt.$$

于是 $\int \frac{1}{x\sqrt{x^2-2x-3}} dx = \int \frac{2(t-1)}{t^2-1} \cdot \frac{2(t-1)}{t^2-2t-3} \cdot \frac{t^2-2t-3}{2(t-1)^2} dt$

$$= \int \frac{2}{t^2+3} dt$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{x+\sqrt{x^2-2x-3}}{\sqrt{3}} + C.$$