§8.2 换元积分法与分部积分法

目标：将求困难的不定积分问题转换为求简单的(例如基本积分公式)不定积分。

一、换元积分法

(复合函数求导运算法则：若 G 可导，φ 可导，则 $G(\varphi(x))$ 可导，并且

$$\frac{d}{dx}G(\varphi(x)) = G'(\varphi(x))\cdot\varphi'(x).)$$

<u>定理1</u>（第一换元积分法理论基础）

设 (1) g 在区间 I 上有定义；（外部函数）

(2) φ 在区间 J 上可导；（内部函数）

(3) $\varphi(J)\subset I$.（保证了 g 和 φ 可复合为 $g(\varphi(x))$）

若 <u>g 在区间 I 上存在原函数 G</u>，则

$$g(\varphi(x))\cdot\varphi'(x)$$

在区间 J 上存在原函数，且

$$\int g(\varphi(x))\cdot\varphi'(x)\,dx = G(\varphi(x)) + C.$$

证：由于 $\frac{d}{dt}G(t)=g(t)$，$t\in I$，并且对 $\forall x\in J$，有 $\varphi(x)\in I$，从而

$$\frac{d}{dx}G(\varphi(x)) = G'(\varphi(x))\cdot\varphi'(x)$$

$$= g(\varphi(x))\cdot\varphi'(x),$$

于是 $\int g(\varphi(x))\cdot\varphi'(x)\,dx = G(\varphi(x)) + C.$

$\boxed{\text{第一换元积分法}}$ 求不定积分 $\int f(x)\,dx$

Step1. 凑微分. 被积函数 $f(x)$ 可分解为两个因子之积:

$$f(x) = g(\varphi(x)) \cdot \varphi'(x) \qquad \left(\int f(x)\,dx = \int g(\varphi(x)) \cdot \varphi'(x)\,dx \right)$$

令 $t = \varphi(x)$, 则 $(dt = \varphi'(x)\,dx)$

$$f(x)\,dx = g(\varphi(x)) \cdot \varphi'(x)\,dx = g(t)\,dt. \quad (\text{形式上})$$

Step2. 能容易地求出不定积分 $\int g(t)\,dt = G(t) + C$.

Step3. 由定理 1,

$$\int f(x)\,dx = \int g(\varphi(x)) \cdot \varphi'(x)\,dx = \int g(t)\,dt = G(t) + C = G(\varphi(x)) + C.$$

例1. $\int \tan x\,dx$

解: $\tan x\,dx = \frac{\sin x}{\cos x}\,dx = \frac{-(\cos x)'}{\cos x}\,dx$. 令 $t = \cos x$, $g(t) = -\frac{1}{t}$, 则

$$\tan x\,dx = g(t)\,dt,$$

由于 $\int g(t)\,dt = \int \left(-\frac{1}{t}\right)dt = -\ln|t| + C$, 则

$$\int \tan x\,dx = -\ln|\cos x| + C.$$

例2. $\int \frac{dx}{x^2+a^2}$ $\left(\int \frac{1}{1+x^2}\,dx = \arctan x \right)$

解: $\frac{1}{x^2+a^2}\,dx = \frac{1}{a^2} \cdot \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx = \frac{1}{a}\left(\frac{x}{a}\right)' \cdot \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx$.

令 $t = \frac{x}{a}$, $g(t) = \frac{1}{a} \cdot \frac{1}{1+t^2}$, 则

$$\frac{1}{x^2+a^2}\,dx = g(t)\,dt.$$

由于 $\int g(t)\,dt = \int \frac{1}{a} \cdot \frac{1}{1+t^2}\,dt = \frac{1}{a}\arctan t + C$, 则

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C.$$

(熟练法)

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot \underbrace{a\,d\left(\frac{x}{a}\right)}_{=dx}$$

$$\underline{\underline{t=\frac{x}{a}}}\ \frac{1}{a^2}\int \frac{a}{1+t^2}dt = \frac{1}{a}\int \frac{1}{1+t^2}dt = \frac{1}{a}\arctan t + C$$

$$= \frac{1}{a}\arctan\frac{x}{a} + C.$$

例13. $\int \frac{dx}{\sqrt{x^2-a^2}}$ $(a>0)$ $\left(\int \frac{1}{\sqrt{x^2-1}}dx = \operatorname{arcsin} x + C\right)$

解: $$\int \frac{dx}{\sqrt{x^2-a^2}} = \frac{1}{a}\int \frac{1}{\sqrt{\left(\frac{x}{a}\right)^2-1}}dx = \int \frac{1}{\sqrt{\left(\frac{x}{a}\right)^2-1}}d\left(\frac{x}{a}\right)$$

$$\underline{\underline{t=\frac{x}{a}}}\ \int \frac{1}{\sqrt{t^2-1}}dt = \operatorname{arcsin} t + C = \operatorname{arcsin}\frac{x}{a} + C.$$

例14. $\int \frac{dx}{x^2-a^2}$ $(a\neq 0)$

$$= \int \frac{1}{(x-a)(x+a)}dx = \int \left(\frac{1}{x-a} - \frac{1}{x+a}\right)\cdot \frac{1}{2a}dx$$

$$= \frac{1}{2a}\int \frac{1}{x-a}dx - \frac{1}{2a}\int \frac{1}{x+a}dx$$

$$= \frac{1}{2a}\int \frac{1}{x-a}d(x-a) - \frac{1}{2a}\int \frac{1}{x+a}d(x+a)$$

第一换元积分法

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

例15 $\int \sec x\,dx$

解: 1°. $$\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{(\sin x)'}{1-\sin^2 x}dx$$

$$\underline{\underline{t=\sin x}}\ \int \frac{1}{1-t^2}dt = -\int \frac{1}{t^2-1}dt = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C.$$

2°. $\int \sec x \, dx$

$$= \int \frac{\sec x\,(\tan x + \sec x)}{\tan x + \sec x}\, dx$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x}\, dx$$

$$= \int \frac{(\sec x + \tan x)'}{\tan x + \sec x}\, dx$$

$$\underline{\underline{t = \sec x + \tan x}} \int \frac{1}{t}\, dt = \ln|t| + C = \ln|\sec x + \tan x| + C.$$

(验证 $2\ln|\sec x + \tan x| = \ln|\sec x + \tan x|^2$

$$= \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right|^2$$

$$= \ln\frac{(1+\sin x)^2}{\cos^2 x}$$

$$= \ln\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}$$

$$= \ln\left|\frac{1+\sin x}{1-\sin x}\right|.$$
)

定理2（第二换元积分法理论基础）

设 (1) f 在区间 I 上有定义；

(2) φ 在区间 J 上可导，

(3) $\varphi(J)=I$，并且 φ 在区间 J 上存在反函数

$$t=\varphi^{-1}(x),\ x\in I$$

① 等价于 $\varphi: J\to I$ 是一一映射

② 例：$\varphi(J)=I$ 并且 $\varphi: J\to I$ 是一个严格单调函数

若 f 在区间 I 存在原函数（存在但不容易求出），则 $f(\varphi(t))\cdot\varphi'(t)$ 在 J 上也存在原函数 G，（容易求出）

并且，

$$\int f(x)dx = G(\varphi^{-1}(x)) + C.$$

证：设 F 是 f 在区间 I 上的一个原函数，对 $\forall t\in J$，从而

$$x=\varphi(t)\in I,$$

由条件(2)可知，$F\circ\varphi$ 在区间 J 上可导，并且

$$\frac{d}{dt}F(\varphi(t)) = F'(\varphi(t))\cdot\varphi'(t)$$

$$= f(\varphi(t))\cdot\varphi'(t)$$

所以 $F\circ\varphi$ 是 $f(\varphi(t))\cdot\varphi'(t)$ 在区间 J 上的一个原函数，令 $G=F\circ\varphi$，

从而 $G\circ\varphi^{-1}=(F\circ\varphi)\circ\varphi^{-1}=F\circ(\varphi\circ\varphi^{-1})=F$，

$$\int f(x)dx = F(x)+C = (G\circ\varphi^{-1})(x)+C = G(\varphi^{-1}(x))+C.$$

$\boxed{\text{第二换元积分法}}$ 求不定积分 $\int f(x)dx$

Step 1. 用可逆可导函数 $x=\varphi(t)$ 代入，将被积表达式 $f(x)dx$ 化为

$$f(\varphi(t))\cdot\varphi'(t)dt$$

Step2. 容易求出 $f(\varphi(t))\cdot\varphi'(t)$ 的不定积分

$$\int f(\varphi(t))\cdot\varphi'(t)\,dt = G(t)+C.$$

Step3. 由定理2. 将 $t=\varphi^{-1}(x)$ 代入. 可得

$$\int f(x)dx = G(\varphi^{-1}(x))+C.$$

注: 定理2中条件"f 在区间 I 存在原函数"不可缺少.

反例. $f(x)=\begin{cases}1, & x\in(0,1]\\ 0, & x=0\end{cases}$, f 在 $[0,1]$ 上存在第一类间断点 $x=0$.

所以 f 在 $[0,1]$ 不存在原函数.

作变量代换 $x=\varphi(t)=t^3$, $t\in[0,1]$, 则

$$t=\varphi^{-1}(x)=x^{\frac{1}{3}},\quad x\in[0,1].$$

$f(\varphi(t))\cdot\varphi'(t) = 3t^2$. $t\in[0,1]$

$$\int 3t^2\,dt = t^3+C.$$

若定理2中不考虑条件"f 在区间 I 存在原函数", 则由结论可得

$$\int f(x)dx = t^3+C = x+C.$$

例6. $\int \frac{du}{\sqrt{u}+\sqrt[3]{u}}$.

解: $f(u)=\frac{1}{\sqrt{u}+\sqrt[3]{u}}$ 定义域为 $(0,+\infty)$. 从而 f 在 $(0,+\infty)$ 上存在原函数.

令 $t=u^{\frac{1}{6}}$, $u\in(0,+\infty)$, 则

$u=t^6$, $t\in(0,+\infty)$, $f(u)=\frac{1}{\sqrt{u}+\sqrt[3]{u}}=\frac{1}{t^3+t^2}$, $du=6t^5dt$, 于是

$t^3+1=(t+1)(t^2-t+1)$

$$f(u)\,du=\frac{1}{t^3+t^2}\cdot 6t^5dt=\frac{6t^3}{t+1}dt$$
$$=\frac{6(t^3+1)-6}{t+1}dt$$
$$=\frac{6(t+1)(t^2-t+1)-6}{t+1}dt$$
$$=\left(6t^2-6t+6-\frac{6}{t+1}\right)dt$$

由于 $\int\left(6t^2-6t+6-\frac{6}{t+1}\right)dt$
$$=2t^3-3t^2+6t-6\ln|t+1|+C$$
则 $\int\frac{1}{\sqrt{u}+\sqrt[3]{u}}du=2t^3-3t^2+6t-6\ln|t+1|+C$
$$=2u^{\frac{1}{2}}-3t^{\frac{1}{3}}+6u^{\frac{1}{6}}-6\ln|u^{\frac{1}{6}}+1|+C.$$

命题. 设

(1) $\mathbb{R}$上的点集D关于原点对称，并且 $D\cap[0,+\infty)$ 是一个区间；

($\Rightarrow D\cap(-\infty,0)$ 也是一个区间)

(2) f 是定义在D上的偶(奇)函数；

(3) f 在区间 $D\cap[0,+\infty)$ 上存在原函数 F.

将F作奇(偶)延拓，即令

$$\tilde{F}(x)=\begin{cases}F(x), & x\in D\cap[0,+\infty)\\ -F(-x), & x\in D\cap(-\infty,0)\end{cases}\rightarrow\text{是奇函数.}$$

$\left(\tilde{F}(x)=\begin{cases}F(x), & x\in D\cap[0,+\infty)\\ F(-x), & x\in D\cap(-\infty,0)\end{cases}\rightarrow\text{偶函数}\right)$, 则 $\tilde{F}'(x)=f(x),\ x\in D.$

证：不妨设 f 在 $D\cap[0,+\infty)$ 上是偶函数，

$$\tilde{F}(x)=\begin{cases} F(x), & x\in D\cap[0,+\infty) \\ -F(-x), & x\in D\cap(-\infty,0). \end{cases}$$

$(F'(x)=f(x),\ x\in D\cap[0,+\infty))$

当 $x\in D\cap(-\infty,0)$ 时，

$$\tilde{F}'(x)=(-F(-x))'=(-1)\cdot F'(-x)\cdot(-1)=F'(-x)=f(-x)\overset{f\text{在}D\text{偶}}{=}f(x),$$

综上，对 $\forall x\in D$，都有

$$\tilde{F}'(x)=f(x).$$

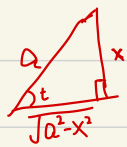
例. 若 f 在 $(-\infty,-a)\cup(a,+\infty)$ 上有定义，$(a>0)$，且 f 是偶(奇)函数，

要求不定积分 $\int f(x)\,dx$，只需要求 f 在 $(a,+\infty)$ 上的原函数 F.

例1. $\int\sqrt{a^2-x^2}\,dx \quad (a>0)$

解：$f(x)=\sqrt{a^2-x^2}$ 的定义域为 $[-a,a]$，则 f 在 $[-a,a]$ 上存在原函数，

令 $x=\underbrace{a\sin t}_{\varphi(t)}$，$t\in[0,\frac{\pi}{2}]$，则



$\cos t=\dfrac{\sqrt{a^2-x^2}}{a}$

$\sin t=\dfrac{x}{a}$. $x=a\sin t$

$t=\arcsin\dfrac{x}{a}$

$t=\arcsin\frac{x}{a}$，$f(x)=\sqrt{a^2-x^2}=a\cos t$，

$dx=a\cos t\,dt$，

$\sqrt{a^2-x^2}\,dx=\underline{a^2\cos^2 t}\,dt$.

由于 $\int a^2\cos^2 t\,dt=\frac{a^2}{2}\int(1+\cos 2t)\,dt=\frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)+C$

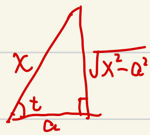
$$=\frac{a^2}{2}(t+\sin t\cos t)+C,$$

所以，$\int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\left(\arcsin\frac{x}{a}+\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right)+C$

$$=\frac{1}{2}\left(a^2\arcsin\frac{x}{a}+x\sqrt{a^2-x^2}\right)+C.$$

例8 $\int \frac{1}{\sqrt{x^2-a^2}}dx \quad (a>0)$

解: $f(x)=\frac{1}{\sqrt{x^2-a^2}}$ 的存在域为 $(-\infty,-a)\cup(a,+\infty)$，则 f 在 $(a,+\infty)$ 上存在原函数.



$\cos t=\frac{a}{x}$, $x=a\sec t$

$\tan t=\frac{\sqrt{x^2-a^2}}{a}$,

$\frac{1}{\sqrt{x^2-a^2}}=\frac{1}{a\tan t}$

令 $x=a\sec t$, $t\in(0,\frac{\pi}{2})$，则

$f(x)=\frac{1}{\sqrt{x^2-a^2}}=\frac{1}{a\tan t}$,

$dx=a\sec t\tan t\,dt$.

$\frac{1}{\sqrt{x^2-a^2}}dx=\frac{a\sec t\tan t}{a\tan t}dt=\sec t\,dt$,

由于 $\int\sec t\,dt=\ln|\sec t+\tan t|+C$，则

$\int\frac{1}{\sqrt{x^2-a^2}}dx=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c=\ln|x+\sqrt{x^2-a^2}|-\ln a+C$

$=\ln|x+\sqrt{x^2-a^2}|+C$.

奇延拓,

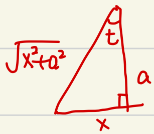
$$F(x)=\begin{cases}\ln|x+\sqrt{x^2-a^2}|, & x>a.\\ -\ln|-x+\sqrt{x^2-a^2}|, & x<-a\end{cases}$$

$-\ln|-x+\sqrt{x^2-a^2}|=\ln\frac{1}{|\sqrt{x^2-a^2}-x|}=\ln\left|\frac{x+\sqrt{x^2-a^2}}{(x-\sqrt{x^2-a^2})(x+\sqrt{x^2-a^2})}\right|$

$=\ln\left|\frac{x+\sqrt{x^2-a^2}}{x^2-x^2+a^2}\right|=\ln\left|\frac{x+\sqrt{x^2-a^2}}{a^2}\right|=\ln|x+\sqrt{x^2-a^2}|-\ln a^2$

例9. $\int \frac{1}{(x^2+a^2)^2} dx \quad (a>0)$

解：$f(x)=\frac{1}{(x^2+a^2)^2}$ 存在域为 $\mathbb{R}$. 则 f 在 $[0,+\infty)$ 上存在原函数，令 $x=a\tan t$, $t\in[0,\frac{\pi}{2})$,



$\cos t=\frac{a}{\sqrt{x^2+a^2}}$,

$\sqrt{x^2+a^2}=a\sec t$,

$(x^2+a^2)^2=a^4\sec^4 t$.

$\tan t=\frac{x}{a}$, $x=a\tan t$.

则 $f(x)=\frac{1}{(x^2+a^2)^2}=\frac{1}{a^4\sec^4 t}$,

$dx=a\sec^2 t\, dt$.

$$\frac{1}{(x^2+a^2)^2}dx=\frac{a\sec^2 t}{a^4\sec^4 t}dt=\frac{1}{a^3\sec^2 t}dt=\frac{1}{a^3}\cos^2 t\, dt.$$

由于 $\int \frac{1}{a^3}\cos^2 t\, dt=\frac{1}{2a^3}\int(1+\cos 2t)dt=\frac{1}{2a^3}\left(t+\frac{1}{2}\sin 2t\right)+C$

$$=\frac{1}{2a^3}(t+\sin t\cos t)+C,$$

所以 $\int\frac{1}{(x^2+a^2)^2}dx=\frac{1}{2a^3}\left(\arctan\frac{x}{a}+\frac{x}{\sqrt{x^2+a^2}}\cdot\frac{a}{\sqrt{x^2+a^2}}\right)+C$

$$=\frac{1}{2a^3}\left(\arctan\frac{x}{a}+\frac{ax}{x^2+a^2}\right)+C$$

总结：例7-例9.

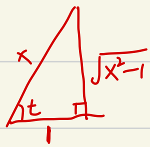
被积函数中出现 $\sqrt{a^2\pm x^2}$，$\sqrt{x^2\pm a^2}$，x^2+a^2 时，

可以尝试利用例7-例9中的辅助直角三角形技巧.

例10. $\int\frac{1}{x^2\sqrt{x^2-1}}dx$

解：$f(x)=\frac{1}{x^2\sqrt{x^2-1}}$ 的存在域为 $(-\infty,-1)\cup(1,+\infty)$. 则 f 在 $(1,+\infty)$ 上存在原函数.

方法1.



$\cos t=\frac{1}{x}$，$x=\sec t$，

$\tan t=\frac{\sqrt{x^2-1}}{1}=\sqrt{x^2-1}$，

令 $x=\sec t$，$t\in(0,\frac{\pi}{2})$．则

$$f(x)=\frac{1}{x^2\sqrt{x^2-1}}=\frac{1}{\sec^2 t\cdot\tan t}，\quad dx=\sec t\cdot\tan t\,dt．$$

$$\frac{1}{x^2\sqrt{x^2-1}}dx=\frac{\sec t\cdot\tan t}{\sec^2 t\cdot\tan t}dt=\frac{1}{\sec t}dt=\cos t\,dt$$

由于 $\int\cos t\,dt=\sin t+C$，则

$$\int\frac{1}{x^2\sqrt{x^2-1}}dx=\frac{\sqrt{x^2-1}}{x}+C$$

$$\left(\frac{1}{x}\right)'=-\frac{1}{x^2}$$

方法2.
(第一换元积分法)

$$\int\frac{1}{x^2\sqrt{x^2-1}}dx=-\int\left(-\frac{1}{x^2}\right)\cdot\frac{1}{\sqrt{x^2-1}}dx=-\int\frac{1}{\sqrt{x^2-1}}d\left(\frac{1}{x}\right)$$

$$(t^2)'=2t$$

$$\xrightarrow[x=\frac{1}{t}]{t=\frac{1}{x}}-\int\frac{1}{\sqrt{\frac{1}{t^2}-1}}dt=-\int\frac{1}{\sqrt{\frac{1-t^2}{t^2}}}dt=-\int\frac{t}{\sqrt{1-t^2}}dt$$

$$=-\int\frac{1}{\sqrt{1-t^2}}\cdot\frac{1}{2}d(t^2)=\frac{1}{2}\int\frac{1}{\sqrt{1-t^2}}d(1-t^2)$$

$$\xlongequal{u=1-t^2}\frac{1}{2}\int u^{-\frac{1}{2}}du=u^{\frac{1}{2}}+C=\sqrt{1-t^2}+C$$

$$=\sqrt{1-\frac{1}{x^2}}+C=\frac{\sqrt{1-x^2}}{x}+C．$$

二. 分部积分法

定理3 (分部积分法理论基础)

若 $u(x)$ 和 $v(x)$ 可导, 并且不定积分 $\int u'(x)v(x)dx$ 存在, 则 $\int u(x)v'(x)dx$ 也存在, 并且

$$\int u(x)\underbrace{v'(x)dx}_{dv} = u(x)v(x) - \int \underbrace{u'(x)\,dx}_{du}\,v(x)$$

(简记为 $\int udv = uv - \int vdu$)

例11. $\int x\cos x\,dx$

解: $\int x\cos x\,dx = \int x(\sin x)'dx = \int x\,d\sin x = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C.$

$\int x\cos x\,dx = \int (\frac{1}{2}x^2)'\cos x\,dx = \frac{1}{2}x^2\cos x - \int \frac{1}{2}x^2\,d\cos x$

$= \frac{1}{2}x^2\cos x + \int \frac{1}{2}x^2\sin x\,dx$

例12. $\int \underline{\arctan x}\,\underline{dx}$

$= x\arctan x - \int x\,d\arctan x$

$= x\arctan x - \int \frac{x}{1+x^2}dx \qquad [\frac{1}{2}(1+x^2)]'$

$= x\arctan x - \frac{1}{2}\int \frac{1}{1+x^2}d(1+x^2)$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

例13. $\int x^3\ln x\,dx$

$= \int (\frac{1}{4}x^4)'\ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^4\,d\ln x = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3dx$

$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$

例. $\int \underline{\ln x}\, d\underline{x}$

$= x\ln x - \int x\, d\ln x = x\ln x - \int 1\, dx = x\ln x - x + C.$

例14. $\int x^2 e^{-x} dx$ $\qquad (-e^{-x})' = e^{-x}$

$= \int x^2 (-e^{-x})' dx = -x^2 e^{-x} + \int e^{-x} d(x^2)$

$= -x^2 e^{-x} + 2\int x e^{-x} dx$

$= -x^2 e^{-x} + 2\int x (-e^{-x})' dx$

$= -x^2 e^{-x} - 2x e^{-x} + 2\int e^{-x} dx$

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$

总结 $\int uv\, dx$

"反对幂三指，后者先凑入"

例： $\int x^2 \underbrace{e^{-x} dx}_{d(-e^{-x})}$ $\qquad \int x^3 \ln x\, dx = \int \ln x\, d(\frac{1}{4}x^4).$

$\int x \cos x\, dx = \int x\, d\sin x.$

"反对幂指三"

例15. $I_1 = \int e^{ax} \cos bx\, dx$ ($a \neq 0$). $\quad I_2 = \int e^{ax} \sin bx\, dx$ ($a \neq 0$)

解: $I_1 = \int e^{ax} \cos bx\, dx$

$= \frac{1}{a} \int \cos bx\, d e^{ax}$

$= \frac{1}{a} e^{ax} \cos bx - \frac{1}{a} \int e^{ax} d\cos bx$

$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx\, dx$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \cdot \frac{1}{a} \int \sin bx \, d e^{ax}$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} \left(e^{ax} \sin bx - \int e^{ax} d \sin bx \right)$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int \underbrace{e^{ax} \cos bx \, dx}$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I_1$$

所以 $I_1 = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I_1$，则

$$I_1 = \frac{a \cos bx + b \sin bx}{a^2 + b^2} e^{ax} + C.$$

同理，可得 $I_2 = \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax} + C.$

专题：递推（迭代）公式法

思路：设法降低 n.

例16 计算 $I_n = \int \frac{x^n}{\sqrt{1-x^2}}\,dx$.

$\left(\sqrt{1-x^2}\right)' = -\frac{x}{\sqrt{1-x^2}}$

解：$I_n = \int \frac{x^n}{\sqrt{1-x^2}}\,dx = \int x^{n-1}\cdot \frac{x}{\sqrt{1-x^2}}\,dx$

$$= -\int x^{n-1}\,d\sqrt{1-x^2} = -x^{n-1}\sqrt{1-x^2} + \int \sqrt{1-x^2}\,d x^{n-1}$$

$$= -x^{n-1}\sqrt{1-x^2} + (n-1)\int \underline{x^{n-2}\sqrt{1-x^2}}\,dx$$

$$= -x^{n-1}\sqrt{1-x^2} + (n-1)\int \frac{x^{n-2}\cdot(1-x^2)}{\sqrt{1-x^2}}\,dx$$

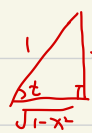
$$= -x^{n-1}\sqrt{1-x^2} + (n-1)\underbrace{\int \frac{x^{n-2}}{\sqrt{1-x^2}}\,dx}_{I_{n-2}} - (n-1)\underbrace{\int \frac{x^{n}}{\sqrt{1-x^2}}\,dx}_{I_n}$$

$$= -x^{n-1}\sqrt{1-x^2} + (n-1)I_{n-2} - (n-1)I_n.$$

所以 $I_n = -\frac{1}{n}x^{n-1}\sqrt{1-x^2} + \left(1-\frac{1}{n}\right)I_{n-2}$. $(n\geq 3)$

当 $n=1$ 时，$I_1 = \int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + C$.

当 $n=2$ 时，$I_2 = \int \frac{x^2}{\sqrt{1-x^2}}\,dx$



$\sin t = x$

$\cos t = \sqrt{1-x^2}$

令 $x = \sin t$，$t\in[0,\frac{\pi}{2})$，则

$\frac{x^2}{\sqrt{1-x^2}} = \frac{\sin^2 t}{\cos t}$，$dx = \cos t\,dt$，

$\frac{x^2}{\sqrt{1-x^2}}\,dx = \sin^2 t\,dt = \underline{\frac{1}{2}(1-\cos 2t)\,dt}$

由于 $\int \frac{1}{2}(1-\cos 2t)\,dt = \frac{1}{2}t - \frac{1}{4}\sin 2t + C = \frac{1}{2}t - \frac{1}{2}\sin t\cos t + C$,

所以，$I_2 = \int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C$.

$(a>0)$

例. 计算 $I_n=\int \frac{1}{(x^2+a^2)^n}dx$.   n=2时. 例9

解: $I_n=\int \frac{1}{(x^2+a^2)^n}dx=\frac{1}{a^2}\int \frac{a^2}{(x^2+a^2)^n}dx=\frac{1}{a^2}\int \frac{(x^2+a^2)-x^2}{(x^2+a^2)^n}dx$

$=\frac{1}{a^2}\int \frac{1}{(x^2+a^2)^{n-1}}dx-\frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^n}dx$

$[(x^2+a^2)^{1-n}]'$

$=(1-n)\cdot\frac{2x}{(x^2+a^2)^n}$

$=2(1-n)\cdot\frac{x}{(x^2+a^2)^n}$

$=\frac{1}{a^2}I_{n-1}-\frac{1}{a^2}\int x\cdot\frac{x}{(x^2+a^2)^n}dx$

$=\frac{1}{a^2}I_{n-1}-\frac{1}{a^2}\int x\cdot\frac{1}{2(1-n)}d[(x^2+a^2)^{1-n}]$

$=\frac{1}{a^2}I_{n-1}+\frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}}-\underbrace{\int\frac{1}{(x^2+a^2)^{n-1}}dx}_{I_{n-1}}\right]$

$=\frac{1}{a^2}I_{n-1}+\frac{1}{2(n-1)a^2}\cdot\frac{x}{(x^2+a^2)^{n-1}}-\frac{1}{2(n-1)a^2}I_{n-1}$

$=\frac{1}{2(n-1)a^2}\cdot\frac{x}{(x^2+a^2)^{n-1}}+\frac{2n-3}{2a^2(n-1)}I_{n-1}.$

当 $n=1$ 时, $I_1=\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\arctan\frac{x}{a}+C.$

$I_2=\frac{1}{2\cdot1\cdot a^2}\cdot\frac{x}{(x^2+a^2)^1}+\frac{1}{2a^2\cdot1}\cdot\frac{1}{a}\arctan\frac{x}{a}+C$

$=\frac{1}{2a^3}\left(\arctan\frac{x}{a}+\frac{ax}{x^2+a^2}\right)+C.$ 同例9.

Ex4. (1) $I_n = \int \tan^n x\,dx$, $n=2,3,\cdots$. 则

$$I_n = \frac{1}{n-1}\tan^{n-1}x - I_{n-2}.$$

(2) $I(m,n) = \int \cos^m x \sin^n x\,dx$, 则

$\cos x$ 的次数　$\sin x$ 的次数.

$$I(m,n) = \frac{\cos^{m-1}x \cdot \sin^{n+1}x}{m+n} + \frac{m-1}{m+n} I(m-2,n)$$

$$= -\frac{\cos^{m+1}x \cdot \sin^{n-1}x}{m+n} + \frac{n-1}{m+n} I(m,n-2)$$

证: (1) $I_n = \int \tan^n x\,dx = \int \tan^{n-2}x \tan^2 x\,dx$

$$= \int \tan^{n-2}x \cdot \frac{\sin^2 x}{\cos^2 x}\,dx$$

$$= \int \tan^{n-2}x \cdot \frac{1-\cos^2 x}{\cos^2 x}\,dx$$

$$= \int \tan^{n-2}x \cdot \sec^2 x\,dx - \int \tan^{n-2}x\,dx$$

$$= \int \tan^{n-2}x\,d\tan x - I_{n-2}$$

第一换元积分法

$$= \frac{1}{n-1}\tan^{n-1}x - I_{n-2}.$$

(2) ① $I(m,n) = \int \cos^m x \sin^n x\,dx$　　$(\sin^{n+1}x)' = (n+1)\cdot\sin^n x\cdot\cos x$

$$= \int \cos^{m-1}x \cdot \cos x \sin^n x\,dx$$

$$= \frac{1}{n+1}\int \cos^{m-1}x\,d\sin^{n+1}x$$

$(m-1)\cos^{m-2}x\cdot(-\sin x)$

$$= \frac{1}{n+1}\cos^{m-1}x\sin^{n+1}x - \frac{1}{n+1}\int \sin^{n+1}x\,d\cos^{m-1}x$$

$$= \frac{1}{n+1}\cos^{m-1}x\sin^{n+1}x + \frac{m-1}{n+1}\int \sin^{n+2}x\cos^{m-2}x\,dx$$

$$= \frac{1}{n+1}\cos^{m-1}x\sin^{n+1}x + \frac{m-1}{n+1}\int \sin^n x\cdot(1-\cos^2 x)\cos^{m-2}x\,dx$$

$$= \frac{1}{n+1}\cos^{m-1}x\sin^{n+1}x + \frac{m-1}{n+1}\underbrace{\int \sin^n x\cos^{m-2}x\,dx}_{=I(m-2,n)}$$

$$- \frac{m-1}{n+1}\underbrace{\int \sin^n x\cos^m x\,dx}_{I(m,n)}$$

$$= \frac{1}{n+1}\cos^{m-1}x\sin^{n+1}x + \frac{m-1}{n+1}I(m-2,n) - \frac{m-1}{n+1}I(m,n)$$

所以, $I(m,n) = \frac{1}{n+m}\cos^{m-1}x\sin^{n+1}x + \frac{m-1}{n+m}I(m-2,n)$.

② $I(m,n) = \int \cos^m x\sin^n x\,dx$ $\quad (\cos^{m+1}x)' = (m+1)\cdot\cos^m x\cdot(-\sin x)$

$$= \int \sin^{n-1}x\cdot\underline{\sin x\cos^m x\,dx}$$

$$= -\frac{1}{m+1}\int \sin^{n-1}x\,d\cos^{m+1}x$$

$$= -\frac{1}{m+1}\cos^{m+1}x\sin^{n-1}x + \frac{1}{m+1}\int \cos^{m+1}x\,\underbrace{d\sin^{n-1}x}_{(n-1)\sin^{n-2}x\cdot\cos x\,dx}$$

$$= -\frac{1}{m+1}\cos^{m+1}x\sin^{n-1}x + \frac{n-1}{m+1}\int \cos^{m+2}x\sin^{n-2}x\,dx$$

$$= -\frac{1}{m+1}\cos^{m+1}x\sin^{n-1}x + \frac{n-1}{m+1}\int \cos^m x\cdot(1-\sin^2 x)\cdot\sin^{n-2}x\,dx$$

$$= -\frac{1}{m+1}\cos^{m+1}x\sin^{n-1}x + \frac{n-1}{m+1}\underbrace{\int \cos^m x\sin^{n-2}x\,dx}_{=I(m,n-2)}$$

$$- \frac{n-1}{m+1}\underbrace{\int \cos^m x\sin^n x\,dx}_{I(m,n)}$$

$$= -\frac{1}{m+1}\cos^{m+1}x\sin^{n-1}x + \frac{n-1}{m+1}I(m,n-2) - \frac{n-1}{m+1}I(m,n)$$

所以 $I(m,n) = -\frac{\cos^{m+1}x\sin^{n-1}x}{m+n} + \frac{n-1}{m+n}I(m,n-2)$.

Ex6 (1) $I_n = \int x^n e^{kx} dx \quad (k \neq 0, n \in \mathbb{N}_+)$.

$$= \frac{1}{k} \int x^n d e^{kx}$$

$$= \frac{1}{k} x^n e^{kx} - \frac{n}{k} \underbrace{\int e^{kx} x^{n-1} dx}_{= I_{n-1}}$$

$$= \frac{1}{k} x^n e^{kx} - \frac{n}{k} I_{n-1}.$$

(2) $I_n = \int (\ln x)^n dx$

$$= x(\ln x)^n - \int x \, d(\ln x)^n$$

$$= x(\ln x)^n - \int n \cdot (\ln x)^{n-1} dx$$

$$= x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

$$= x(\ln x)^n - n I_{n-1}.$$

(3) $I_n = \int (\arcsin x)^n dx$

$$= x(\arcsin x)^n - \int x \, d(\arcsin x)^n \qquad \left(\sqrt{1-x^2}\right)' = -\frac{x}{\sqrt{1-x^2}}$$

$$= x(\arcsin x)^n - n \int (\arcsin x)^{n-1} \cdot \underline{\frac{x}{\sqrt{1-x^2}} dx}$$

$$= x(\arcsin x)^n + n \int (\arcsin x)^{n-1} d\sqrt{1-x^2}$$

$$= x(\arcsin x)^n + n(\arcsin x)^{n-1}\sqrt{1-x^2} - n \int \sqrt{1-x^2} \, d(\arcsin x)^{n-1}$$

$$= x(\arcsin x)^n + n(\arcsin x)^{n-1}\sqrt{1-x^2} - n \int \sqrt{1-x^2} \cdot (n-1) \cdot (\arcsin x)^{n-2} \cdot \frac{1}{\sqrt{1-x^2}} dx$$

$$= x(\arcsin x)^n + n(\arcsin x)^{n-1}\sqrt{1-x^2} - n(n-1) \underbrace{\int (\arcsin x)^{n-2} dx}_{I_{n-2}}$$

(4) $I_n = \int e^{ax}\sin^n x\,dx \quad (a \neq 0)$

$$= \frac{1}{a}\int \sin^n x \, d\,e^{ax}$$

$$= \frac{1}{a} e^{ax}\sin^n x - \frac{1}{a}\int e^{ax}\, \overbrace{d \sin^n x}^{n\cdot \sin^{n-1}x \cdot \cos x}$$

$$= \frac{1}{a} e^{ax}\sin^n x - \frac{n}{a}\int e^{ax}\sin^{n-1}x \cdot \cos x \, dx$$

$$= \frac{1}{a} e^{ax}\sin^n x - \frac{n}{a^2}\int \sin^{n-1}x \cos x \, d\,e^{ax}$$

$$= \frac{1}{a} e^{ax}\sin^n x - \frac{n}{a^2}\left[e^{ax}\sin^{n-1}x\cos x - \int e^{ax} d(\sin^{n-1}x\cos x)\right]$$

$$= \frac{1}{a} e^{ax}\sin^n x - \frac{n}{a^2}\left[e^{ax}\sin^{n-1}x\cos x - \int e^{ax}\left((n-1)\sin^{n-2}x\cos^2 x - \sin^n x\right) dx\right]$$

$$= \frac{1}{a} e^{ax}\sin^n x - \frac{n}{a^2} e^{ax}\sin^{n-1}x\cos x$$
$$+ \frac{n(n-1)}{a^2}\int e^{ax}\sin^{n-2}x\underbrace{\cos^2 x}_{(1-\sin^2 x)}\,dx - \frac{n}{a^2}\underbrace{\int e^{ax}\sin^n x\,dx}_{I_n}$$

$$= \frac{1}{a} e^{ax}\sin^n x - \frac{n}{a^2} e^{ax}\sin^{n-1}x\cos x$$
$$+ \frac{n(n-1)}{a^2}\underbrace{\int e^{ax}\sin^{n-2}x\,dx}_{I_{n-2}} - \frac{n(n-1)}{a^2}\underbrace{\int e^{ax}\sin^n x\,dx}_{I_n}$$
$$- \frac{n}{a^2}\underbrace{\int e^{ax}\sin^n x\,dx}_{I_n}$$

所以 $I_n = \frac{1}{n^2+a^2}\left[a e^{ax}\sin^n x - n e^{ax}\sin^{n-1}x\cos x + n(n-1) I_{n-2}\right]$

总 Ex 1. (20). $I_n=\int \frac{V^n}{\sqrt{u}}dx$. 其中 $u=a_1+b_1x$, $V=a_2+b_2x$, $(b_1, b_2\neq 0)$

解: $I_n=\int\frac{V^n}{\sqrt{u}}dx$ $\frac{1}{\sqrt{a_1+b_1x}}dx$. $(\sqrt{a_1+b_1x})'=\frac{b_1}{2}\cdot\frac{1}{\sqrt{a_1+b_1x}}$

$=\frac{2}{b_1}\int V^n d\sqrt{a_1+b_1x}=\frac{2}{b_1}\int V^n d\sqrt{u}$

$=\frac{2}{b_1}\sqrt{u}V^n-\frac{2}{b_1}\int\sqrt{u}\,d(V^n)$ $d(V^n)=nV^{n-1}\cdot b_2dx$

$=\frac{2}{b_1}\sqrt{u}V^n-\frac{2nb_2}{b_1}\int\sqrt{u}\cdot V^{n-1}dx$

$=\frac{2}{b_1}\sqrt{u}V^n-\frac{2nb_2}{b_1}\int\frac{\sqrt{u}\cdot V^{n-1}\cdot\sqrt{u}}{\sqrt{u}}dx$

$u=a_1+b_1x$, $V=a_2+b_2x$,

$=\frac{2}{b_1}\sqrt{u}V^n-\frac{2nb_2}{b_1}\int\frac{u\,V^{n-1}}{\sqrt{u}}dx$ $x=\frac{u-a_1}{b_1}=\frac{V-a_2}{b_2}$

$\therefore\ u=\frac{b_1}{b_2}(V-a_2)+a_1$

$=\frac{2}{b_1}\sqrt{u}V^n-\frac{2nb_2}{b_1}\int\frac{V^{n-1}}{\sqrt{u}}\left[\frac{b_1}{b_2}V+(a_1-\frac{a_2b_1}{b_2})\right]dx$ $=\frac{b_1}{b_2}V+(a_1-\frac{a_2b_1}{b_2})$

$=\frac{2}{b_1}\sqrt{u}V^n-\frac{2nb_2}{b_1}\int\frac{b_1}{b_2}\cdot\frac{V^n}{\sqrt{u}}dx-\frac{2nb_2}{b_1}\int(a_1-\frac{a_2b_1}{b_2})\cdot\frac{V^{n-1}}{\sqrt{u}}dx$

$=\frac{2}{b_1}\sqrt{u}V^n-2n\underbrace{\int\frac{V^n}{\sqrt{u}}dx}_{I_n}-\frac{2n}{b_1}(a_1b_2-a_2b_1)\underbrace{\int\frac{V^{n-1}}{\sqrt{u}}dx}_{I_{n-1}}$

$=\frac{2}{b_1}\sqrt{u}V^n-2nI_n-\frac{2n}{b_1}(a_1b_2-a_2b_1)I_{n-1}$

所以 $I_n=\frac{2}{(2n+1)b_1}\left[\sqrt{u}V^n-n(a_1b_2-a_2b_1)I_{n-1}\right]$

总 Ex 5. (1) $I_n=\int\frac{1}{\cos^nx}dx$. (2) $I_n=\int\frac{\sin nx}{\sin x}dx$

解: (1) $I_n=\int\frac{1}{\cos^nx}dx=\int\sec^nx\,dx=\int\sec^{n-2}x\cdot\underbrace{\sec^2x\,dx}_{d\tan x}$

$=\int\sec^{n-2}x\,d\tan x$

$= (n-2)\sec^{n-3}x \cdot \sec x \tan x$

$= (n-2)\sec^{n-2}x \tan x$

$$= \sec^{n-2}x \tan x - \int \tan x \, d\sec^{n-2}x$$

$$= \sec^{n-2}x \tan x - (n-2)\int \tan^2 x \sec^{n-2}x \, dx$$

$\frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \sec^2 x - 1$

$$= \sec^{n-2}x \tan x - (n-2)\underbrace{\int \sec^n x \, dx}_{= I_n} + (n-2)\underbrace{\int \sec^{n-2}x \, dx}_{I_{n-2}}$$

$$= \sec^{n-2}x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

所以 $I_n = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1} I_{n-2}.$

(2) $I_n = \int \frac{\sin nx}{\sin x} dx$

① $\sin nx = \sin[(n-1)x + x]$

② $\sin nx = \sin[(n-2)x + 2x]$

$$= \int \frac{\sin[(n-2)x + 2x]}{\sin x} dx$$

$$= \int \frac{\sin(n-2)x \cos 2x + \cos(n-2)x \sin 2x}{\sin x} dx$$

$$= \int \frac{\sin(n-2)x \, (1-2\sin^2 x)}{\sin x} dx + \int \frac{\cos(n-2)x \cdot 2\sin x \cos x}{\sin x} dx$$

$$= \int \frac{\sin(n-2)x}{\sin x} dx - 2\int \sin(n-2)x \sin x \, dx + 2\int \cos(n-2)x \cos x \, dx$$

$$= I_{n-2} + 2\int [\cos(n-2)x \cos x - \sin(n-2)x \sin x] \, dx$$

$$= I_{n-2} + 2\int \cos(n-1)x \, dx$$

$$= I_{n-2} + \frac{2}{n-1}\sin(n-1)x .$$